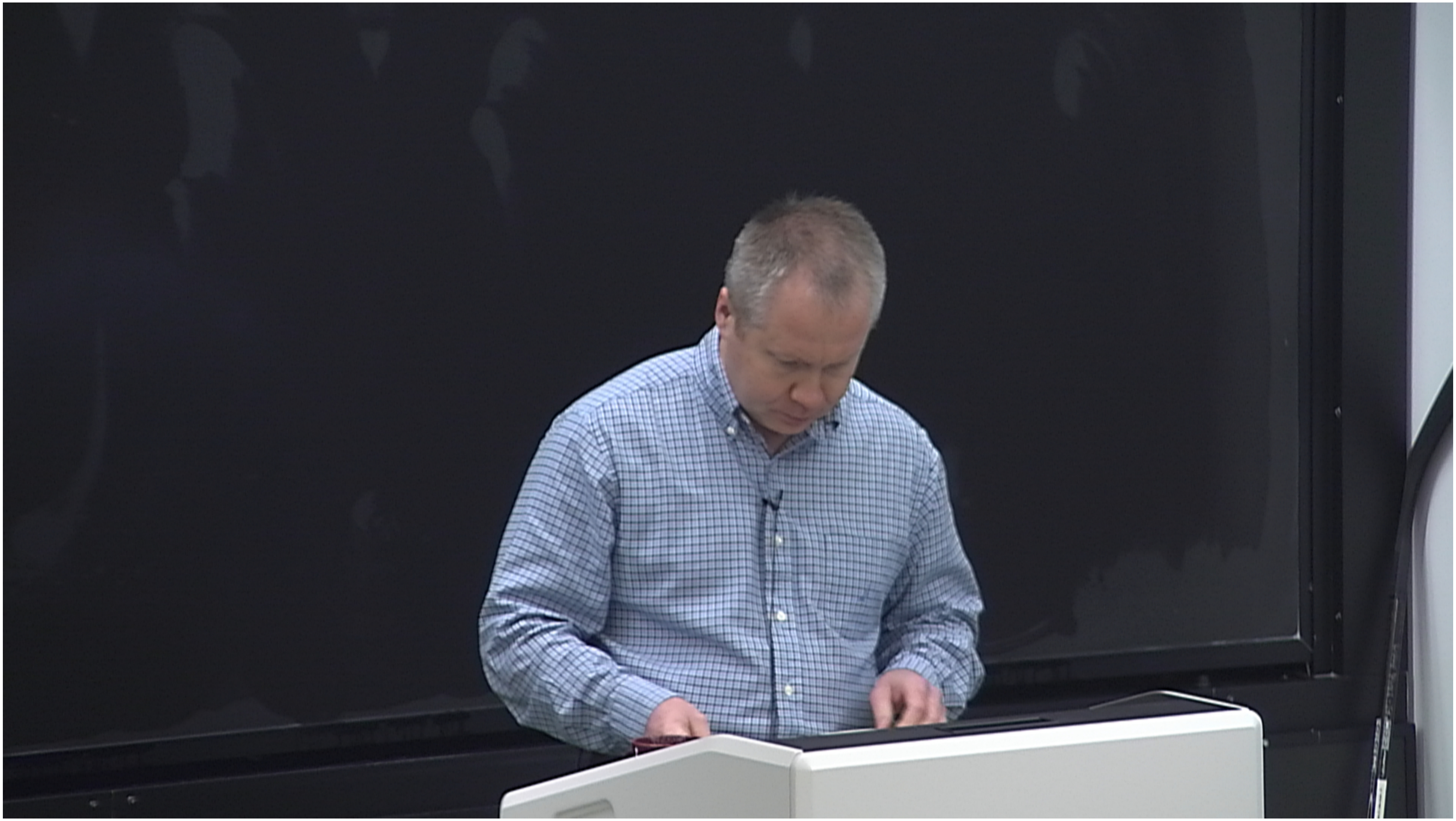
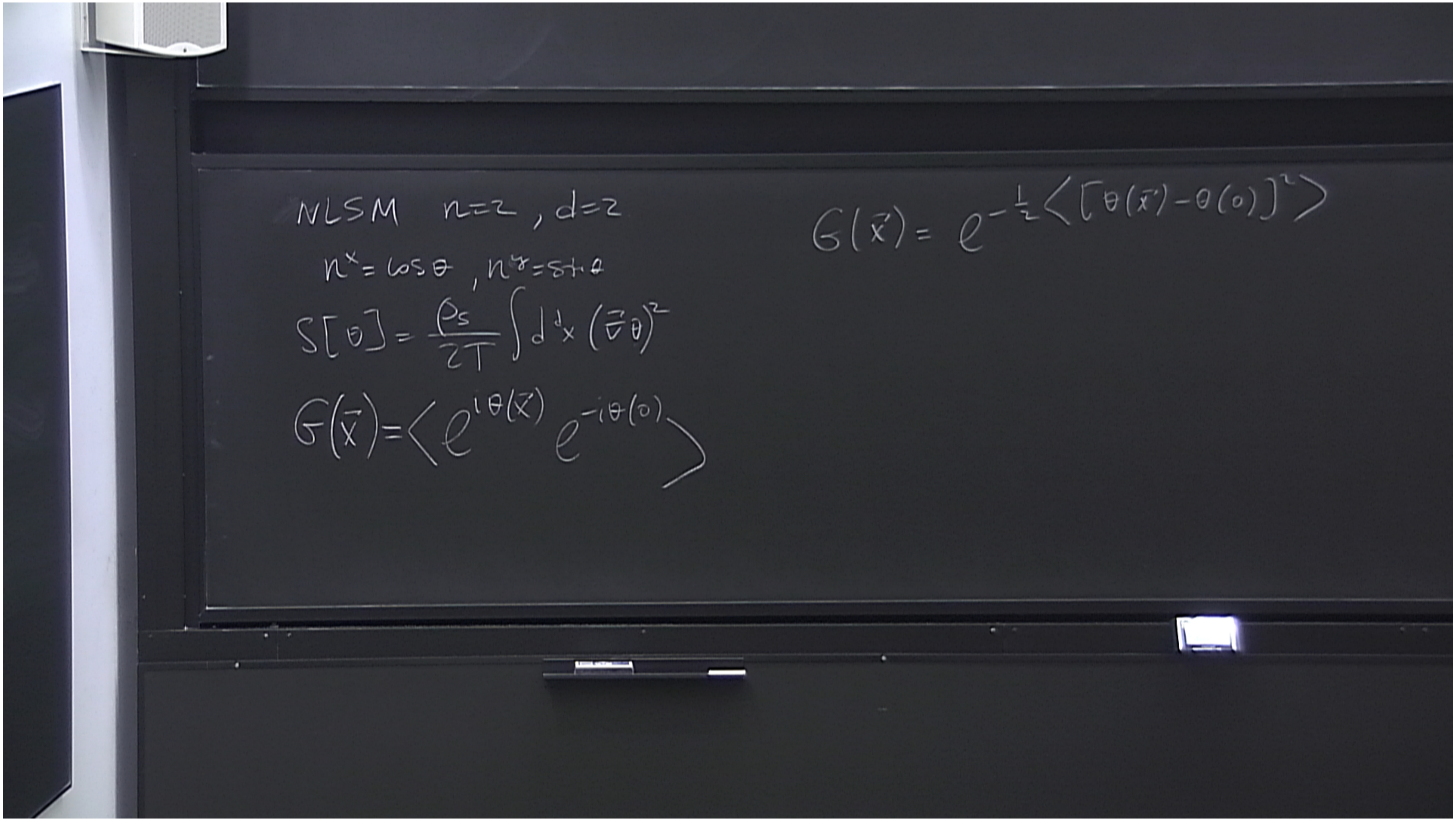
Title: 14/15 PSI - Statistical Mechanics - Lecture 12

