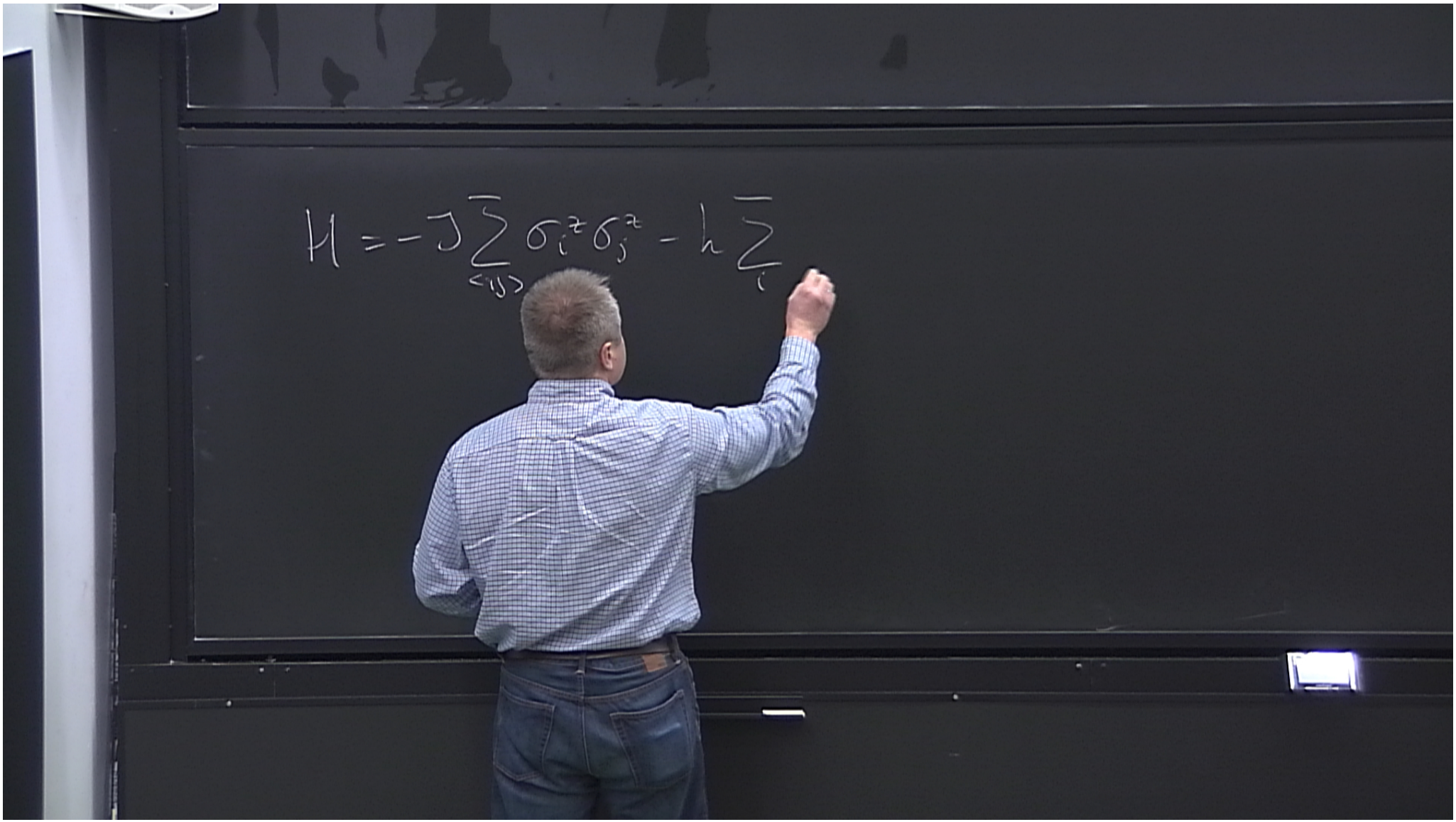
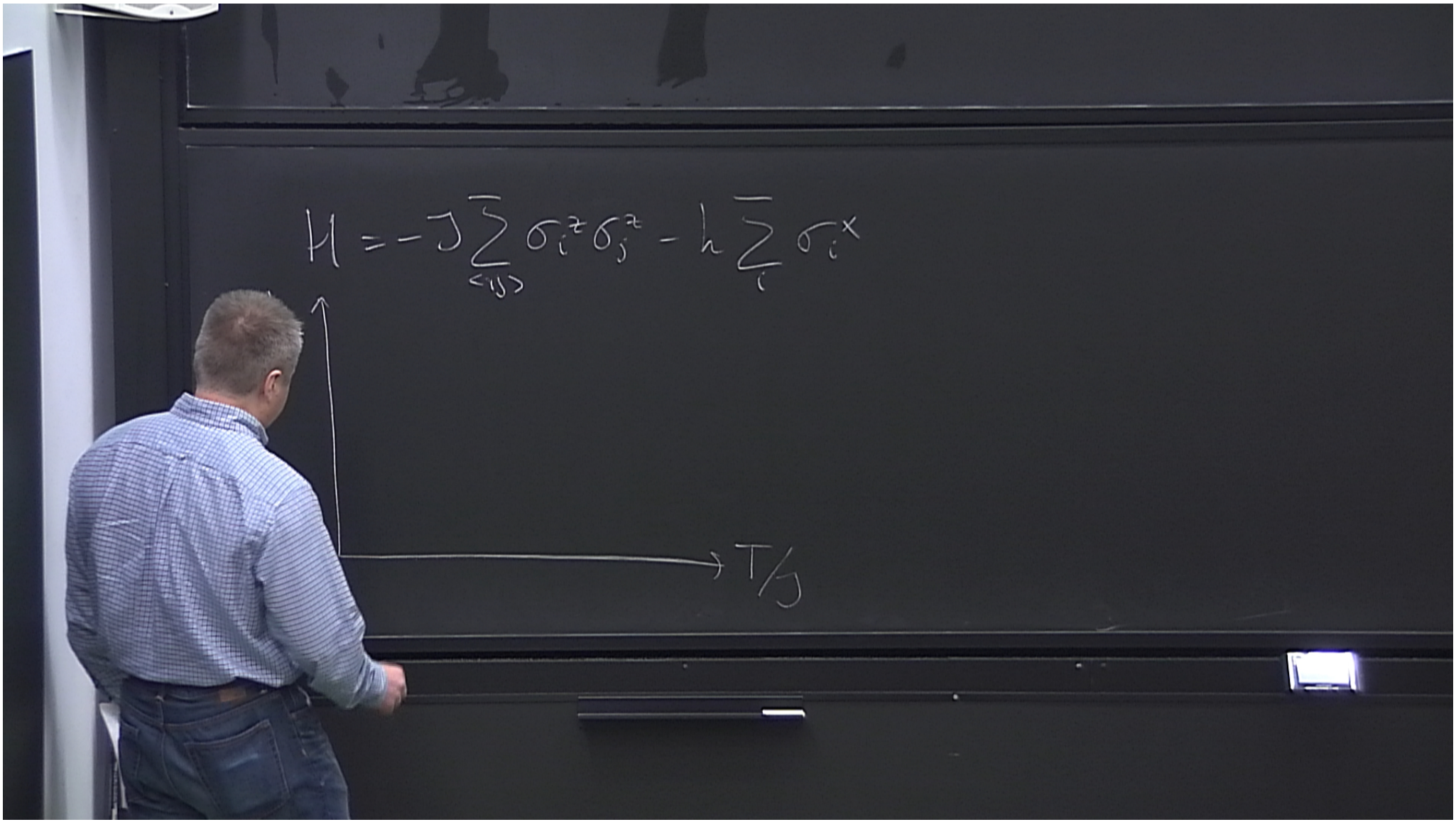
Title: 14/15 PSI - Statistical Mechanics - Lecture 13

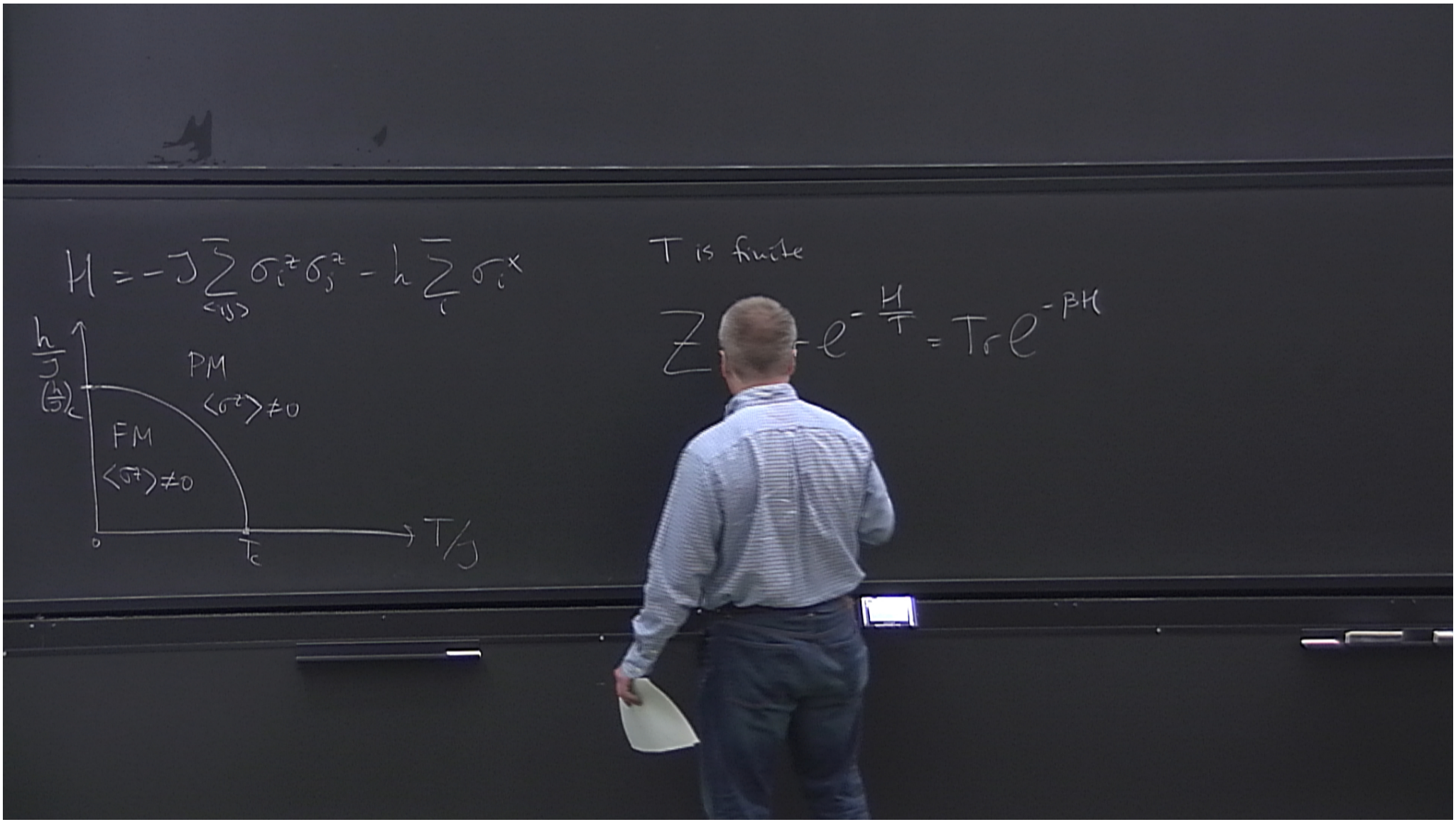
Date: Oct 23, 2014 10:45 AM

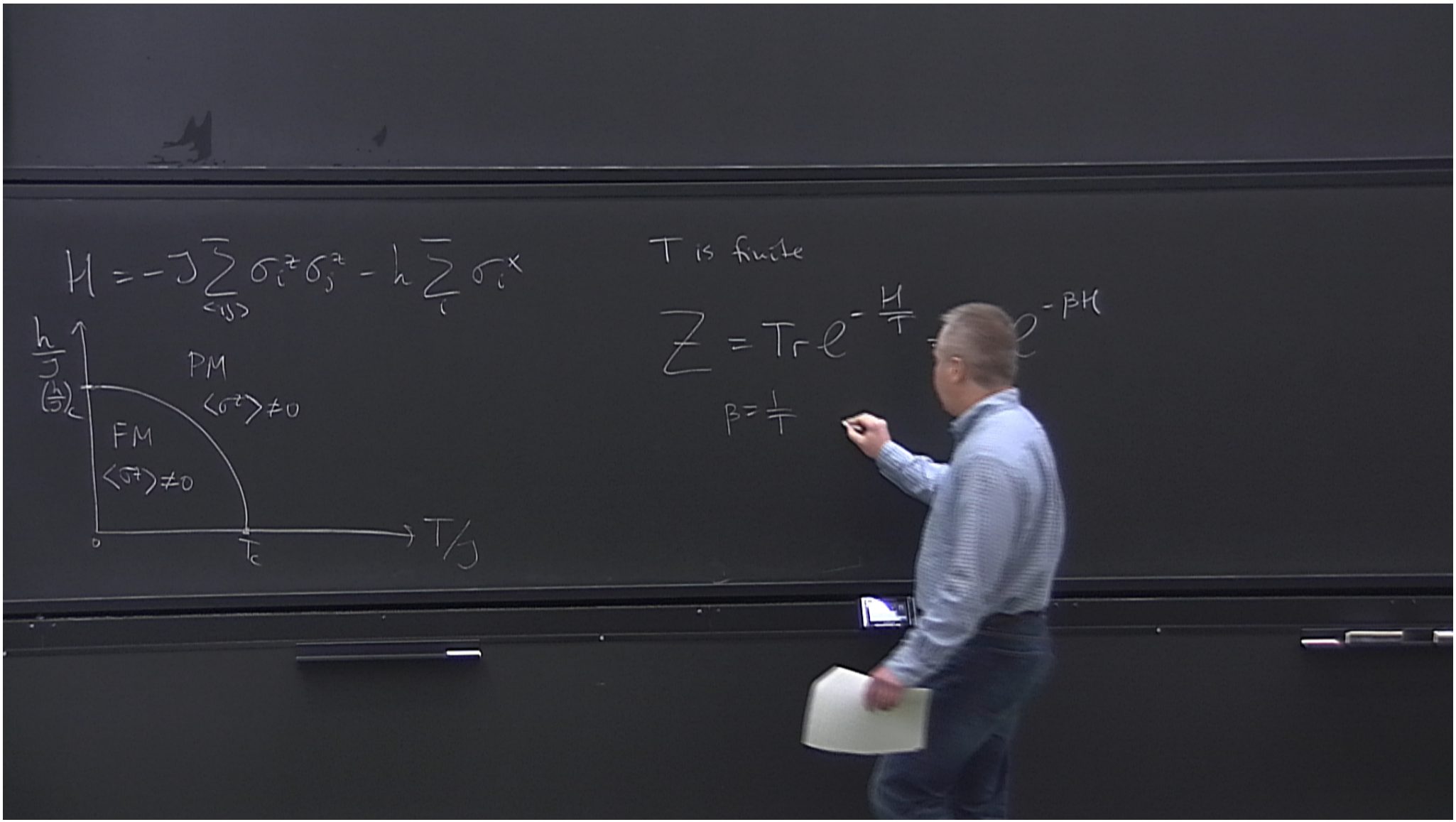
URL: http://pirsa.org/14100101

Abstract:


$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i$


$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$
T/J


$H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x$
T is finite
$Z = e^{-\frac{H}{T}} = \mathrm{Tr}\, e^{-\beta H}$
$\frac{h}{J}$
$\left(\frac{h}{J}\right)_c$
PM
$\langle\sigma^z\rangle \neq 0$
FM
$\langle\sigma^z\rangle \neq 0$
0
T_c
T/J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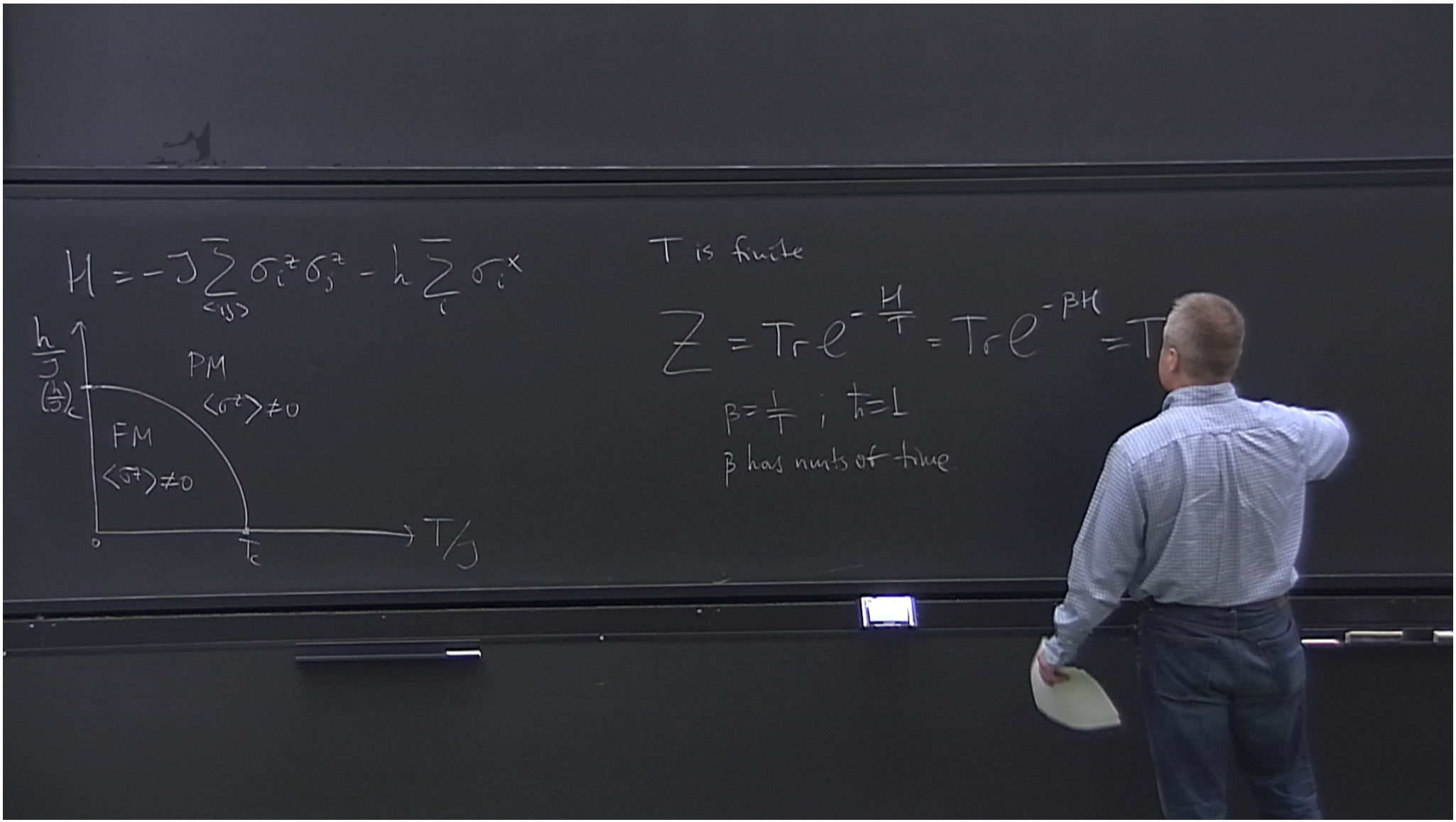
$\beta = \frac{1}{T}$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T is finite
$Z = \mathrm{Tr}\, e^{-\frac{H}{T}} = \mathrm{Tr}\, e^{-\beta H} = \mathrm{T}$
$\beta = \frac{1}{T}$; $\hbar = 1$
β has units of time

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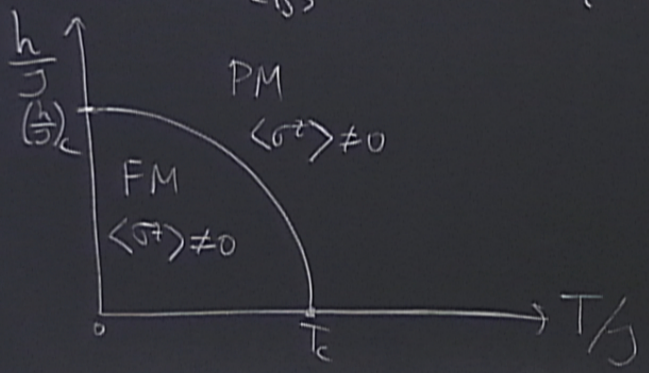


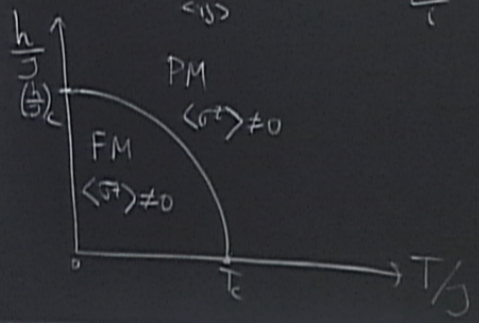
T is finite

$$Z = \mathrm{Tr}\, e^{-\frac{H}{T}} = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{\beta J \sum_{\langle ij\rangle} \sigma_i^z \sigma_j^z + \cdots}$$

$\beta = \frac{1}{T}$; $\hbar = 1$

β has units of time

$$H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x$$



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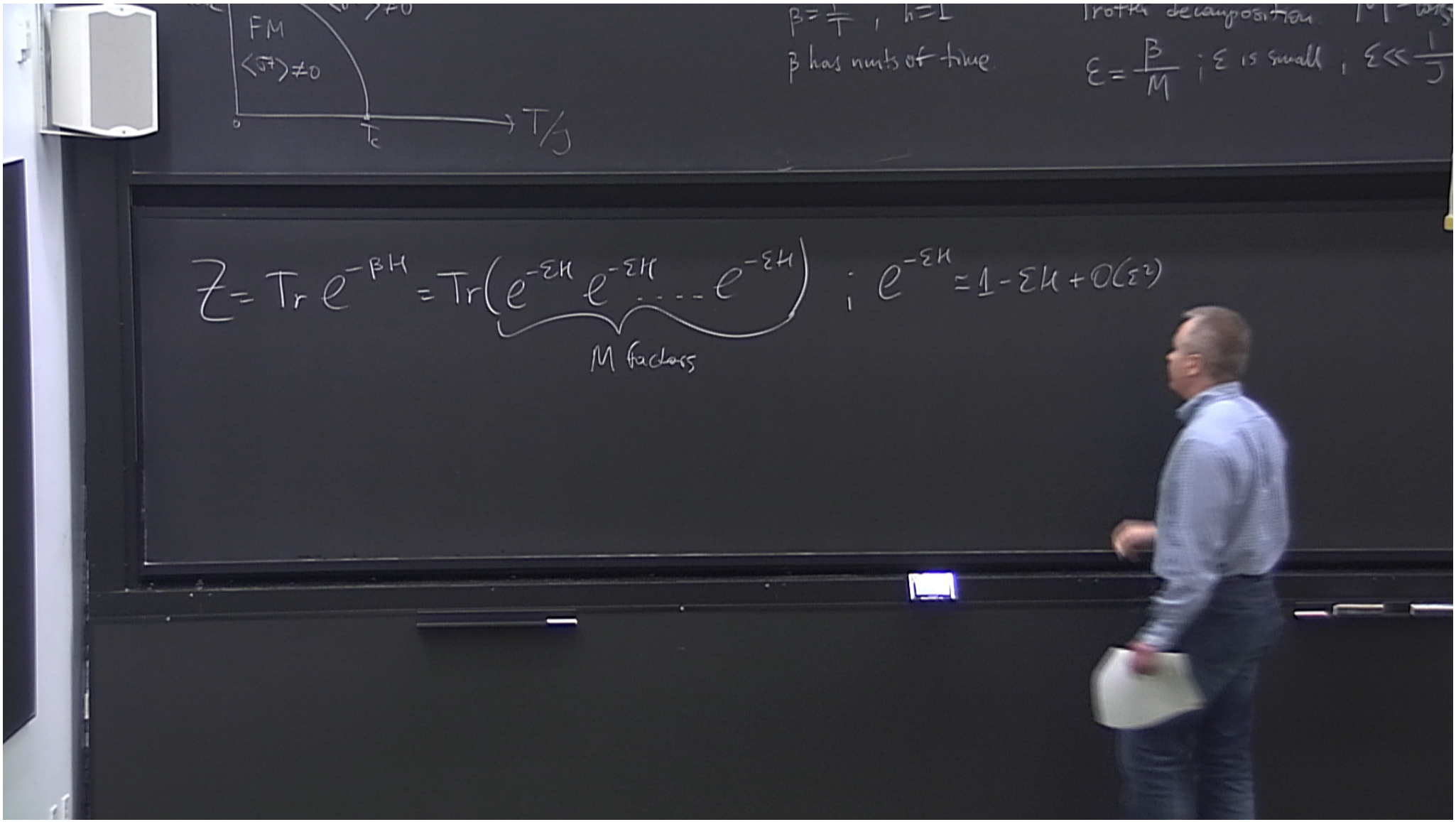
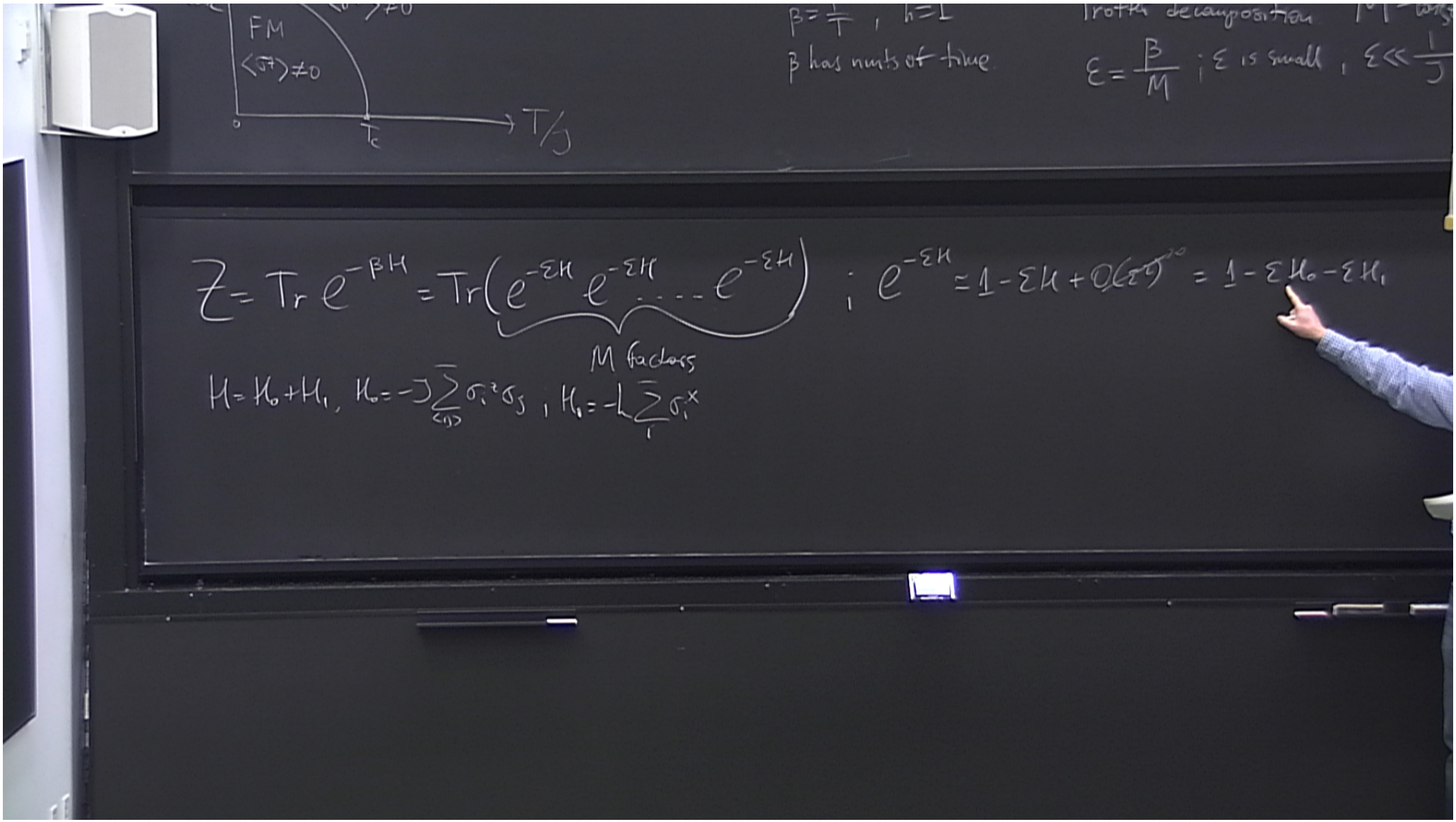

FM
$\langle \sigma^z \rangle \neq 0$
T_c
T/J
$\beta = \frac{1}{T}$, $\hbar = 1$
β has units of time
$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}$
$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left(\underbrace{e^{-\varepsilon H} e^{-\varepsilon H} \cdots e^{-\varepsilon H}}_{M \text{ factors}}\right)$; $e^{-\varepsilon H} = 1 - \varepsilon H + O(\varepsilon^2)$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$\beta = \frac{1}{T}$, $\hbar = 1$
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$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left(\underbrace{e^{-\varepsilon H} e^{-\varepsilon H} \cdots e^{-\varepsilon H}}_{M \text{ factors}}\right)$; $e^{-\varepsilon H} = 1 - \varepsilon H + O(\varepsilon^2) = 1 - \varepsilon H_0 - \varepsilon H_1$
$H = H_0 + H_1$, $H_0 = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j$, $H_1 = -h \sum_i \sigma_i^x$

$\beta = \frac{1}{T}$, $\hbar = 1$

β has units of time.

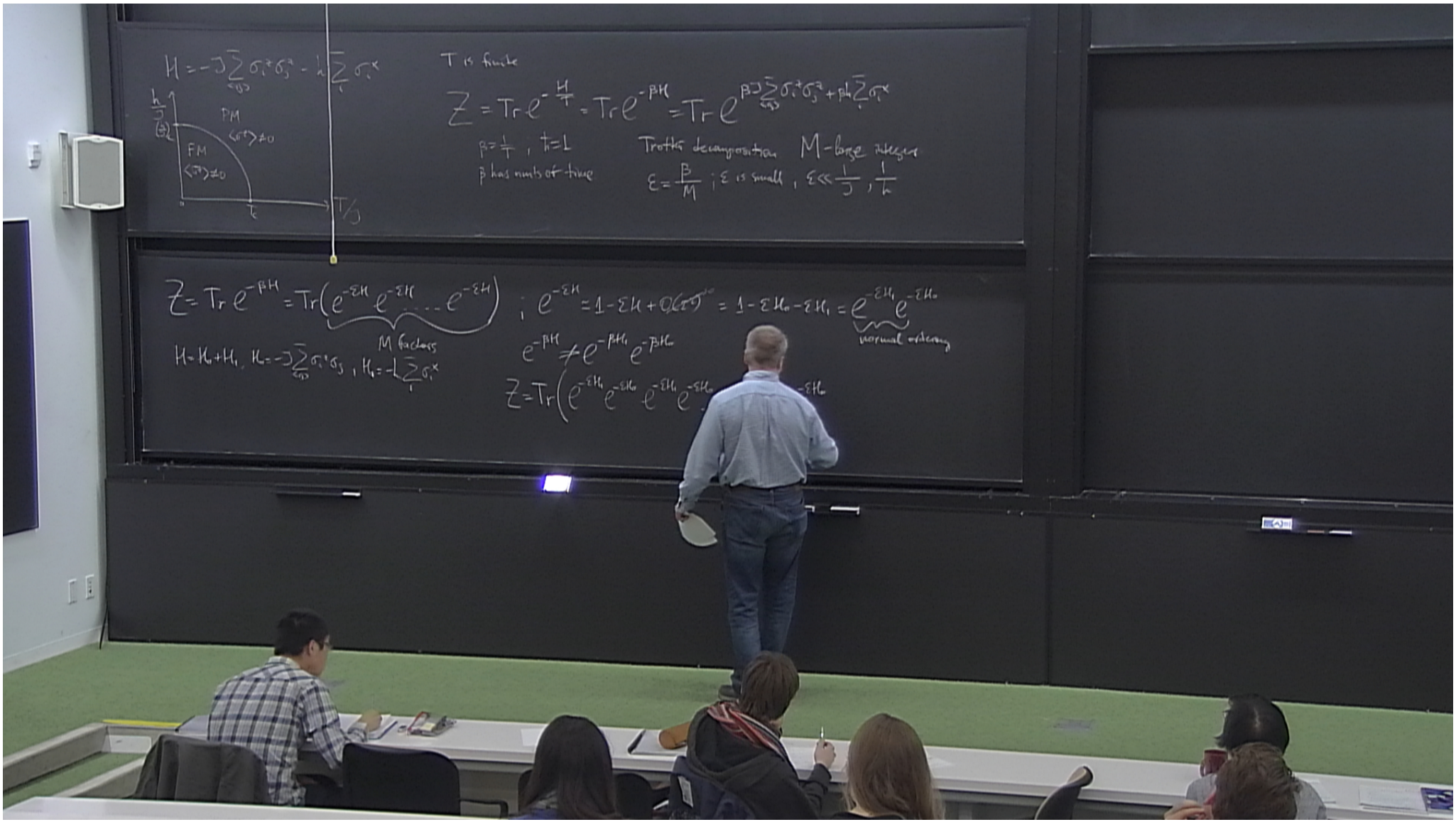
Trotter decomposition. M - large integer.