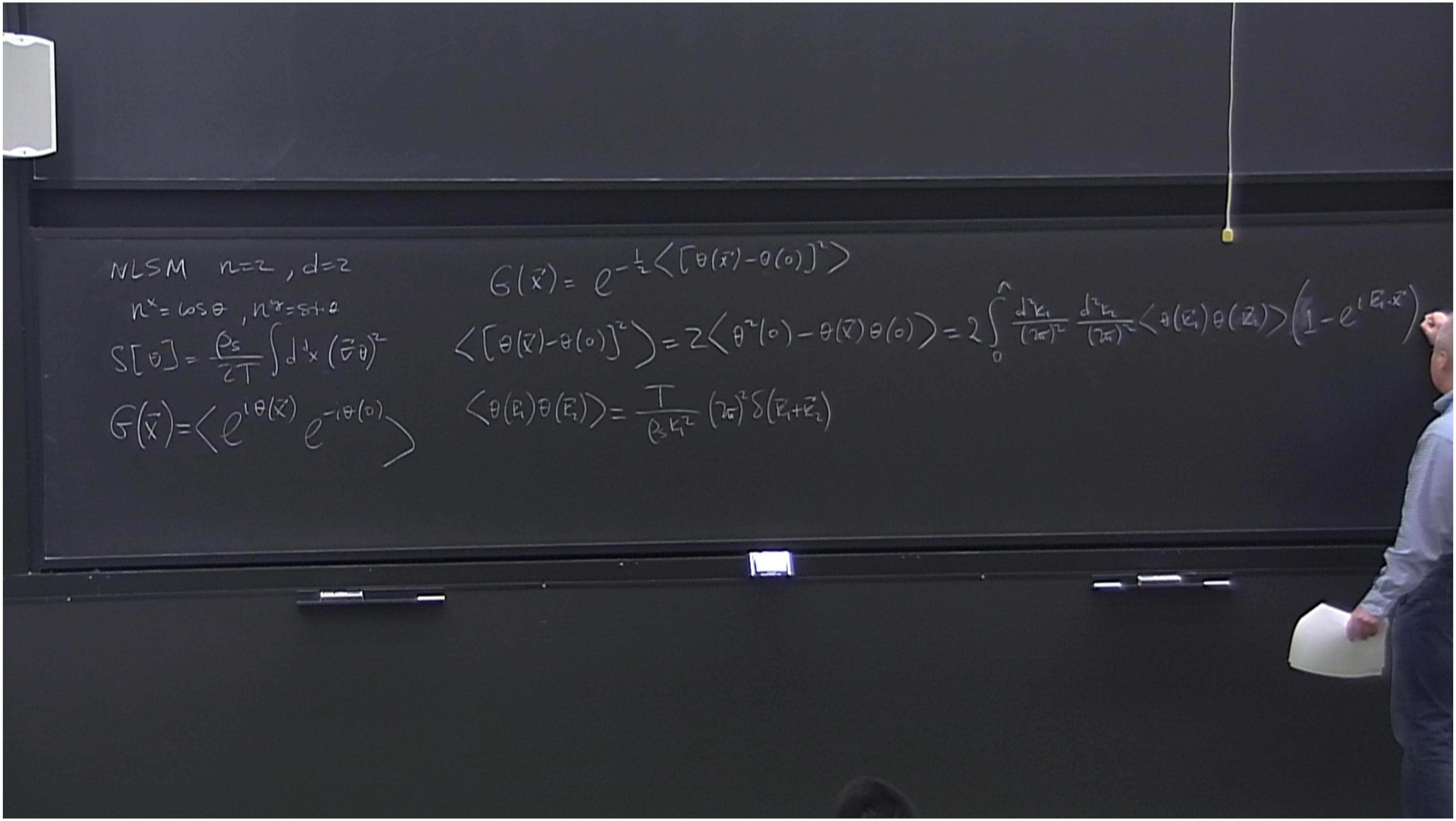
Date: Oct 22, 2014 10:45 AM

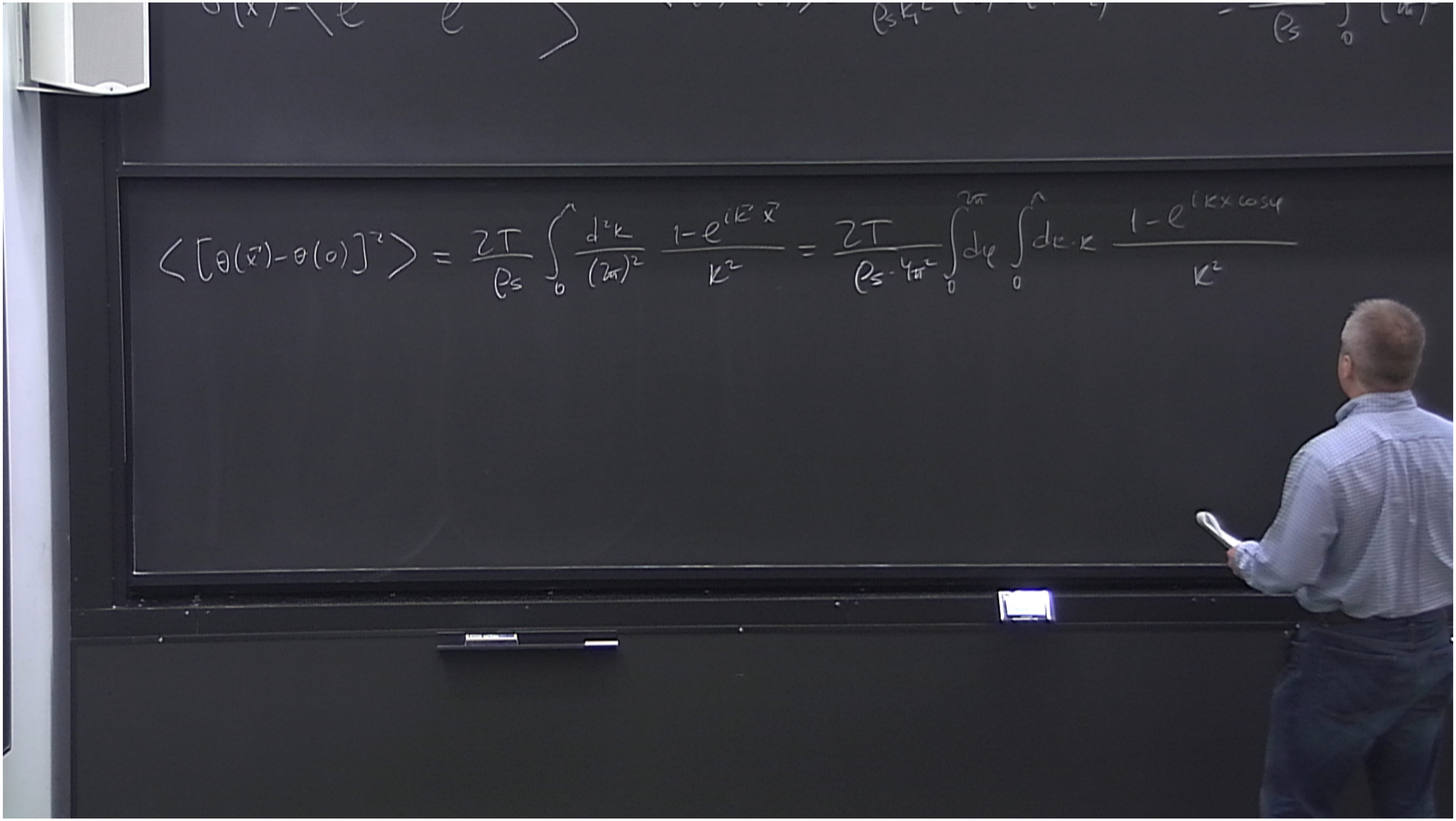