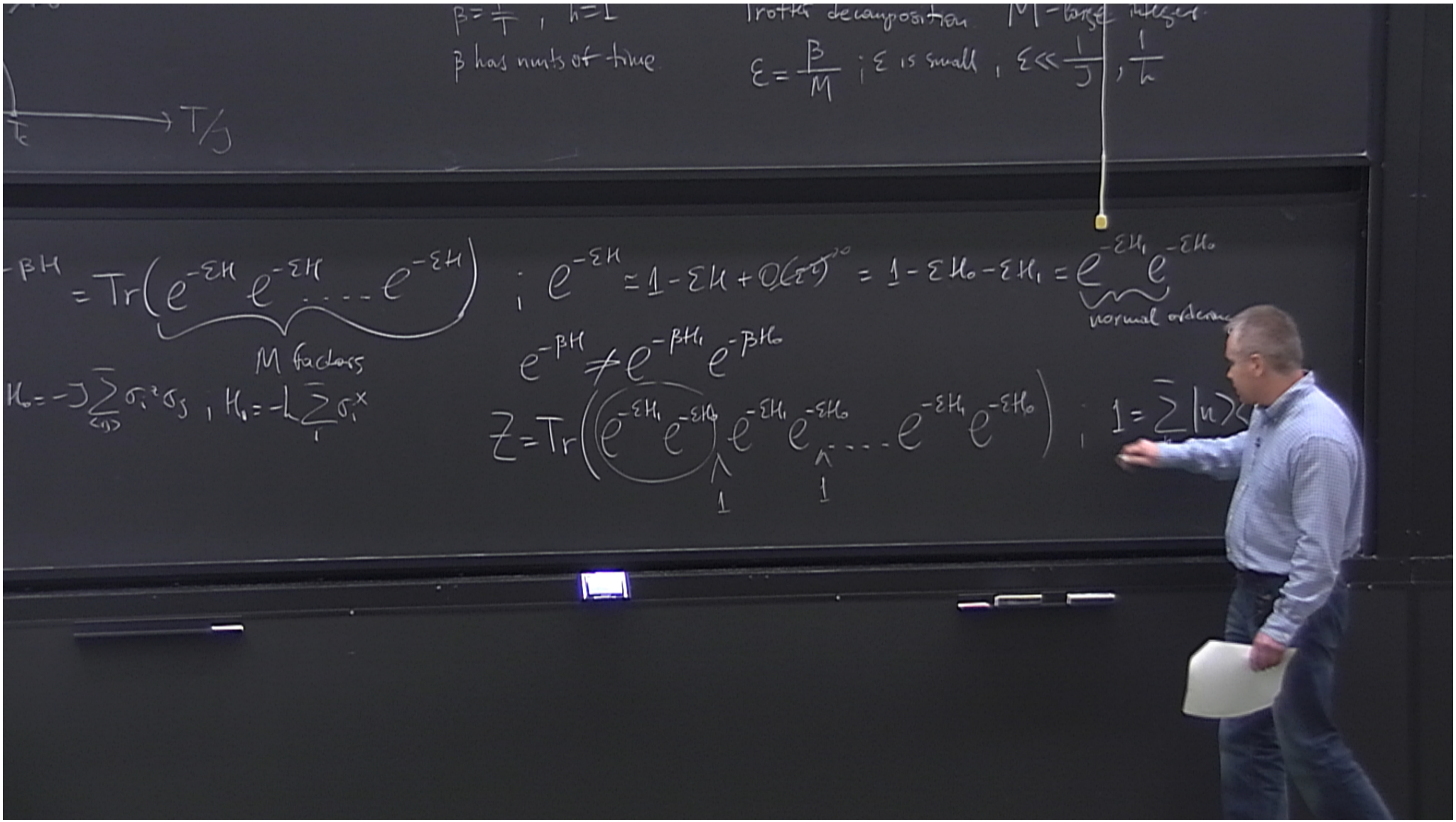
$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}, \frac{1}{h}$

T_c $\rightarrow T/J$

$$e^{-\beta H} = \mathrm{Tr}\Big(\underbrace{e^{-\varepsilon H} e^{-\varepsilon H} \cdots e^{-\varepsilon H}}_{M \text{ factors}}\Big) \; ; \; e^{-\varepsilon H} = 1 - \varepsilon H + O(\varepsilon^2) = 1 - \varepsilon H_0 - \varepsilon H_1 =$$

$$H_0 = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j \; , \; H_1 = -h \sum_i \sigma_i^x$$


T is finite
PM
FM
Trotter decomposition
M - large integer
β has units of time
M factors
normal ordering


β = 1/T , ħ = 1
β has units of time
Trotter decomposition
M - large integer
ε = β/M ; ε is small , ε ≪ 1/J , 1/h
T/J
= Tr(e^{-εH} e^{-εH} ... e^{-εH})
M factors
; e^{-εH} ≃ 1 - εH + O(ε²) = 1 - εH₀ - εH₁ = e^{-εH₁} e^{-εH₀}
normal ordering
e^{-βH} ≠ e^{-βH₁} e^{-βH₀}
H₀ = -J Σ_{⟨ij⟩} σ_i^z σ_j , H₁ = -h Σ_i σ_i^x
Z = Tr(e^{-εH₁} e^{-εH₀} e^{-εH₁} e^{-εH₀} ... e^{-εH₁} e^{-εH₀})
1
1
1 = Σ |n⟩⟨

$\beta = \frac{1}{T}$, $\hbar = 1$

β has units of time

Trotter decomposition M - large integer

$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}, \frac{1}{h}$

T_c T/J

$$e^{-\beta H} = \mathrm{Tr}\Big(\underbrace{e^{-\varepsilon H} e^{-\varepsilon H} \cdots e^{-\varepsilon H}}_{M \text{ factors}}\Big) \; ; \; e^{-\varepsilon H} = 1 - \varepsilon H + \cancel{O(\varepsilon^2)}^{\to 0} = 1 - \varepsilon H_0 - \varepsilon H_1 = \underbrace{e^{-\varepsilon H_1} e^{-\varepsilon H_0}}_{\text{normal ordering}}$$

$$H_0 = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j \; , \; H_1 = -h \sum_i \sigma_i^x$$

$$e^{-\beta H} \neq e^{-\beta H_1} e^{-\beta H_0}$$

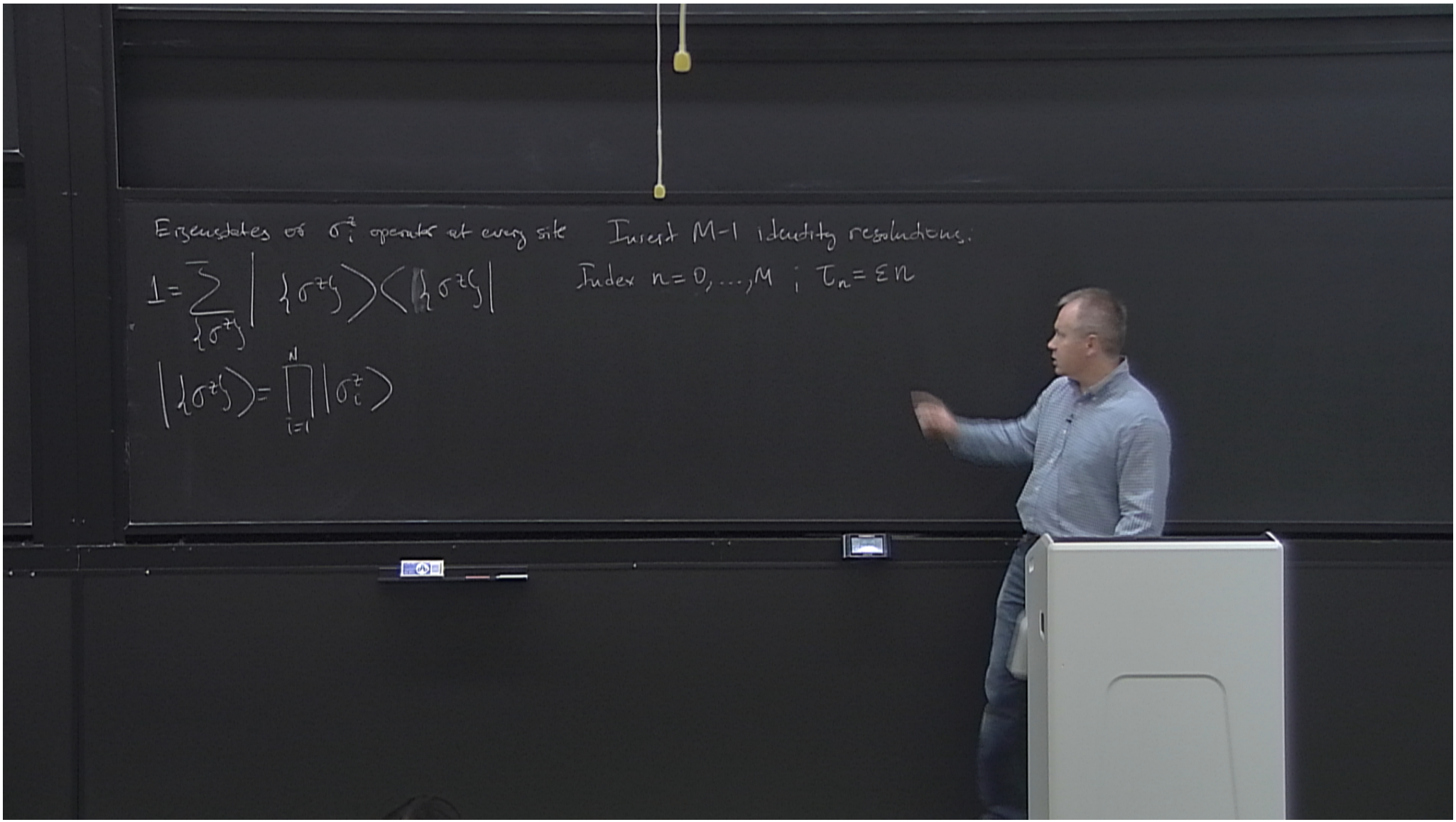
$$Z = \mathrm{Tr}\Big(e^{-\varepsilon H_1} e^{-\varepsilon H_0} \underset{1}{\wedge} e^{-\varepsilon H_1} e^{-\varepsilon H_0} \underset{1}{\wedge} \cdots e^{-\varepsilon H_1} e^{-\varepsilon H_0}\Big) \; ; \; 1 = \sum_n |n\rangle\langle n|$$

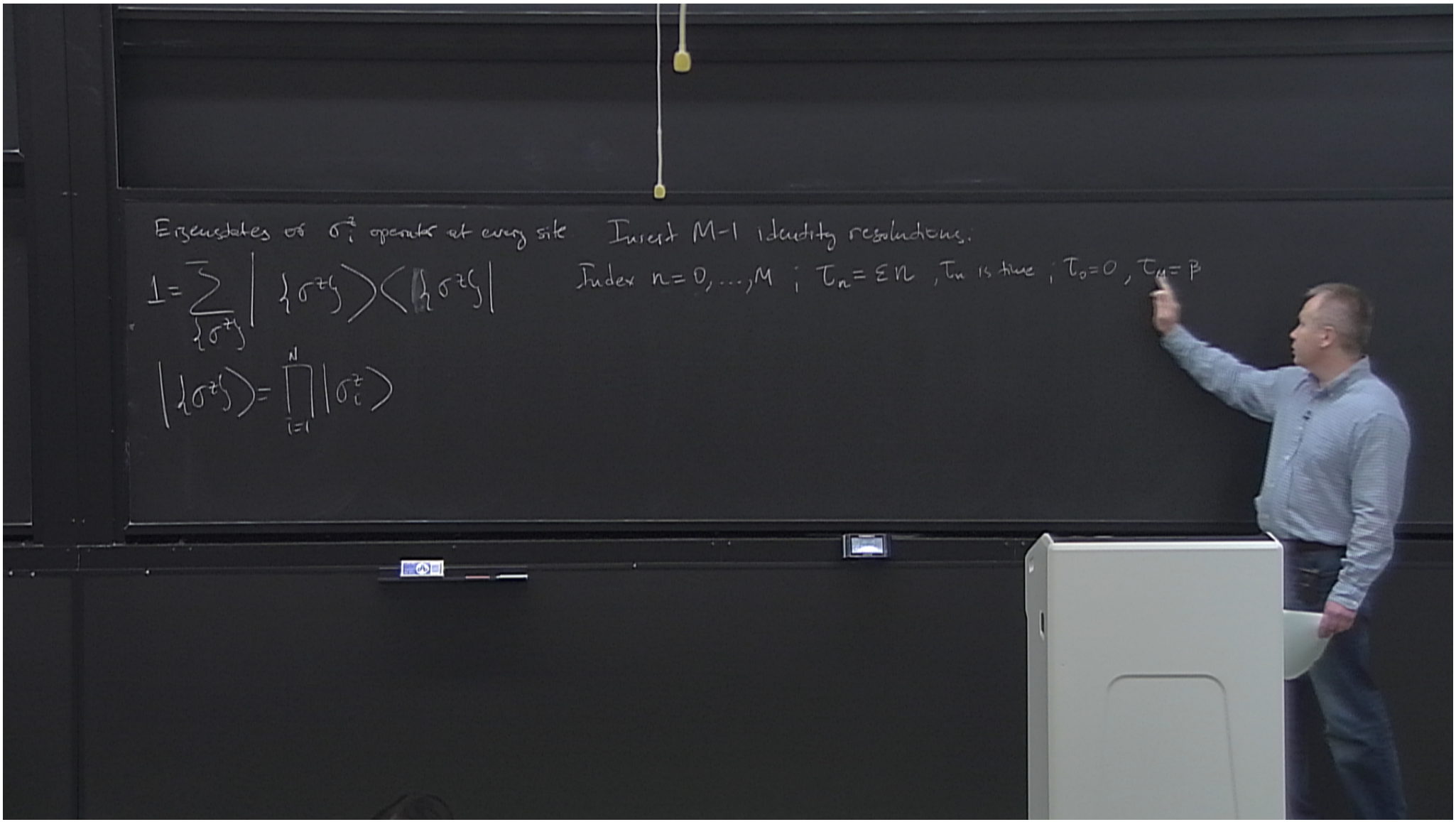
$|n\rangle$ - any complete set of states

Eigenstates of σ_i^z operator at every site Insert M-1 identity resolutions:

$$\mathbb{1} = \sum_{\{\sigma^z\}} |\{\sigma^z\}\rangle\langle\{\sigma^z\}|$$

$$|\{\sigma^z\}\rangle = \prod_{i=1}^{N} |\sigma_i^z\rangle$$


Eigenstates of σ_i^z operator at every site
Insert M-1 identity resolutions:
$1 = \sum_{\{\sigma^z\}} |\{\sigma^z\}\rangle\langle\{\sigma^z\}|$
Index $n = 0, \ldots, M$; $\tau_n = \varepsilon n$
$|\{\sigma^z\}\rangle = \prod_{i=1}^{N} |\sigma_i^z\rangle$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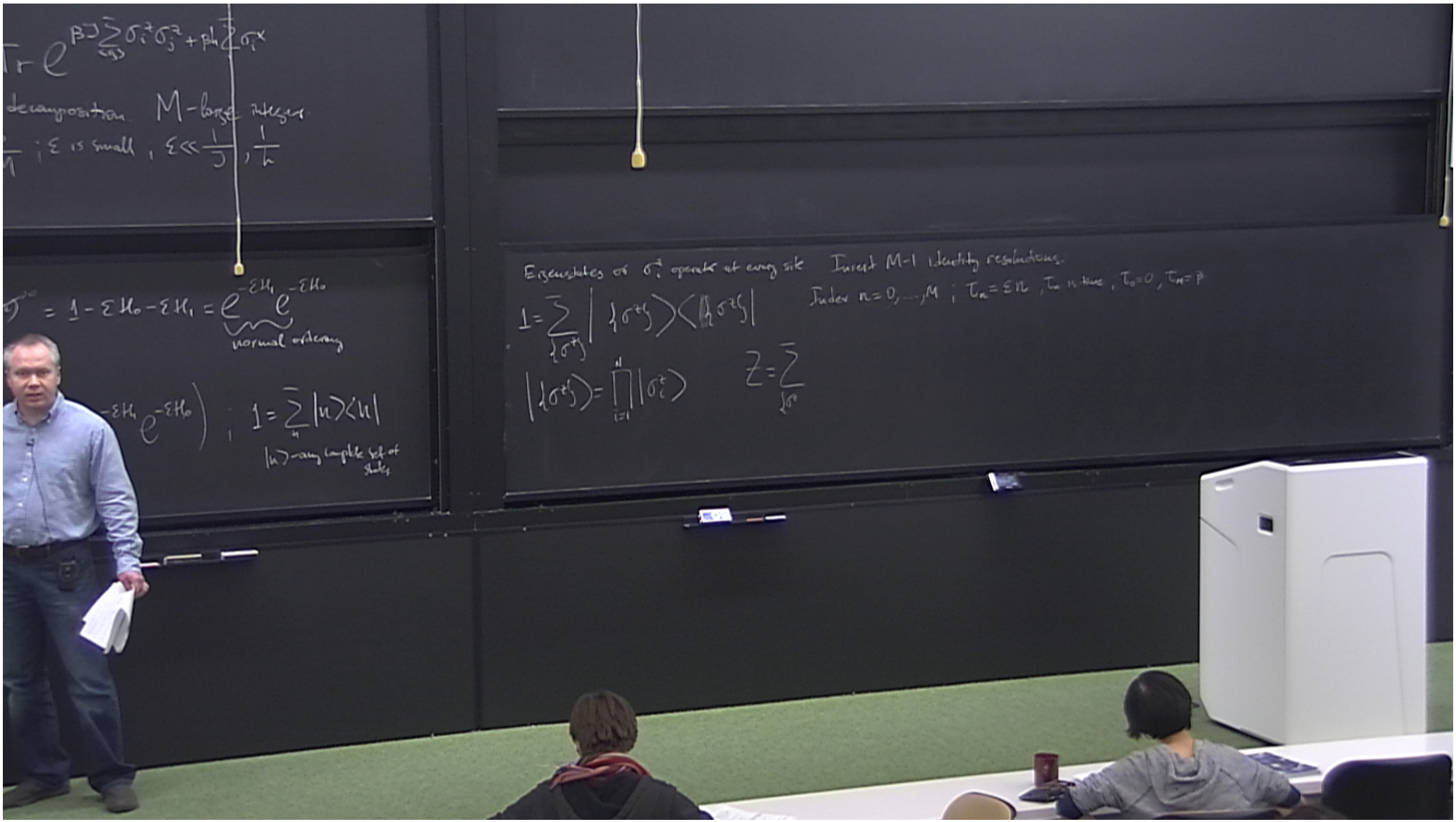
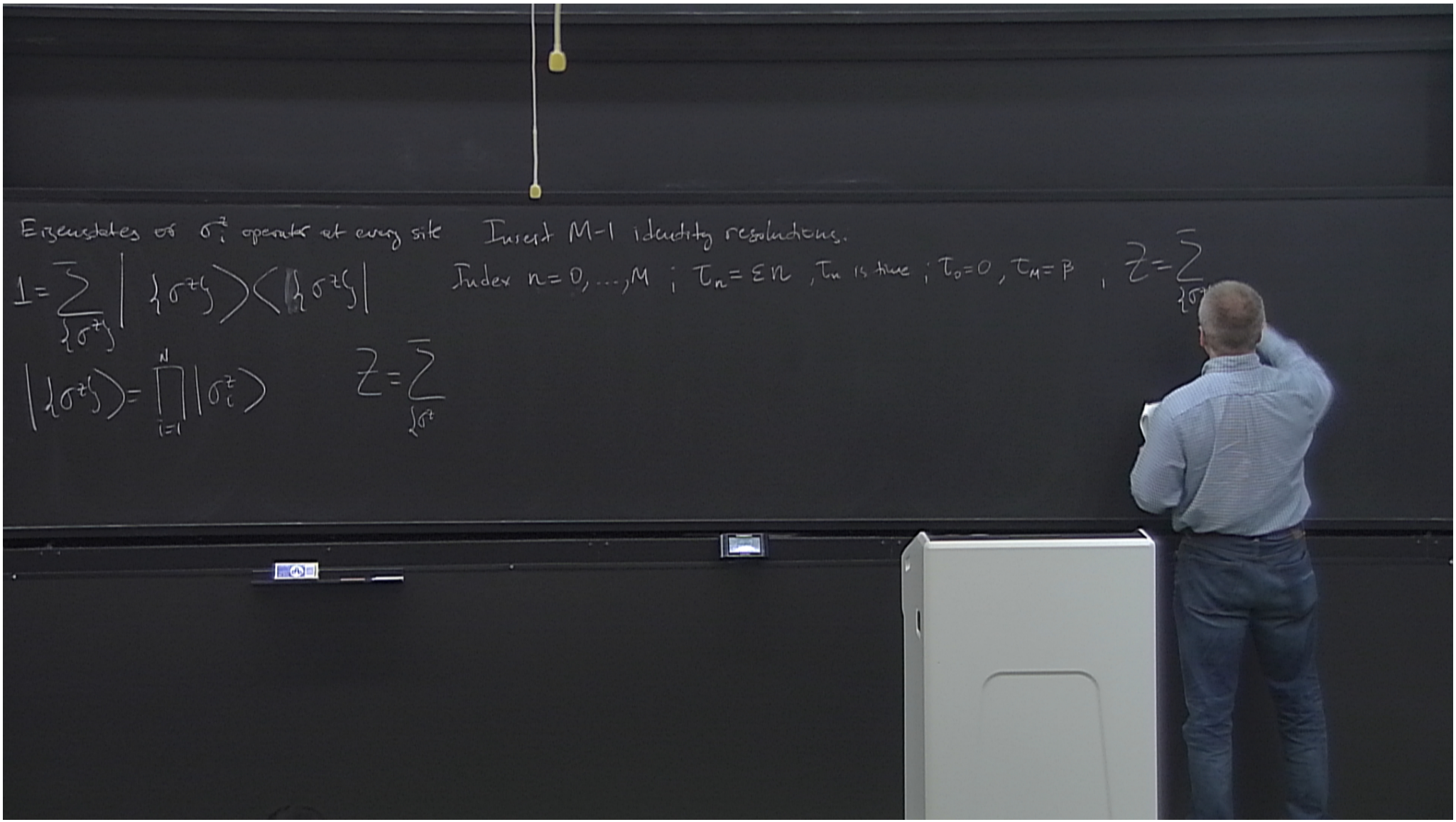

decomposition M - large integer
; ε is small , ε ≪ 1/J , 1/h
= 1 - εH₀ - εH₁ = e^{-εH₁} e^{-εH₀}
normal ordering
; 1 = Σ_n |n⟩⟨n|
|n⟩ - any complete set of states
Eigenstates of σ_i^z operator at every site
Insert M-1 identity resolutions.
Index n = 0, ..., M ; τ_n = εn , τ_n is time , τ_0 = 0 , τ_M = β
1 = Σ_{{σ^z}} |{σ^z}⟩⟨{σ^z}|
|{σ^z}⟩ = Π_{i=1}^{N} |σ_i^z⟩
Z = Σ_{{σ^z}}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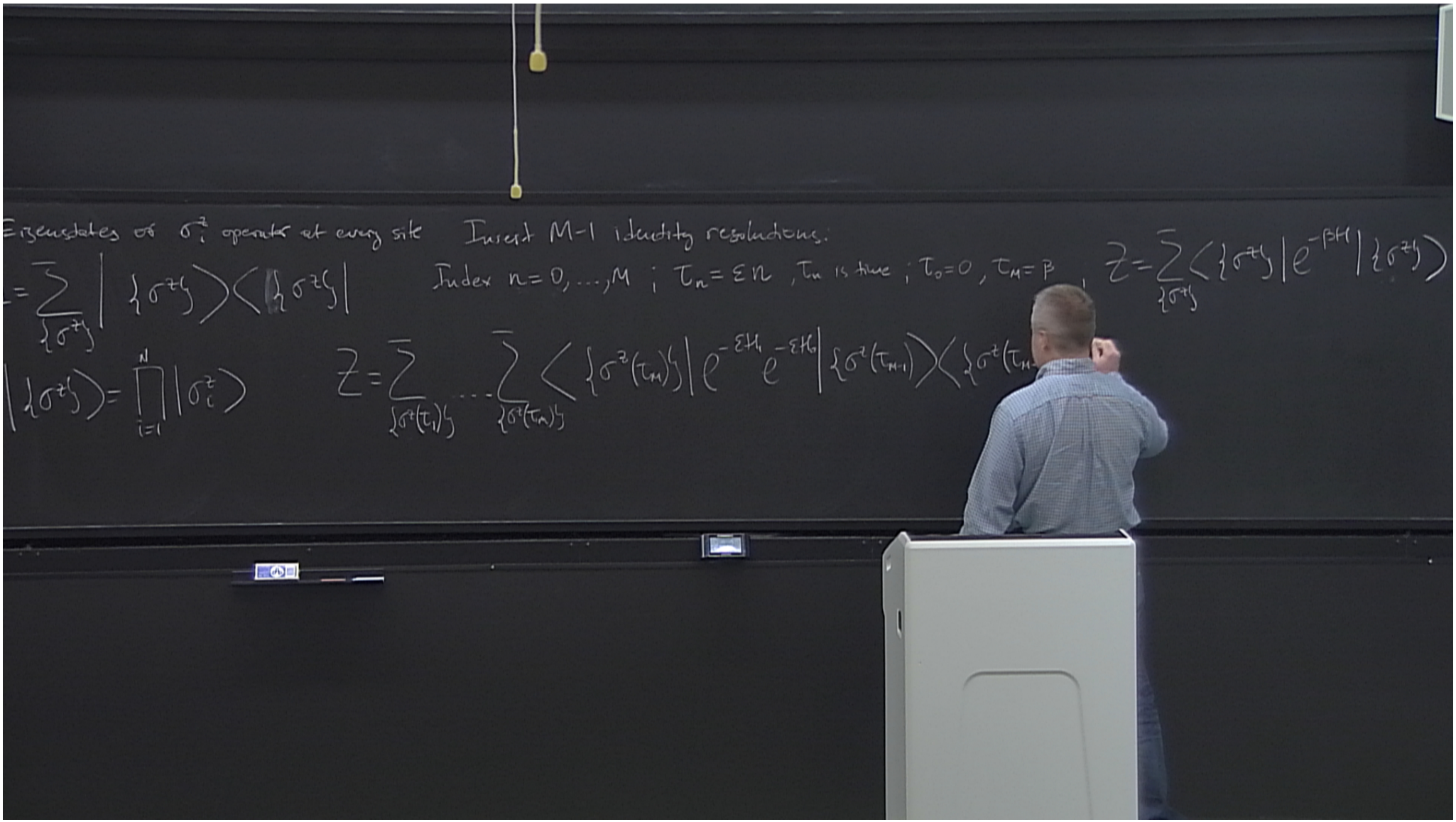
$|\{\sigma^z\}\rangle = \prod_{i=1}^{N} |\sigma_i^z\rangle$
$Z = \sum_{\{\sigma^z}$


Eigenstates of σ_i^z operator at every site
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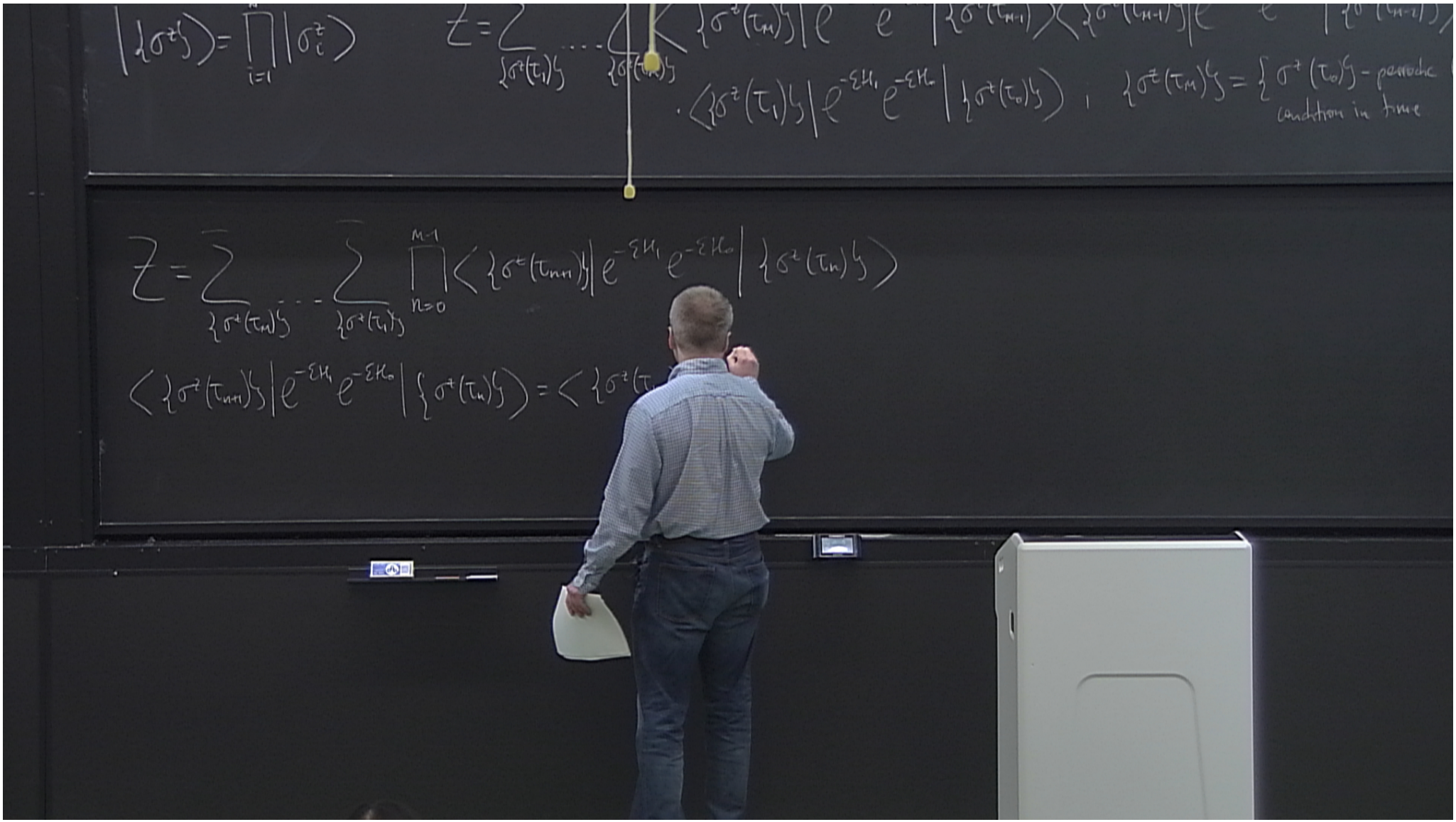
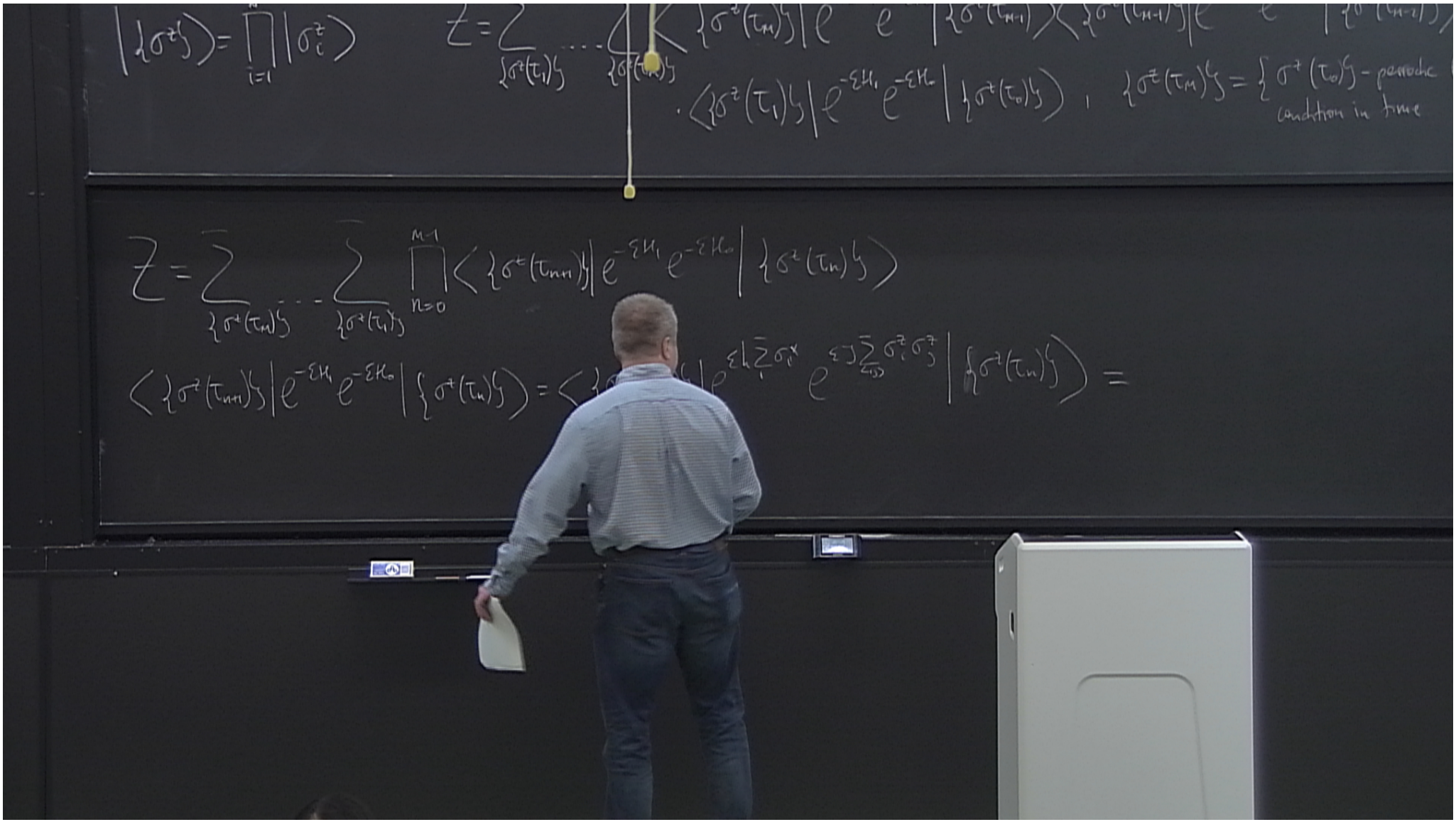
$$Z = \sum_{\{\sigma^z(\tau_1)\}} \cdots \sum_{\{\sigma^z(\tau_M)\}} \langle\{\sigma^z(\tau_M)\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_{M-1})\}\rangle\langle\{\sigma^z(\tau_{M-1})\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_{M-2})\}\rangle\langle\sigma^z(\tau_{M-2})| \cdots$$

$$\cdot \langle\{\sigma^z(\tau_1)\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_0)\}\rangle$$