URL: http://pirsa.org/14100100

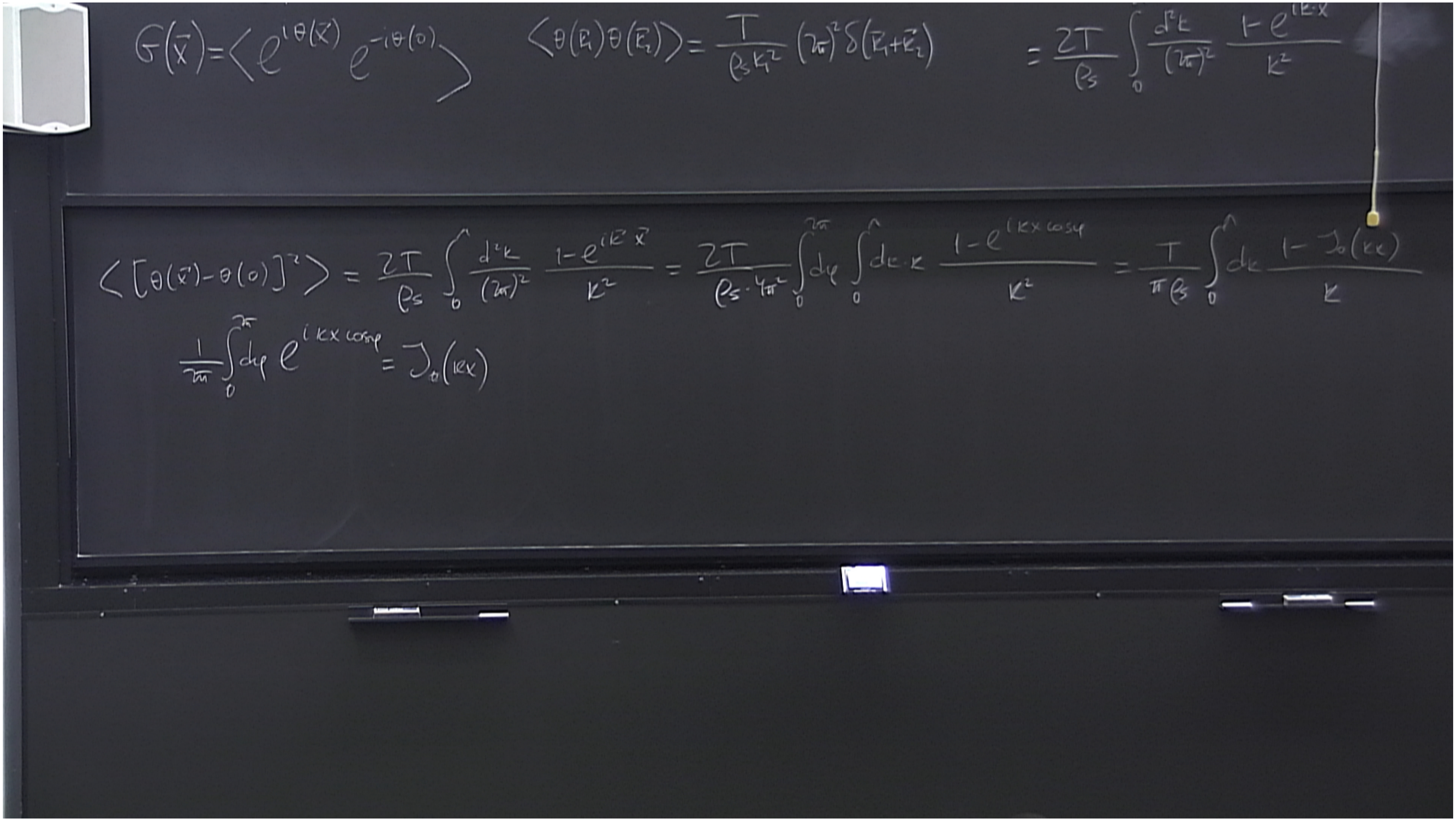
Abstract:




NLSM $n=2$, $d=2$
$n^x = \cos\theta$, $n^y = \sin\theta$
$S[\theta] = \frac{\rho_s}{2T}\int d^dx\,(\vec{\nabla}\theta)^2$
$G(\vec{x}) = \langle e^{i\theta(\vec{x})}\, e^{-i\theta(0)} \rangle$
$G(\vec{x}) = e^{-\frac{1}{2}\langle [\theta(\vec{x})-\theta(0)]^2 \rangle}$


NLSM n=2 , d=2
n^x = cos θ , n^y = sin θ
S[θ] = ρs/2T ∫ d^2x (∇θ)^2
G(x) = < e^{iθ(x)} e^{-iθ(0)} >
G(x) = e^{-1/2 <[θ(x) - θ(0)]^2>}
<[θ(x) - θ(0)]^2> = 2 <θ^2(0) - θ(x)θ(0)> = 2 ∫_0^Λ d^2k1/(2π)^2 d^2k2/(2π)^2 <θ(k1)θ(k2)> (1 - e^{i k1·x})
<θ(k1)θ(k2)> = T/(ρs k1^2) (2π)^2 δ(k1 + k2)


$\langle [\theta(\vec{x})-\theta(0)]^2 \rangle = \frac{2T}{\rho_s}\int_0^\Lambda \frac{d^2k}{(2\pi)^2}\frac{1-e^{i\vec{k}\cdot\vec{x}}}{k^2} = \frac{2T}{\rho_s\cdot 4\pi^2}\int_0^{2\pi} d\varphi \int_0^\Lambda dk\cdot k \frac{1-e^{ikx\cos\varphi}}{k^2}$


$G(\vec{x}) = \langle e^{i\theta(\vec{x})} e^{-i\theta(0)} \rangle$
$\langle \theta(\vec{k}_1)\theta(\vec{k}_2)\rangle = \frac{T}{\rho_s k_1^2}(2\pi)^2 \delta(\vec{k}_1+\vec{k}_2)$
$= \frac{2T}{\rho_s}\int_0 \frac{d^2k}{(2\pi)^2}\frac{1-e^{i\vec{k}\cdot\vec{x}}}{k^2}$
$\langle[\theta(\vec{x})-\theta(0)]^2\rangle = \frac{2T}{\rho_s}\int_0^\Lambda \frac{d^2k}{(2\pi)^2}\frac{1-e^{i\vec{k}\cdot\vec{x}}}{k^2} = \frac{2T}{\rho_s\cdot 4\pi^2}\int_0^{2\pi}d\varphi\int_0^\Lambda dk\cdot k\,\frac{1-e^{ikx\cos\varphi}}{k^2} = \frac{T}{\pi\rho_s}\int_0^\Lambda dk\,\frac{1-J_0(kx)}{k}$
$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\, e^{ikx\cos\varphi} = J_0(kx)$

$$G(\vec{x}) = \langle e^{i\theta(\vec{x})} e^{-i\theta(0)} \rangle \qquad \langle \theta(\vec{k}_1)\theta(\vec{k}_2)\rangle = \frac{T}{\rho_s k_1^2}(2\pi)^2 \delta(\vec{k}_1+\vec{k}_2) \qquad = \frac{2T}{\rho_s}\int_0^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1-e^{i\vec{k}\cdot\vec{x}}}{k^2}$$

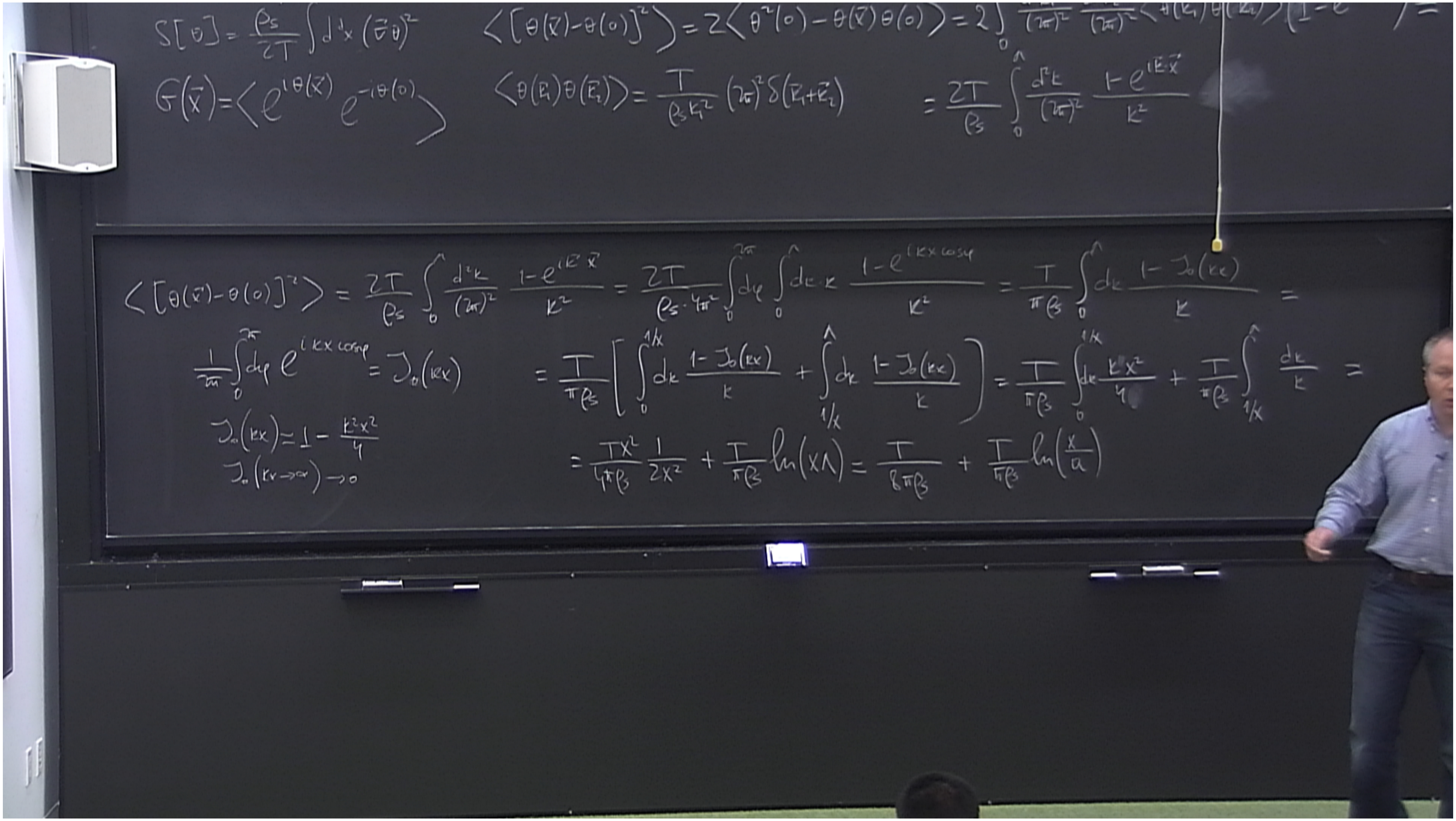
$$\langle [\theta(\vec{x})-\theta(0)]^2 \rangle = \frac{2T}{\rho_s}\int_0^{\Lambda} \frac{d^2k}{(2\pi)^2}\frac{1-e^{i\vec{k}\cdot\vec{x}}}{k^2} = \frac{2T}{\rho_s \cdot 4\pi^2}\int_0^{2\pi} d\varphi \int_0^{\Lambda} dk \cdot k \frac{1-e^{ikx\cos\varphi}}{k^2} = \frac{T}{\pi\rho_s}\int_0^{\Lambda} dk \frac{1-J_0(kx)}{k} =$$

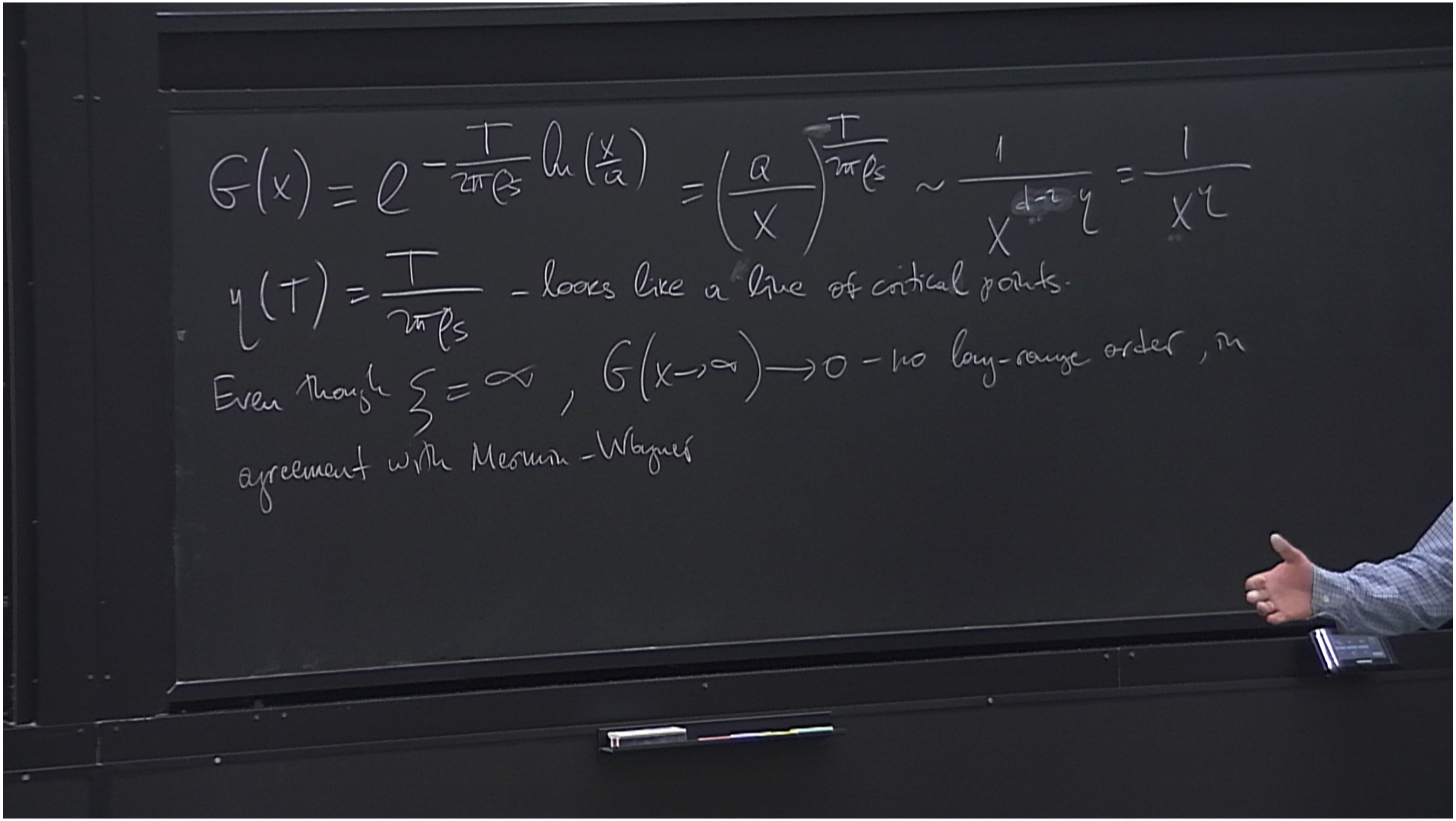
$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{ikx\cos\varphi} = J_0(kx)$$