$|\{\sigma^z\}\rangle = \prod_{i=1}^{n} |\sigma_i^z\rangle$
$Z = \sum_{\{\sigma^z(\tau_1)\}} \cdots \sum_{\{\sigma^z(\tau_M)\}}$
$\langle \{\sigma^z(\tau_1)\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_0)\} \rangle$; $\{\sigma^z(\tau_M)\} = \{\sigma^z(\tau_0)\}$ - periodic condition in time
$Z = \sum_{\{\sigma^z(\tau_M)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} \langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle =$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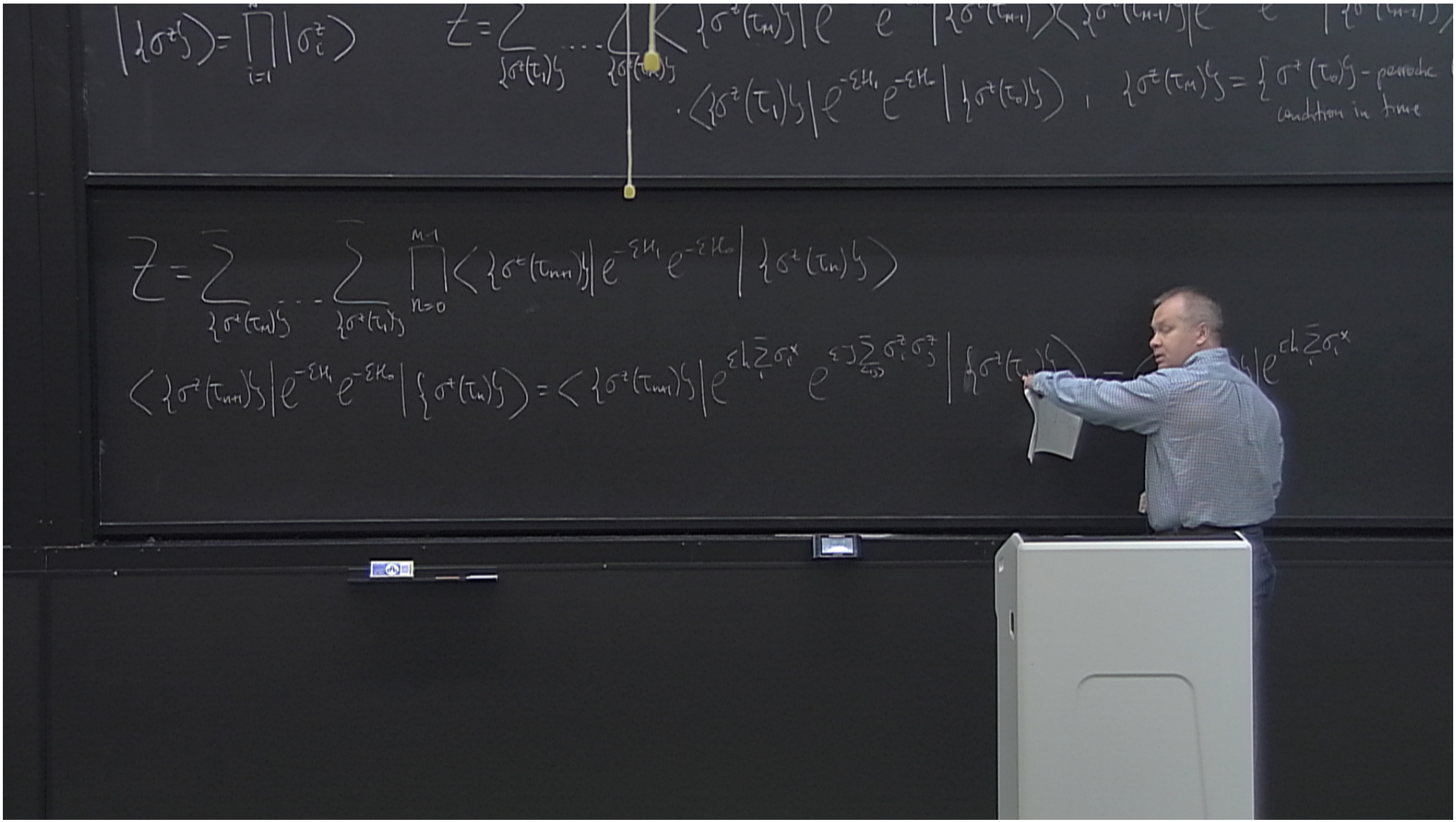
$\langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle = \langle$
$e^{\varepsilon h \sum_i \sigma_i^x} e^{\varepsilon J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z} | \{\sigma^z(\tau_n)\} \rangle =$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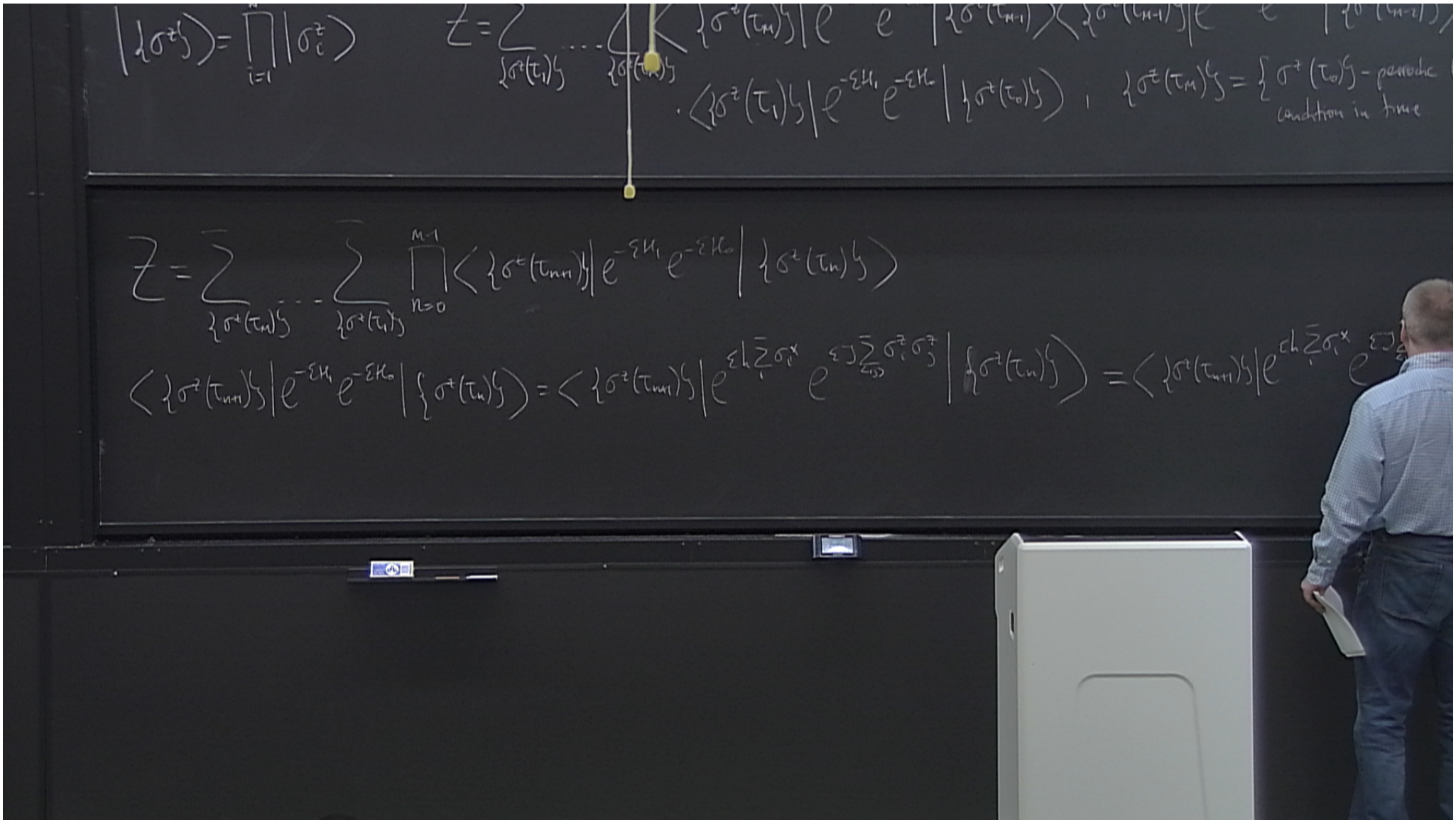
$\langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle = \langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} e^{\varepsilon J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z}$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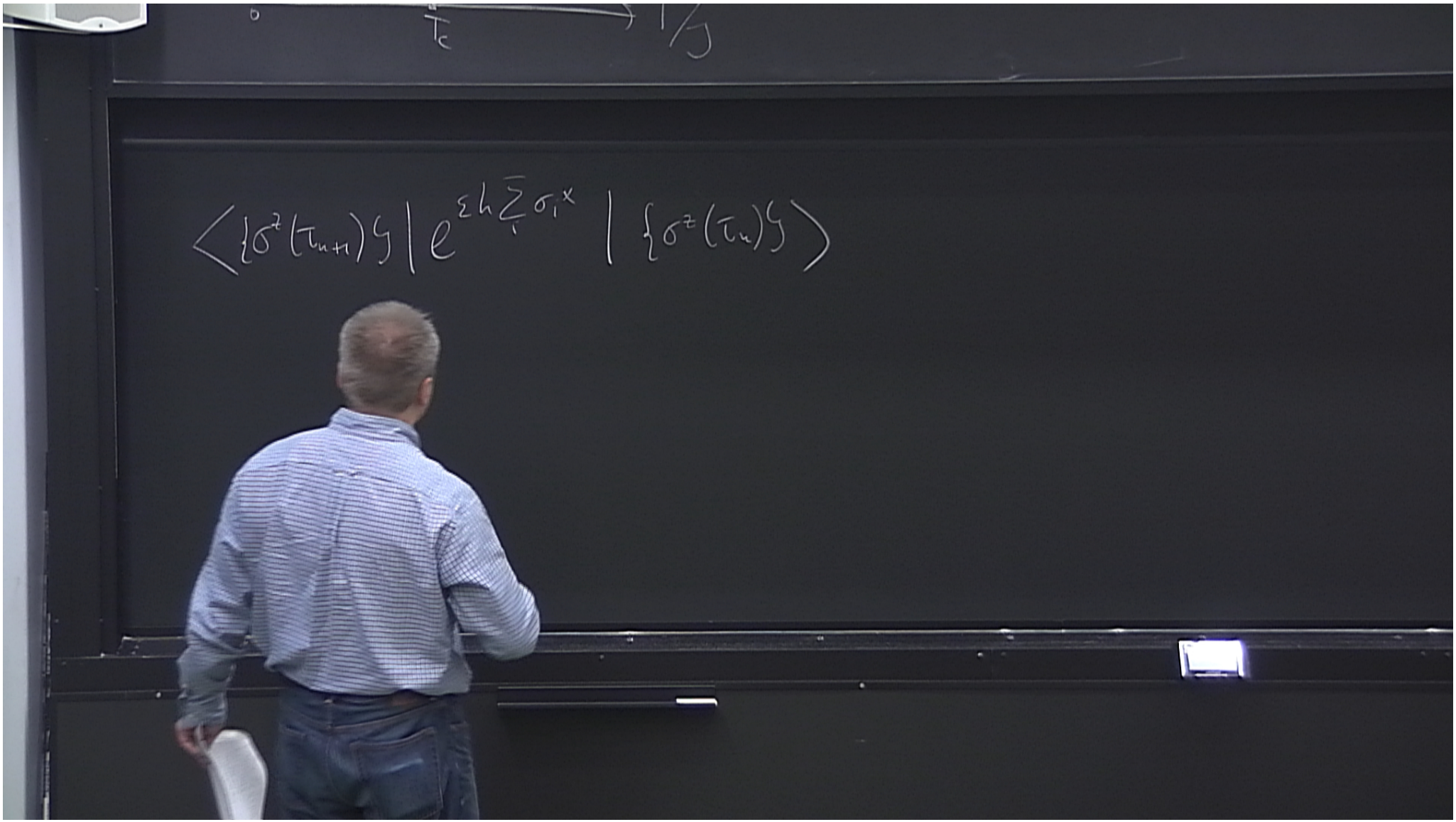
$\langle\{\sigma^z(\tau_1)\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_0)\}\rangle$; $\{\sigma^z(\tau_M)\} = \{\sigma^z(\tau_0)\}$ - periodic condition in time
$Z = \sum_{\{\sigma^z(\tau_M)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} \langle\{\sigma^z(\tau_{n+1})\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_n)\}\rangle$
$\langle\{\sigma^z(\tau_{n+1})\}| e^{-\varepsilon H_1} e^{-\varepsilon H_0} |\{\sigma^z(\tau_n)\}\rangle = \langle\{\sigma^z(\tau_{n+1})\}| e^{\varepsilon h \sum_i \sigma_i^x} e^{\varepsilon J \sum_{\langle ij\rangle} \sigma_i^z \sigma_j^z} |\{\sigma^z(\tau_n)\}\rangle = \langle\{\sigma^z(\tau_{n+1})\}| e^{\varepsilon h \sum_i \sigma_i^x}$

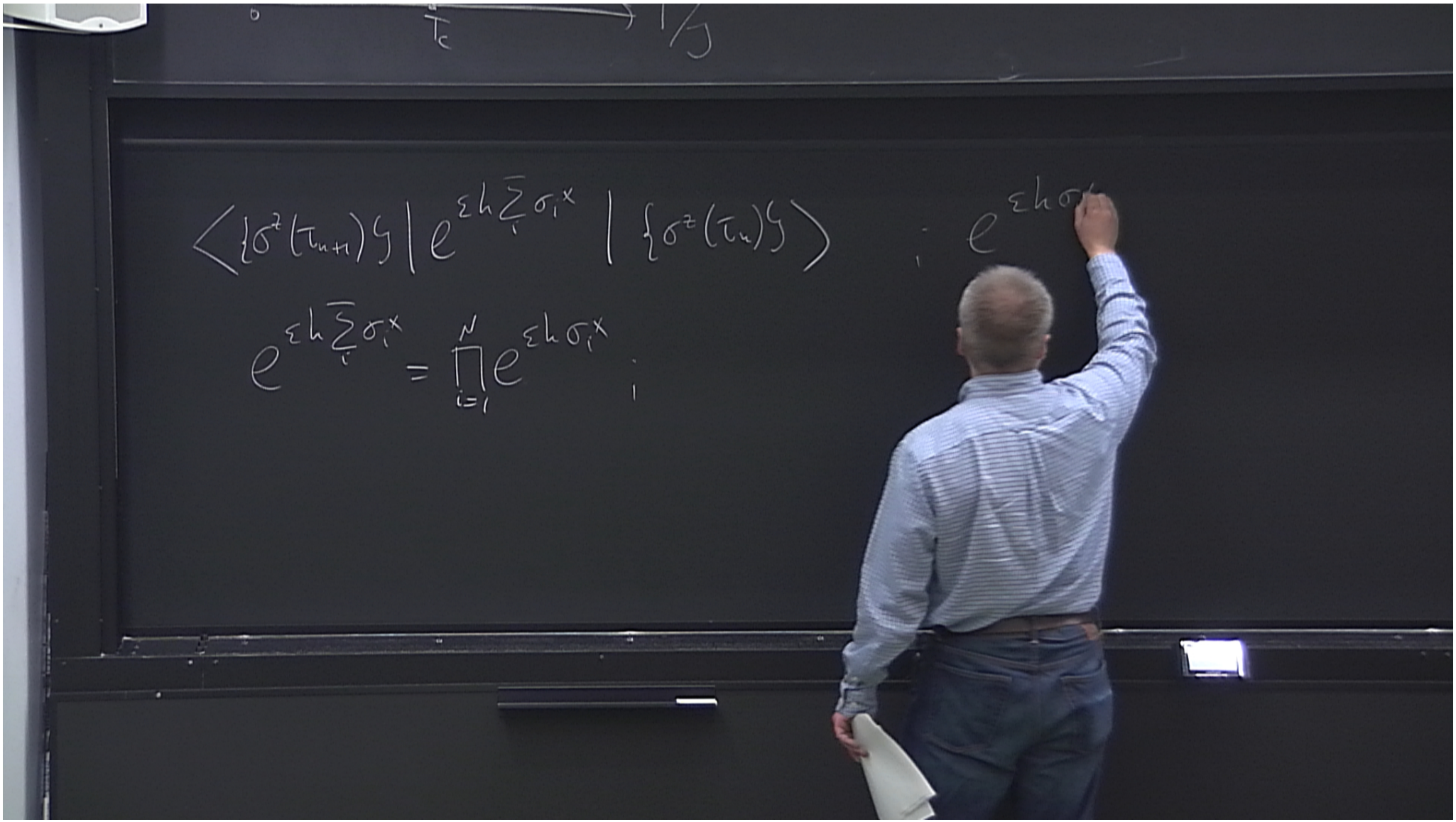
$$= \prod_{i=1} |\sigma_i^z\rangle \qquad Z = \sum_{\{\sigma^z(\tau_1)\}} \cdots \sum_{\{\sigma^z(\tau_M)\}} \langle \{\sigma^z(\tau_M)\} | e^{\quad} e^{\quad} | \{\sigma^z(\tau_{M-1})\} \rangle \langle \{\sigma^z(\tau_{M-1})\} | e^{\quad} e^{\quad} | \{\sigma^z(\tau_{M-2})\} \rangle \langle \quad$$

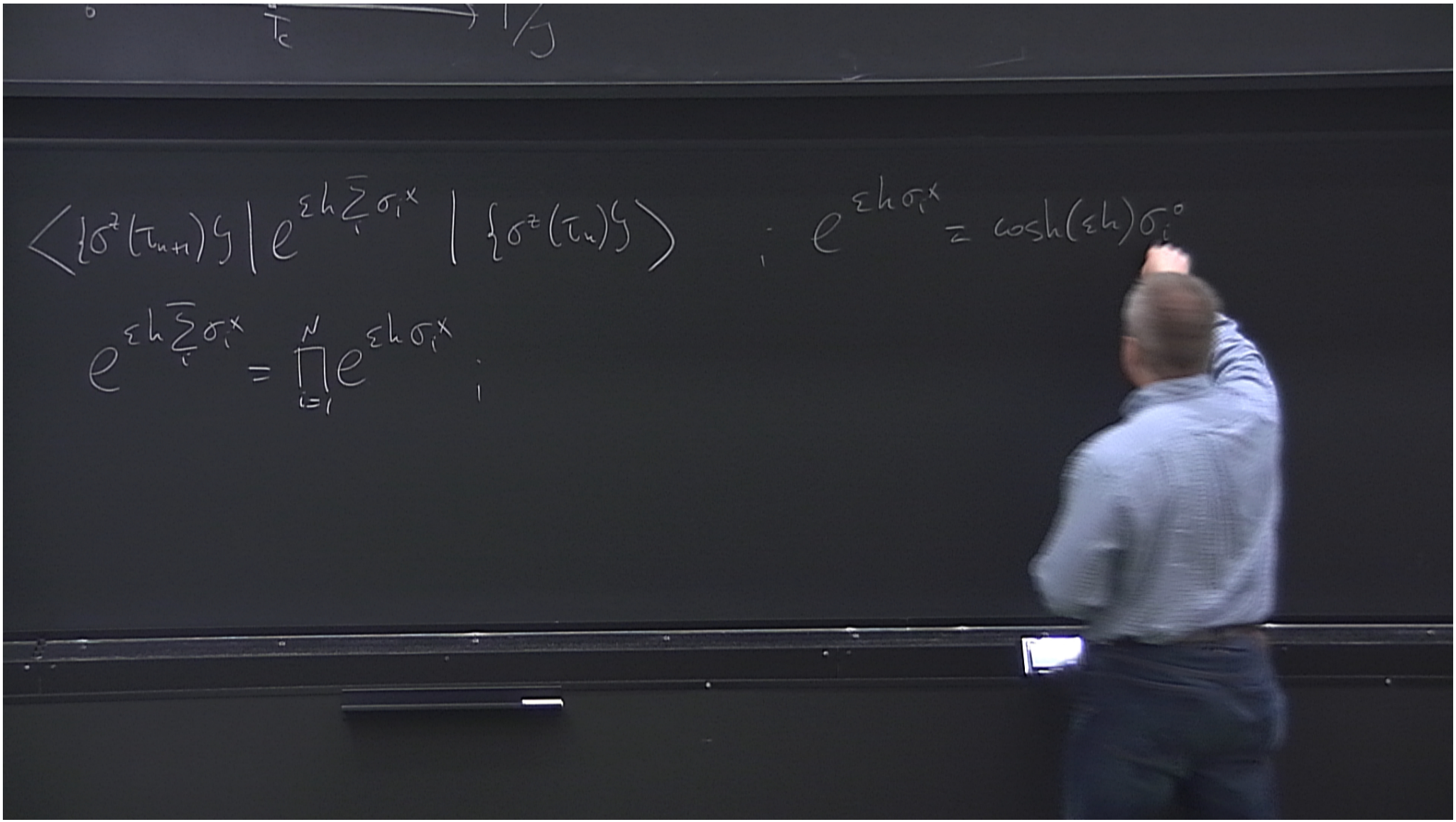
$$\cdot \langle \{\sigma^z(\tau_1)\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_0)\} \rangle \,, \quad \{\sigma^z(\tau_M)\} = \{\sigma^z(\tau_0)\}$$ – periodic boundary condition in time

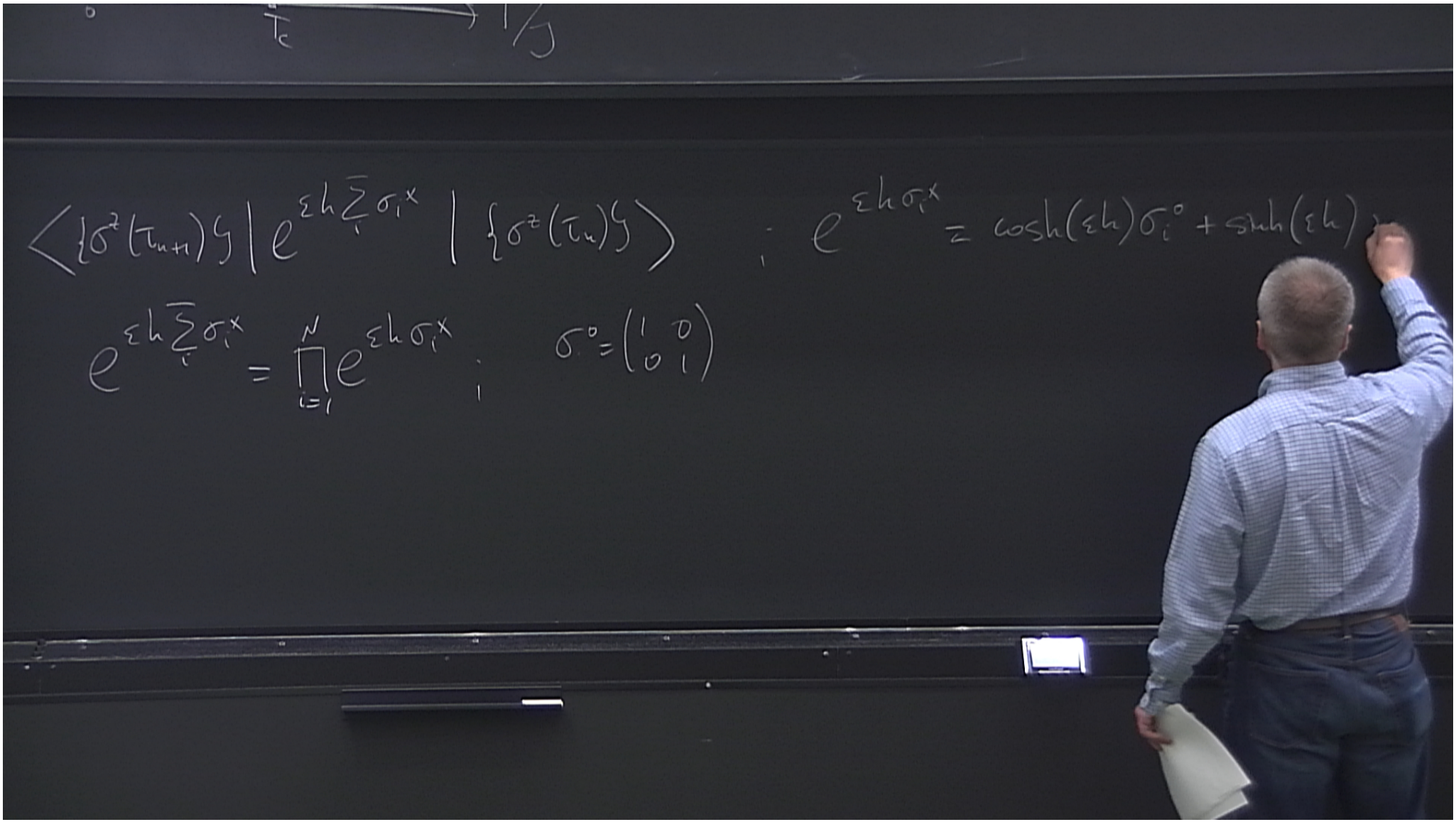
$$\sum_{\{\sigma^z(\tau_M)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} \langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle$$

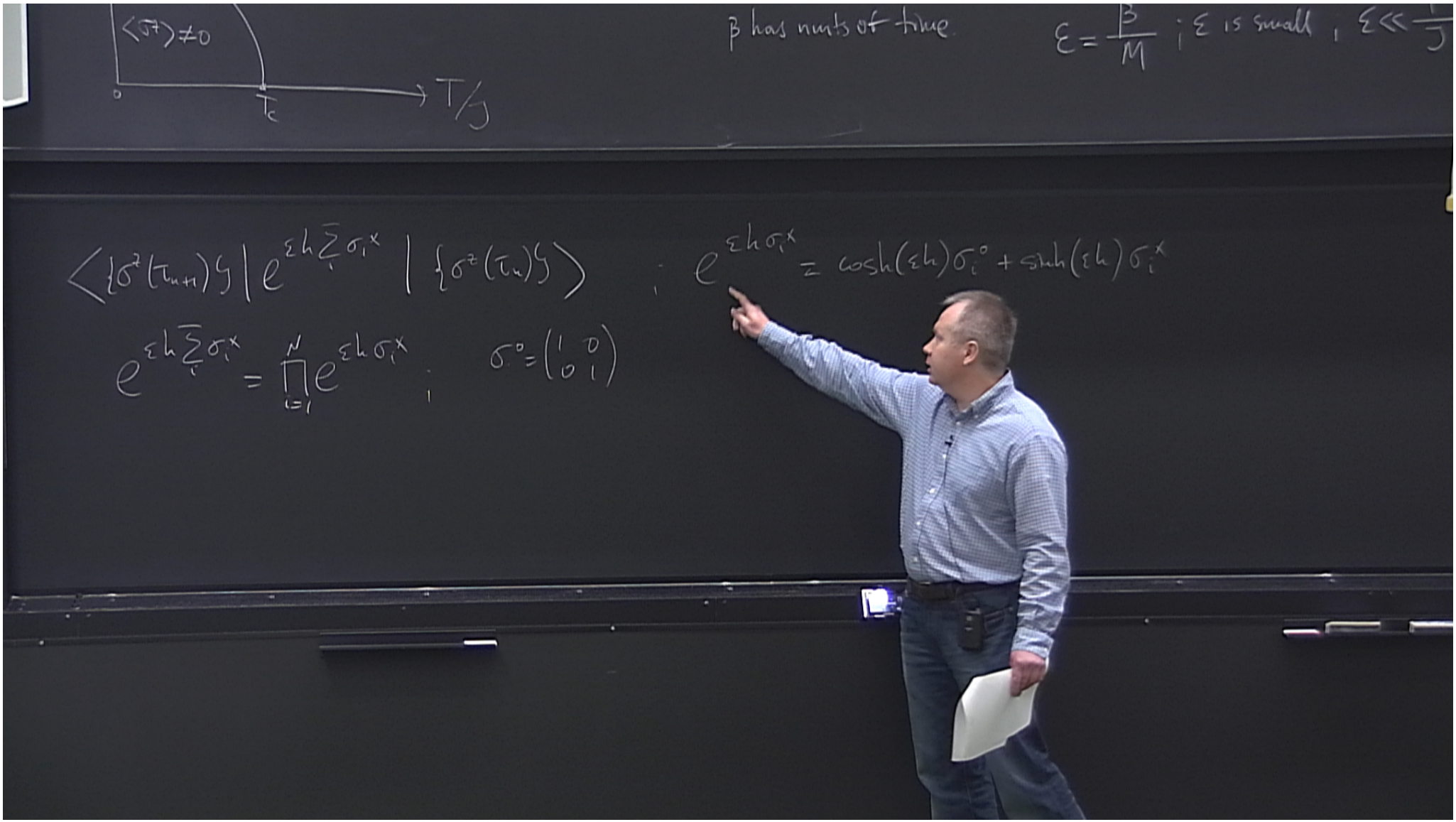
$$\langle \{\sigma^z(\tau_{n+1})\} | e^{-\varepsilon H_1} e^{-\varepsilon H_0} | \{\sigma^z(\tau_n)\} \rangle = \langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} e^{\varepsilon J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z} | \{\sigma^z(\tau_n)\} \rangle = \langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} e^{\varepsilon J \sum_{\langle ij \rangle} \sigma_i^z(\tau_n) \sigma_j^z(\tau_n)} | \{\sigma^z(\tau_n)\} \rangle$$

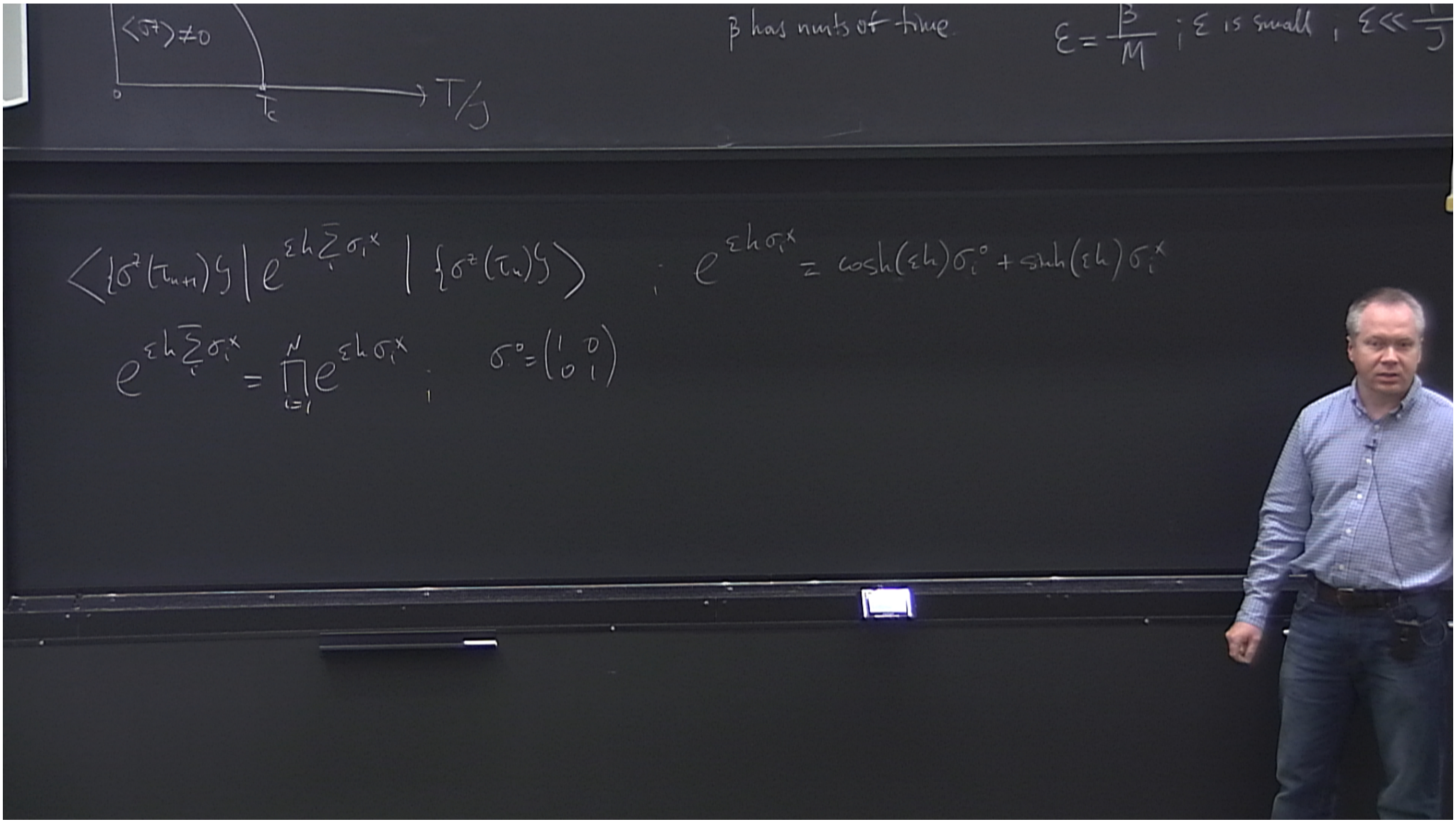

$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$

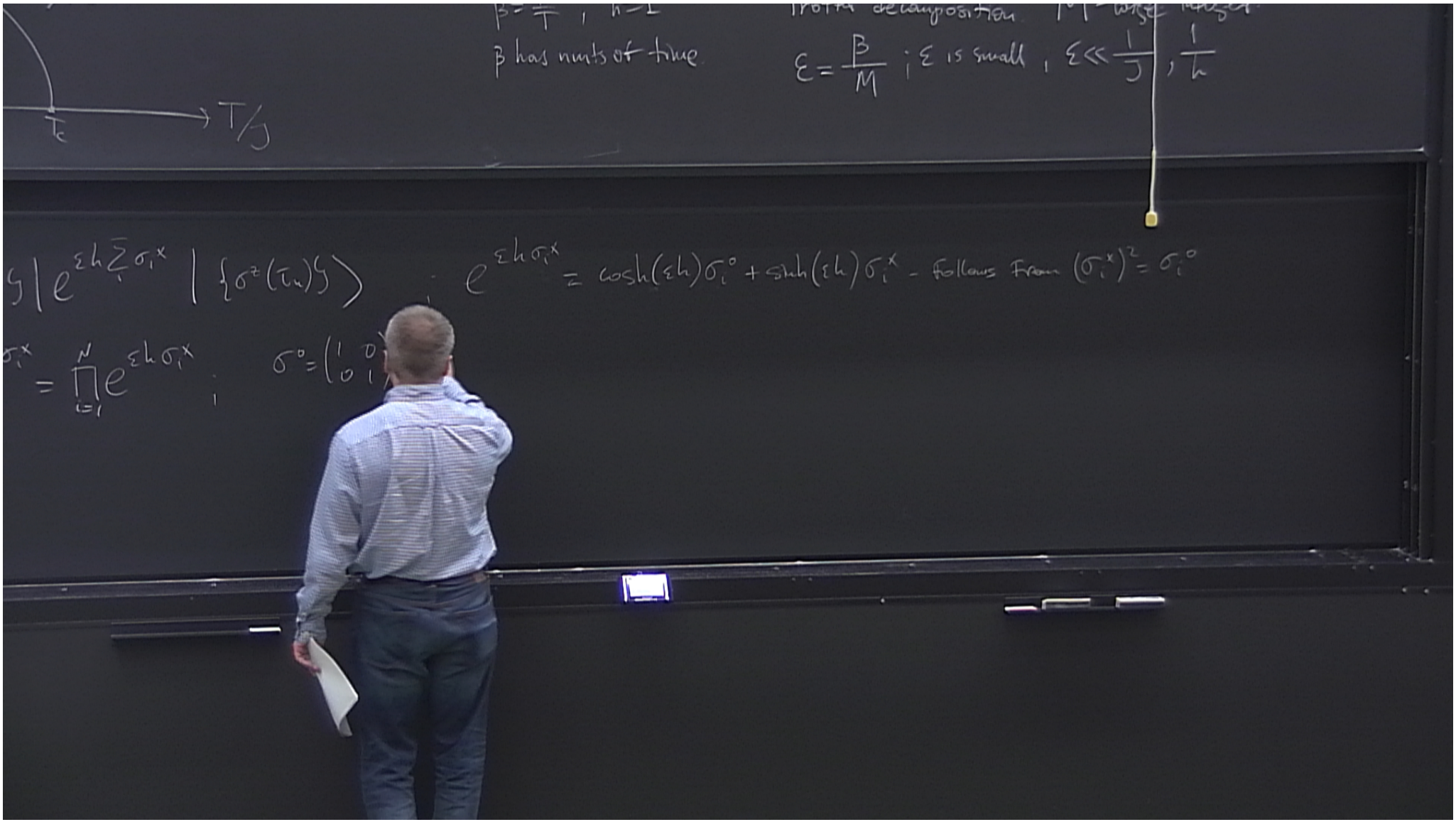

T_c
$1/J$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$
$e^{\varepsilon h \sigma_i^x}$
$e^{\varepsilon h \sum_i \sigma_i^x} = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$