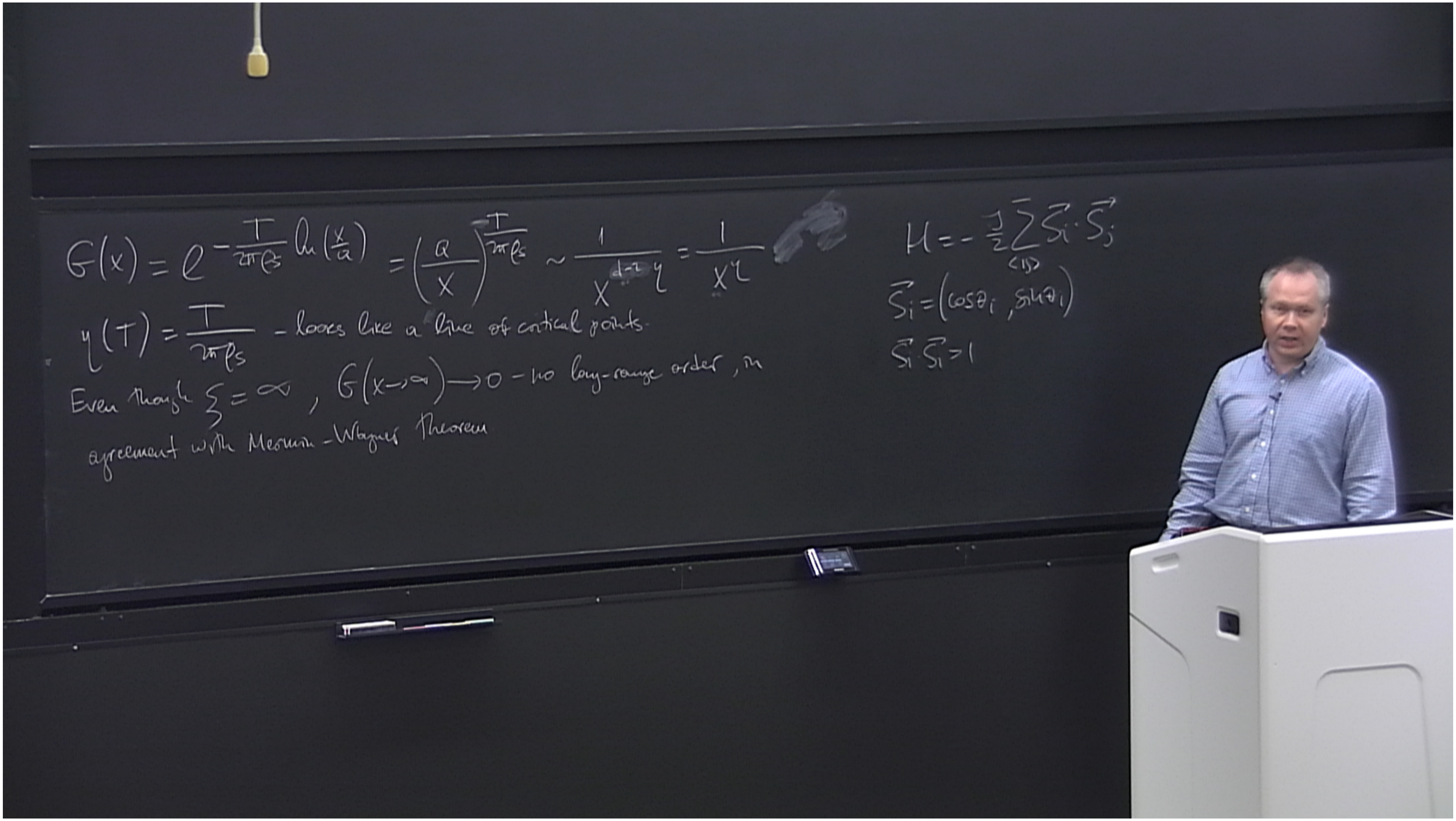
$$= \frac{T}{\pi\rho_s}\left[\int_0^{1/x} dk \frac{1-J_0(kx)}{k} + \int_{1/x}^{\Lambda} dk \frac{1-J_0(kx)}{k}\right] = \frac{T}{\pi\rho_s}\int_0^{1/x} dk \frac{k^2x^2}{4} + \frac{T}{\pi\rho_s}\int_{1/x}^{\Lambda} \frac{dk}{k}$$