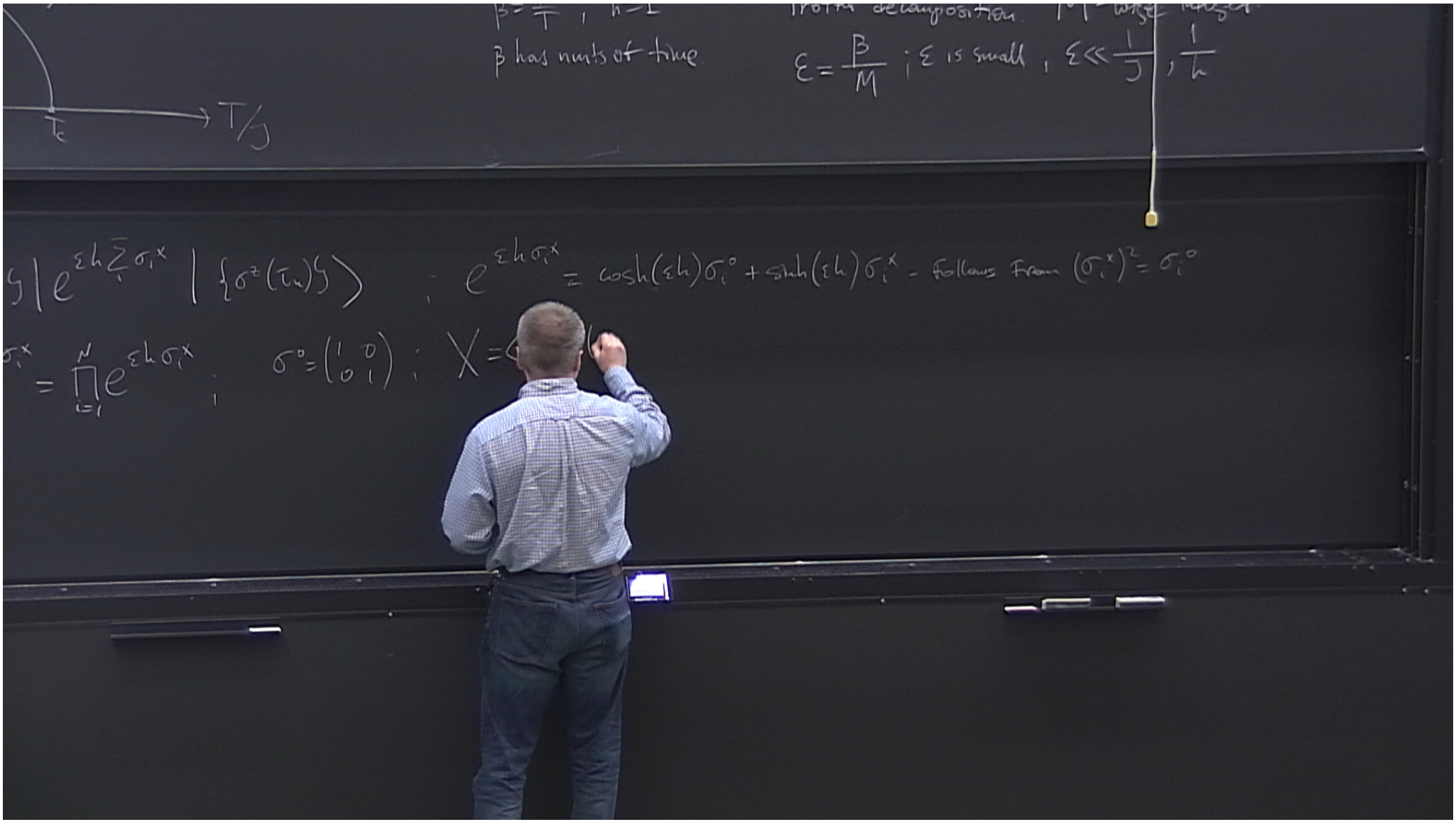

T_c
$1/J$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$
$e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0$
$e^{\varepsilon h \sum_i \sigma_i^x} = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$

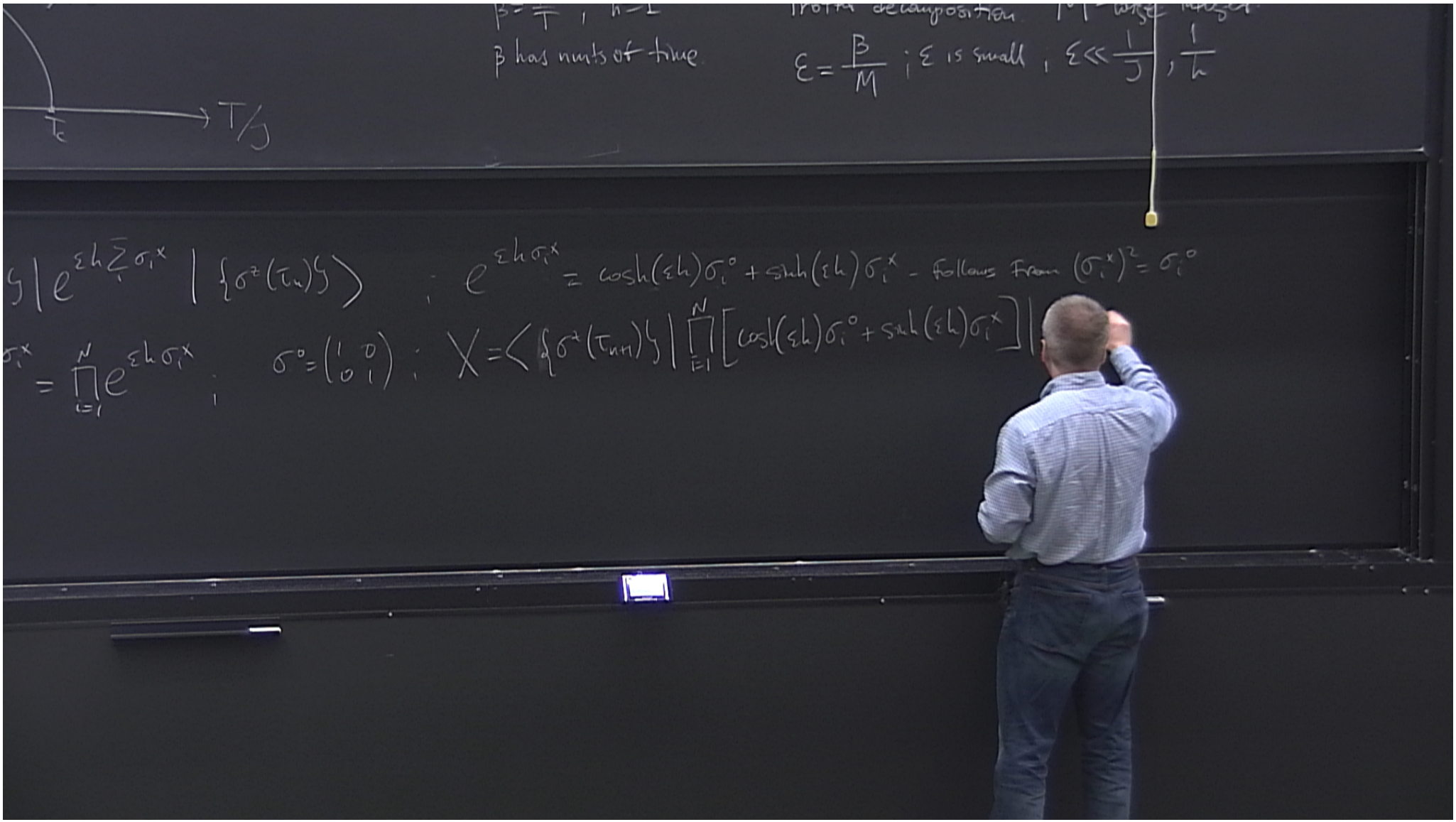

T_c
$1/J$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$
$e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)$
$e^{\varepsilon h \sum_i \sigma_i^x} = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$
$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

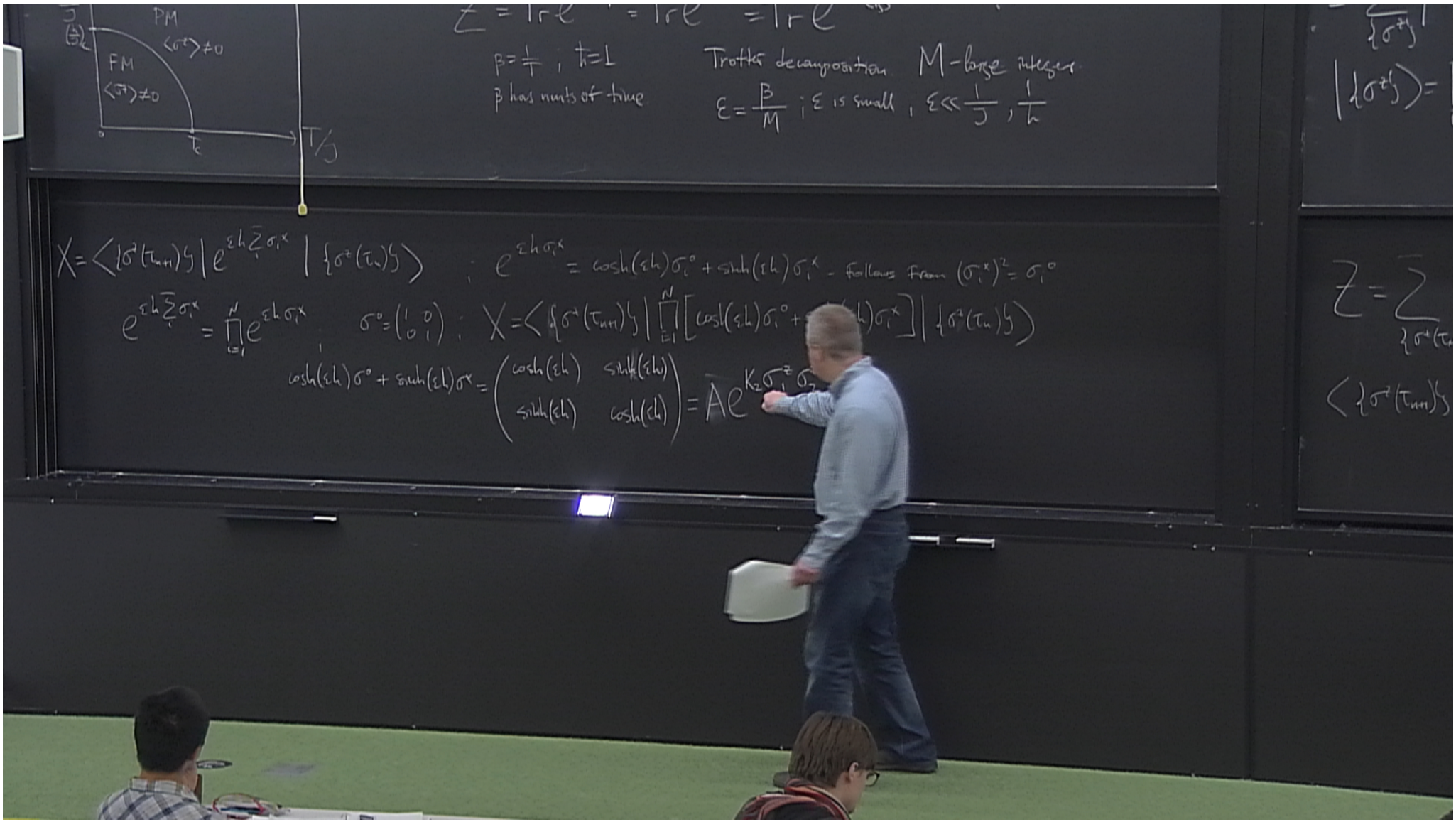

$\langle \sigma^z \rangle \neq 0$
0
T_c
T/J
β has units of time
$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$; $e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x$
$e^{\varepsilon h \sum_i \sigma_i^x} = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$; $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$


$\langle \sigma^z \rangle \neq 0$
0
T_c
T/J
β has units of time
$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}$
$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$; $e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x$
$e^{\varepsilon h \sum_i \sigma_i^x} = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$; $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$


β has units of time
ε = β/M ; ε is small , ε ≪ 1/J , 1/h
T/J
T_c
e^{εh σ_i^x} = cosh(εh) σ_i^0 + sinh(εh) σ_i^x - follows from (σ_i^x)^2 = σ_i^0
= ∏_{i=1}^{N} e^{εh σ_i^x}


β has units of time
ε = β/M ; ε is small , ε ≪ 1/J , 1/h
T/J
T_c
e^{εh Σ_i σ_i^x} = cosh(εh) σ_i^0 + sinh(εh) σ_i^x - follows from (σ_i^x)^2 = σ_i^0
σ_i^x = Π_{i=1}^N e^{εh σ_i^x} ; σ^0 = (1 0 / 0 1) ; X =


T_c
T/J
$\beta = \frac{1}{T}$, $\hbar = 1$
β has units of time
$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}, \frac{1}{h}$
$\langle \{\sigma^z(\tau_n)\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle$; $e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x$ - follows from $(\sigma_i^x)^2 = \sigma_i^0$
$\sigma_i^x = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x}$; $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $X = \langle \{\sigma^z(\tau_{n+1})\} | \prod_{i=1}^{N} [\cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x] |$

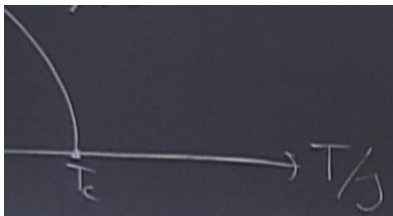

PM
FM
T/J
β = 1/T ; ħ=1
β has units of time
Trotter decomposition M-large integer
ε = β/M ; ε is small , ε ≪ 1/J , 1/h
follows from

$\beta = \frac{1}{T}$, $h = 1$

β has units of time

Trotter decomposition

$\varepsilon = \frac{\beta}{M}$; ε is small , $\varepsilon \ll \frac{1}{J}, \frac{1}{h}$

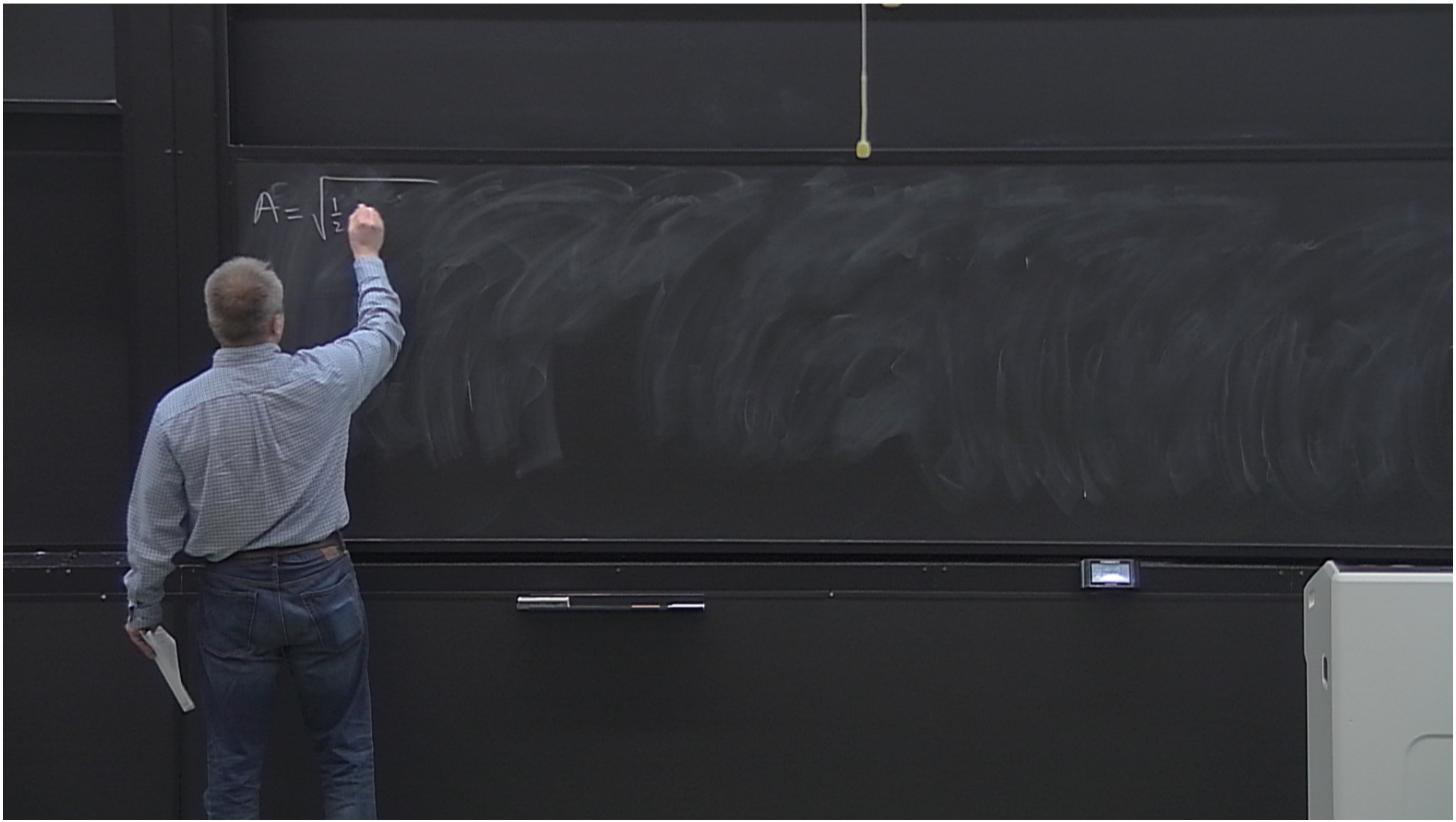


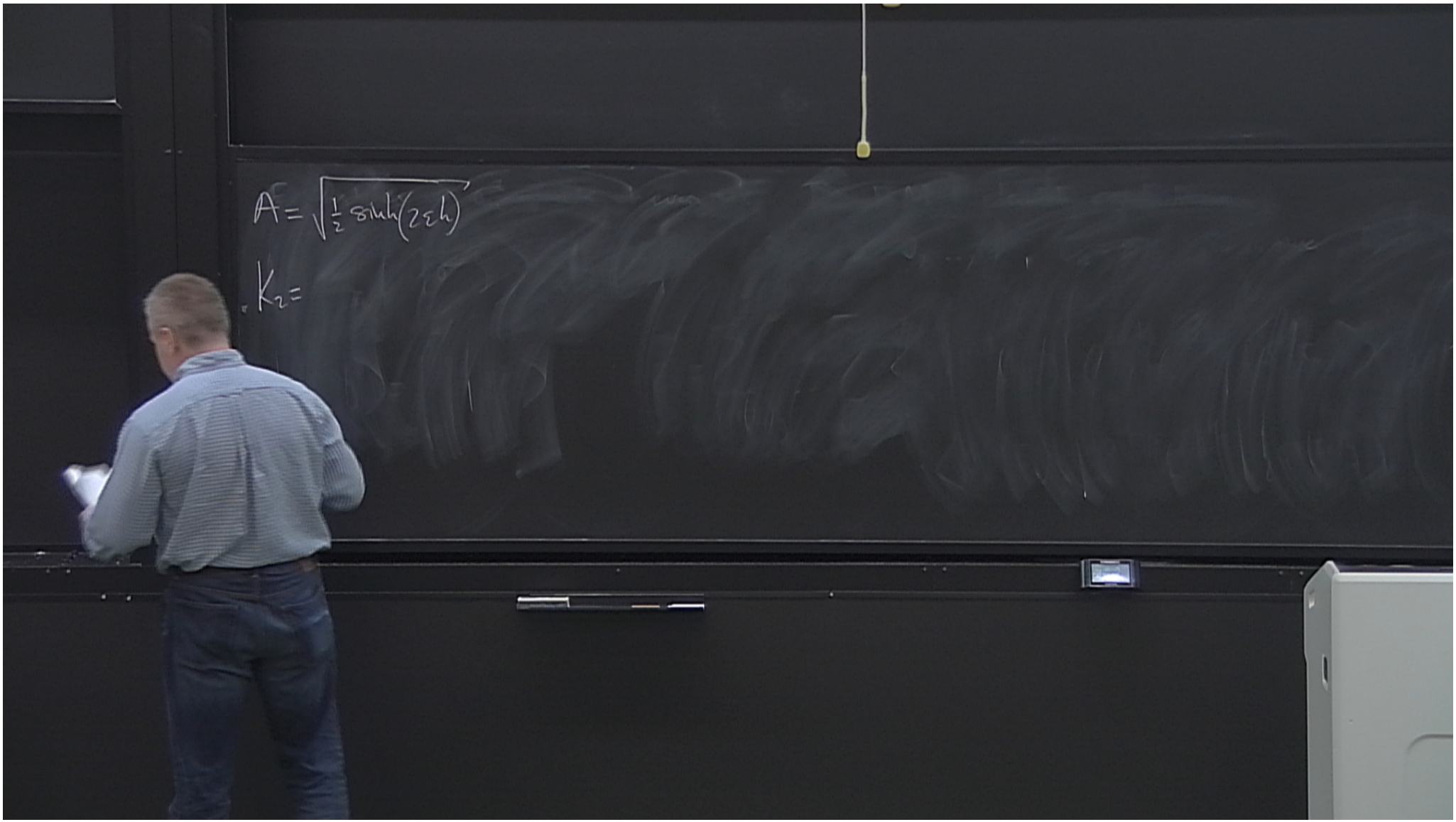
$$\langle \{\sigma^z(\tau_{n+1})\} | e^{\varepsilon h \sum_i \sigma_i^x} | \{\sigma^z(\tau_n)\} \rangle \quad ; \quad e^{\varepsilon h \sigma_i^x} = \cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x$$

follows from $(\sigma_i^x)^2 = \sigma_i^0$

$$\sigma_i^x = \prod_{i=1}^{N} e^{\varepsilon h \sigma_i^x} \quad ; \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad X = \langle \{\sigma^z(\tau_{n+1})\} | \prod_{i=1}^{N} \left[\cosh(\varepsilon h)\sigma_i^0 + \sinh(\varepsilon h)\sigma_i^x\right] | \{\sigma^z(\tau_n)\} \rangle$$

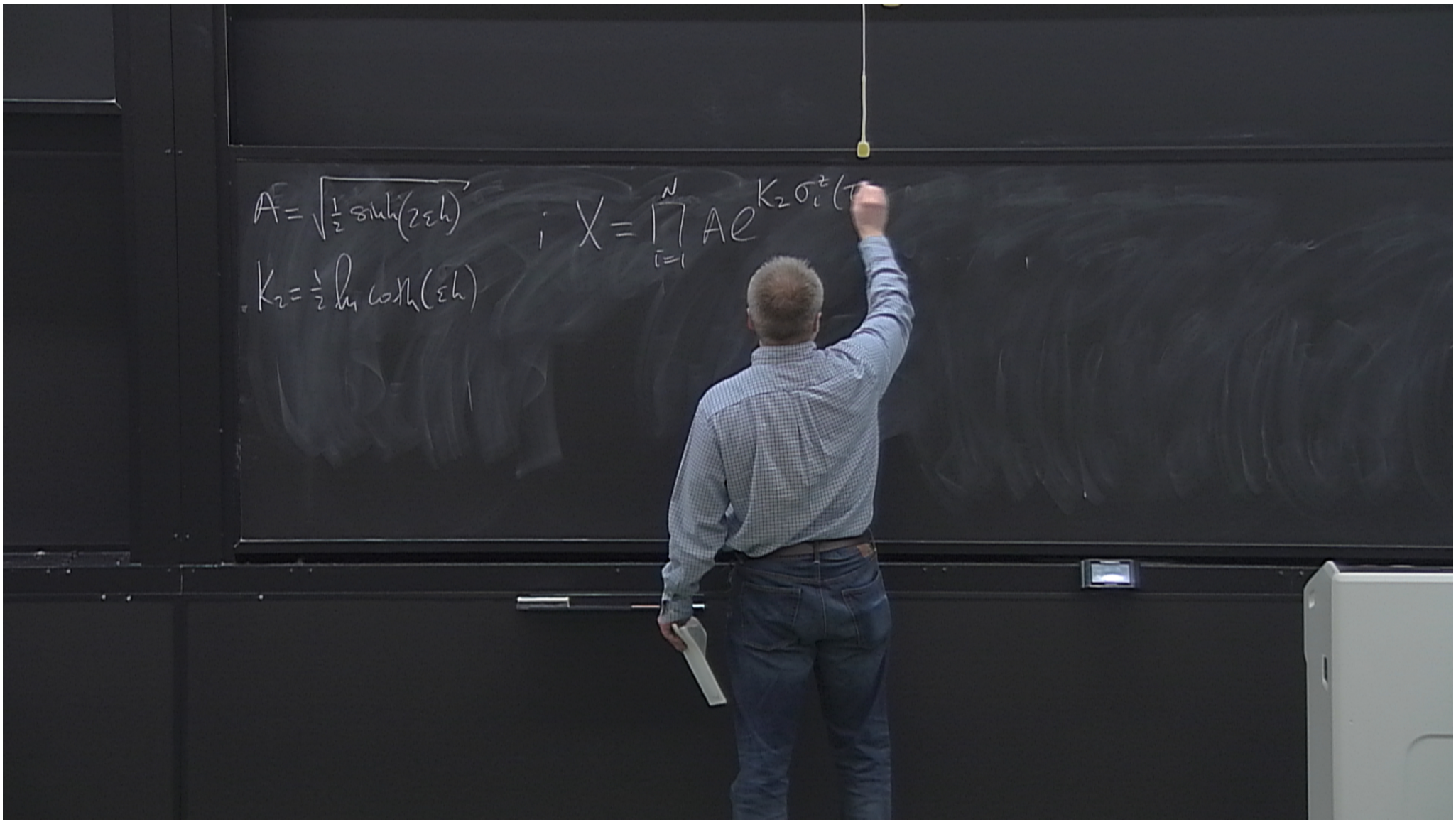
$$\cosh(\varepsilon h)\sigma^0 + \sinh(\varepsilon h)\sigma^x = \begin{pmatrix} \cosh(\varepsilon h) & \sinh(\varepsilon h) \\ \sinh(\varepsilon h) & \cosh(\varepsilon h) \end{pmatrix} = A e^{K_2 \sigma_1^z \sigma_2^z} \quad ; \quad \begin{aligned} A e^{K_2} &= \cosh(\varepsilon h) \\ A e^{-K_2} &= \sinh(\varepsilon h) \end{aligned}$$

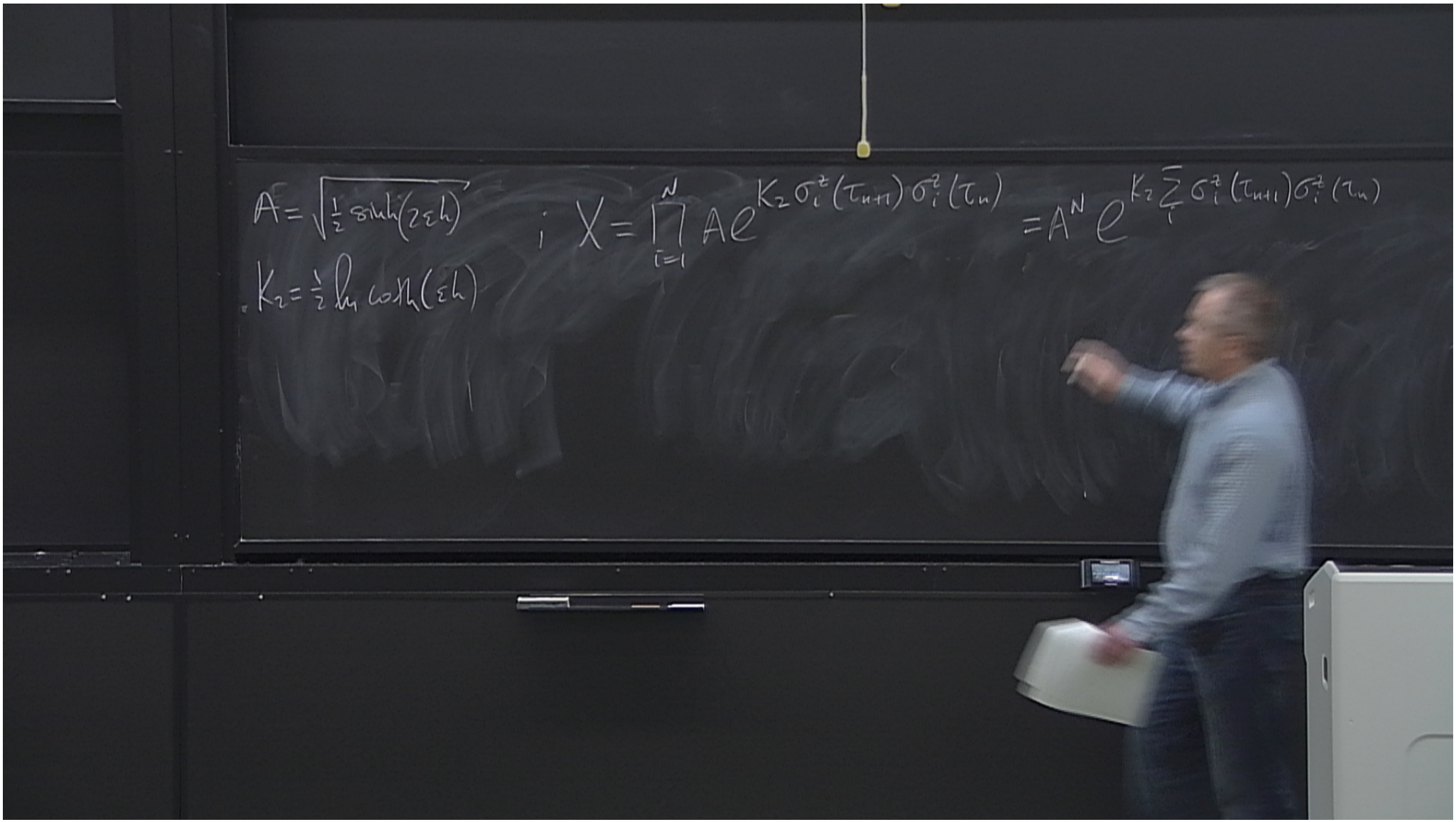

A = √(1/2

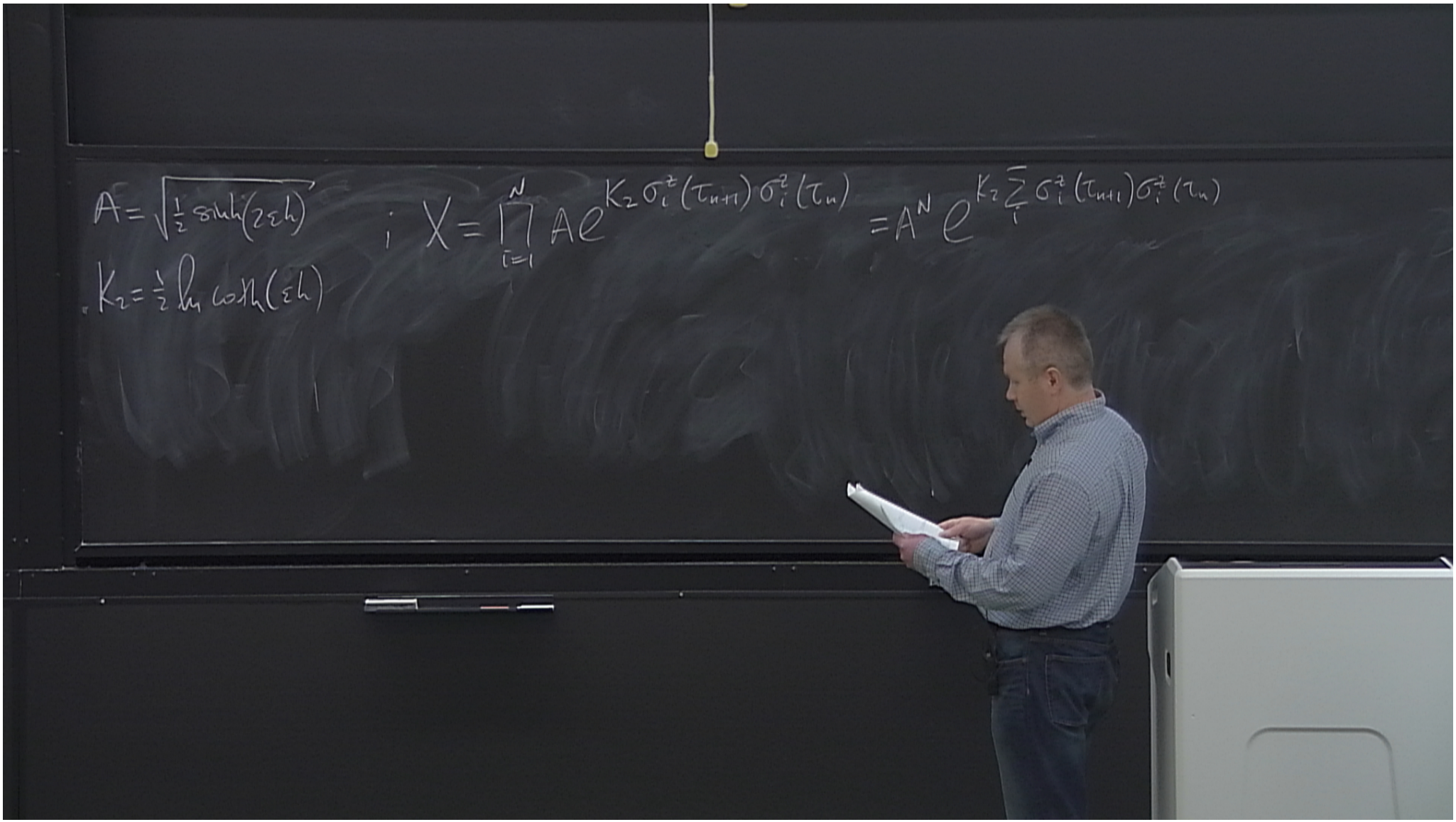

$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$
$K_2 =$

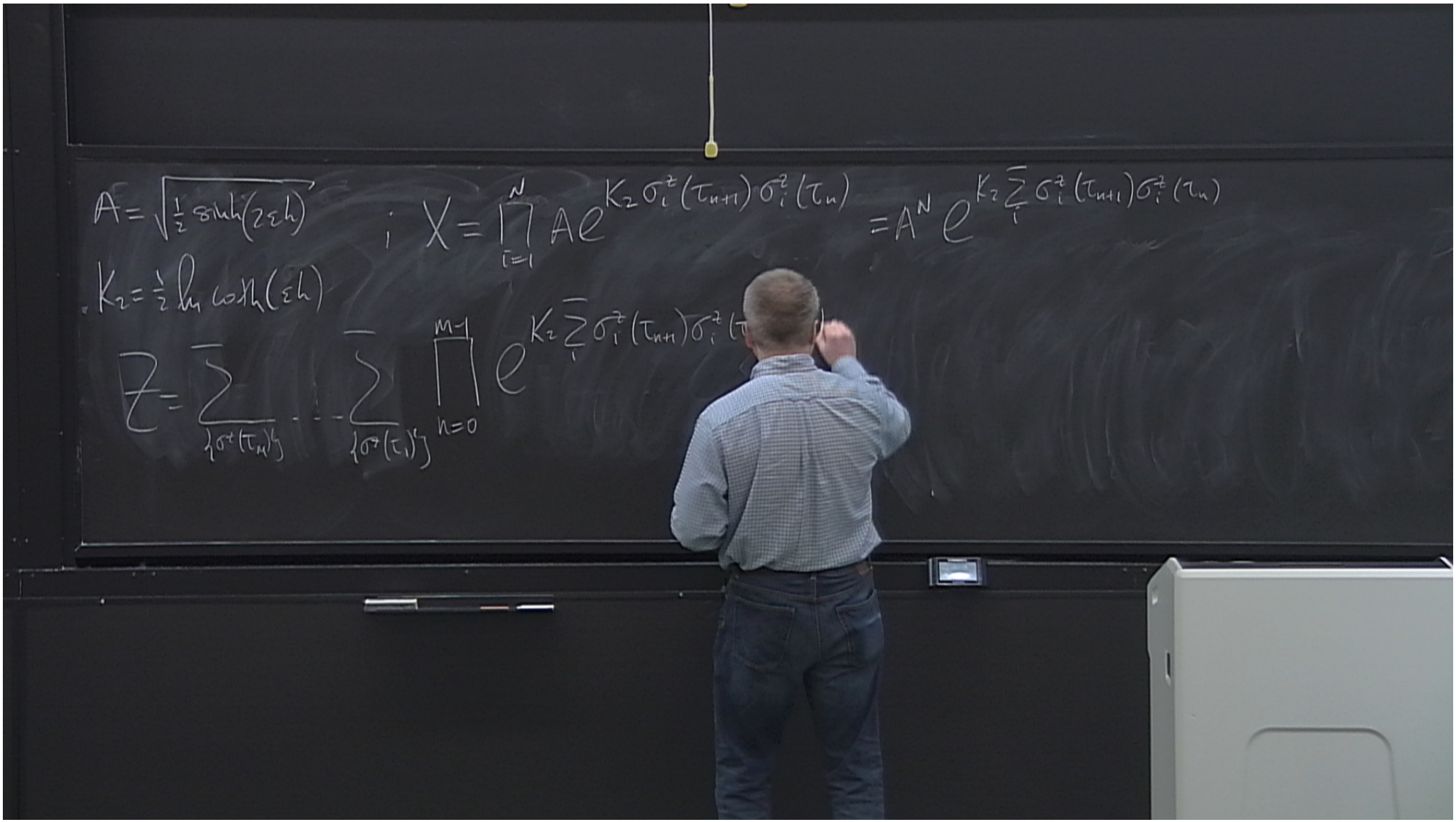
$$A = \sqrt{\tfrac{1}{2}\sinh(2\varepsilon h)} \quad ; \quad X = 17$$

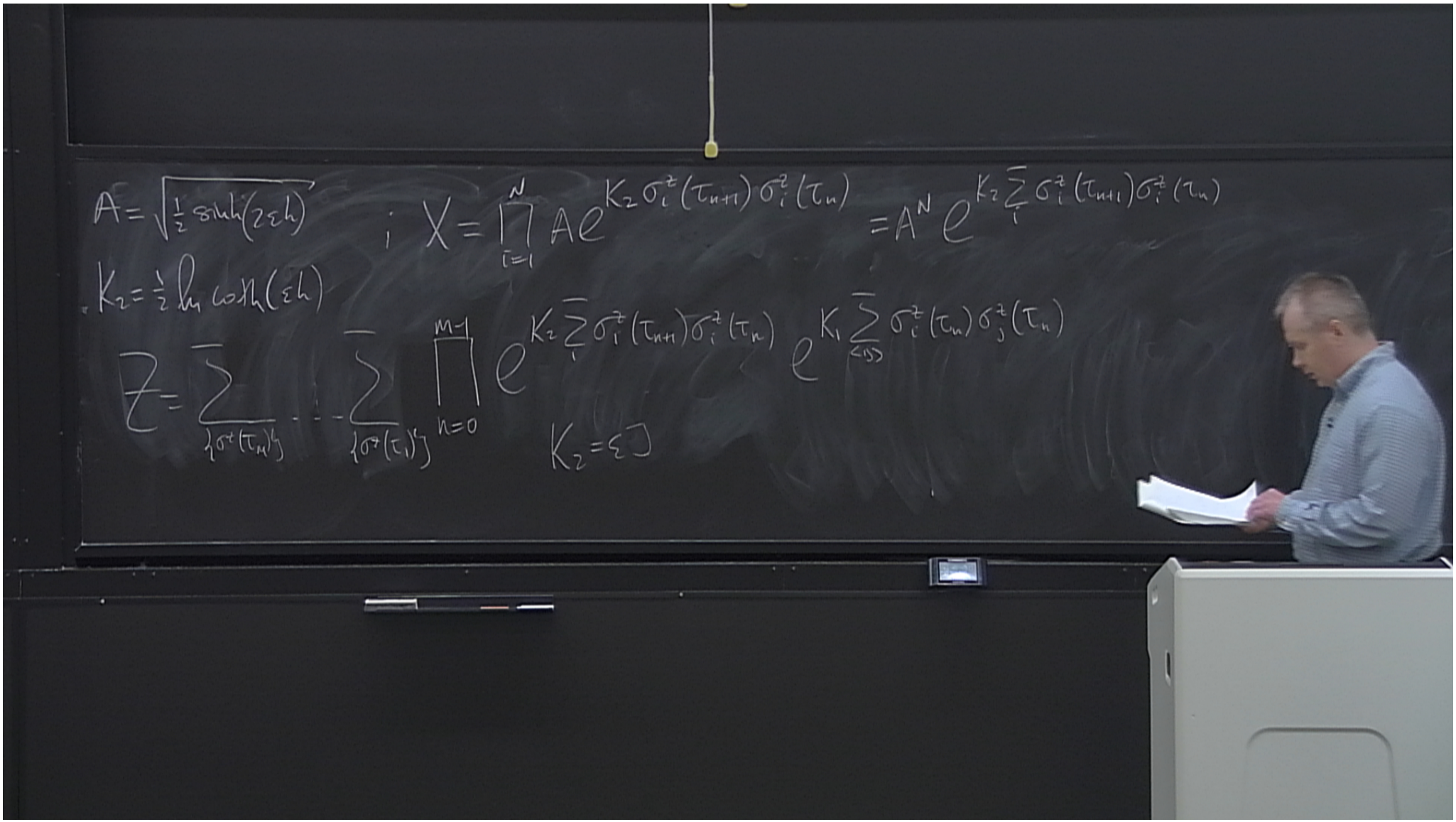
$$K_2 = \tfrac{1}{2}\ln\coth(\varepsilon h)$$


$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$; $X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z (\tau}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$


$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$
$X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$


$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$
$X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$


$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$; $X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$
$Z = \sum_{\{\sigma^z(\tau_N)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$


$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$; $X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$
$Z = \sum_{\{\sigma^z(\tau_N)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} e^{K_1 \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)\sigma_j^z(\tau_n)}$
$K_2 = \varepsilon J$

$$A = \sqrt{\tfrac{1}{2}\sinh(2\varepsilon h)} \quad ; \quad X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$$

$$K_2 = \tfrac{1}{2}\ln\coth(\varepsilon h)$$

$$Z = \sum_{\{\sigma^z(\tau_N)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} \, e^{K_1 \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)\sigma_j^z(\tau_n)}$$

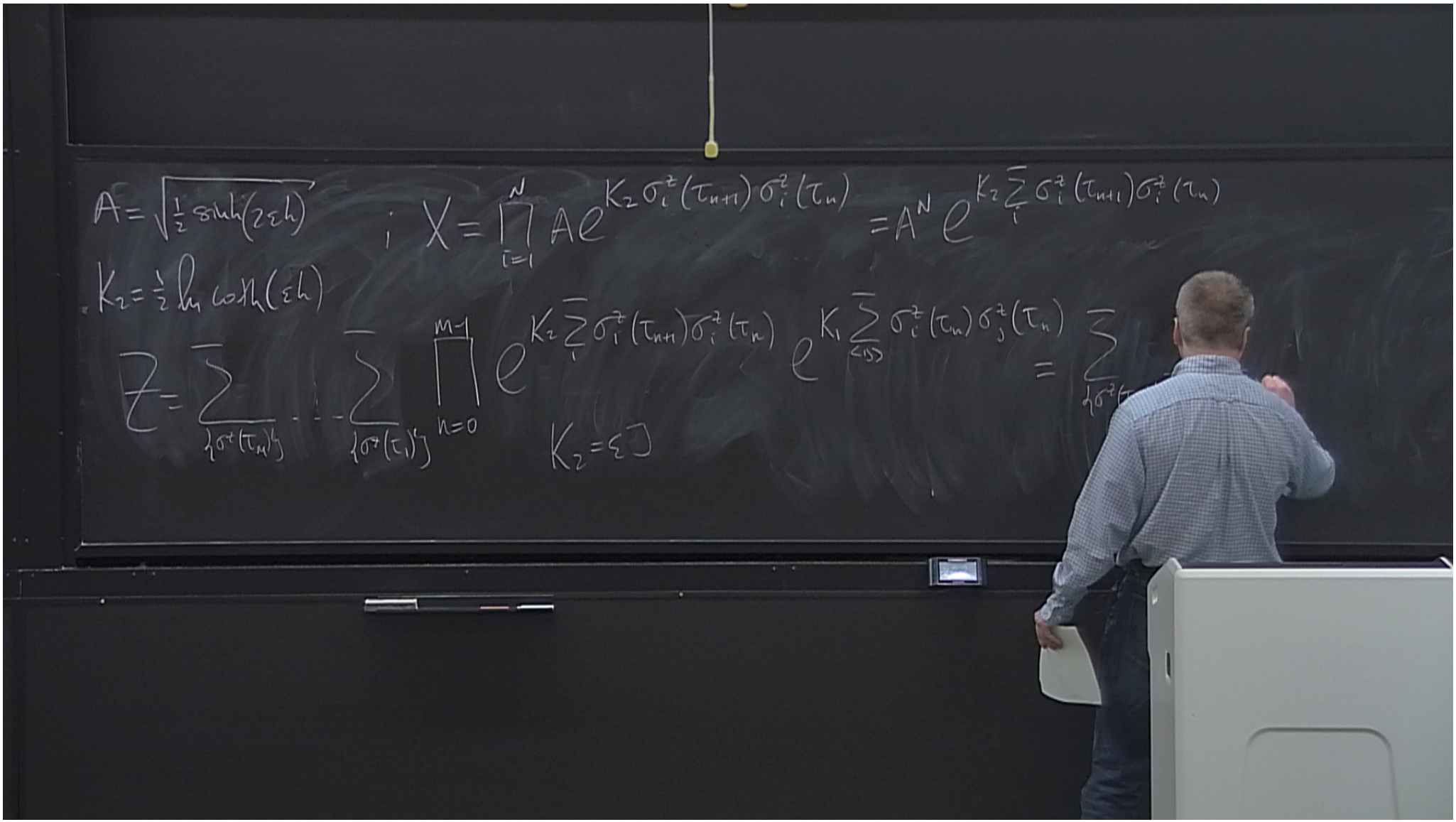
$$K_2 = \varepsilon J$$

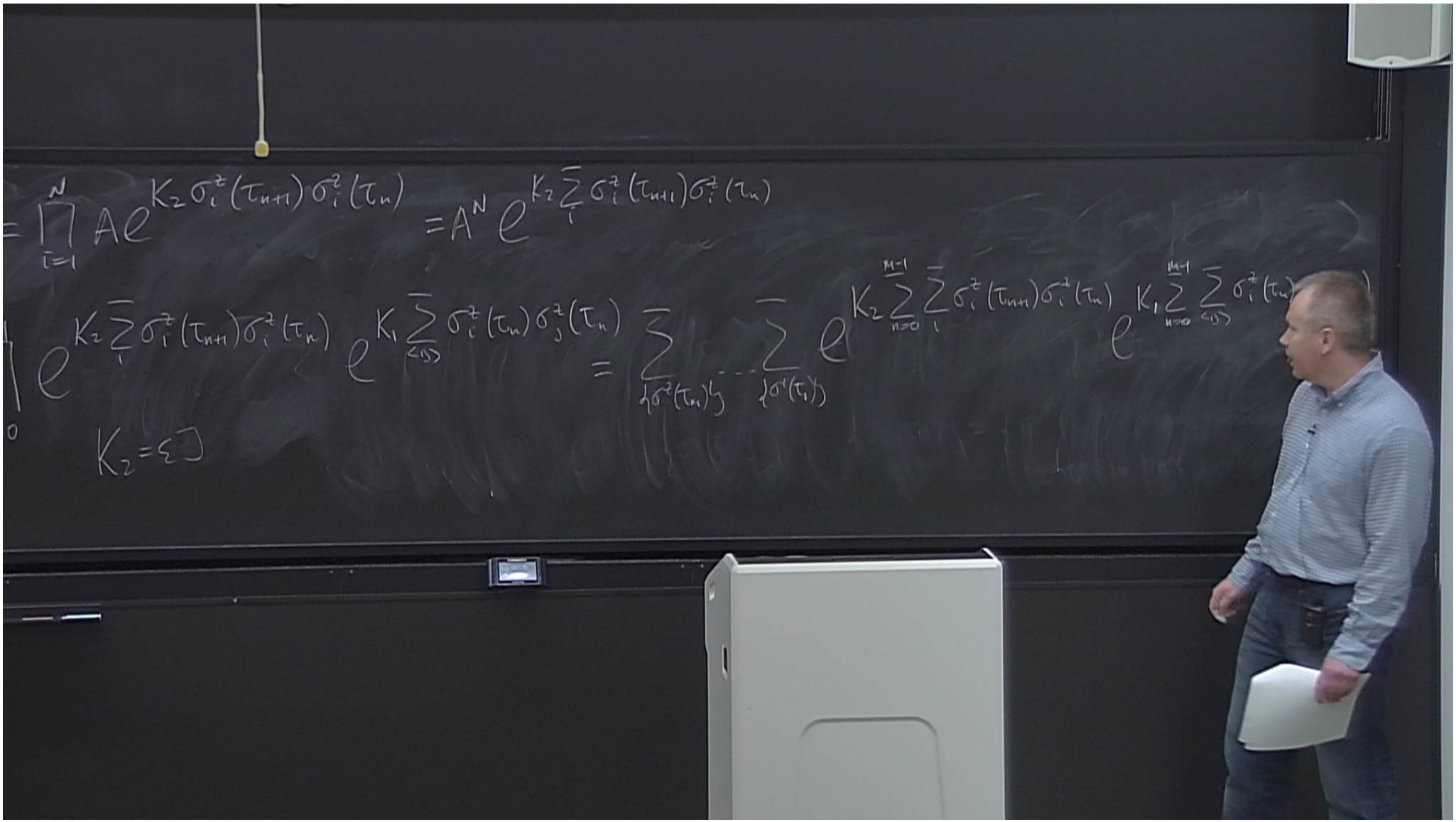
$$A = \sqrt{\tfrac{1}{2}\sinh(2\varepsilon h)} \quad ; \quad X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$$

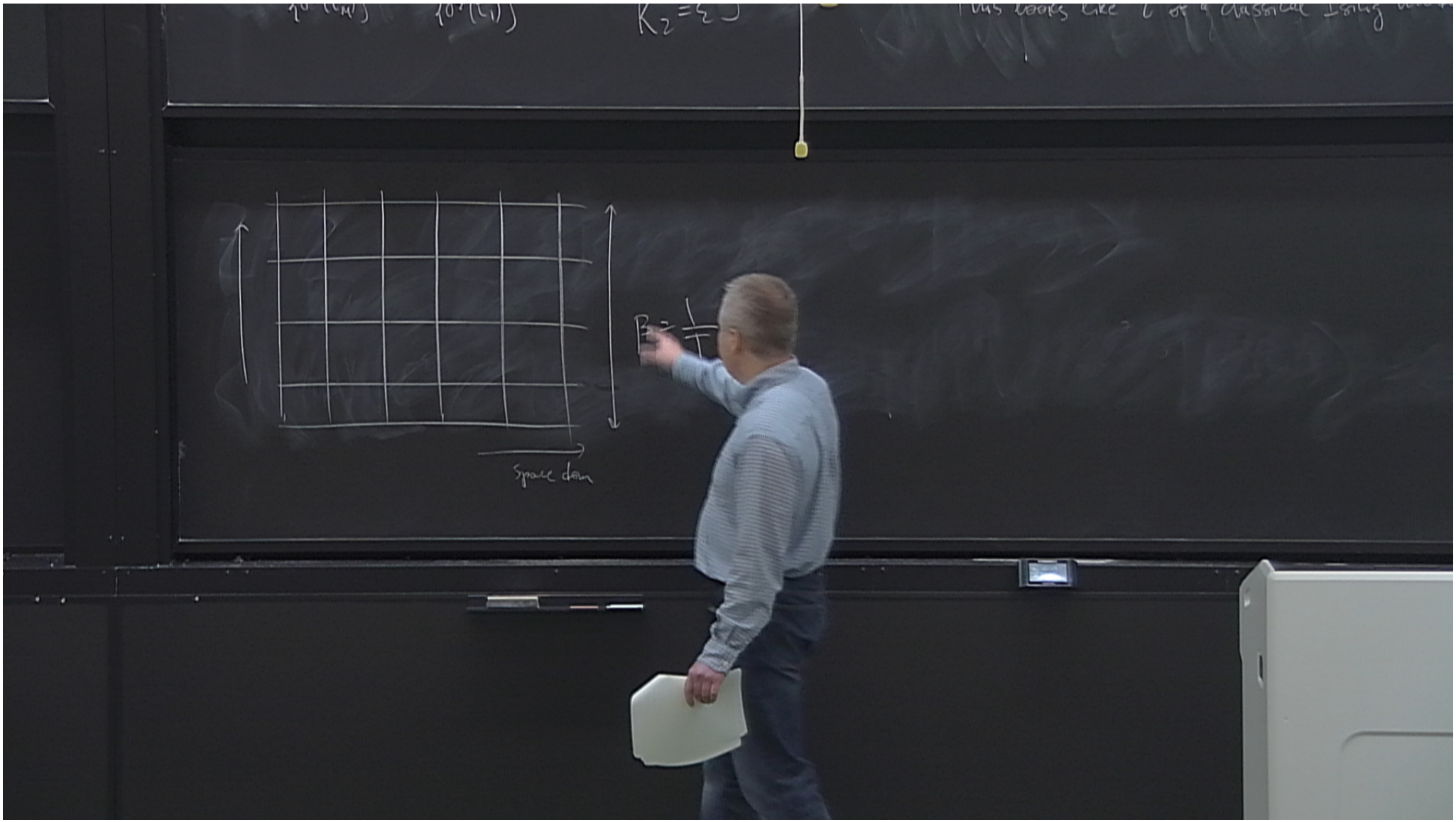
$$K_2 = \tfrac{1}{2}\ln\coth(\varepsilon h)$$

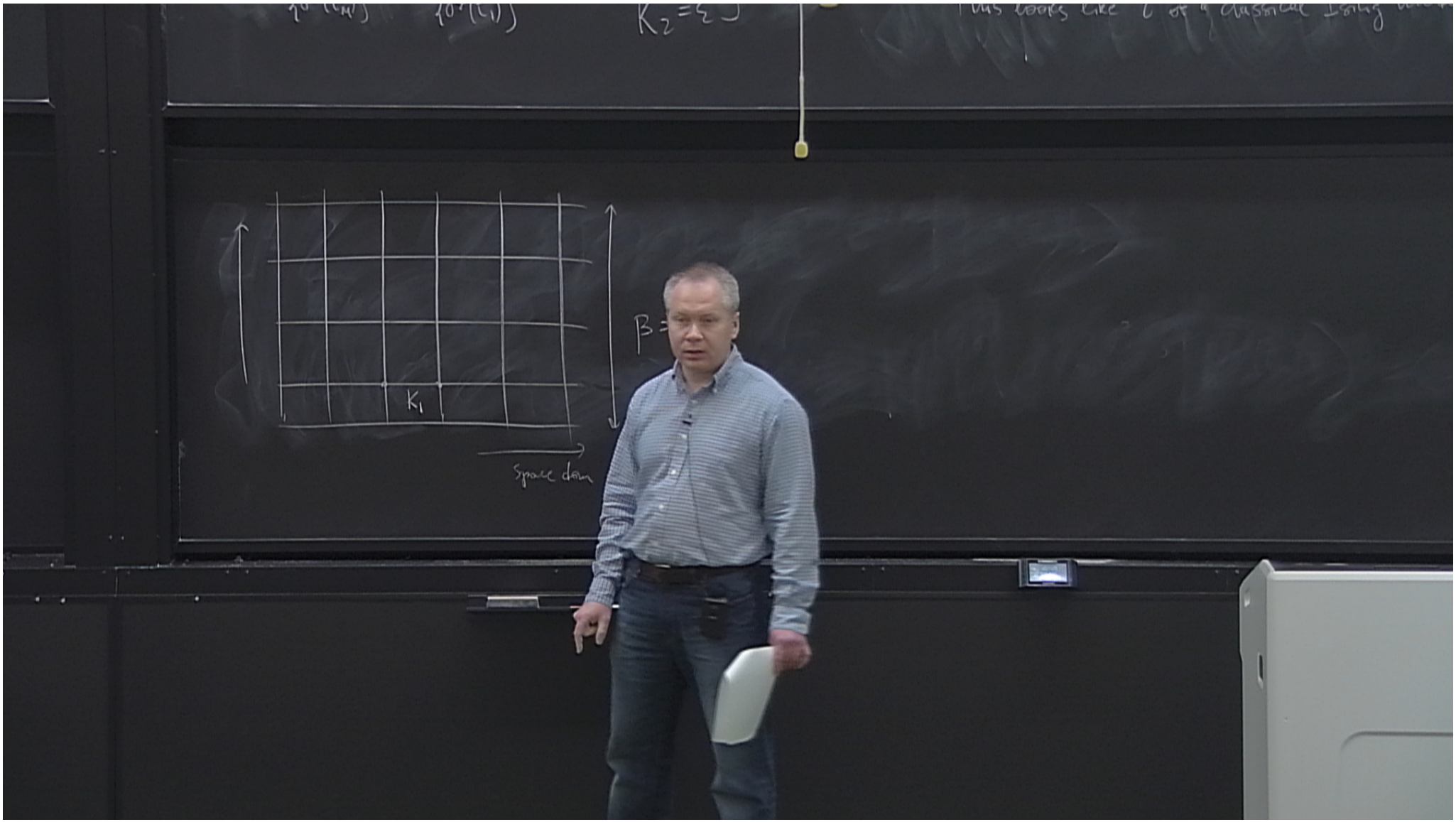
$$Z = \sum_{\{\sigma^z(\tau_N)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} \, e^{K_1 \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)\sigma_j^z(\tau_n)}$$

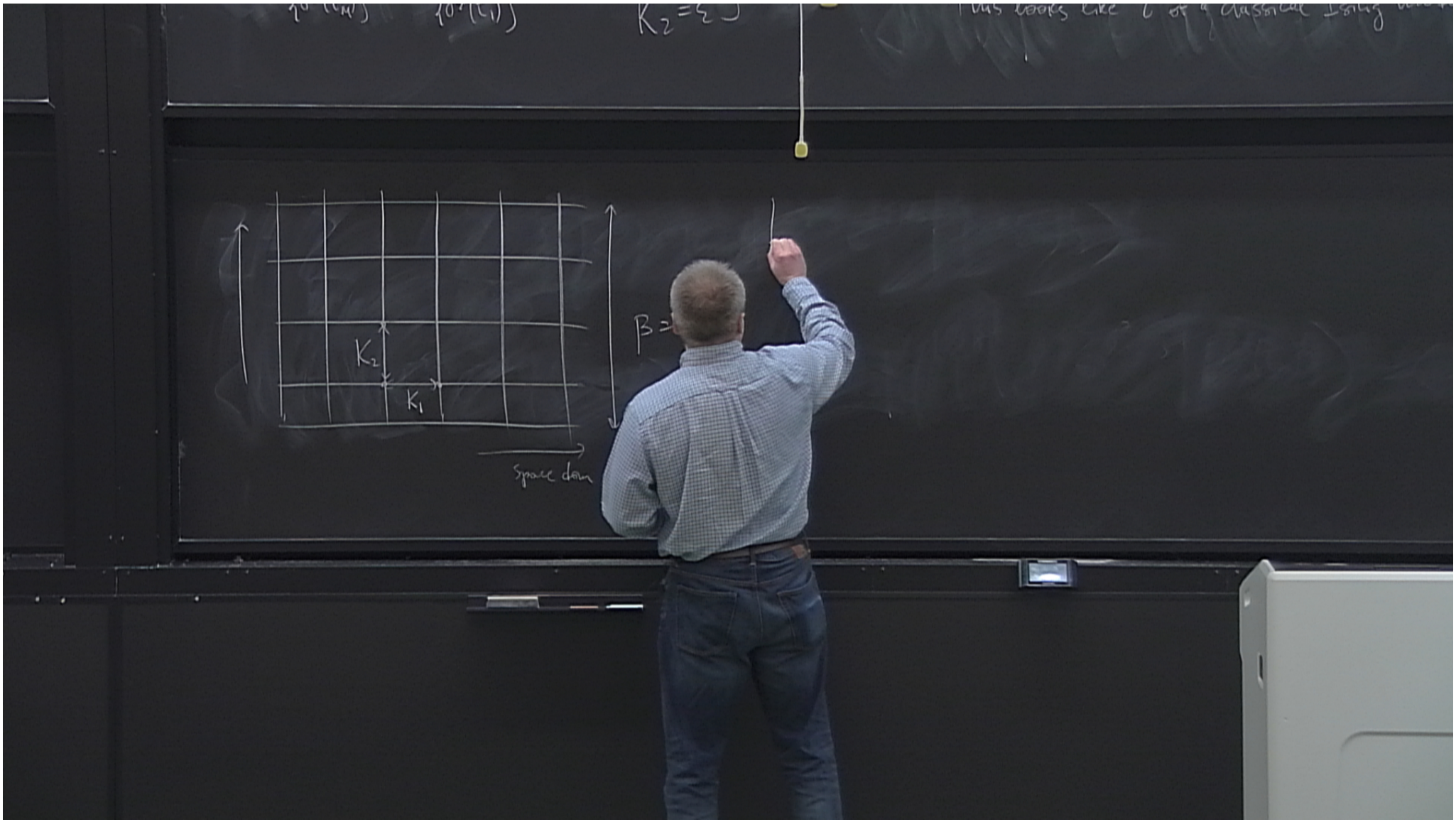
$$K_2 = \varepsilon J$$

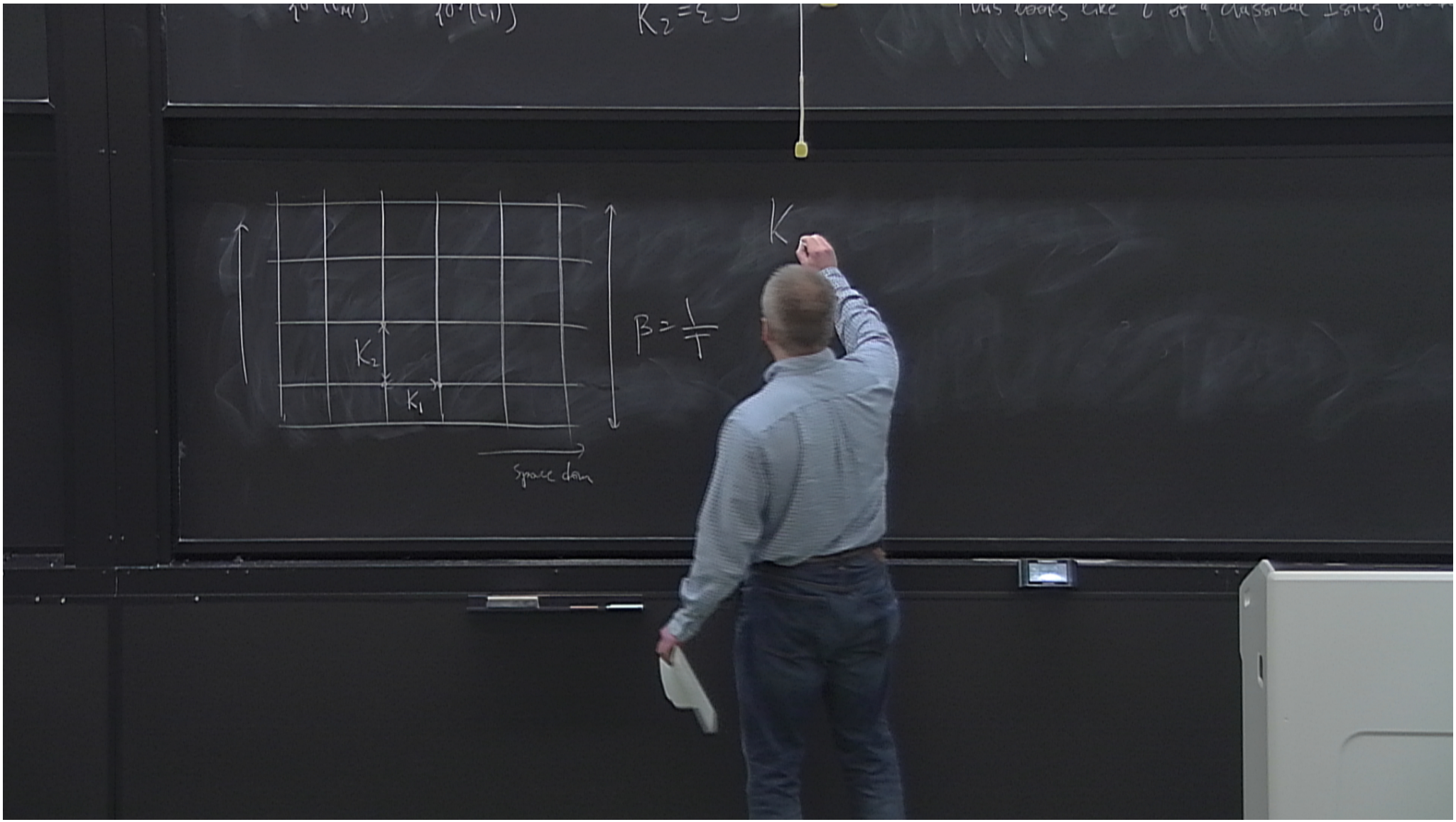

$A = \sqrt{\frac{1}{2}\sinh(2\varepsilon h)}$; $X = \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$
$K_2 = \frac{1}{2}\ln\coth(\varepsilon h)$
$Z = \sum_{\{\sigma^z(\tau_N)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0}^{M-1} e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} e^{K_1 \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)\sigma_j^z(\tau_n)} = \sum$
$K_2 = \varepsilon J$

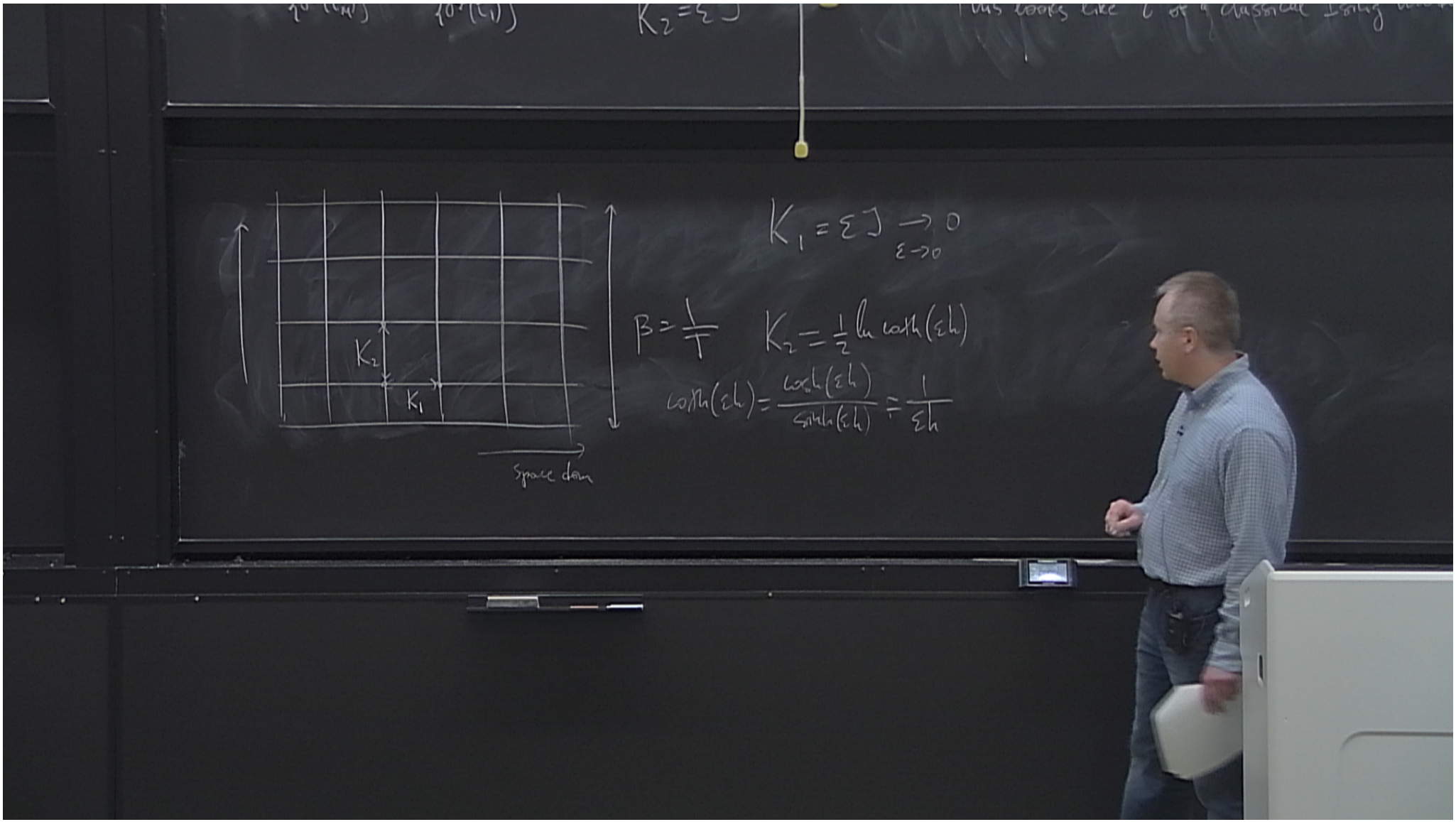

$= \prod_{i=1}^{N} A e^{K_2 \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} = A^N e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)}$
$e^{K_2 \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} e^{K_1 \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)\sigma_j^z(\tau_n)} = \sum_{\{\sigma^z(\tau_n)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} e^{K_2 \sum_{n=0}^{M-1} \sum_i \sigma_i^z(\tau_{n+1})\sigma_i^z(\tau_n)} e^{K_1 \sum_{n=0}^{M-1} \sum_{\langle ij \rangle} \sigma_i^z(\tau_n)}$
$K_2 = \xi J$