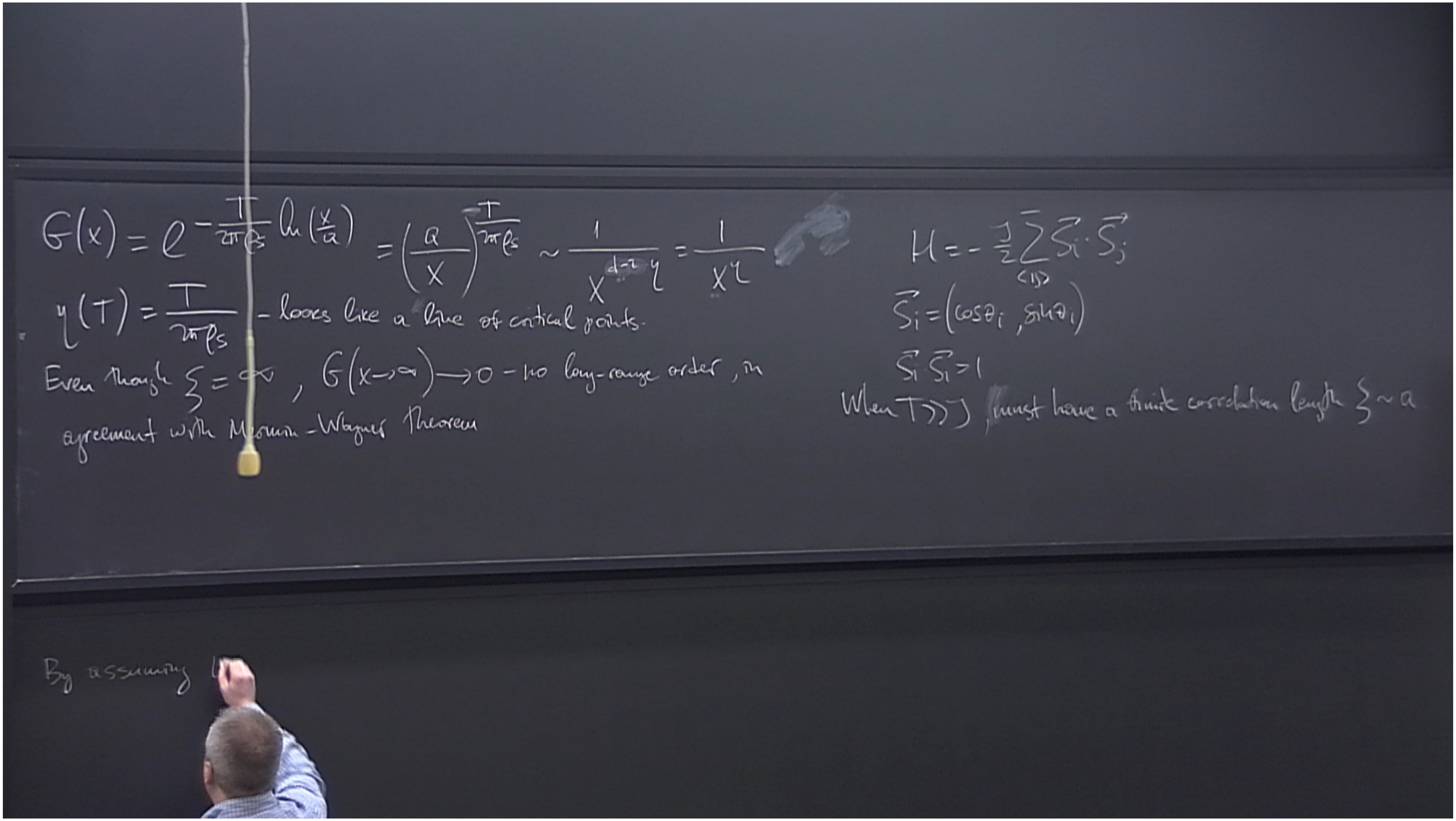
$$J_0(kx) = 1 - \frac{k^2x^2}{4}$$

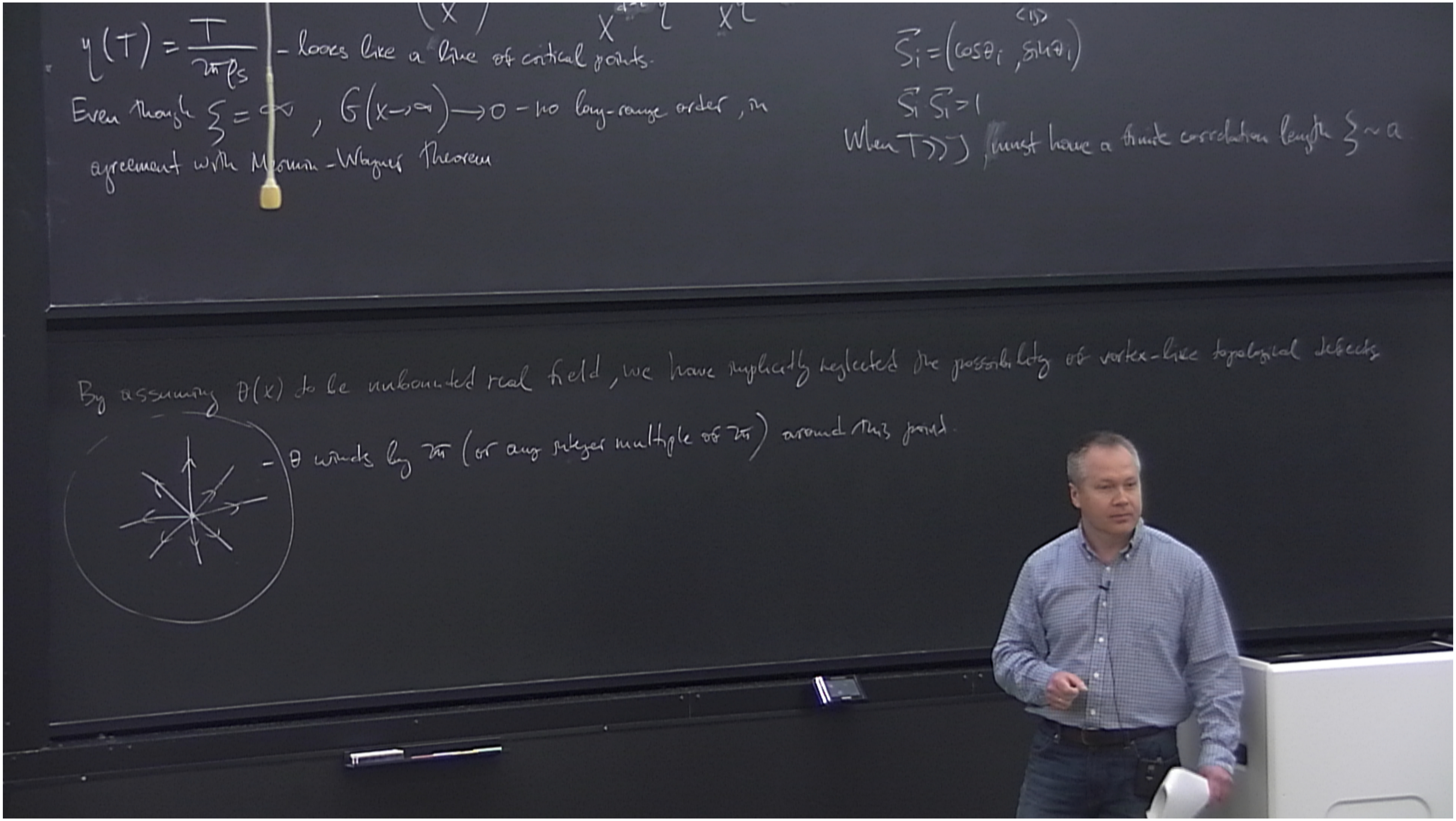
$$J_0(kx \to \infty) \to 0$$

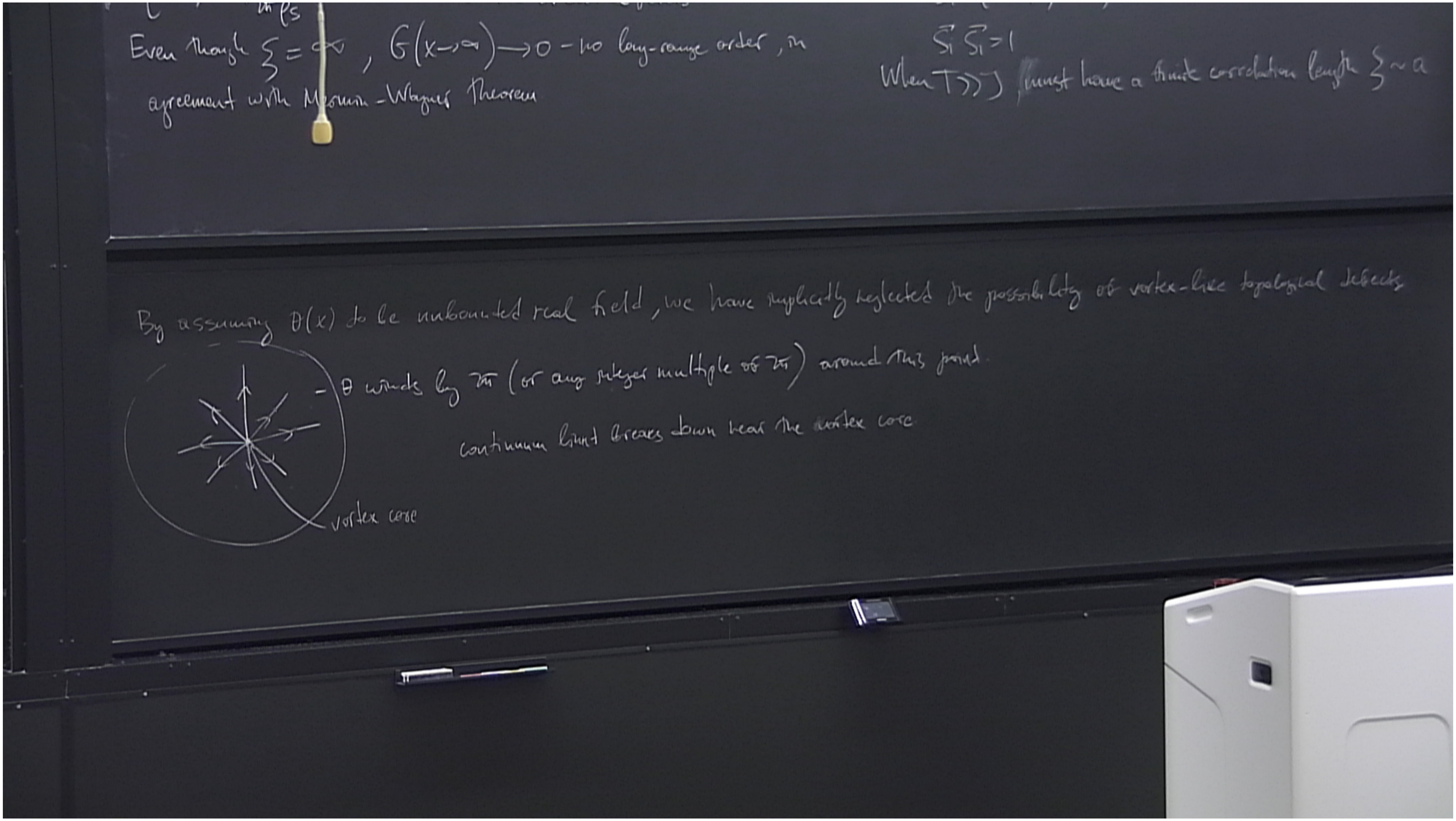

S[θ] = ρs/2T ∫d²x (∇θ)²
G(x) = ⟨e^{iθ(x)} e^{-iθ(0)}⟩
⟨θ(k1)θ(k2)⟩ = T/(ρs k1²) (2π)² δ(k1+k2)
⟨[θ(x)−θ(0)]²⟩ = 2⟨θ²(0) − θ(x)θ(0)⟩ = 2∫
= 2T/ρs ∫₀^Λ d²k/(2π)² (1−e^{ik·x})/k²
⟨[θ(x)−θ(0)]²⟩ = 2T/ρs ∫₀^Λ d²k/(2π)² (1−e^{ik·x})/k² = 2T/(ρs·4π²) ∫₀^{2π}dφ ∫₀^Λ dk·k (1−e^{ikx cosφ})/k² = T/(πρs) ∫₀^Λ dk (1−J₀(kx))/k =
1/2π ∫₀^{2π} dφ e^{ikx cosφ} = J₀(kx)
= T/(πρs) [∫₀^{1/x} dk (1−J₀(kx))/k + ∫_{1/x}^Λ dk (1−J₀(kx))/k] = T/(πρs) ∫₀^{1/x} dk k²x²/4 + T/(πρs) ∫_{1/x}^Λ dk/k =
J₀(kx) ≃ 1 − k²x²/4
J₀(kx→∞) → 0
= Tx²/(4πρs) 1/(2x²) + T/(πρs) ln(xΛ) = T/(8πρs) + T/(πρs) ln(x/a)


$G(x) = e^{-\frac{T}{2\pi\rho_s}\ln\left(\frac{x}{a}\right)} = \left(\frac{a}{x}\right)^{\frac{T}{2\pi\rho_s}} \sim \frac{1}{x^{d-2+\eta}} = \frac{1}{x^{\eta}}$
$\eta(T) = \frac{T}{2\pi\rho_s}$ - looks like a line of critical points.