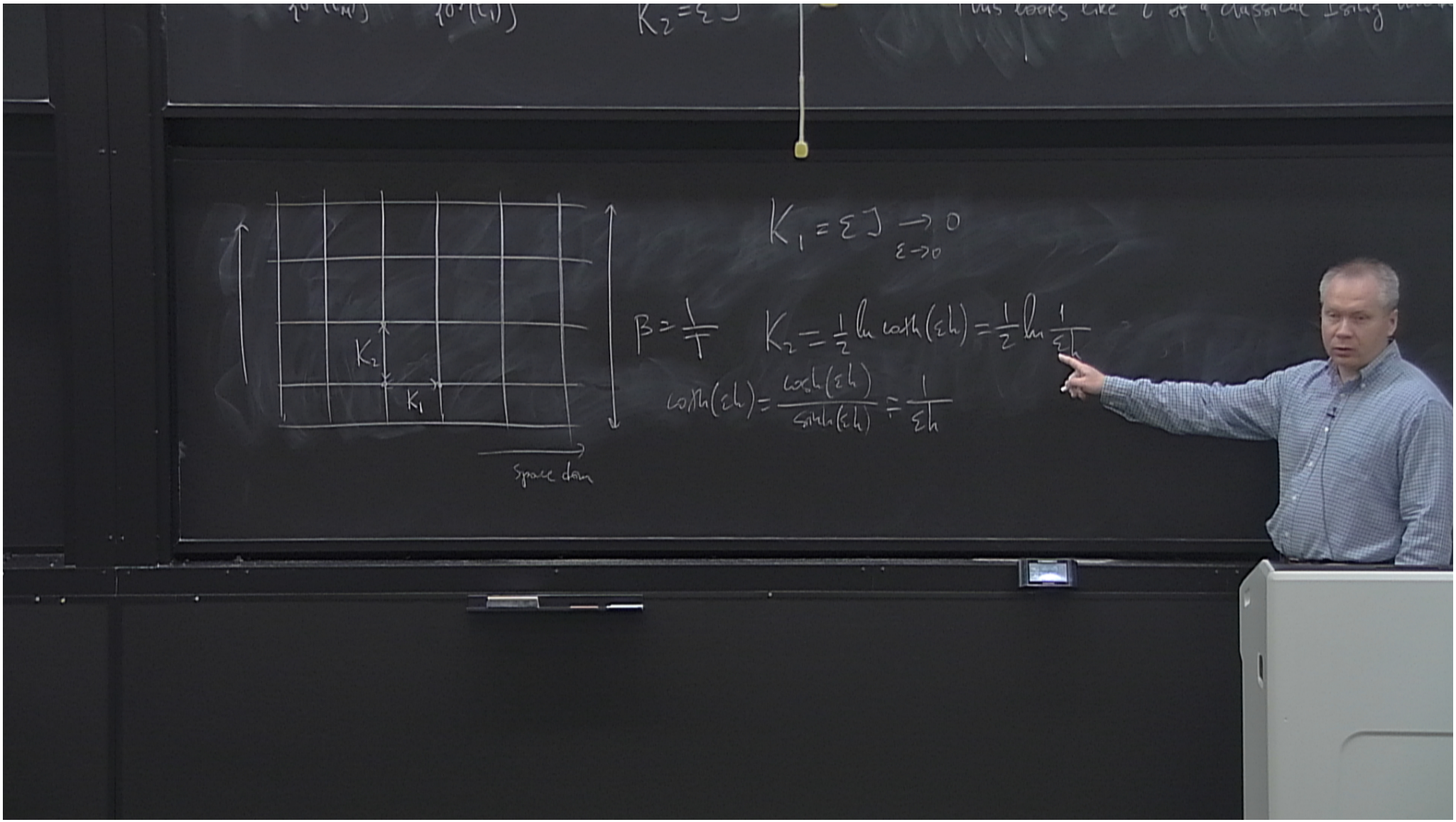

Space down

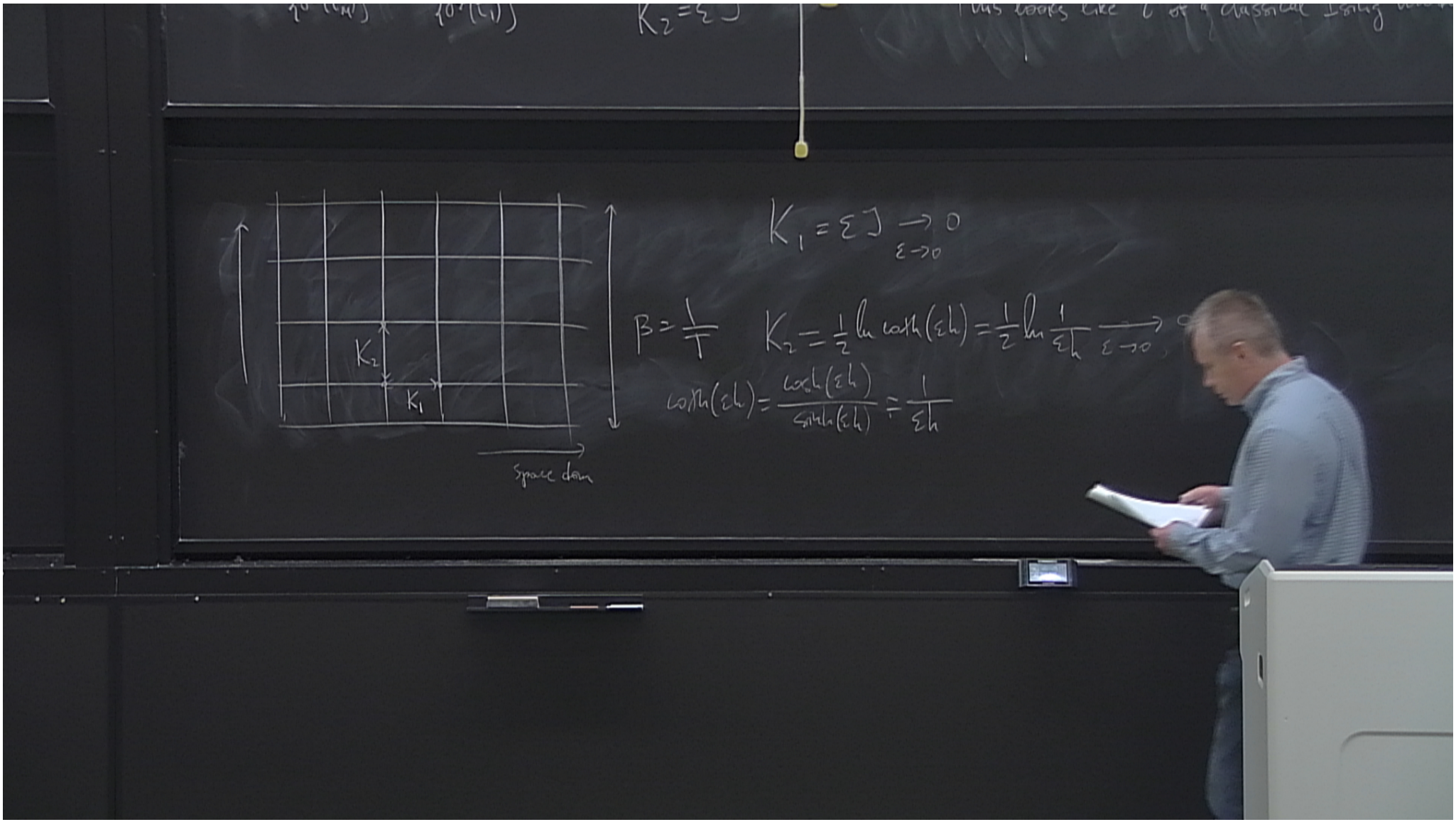

K_1
Space down

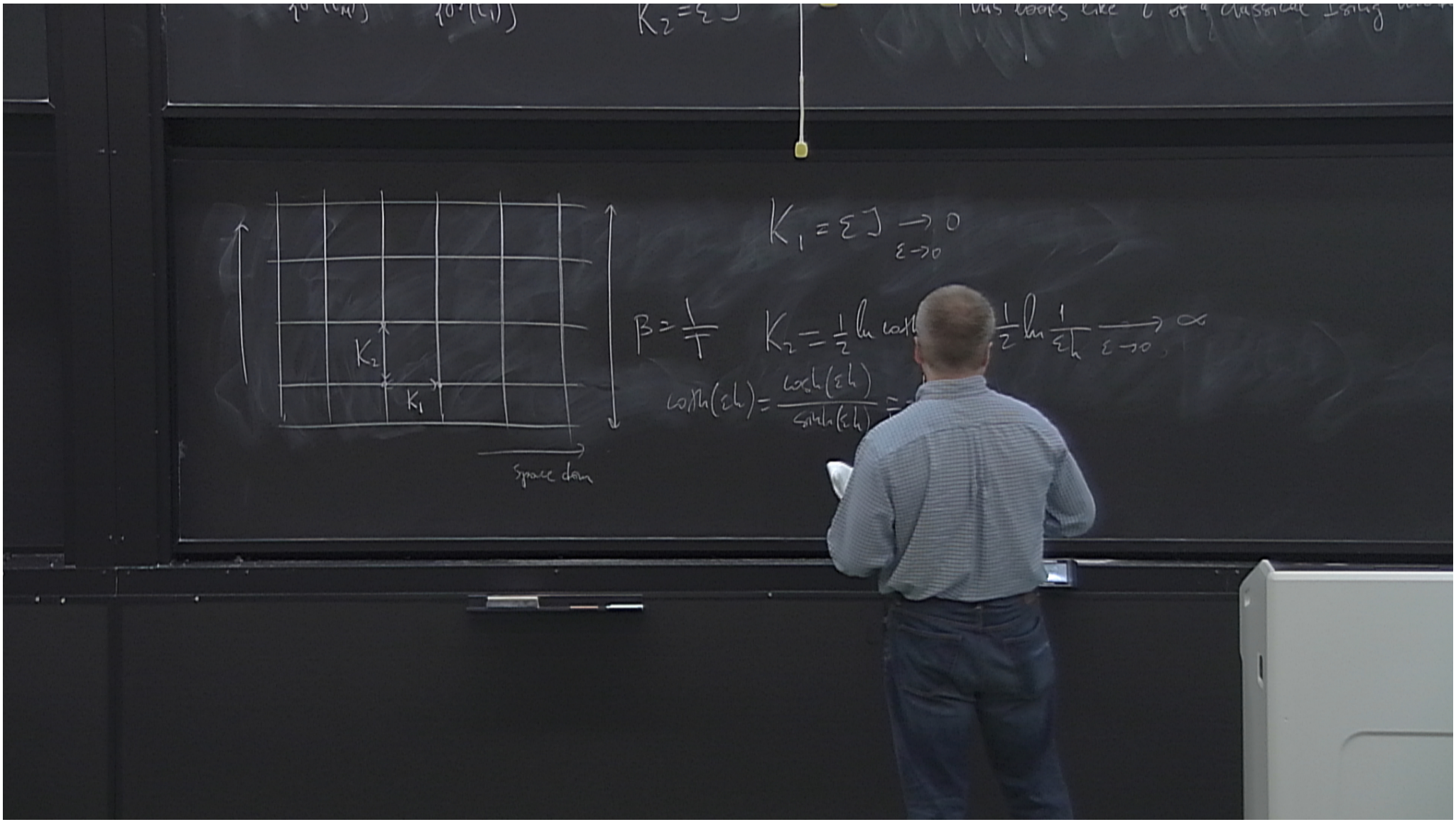

K_2
K_1
Space down


K_2
K_1
$\beta = \frac{1}{T}$
K
Space dom

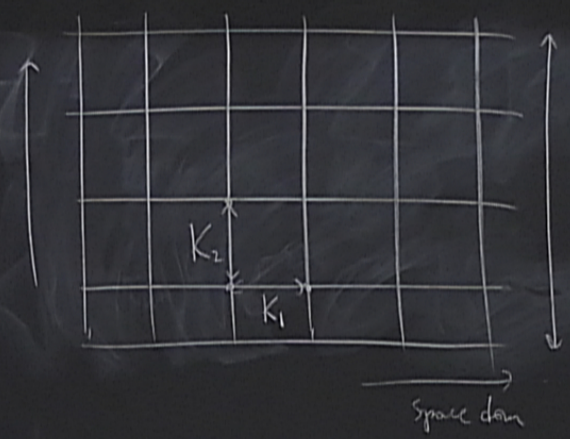

$K_2 = \varepsilon J$
$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$
$\beta = \frac{1}{T}$
$K_2 = \frac{1}{2} \ln \coth(\varepsilon h)$
$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \approx \frac{1}{\varepsilon h}$
K_2
K_1
Space down


$K_2 = \varepsilon J$
$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$
$\beta = \frac{1}{T}$
$K_2 = \frac{1}{2} \ln \coth(\varepsilon h) = \frac{1}{2} \ln \frac{1}{\varepsilon h}$
$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \approx \frac{1}{\varepsilon h}$
K_2
K_1
Space down


$K_2 = \varepsilon J$
$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$
$\beta = \frac{1}{T}$
$K_2 = \frac{1}{2} \ln \coth(\varepsilon h) = \frac{1}{2} \ln \frac{1}{\varepsilon h} \xrightarrow[\varepsilon \to 0]{}$
$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \approx \frac{1}{\varepsilon h}$
K_2
K_1
Space down


$K_2 = \varepsilon J$
$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$
$\beta = \frac{1}{T}$
$K_2 = \frac{1}{2} \ln \coth \ldots \frac{1}{2} \ln \frac{1}{\varepsilon h} \xrightarrow[\varepsilon \to 0]{} \infty$
$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)}$
K_2
K_1
space dim

$K_2 = \varepsilon J$

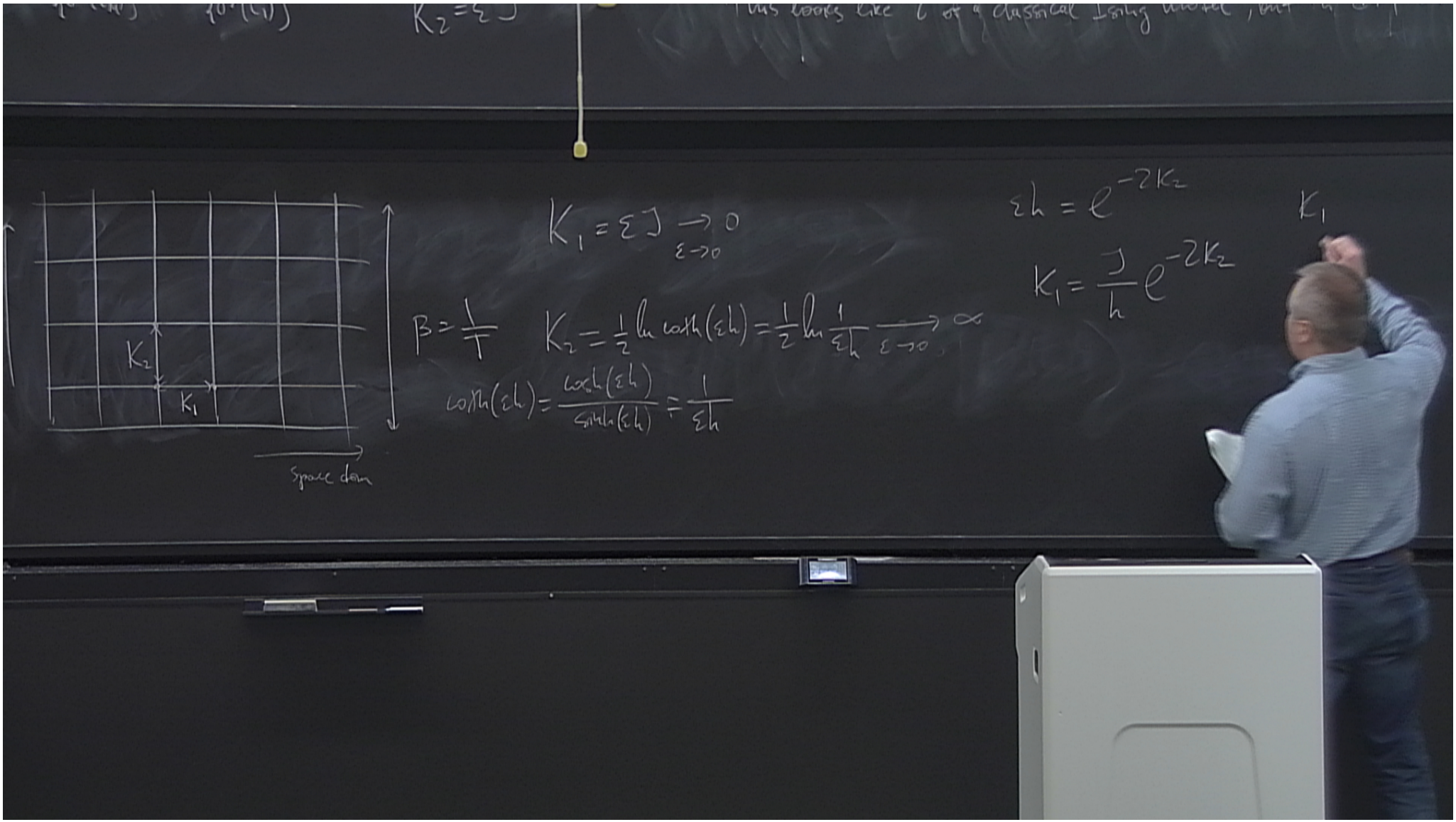


$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$

$\varepsilon h = e^{-2K_2}$

$\beta = \frac{1}{T}$ $\quad K_2 = \frac{1}{2} \ln \coth(\varepsilon h) = \frac{1}{2} \ln \frac{1}{\varepsilon h} \xrightarrow[\varepsilon \to 0]{} \infty$

$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \approx \frac{1}{\varepsilon h}$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This looks like 2 of a classical Ising model, but
K_2
K_1
Space dim
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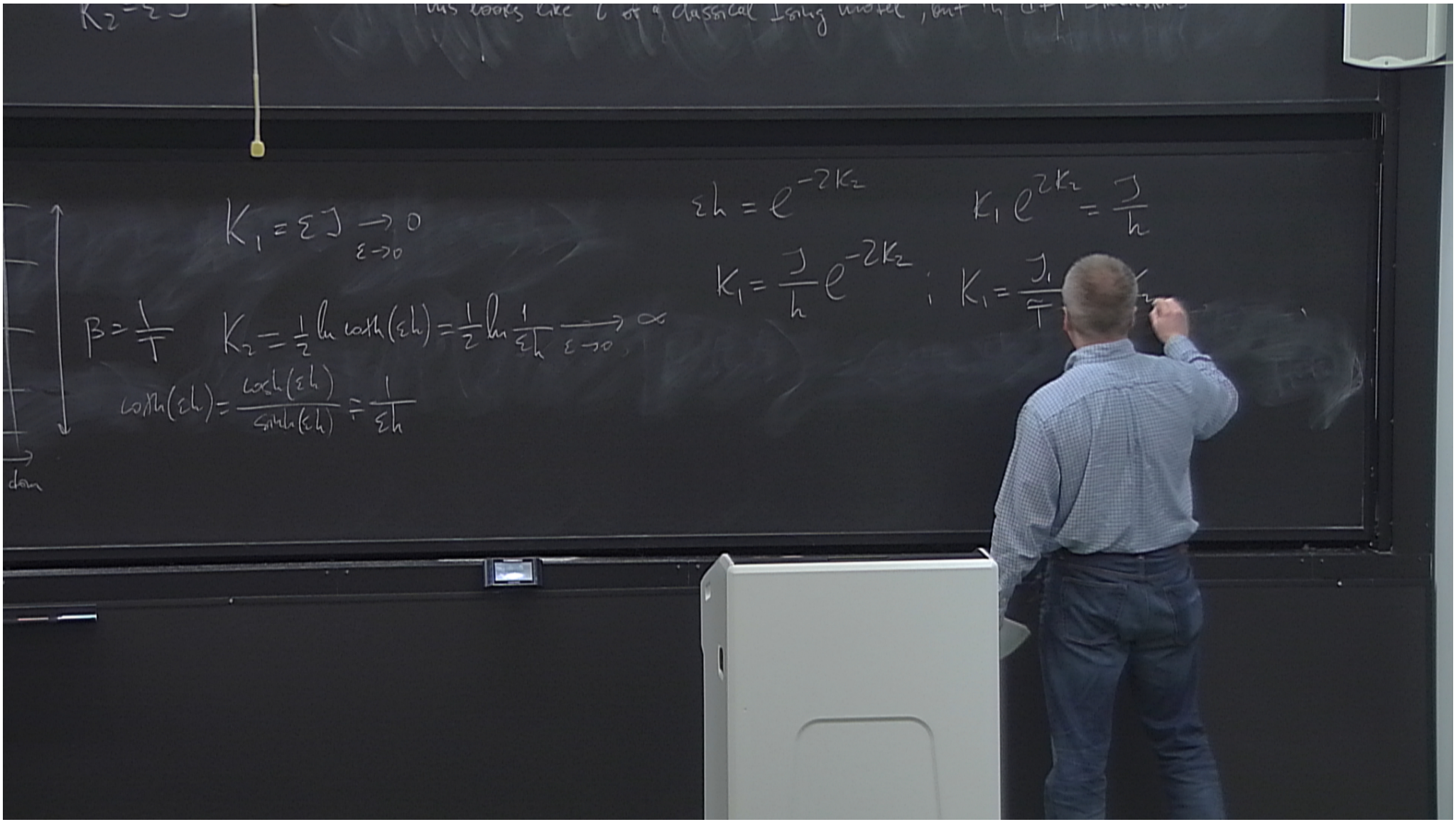
K_1
$K_1 = \frac{J}{h} e^{-2K_2}$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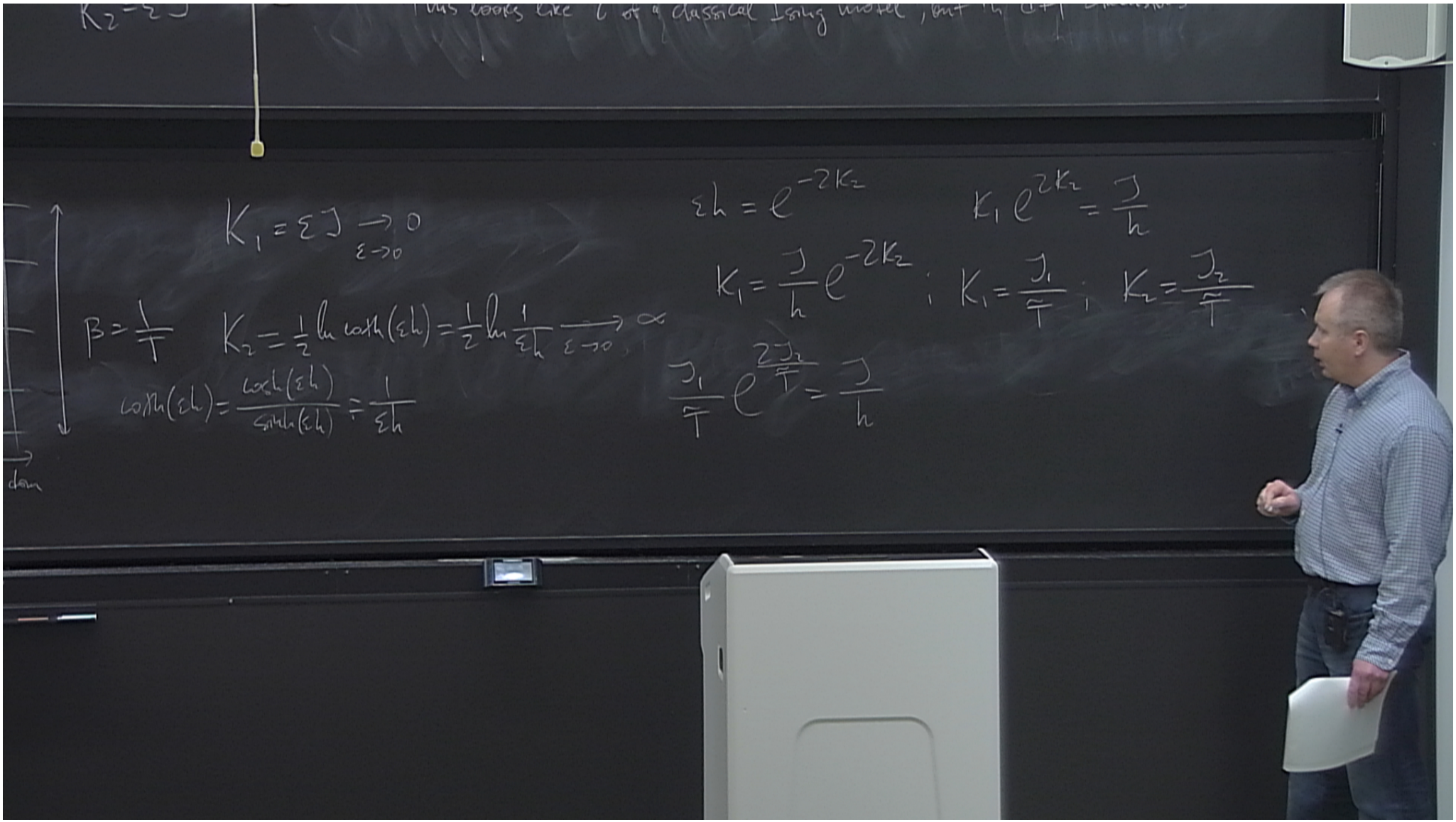
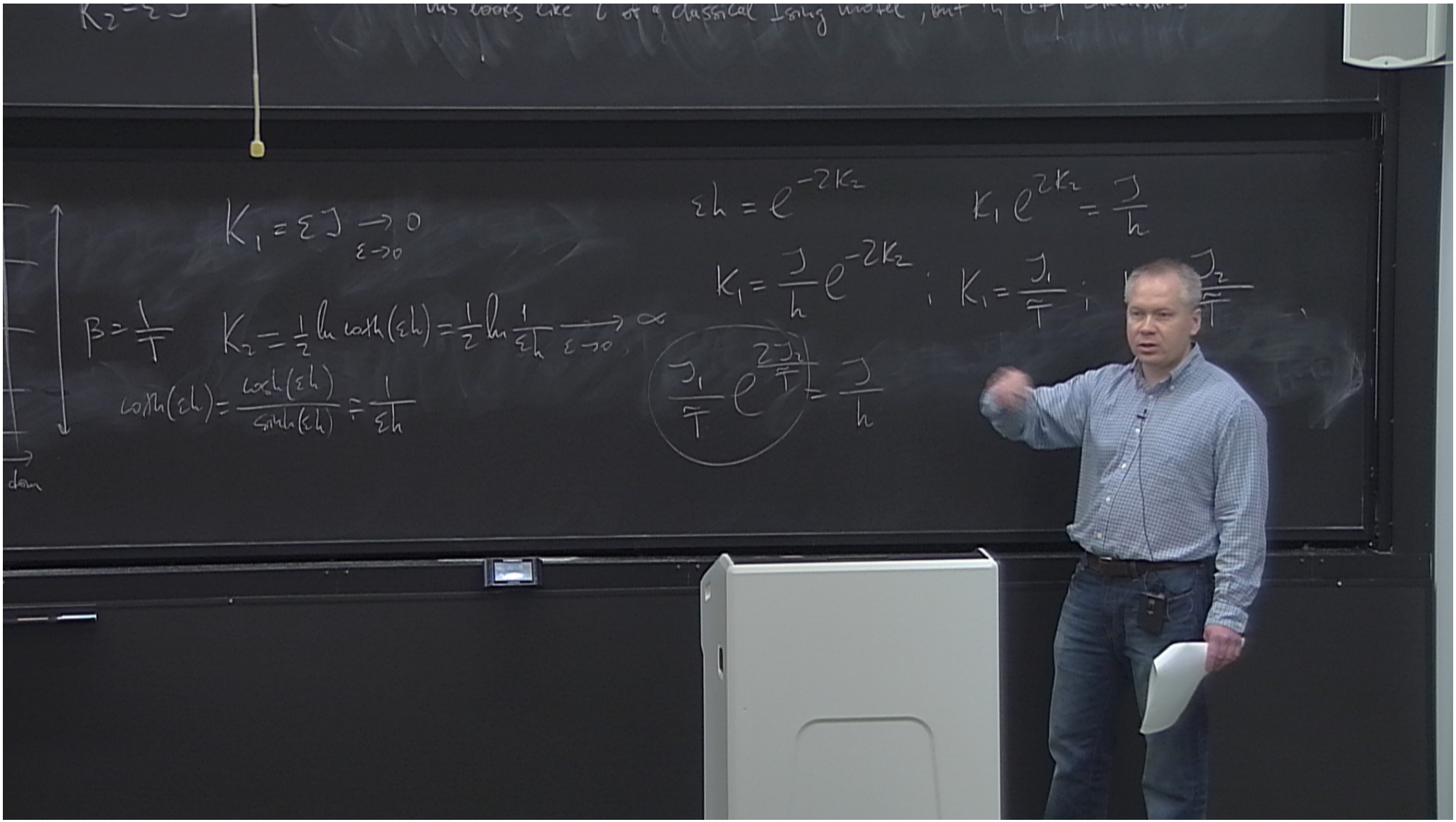
$K_1 e^{2K_2} = \frac{J}{h}$
$K_1 = \frac{J}{h} e^{-2K_2}$


$K_2 = \varepsilon J$
This looks like 2 of a classical Ising model, but in 1+1 dimensions
$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$
$\varepsilon h = e^{-2K_2}$
$K_1 e^{2K_2} = \frac{J}{h}$
$K_1 = \frac{J}{h} e^{-2K_2}$; $K_1 = \frac{J_1}{T}$; $K_2 = \frac{J_2}{T}$
$\beta = \frac{1}{T}$
$K_2 = \frac{1}{2} \ln \coth(\varepsilon h) = \frac{1}{2} \ln \frac{1}{\varepsilon h} \xrightarrow[\varepsilon \to 0]{} \infty$
$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \doteq \frac{1}{\varepsilon h}$
$\frac{J_1}{T} e^{\frac{2J_2}{T}} = \frac{J}{h}$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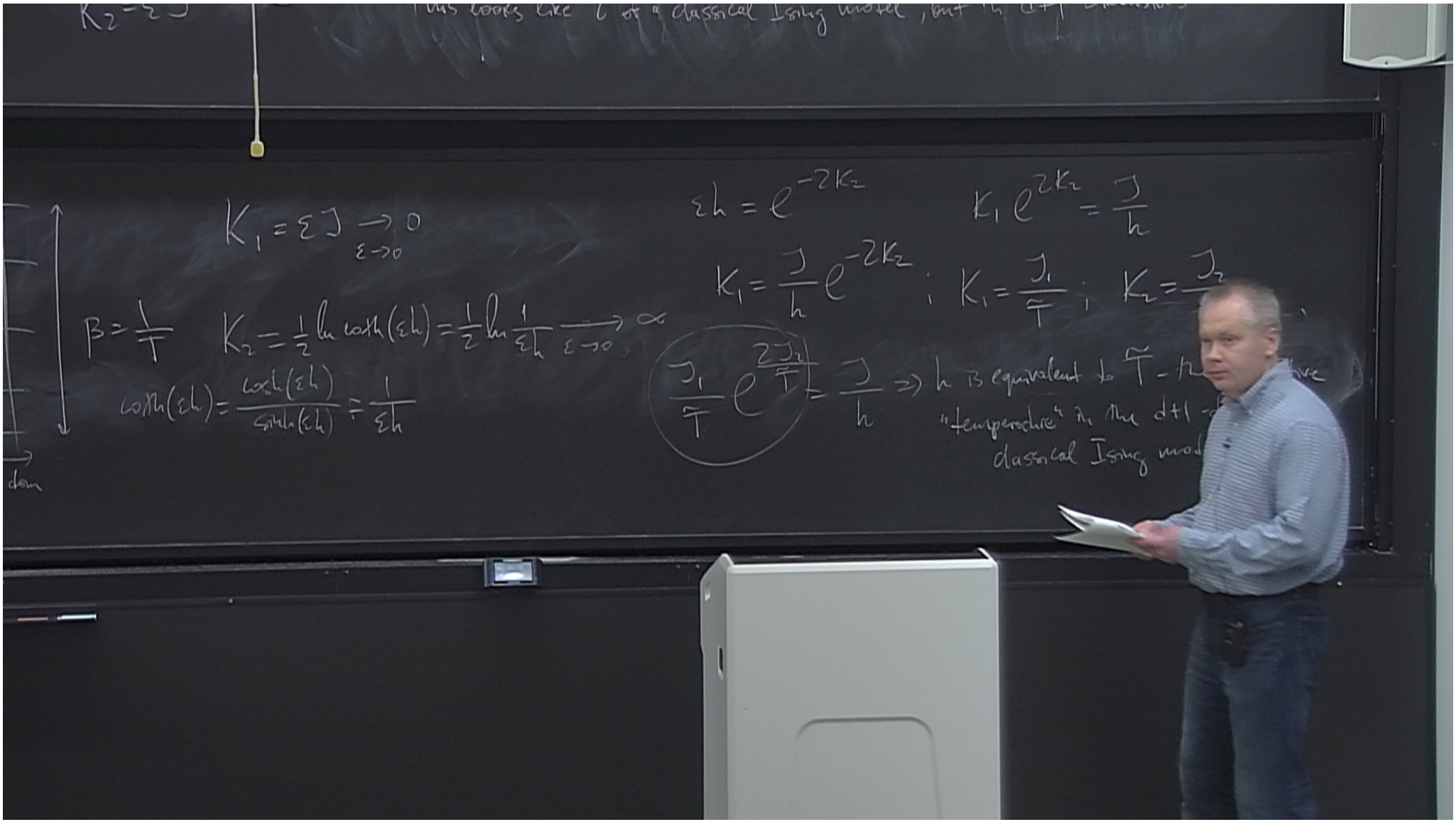
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$\frac{J_1}{T} e^{\frac{2J_2}{T}} = \frac{J}{h}$ ⇒ h is equivalent to T
"temperature" in the d+1
classical Ising mod

$K_2 = \varepsilon J$

This looks like [illegible] of a classical Ising model, but in d+1 dimensions

$$K_1 = \varepsilon J \xrightarrow[\varepsilon \to 0]{} 0$$

$$\beta = \frac{1}{T} \qquad K_2 = \frac{1}{2} \ln \coth(\varepsilon h) = \frac{1}{2} \ln \frac{1}{\varepsilon h} \xrightarrow[\varepsilon \to 0]{} \infty$$

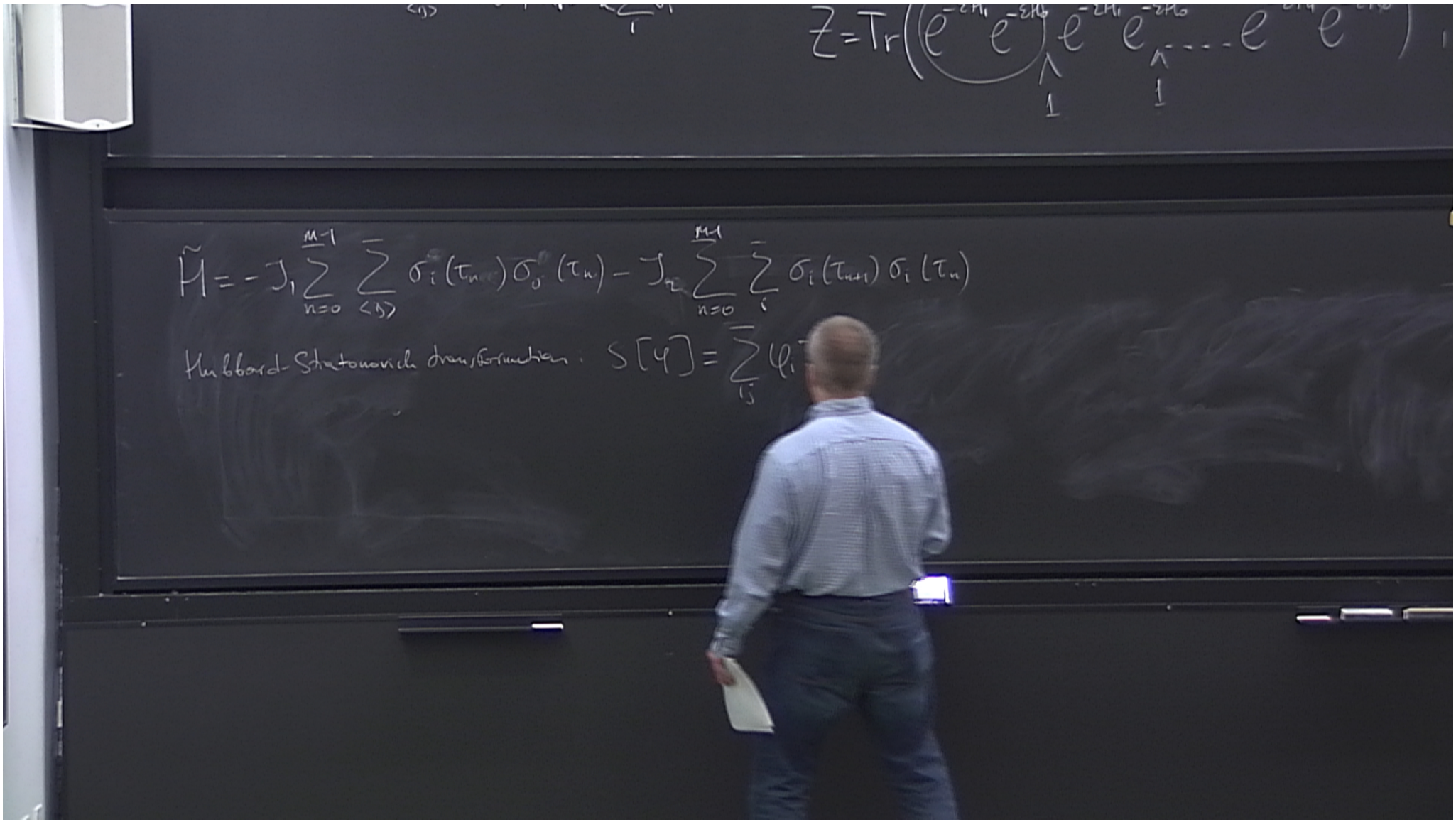
$$\coth(\varepsilon h) = \frac{\cosh(\varepsilon h)}{\sinh(\varepsilon h)} \doteq \frac{1}{\varepsilon h}$$

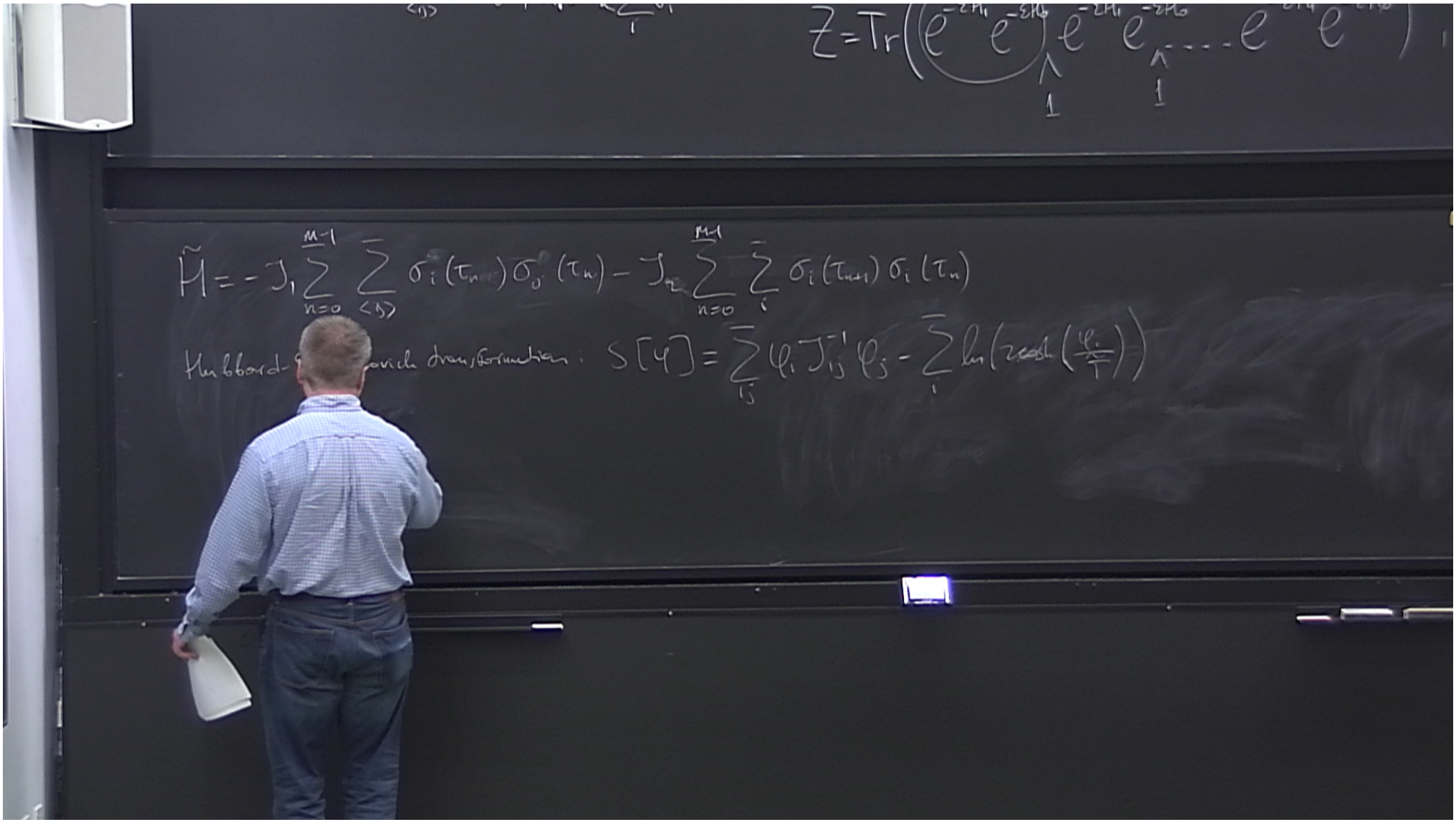
$$\varepsilon h = e^{-2K_2} \qquad K_1 e^{2K_2} = \frac{J}{h}$$

$$K_1 = \frac{J}{h} e^{-2K_2} \; ; \; K_1 = \frac{J_1}{\tilde{T}} \; ; \; K_2 = \frac{J_2}{\tilde{T}}$$

$$\frac{J_1}{\tilde{T}} e^{\frac{2J_2}{\tilde{T}}} = \frac{J}{h} \Rightarrow h \text{ is equivalent to } \tilde{T} \text{ - the effective}$$

"temperature" in the d+1-dimensional classical Ising model.


$Z = \mathrm{Tr}\left(e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} \cdots e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} \right)$
$\tilde{H} = -J_1 \sum_{n=0}^{M-1} \sum_{\langle ij \rangle} \sigma_i(\tau_n)\sigma_j(\tau_n) - J_2 \sum_{n=0}^{M-1} \sum_i \sigma_i(\tau_{n+1})\sigma_i(\tau_n)$
Hubbard-Stratonovich transformation: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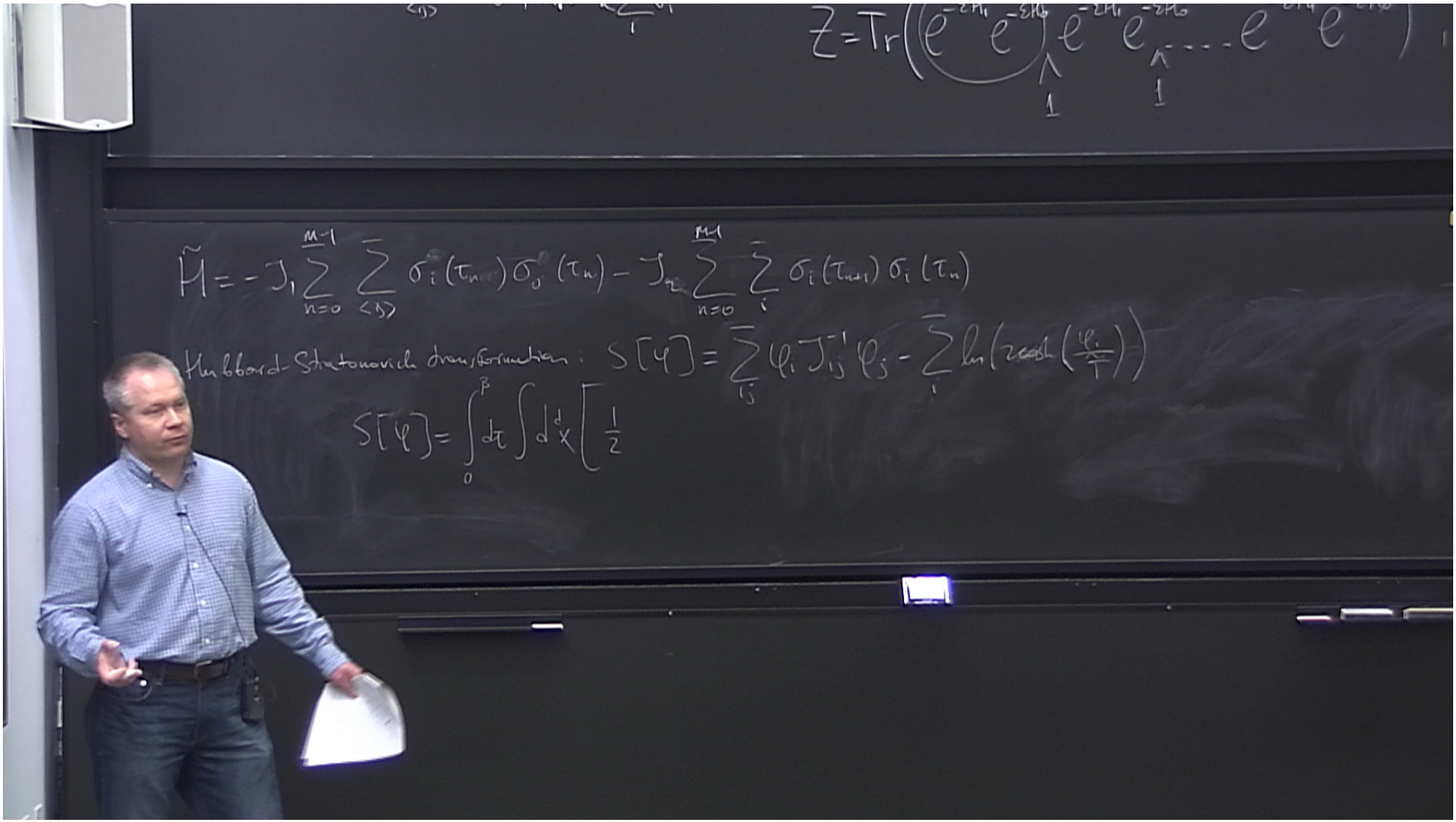
Hubbard-Stratonovich transformation: $S[\varphi] = \sum_{ij} \varphi_i J^{-1}_{ij} \varphi_j - \sum_i \ln\left(2\cosh\left(\frac{\varphi_i}{T}\right)\right)$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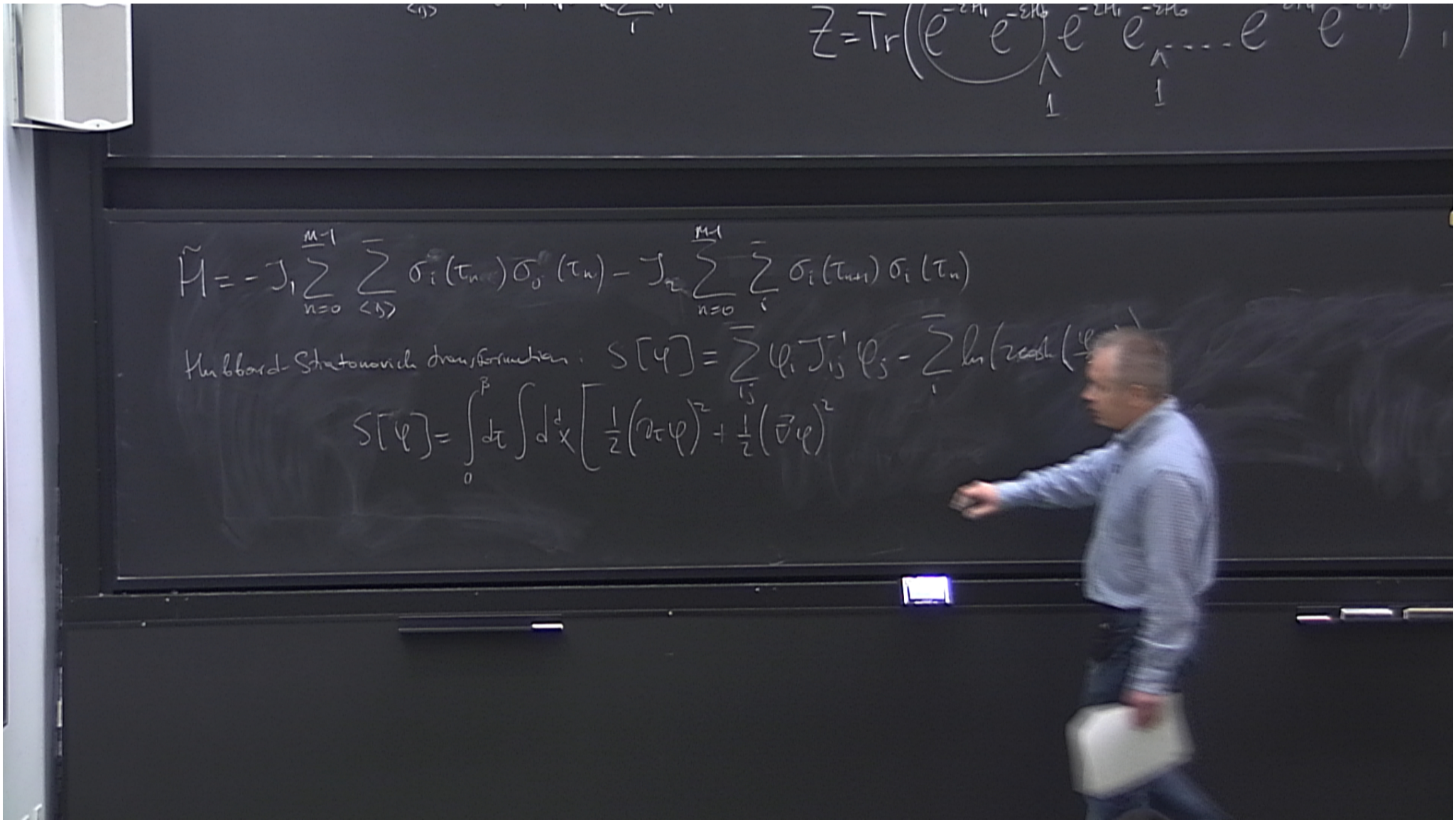
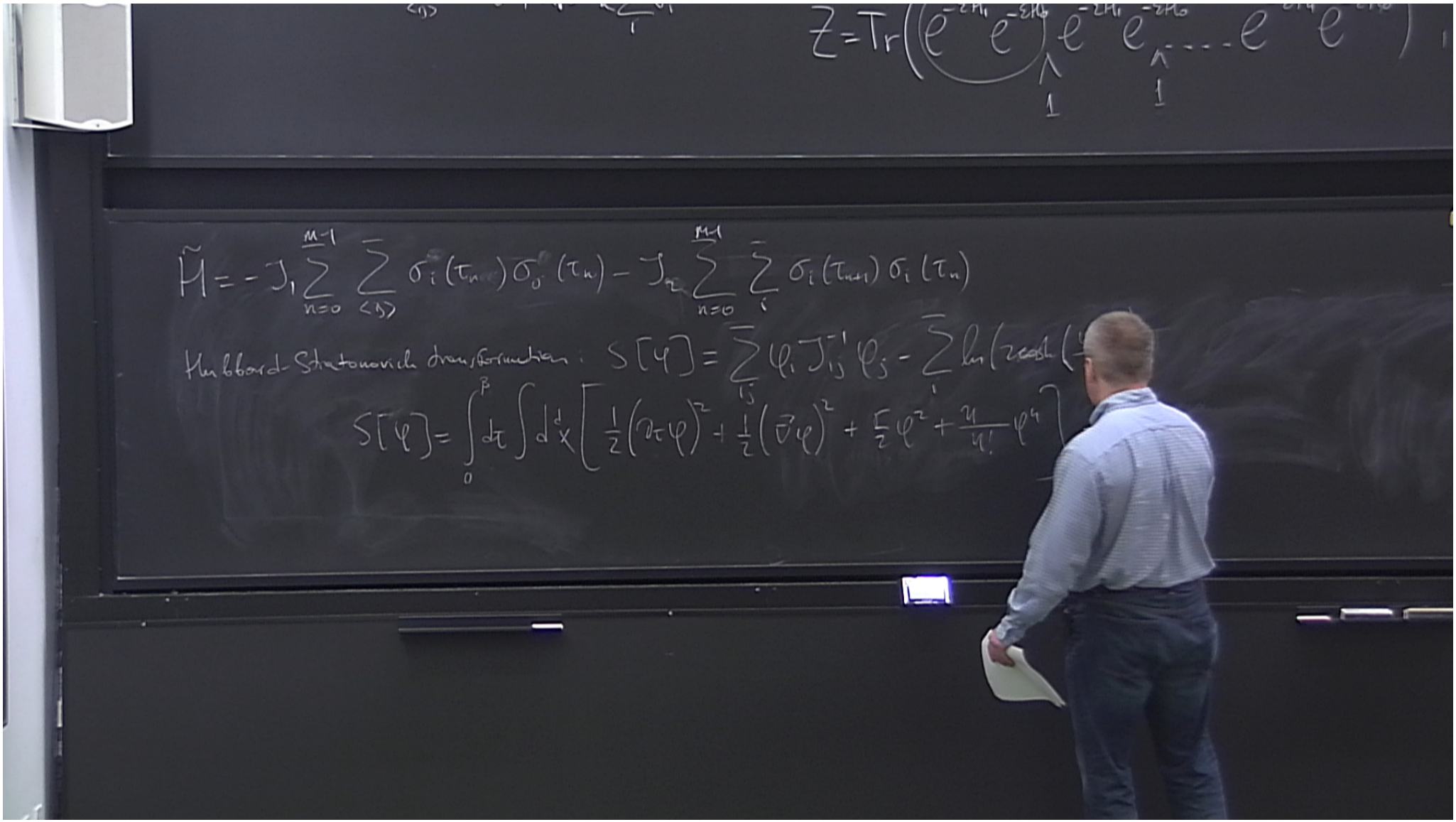
$S[\varphi] = \int_0^\beta d\tau \int d^d x \left[\frac{1}{2} \right.$


Hubbard-Stratonovich transformation:


$Z = \mathrm{Tr}\left(e^{-\varepsilon H_1} e^{-\varepsilon H_0} e^{-\varepsilon H_1} e^{-\varepsilon H_0} \cdots e^{-\varepsilon H_1} e^{-\varepsilon H_0}\right)$
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$$Z = \mathrm{Tr}\left(e^{-\varepsilon H_1} e^{-\varepsilon H_0} \mathbb{1}\, e^{-\varepsilon H_1} e^{-\varepsilon H_0} \mathbb{1} \cdots e^{-\varepsilon H_1} e^{-\varepsilon H_0} \right), \quad \mathbb{1} = \sum_n |n\rangle$$

$|n\rangle$ – any complete

$$\tilde{H} = -J_1 \sum_{n=0}^{M-1} \sum_{\langle ij \rangle} \sigma_i(\tau_n)\sigma_j(\tau_n) - J_2 \sum_{n=0}^{M-1} \sum_i \sigma_i(\tau_{n+1})\sigma_i(\tau_n)$$

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$$Z = \int D\varphi\, e^{-S[\varphi]}$$

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$$Z = \mathrm{Tr}\left(\left(e^{-\varepsilon H_1} e^{-\varepsilon H_0}\right) e^{-\varepsilon H_1} e^{-\varepsilon H_0} \cdots e^{-\varepsilon H_1} e^{-\varepsilon H_0}\right)$$

$\wedge$ 1 $\quad \wedge$ 1

$$1 = \sum_n |n\rangle\langle n|$$

$|n\rangle$ – any complete set of states

$$\ldots)\,\sigma_0(\tau_n) - J_2 \sum_{n=0}^{M-1} \sum_i \sigma_i(\tau_{n+1})\,\sigma_i(\tau_n)$$

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$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$\ldots d\tau \int d^d x \left[\frac{1}{2}(\partial_\tau \varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4!}\varphi^4\right]$$ – imaginary-time action

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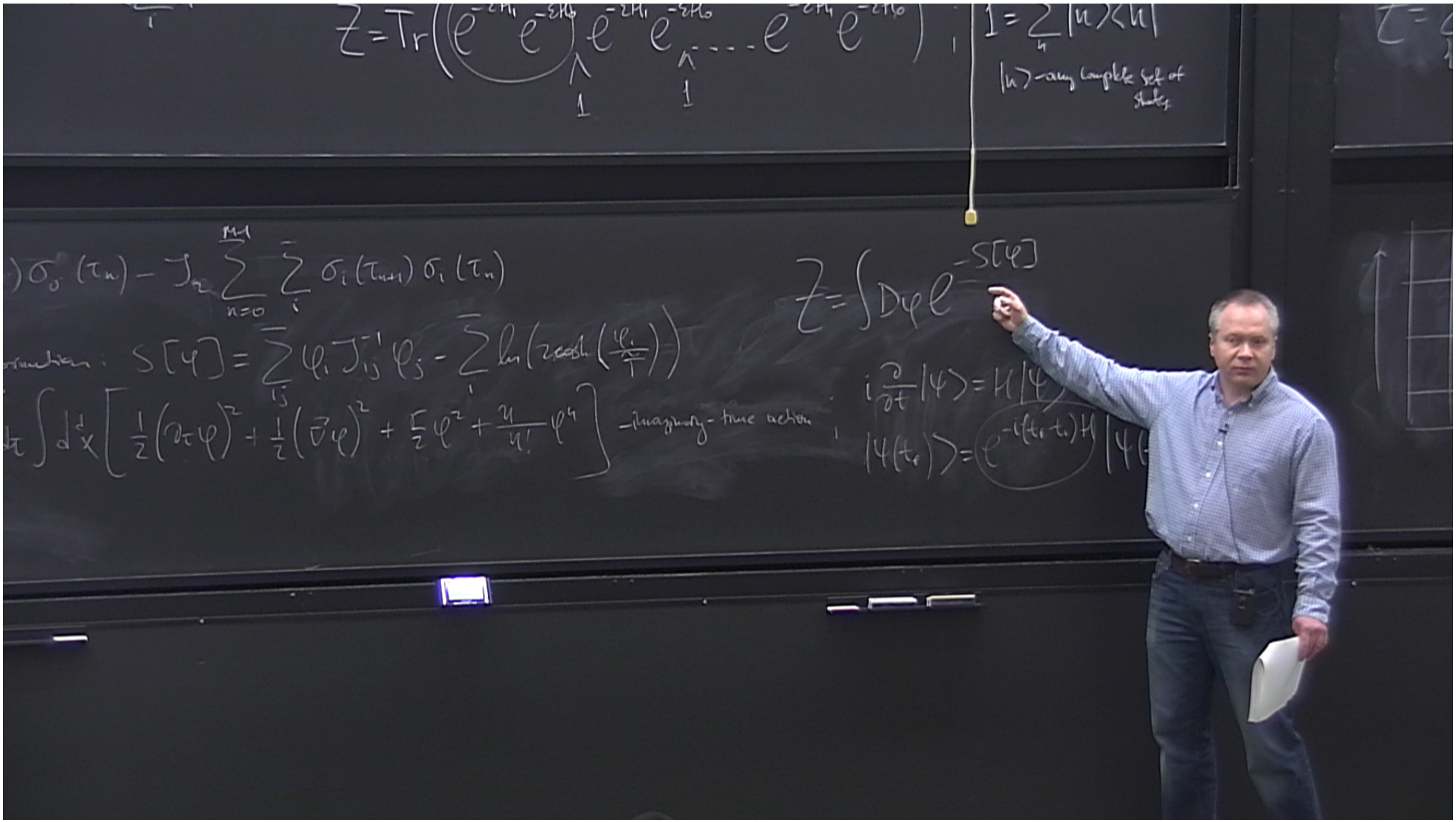
$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$|\psi(t_f)\rangle = e^{-i(t_f - t_i)H}|\psi(t_i)\rangle$$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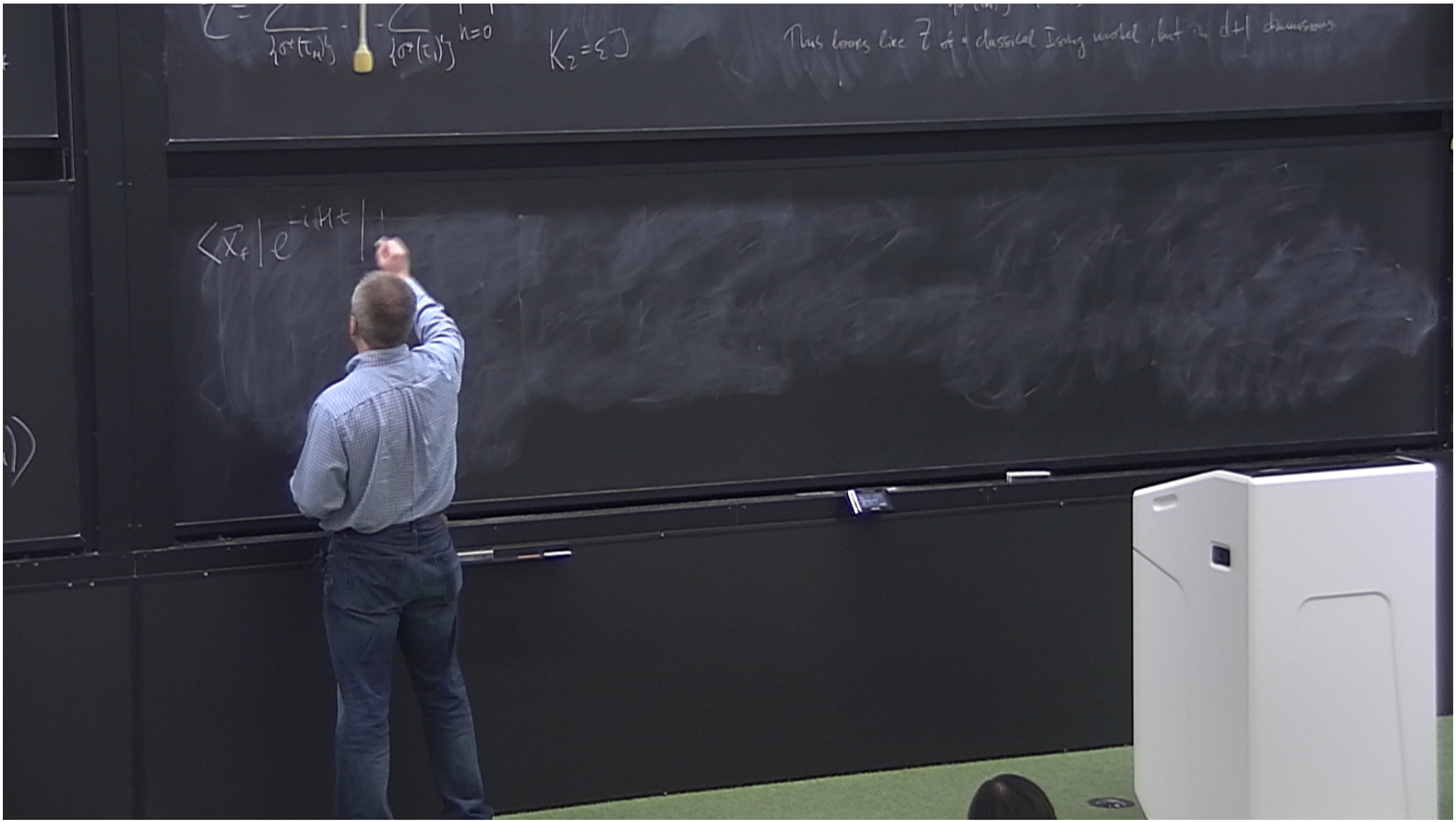
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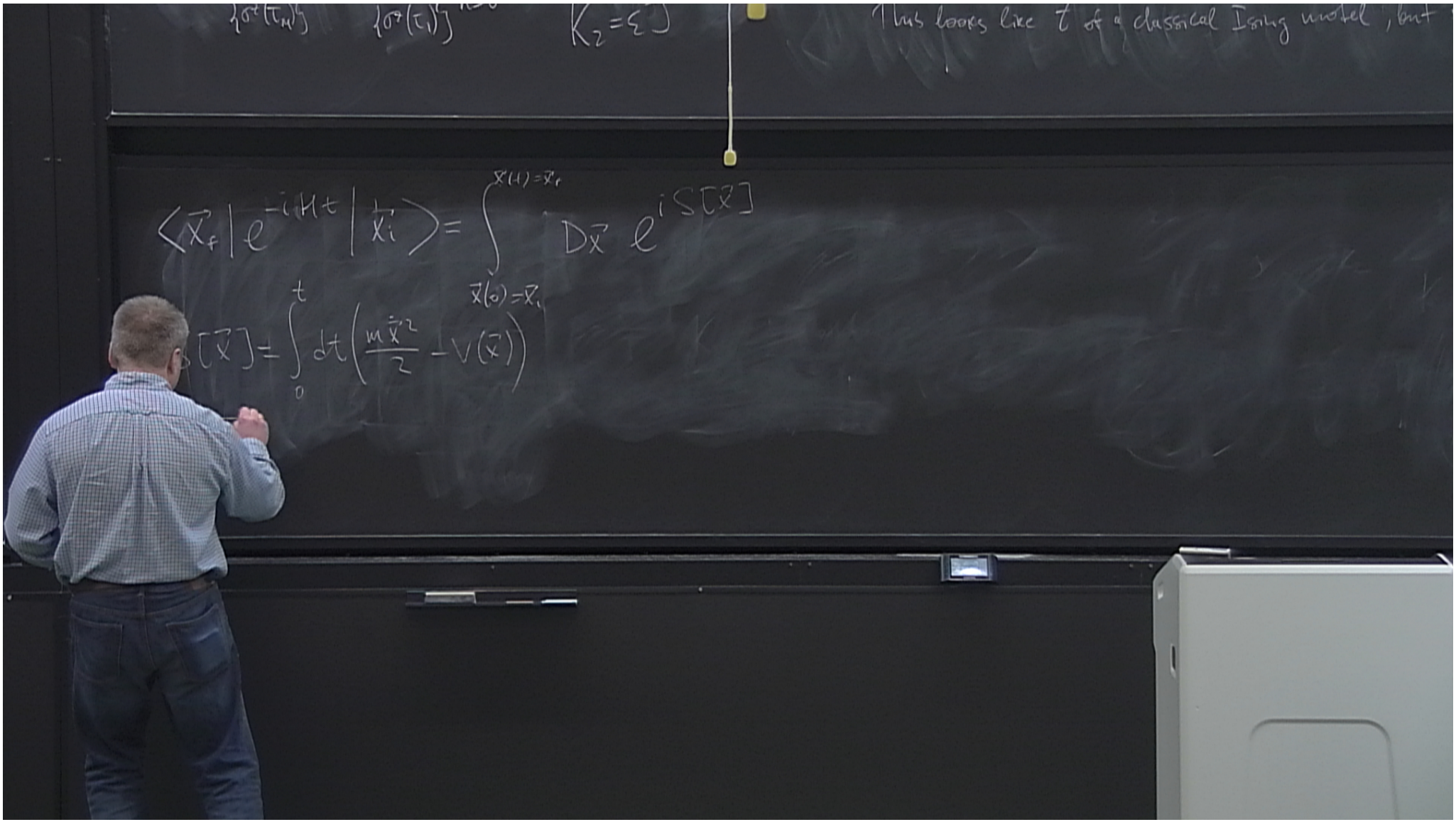
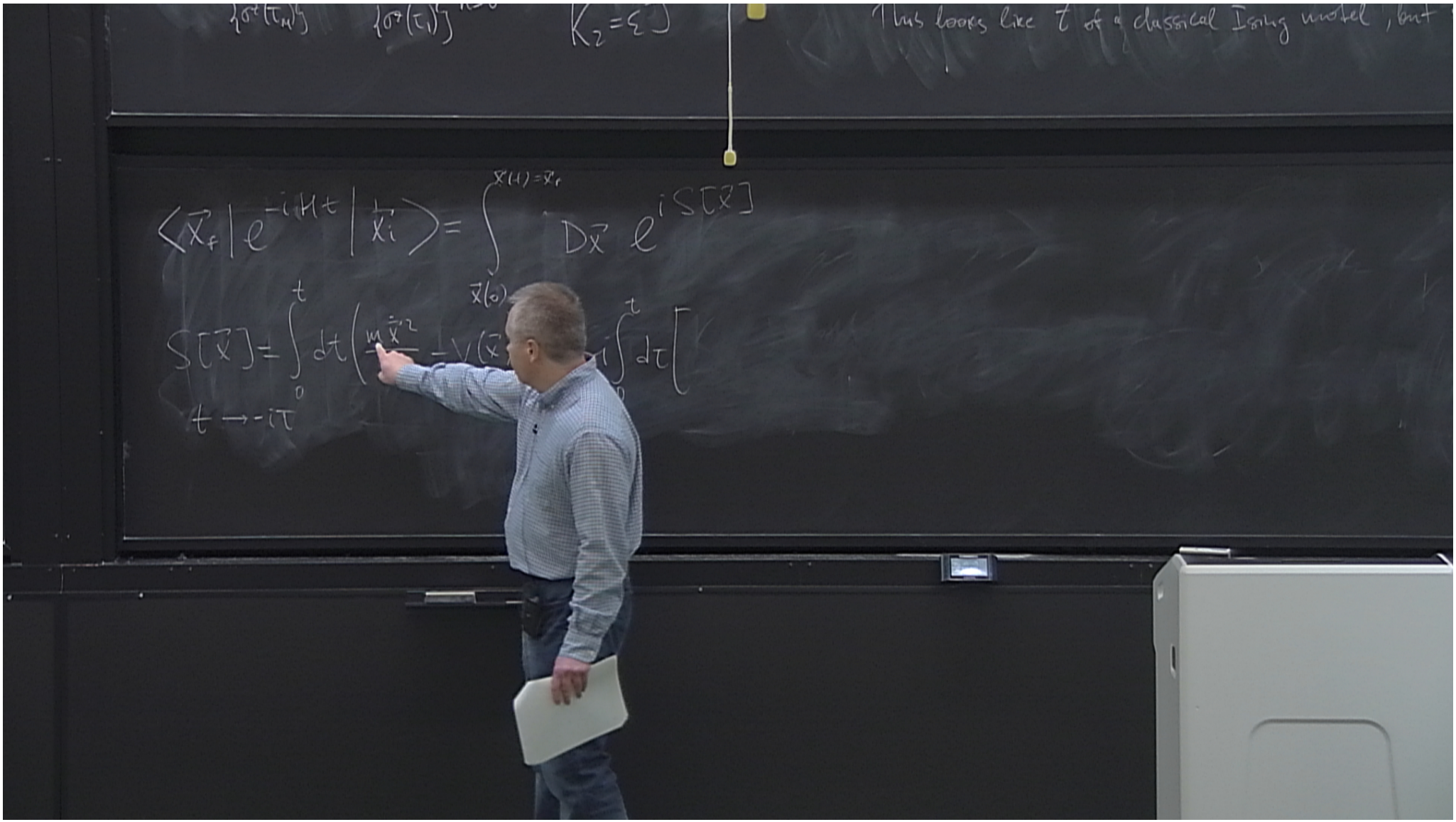
This looks like Z of a classical Ising model, but in d+1 dimensions
$\langle \vec{x}_f | e^{-iHt} |$


$\{\sigma^z(\tau_N)\}$
$\{\sigma^z(\tau_1)\}$
$K_2 = \xi$
This looks like Z of a classical Ising model, but
$\langle \vec{x}_f | e^{-iHt} | \vec{x}_i \rangle = \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(t)=\vec{x}_f} D\vec{x}\, e^{iS[\vec{x}]}$
$S[\vec{x}] = \int_0^t dt \left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) \right)$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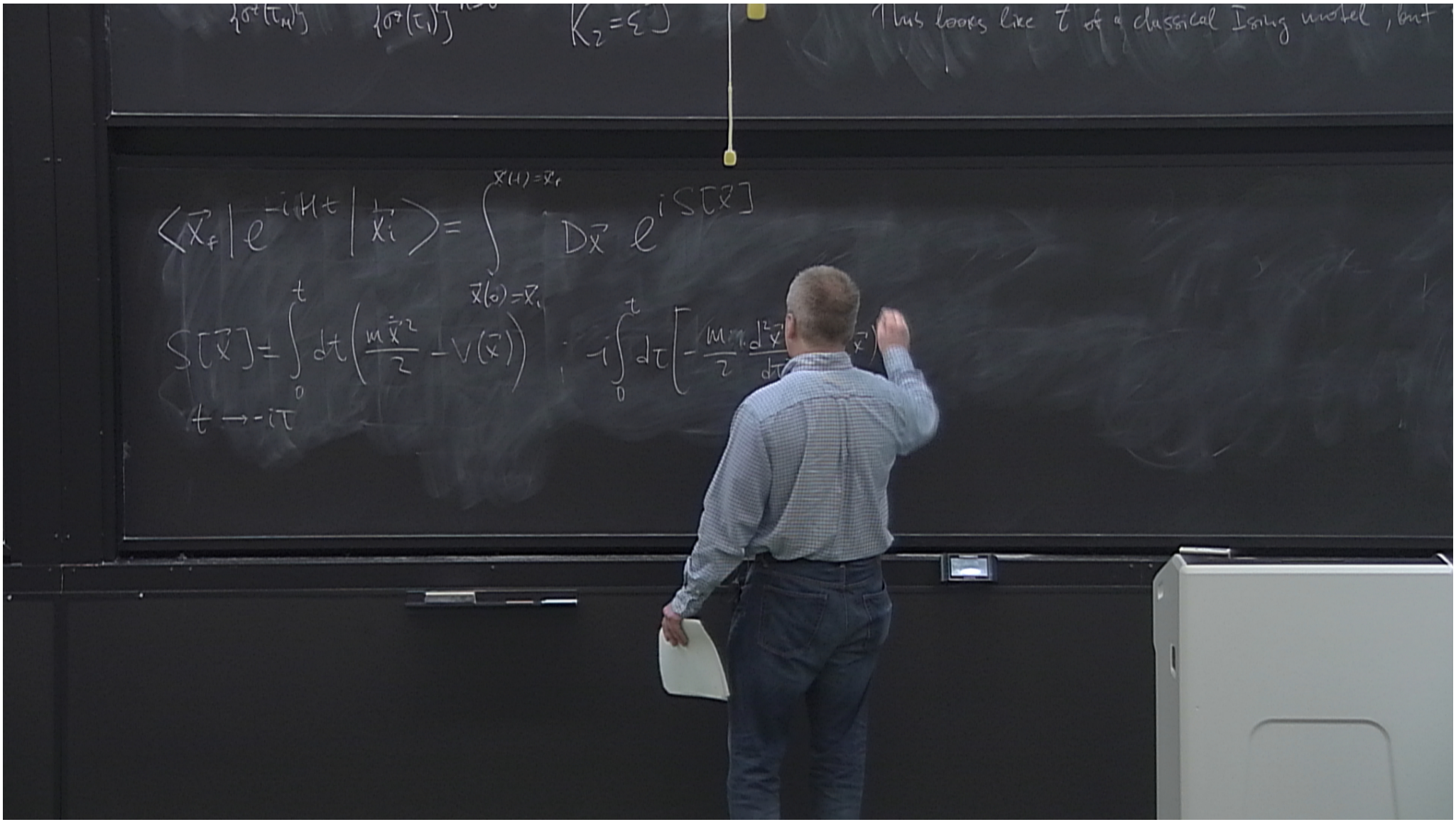
$\langle \vec{x}_f | e^{-iHt} | \vec{x}_i \rangle = \int_{\vec{x}(0)}^{\vec{x}(t)=\vec{x}_f} D\vec{x}\, e^{iS[\vec{x}]}$
$S[\vec{x}] = \int_0^t dt \left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) \right.$
$t \to -i\tau$


$\{\sigma^z(\tau_N)\}$ $\{\sigma^z(\tau_1)\}$
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This looks like τ of a classical Ising model, but
$\langle \vec{x}_f | e^{-iHt} | \vec{x}_i \rangle = \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(t)=\vec{x}_f} D\vec{x}\, e^{iS[\vec{x}]}$
$S[\vec{x}] = \int_0^t dt \left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) \right)$
$t \to -i\tau$
$-i\int_0^t d\tau \left[-\frac{m}{2} \frac{d^2\vec{x}}{d\tau^2} \right.$

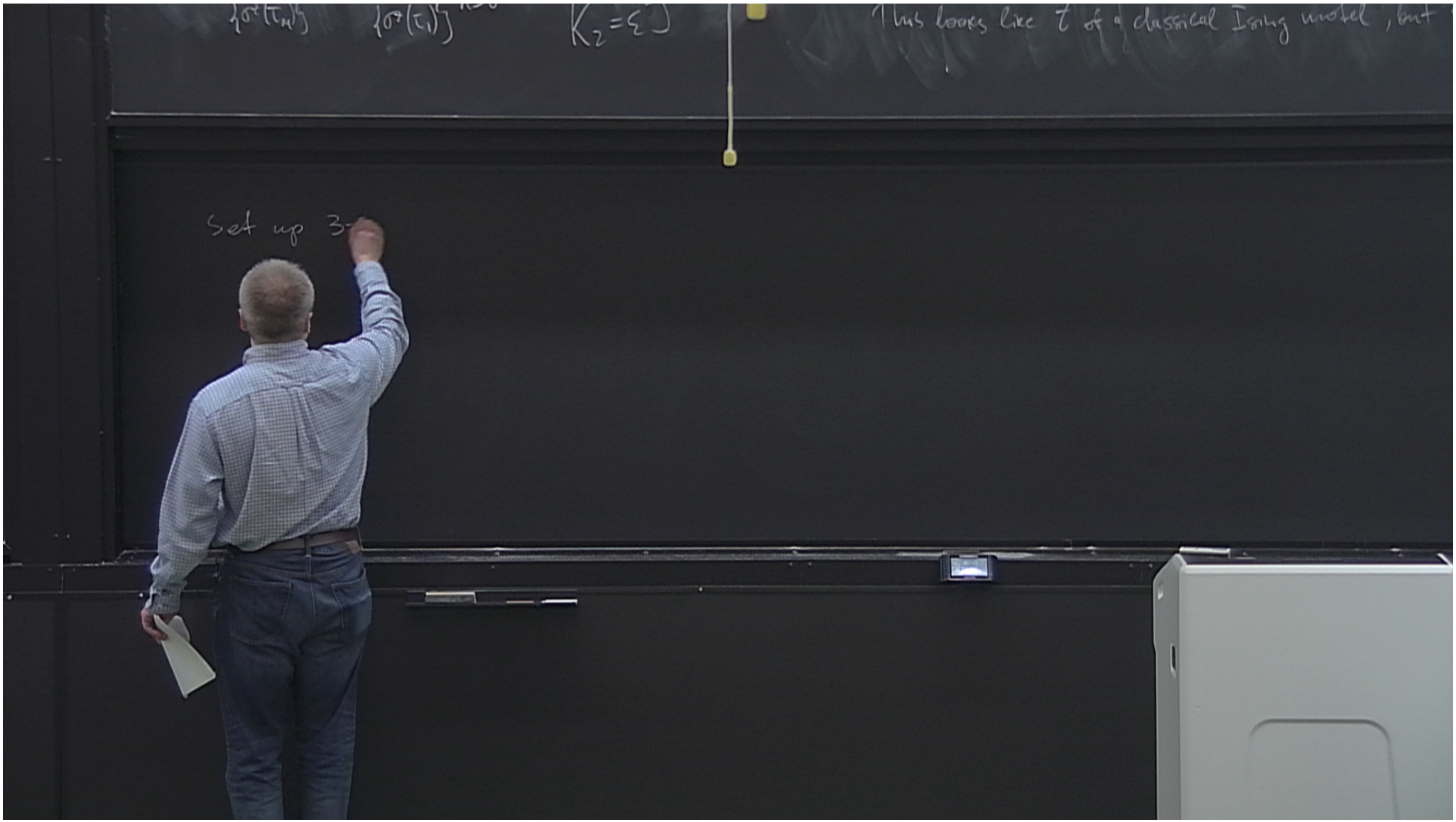
$\{\sigma^z(\tau_N)\}$ $\{\sigma^z(\tau_1)\}$ $K_2 = \xi$

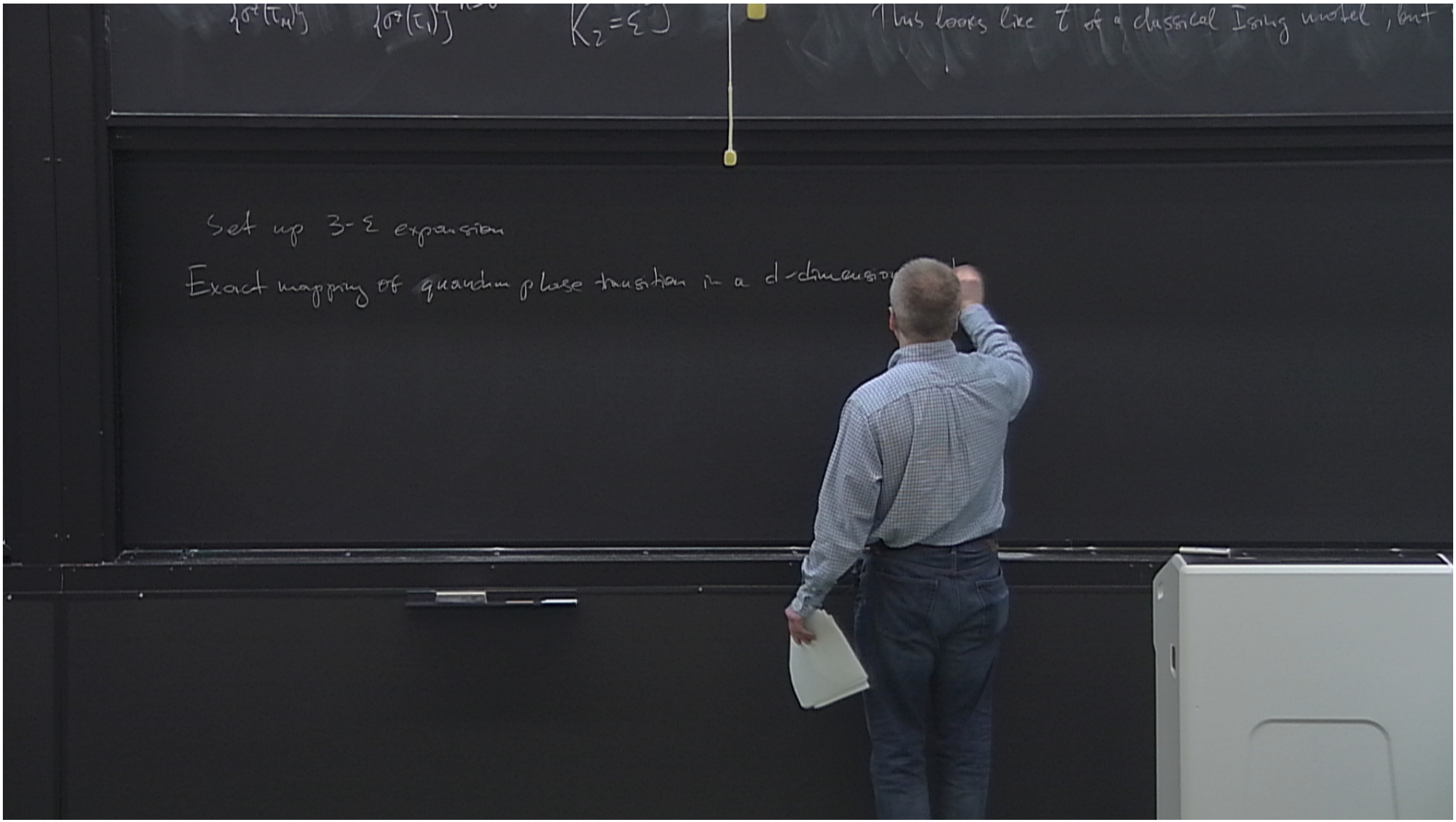
This looks like Z of a classical Ising model, but

$$\langle \vec{x}_f | e^{-iHt} | \vec{x}_i \rangle = \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(t)=\vec{x}_f} D\vec{x}\, e^{iS[\vec{x}]} = \int D\vec{x}\, e^{-S[\vec{x}]}$$

$$S[\vec{x}] = \int_0^t dt' \left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) \right) \quad ; \quad -i\int_0^t d\tau' \left[-\frac{m}{2}\left(\frac{d\vec{x}}{d\tau'}\right)^2 - V(\vec{x}) \right] \quad , \quad S[\vec{x}] = \int_0^t d\tau' \left[\frac{m}{2}\left(\frac{d\vec{x}}{d\tau'}\right)^2 + V(\vec{x}) \right]$$

$t \to -i\tau$


Set up 3-
This looks like Z of a classical Ising model, but


Set up 3-ε expansion
Exact mapping of quantum phase transition in a d-dimensio

$$Z = \sum_{\{\sigma^z(\tau_n)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}} \prod_{n=0} e$$

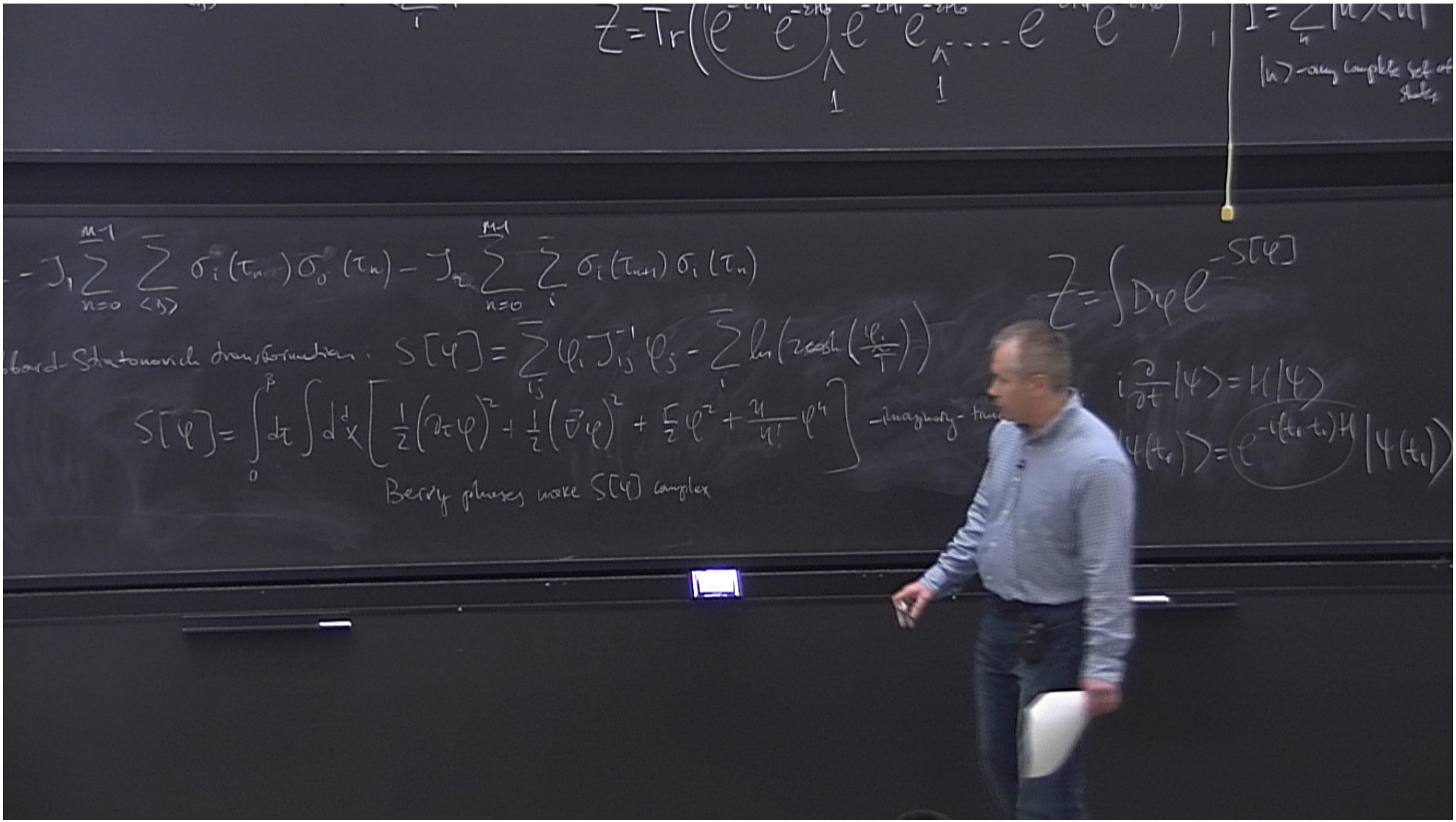
$$K_2 = \varepsilon J$$

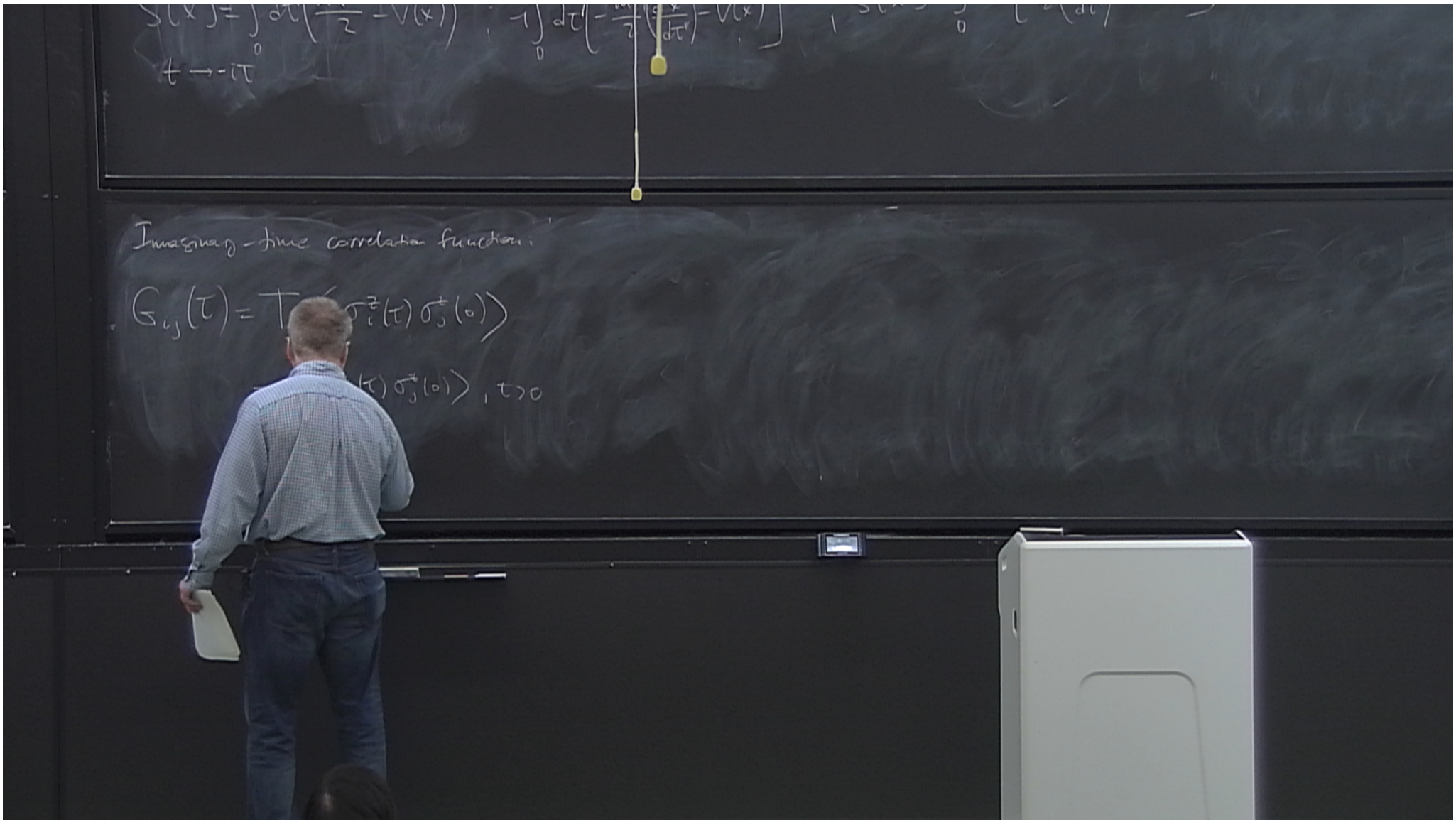
$$= \sum_{\{\sigma^z(\tau_n)\}} \cdots \sum_{\{\sigma^z(\tau_1)\}}$$

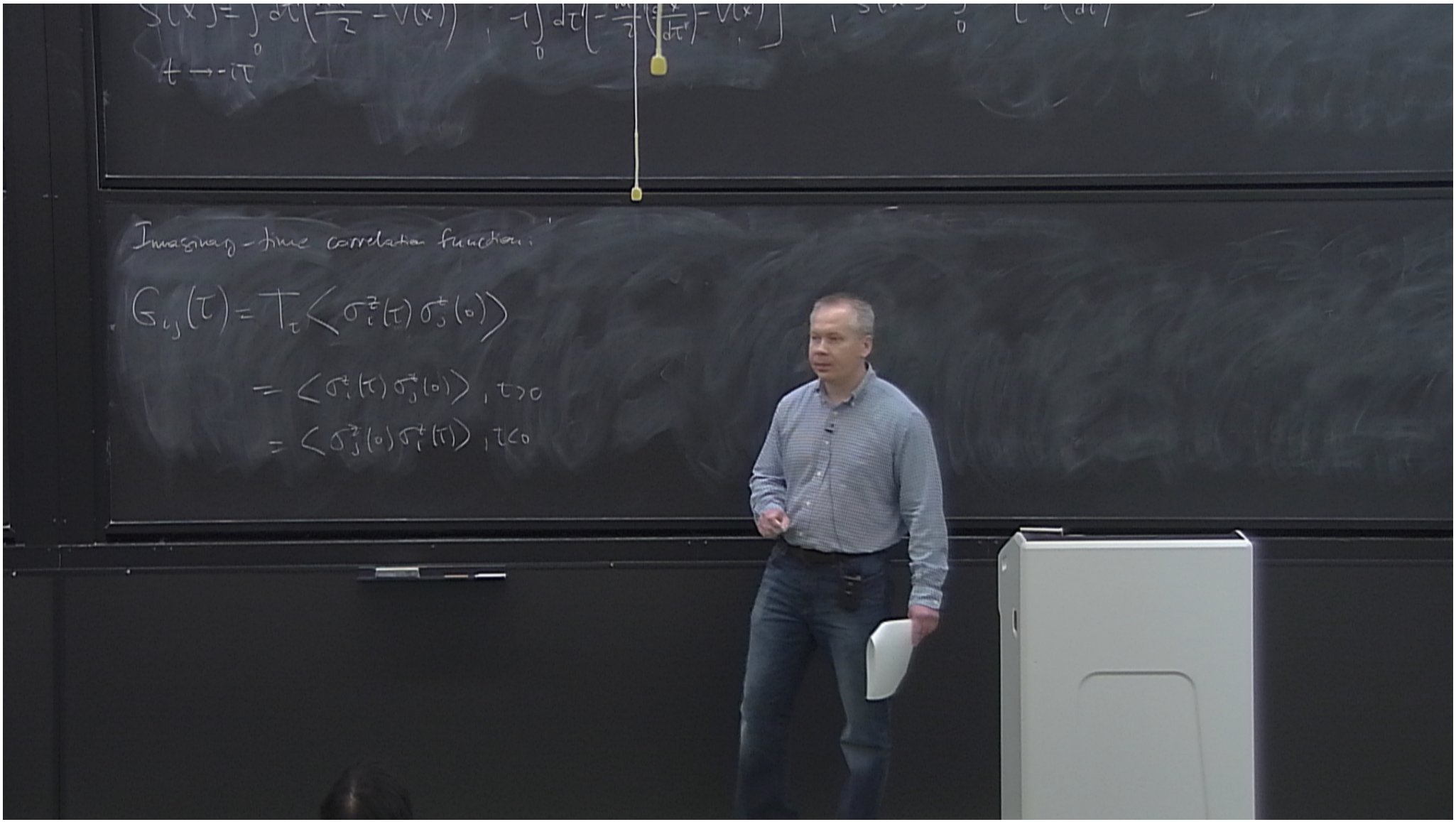
This looks like Z of a classical Ising model, but in $d+1$ dimensions.

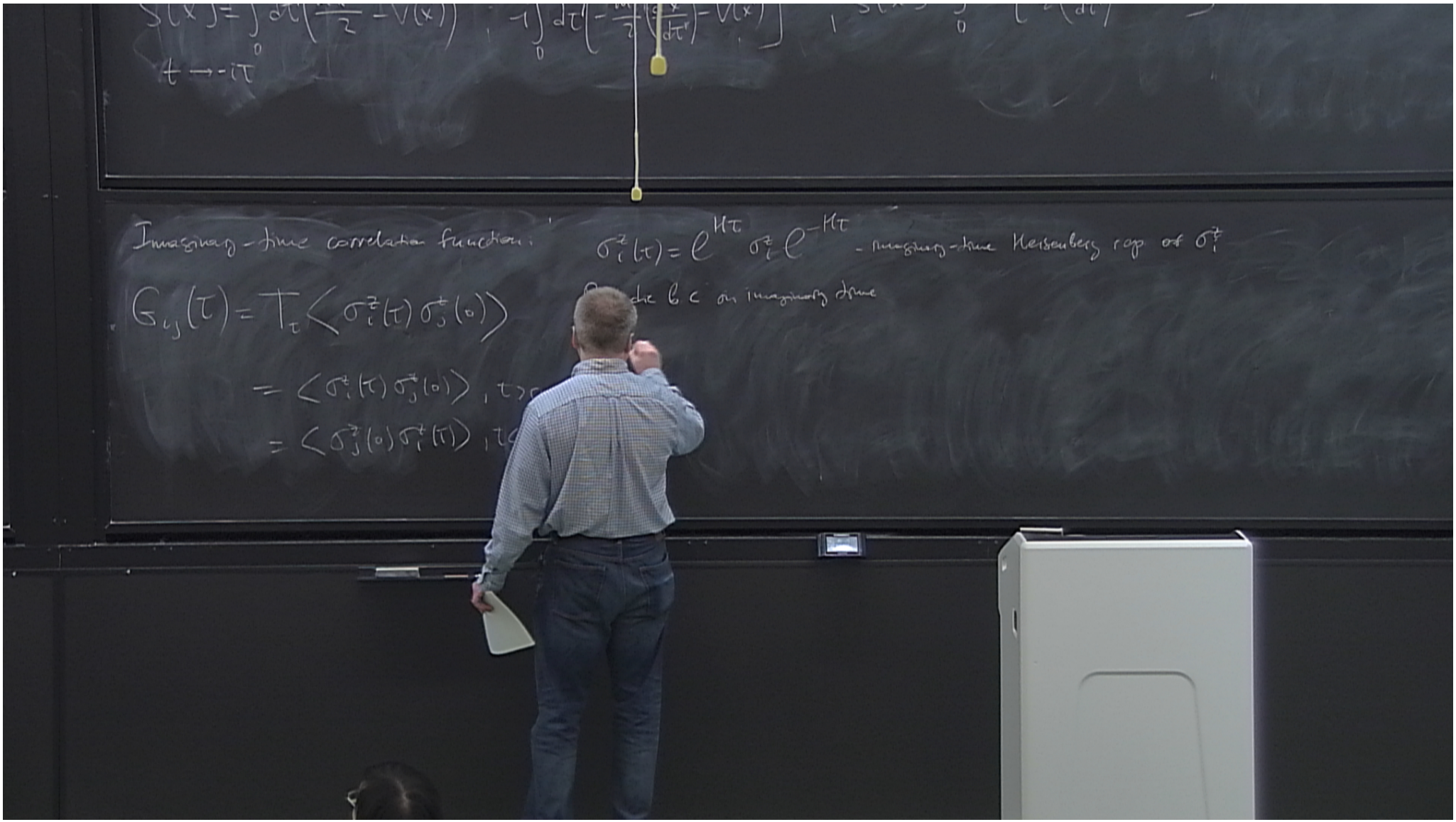
Set up $3-\varepsilon$ expansion

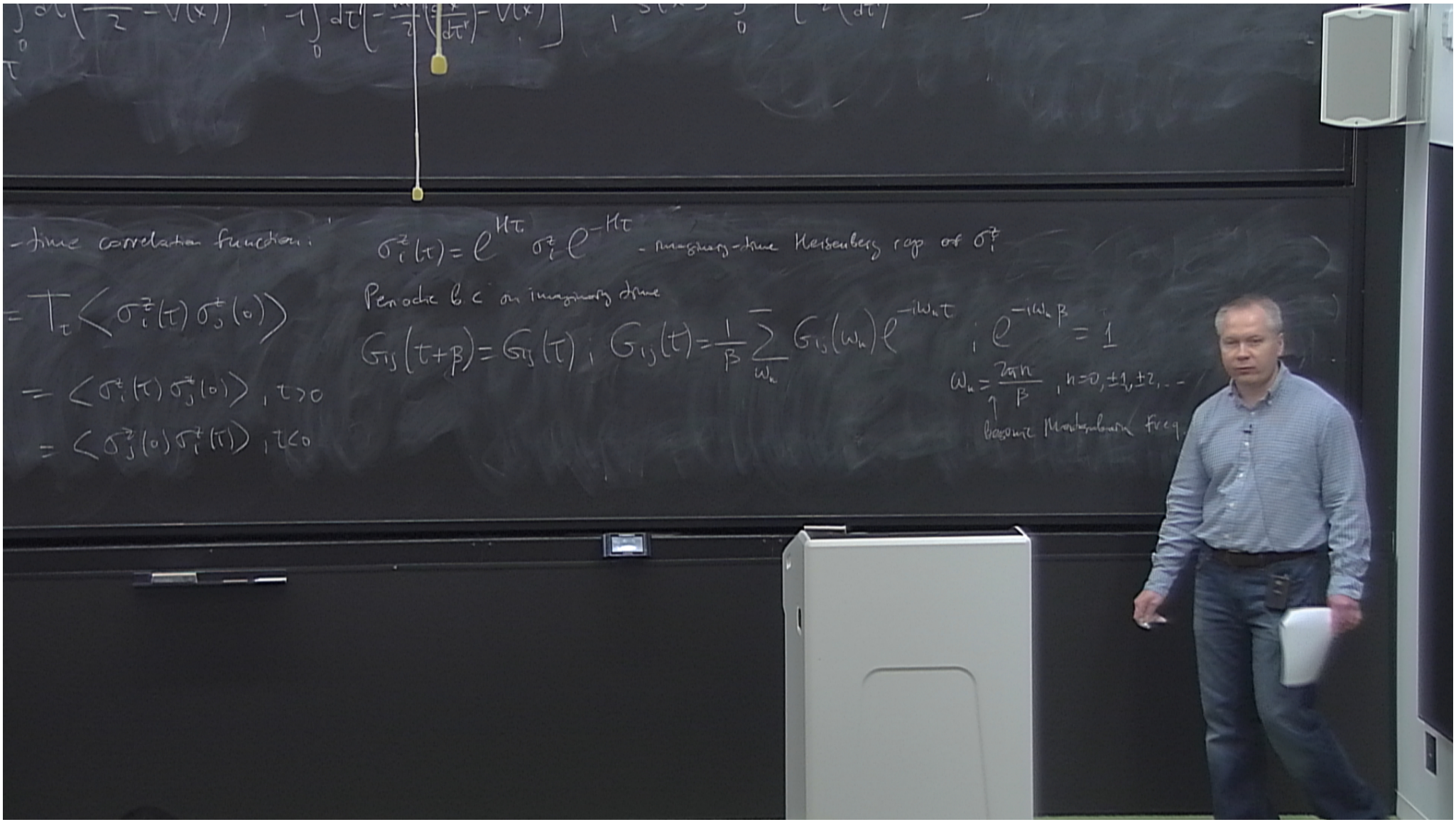
Exact mapping of quantum phase transition in a d-dimensional transverse-field Ising model to classical phase transition in a $d+1$-dimensional classical Ising model


|n> - any complete set of states
Bard-Stratonovich transformation
Berry phases make S[φ] complex


t → -iτ
Imaginary-time correlation function:


$t \to -i\tau$
Imaginary-time correlation function:
$G_{ij}(\tau) = T_\tau \langle \sigma_i^z(\tau)\, \sigma_j^z(0) \rangle$
$= \langle \sigma_i^z(\tau)\, \sigma_j^z(0) \rangle \,,\ \tau > 0$
$= \langle \sigma_j^z(0)\, \sigma_i^z(\tau) \rangle \,,\ \tau < 0$


t → -iτ
Imaginary-time correlation function:
σ_i^z(τ) = e^{Hτ} σ_i^z e^{-Hτ} - imaginary-time Heisenberg rep of σ_i^z
G_ij(τ) = T_τ ⟨σ_i^z(τ) σ_j^z(0)⟩
= ⟨σ_i^z(τ) σ_j^z(0)⟩ , τ>0
= ⟨σ_j^z(0) σ_i^z(τ)⟩ , τ<0


-time correlation function:
$= T_\tau \langle \sigma_i^z(\tau)\, \sigma_j^z(0) \rangle$
$= \langle \sigma_i^z(\tau)\, \sigma_j^z(0) \rangle \,,\ \tau > 0$
$= \langle \sigma_j^z(0)\, \sigma_i^z(\tau) \rangle \,,\ \tau < 0$
$\sigma_i^z(\tau) = e^{H\tau}\, \sigma_i^z\, e^{-H\tau}$ - imaginary-time Heisenberg rep of σ_i^z
Periodic b.c. on imaginary time
$G_{ij}(\tau+\beta) = G_{ij}(\tau) \,;\ G_{ij}(\tau) = \frac{1}{\beta} \sum_{\omega_n} G_{ij}(\omega_n)\, e^{-i\omega_n \tau} \,;\ e^{-i\omega_n \beta} = 1$
$\omega_n = \frac{2\pi n}{\beta} \,,\ n = 0, \pm 1, \pm 2, \ldots$
bosonic Matsubara freq.

$+H_1,\ H_0 = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j,\ H_1 = -h\sum_i \sigma_i^x$

$$Z = \mathrm{Tr}\Big(e^{-\varepsilon H_1}e^{-\varepsilon H_0}\underset{\substack{\wedge\\ 1}}{}e^{-\varepsilon H_1}e^{-\varepsilon H_0}\underset{\substack{\wedge\\ 1}}{}\cdots e^{-\varepsilon H_1}e^{-\varepsilon H_0}\Big);\quad 1 = \sum_n |n\rangle\langle n|$$

$|n\rangle$ – any complete set of states

$$)=\langle\sigma_i^z(\tau)\sigma_j^z(0)\rangle = \langle e^{H\tau}\sigma_i^z e^{-H\tau}\sigma_j^z\rangle = \frac{1}{Z}\mathrm{Tr}\Big[e^{-\beta H}\underset{\substack{\wedge\\ 1=\sum_n|n\rangle\langle n|}}{}e^{H\tau}\sigma_i^z e^{-H\tau}\sigma_j^z\Big] = \frac{1}{Z}\sum_{nm}\langle n|e^{-\beta H}|n\rangle\cdot$$

$\rangle = E_n|n\rangle$

– complete set of eigenstates of H

$$\cdot\langle n|e^{H\tau}|n\rangle\langle n|\sigma_i^z|m\rangle\langle m|e^{-H\tau}|m\rangle\langle m|\sigma_j^z|n\rangle =$$

$$= \frac{1}{Z}\sum_{nm} e^{-\beta E_n}e^{\tau E_n}\langle n|\sigma_i^z|m\rangle e^{-E_m\tau}\langle m|\sigma_j^z|n\rangle = \frac{1}{Z}\sum_{nm} e^{-\beta E_n}e^{-(E_m-E_n)\tau}\langle n|\sigma_i^z|m\rangle\langle m|\sigma_j^z|n\rangle$$