Even though $\xi = \infty$, $G(x \to \infty) \to 0$ - no long-range order, in agreement with Mermin-Wagner


G(x) = e^{-T/(2πρs) ln(x/a)} = (a/x)^{T/(2πρs)} ~ 1/x^{d-2+η} = 1/x^η
η(T) = T/(2πρs) - looks like a line of critical points.
Even though ξ = ∞ , G(x→∞) → 0 - no long-range order, in
agreement with Mermin-Wagner Theorem
H = -J/2 Σ_{<ij>} S_i · S_j
S_i = (cosθ_i , sinθ_i)
S_i S_i = 1


$G(x) = e^{-\frac{T}{2\pi\rho_s}\ln\left(\frac{x}{a}\right)} = \left(\frac{a}{x}\right)^{\frac{T}{2\pi\rho_s}} \sim \frac{1}{x^{d-2+\eta}} = \frac{1}{x^{\eta}}$
$\eta(T) = \frac{T}{2\pi\rho_s}$ - looks like a line of critical points.
Even though $\xi = \infty$, $G(x\to\infty) \to 0$ - no long-range order, in agreement with Mermin-Wagner Theorem
$H = -\frac{J}{2}\sum_{\langle ij\rangle} \vec{S}_i\cdot\vec{S}_j$
$\vec{S}_i = (\cos\theta_i, \sin\theta_i)$
$\vec{S}_i\cdot\vec{S}_i = 1$
When $T \gg J$ (i.e.) must have a finite correlation length $\xi \sim a$
By assuming

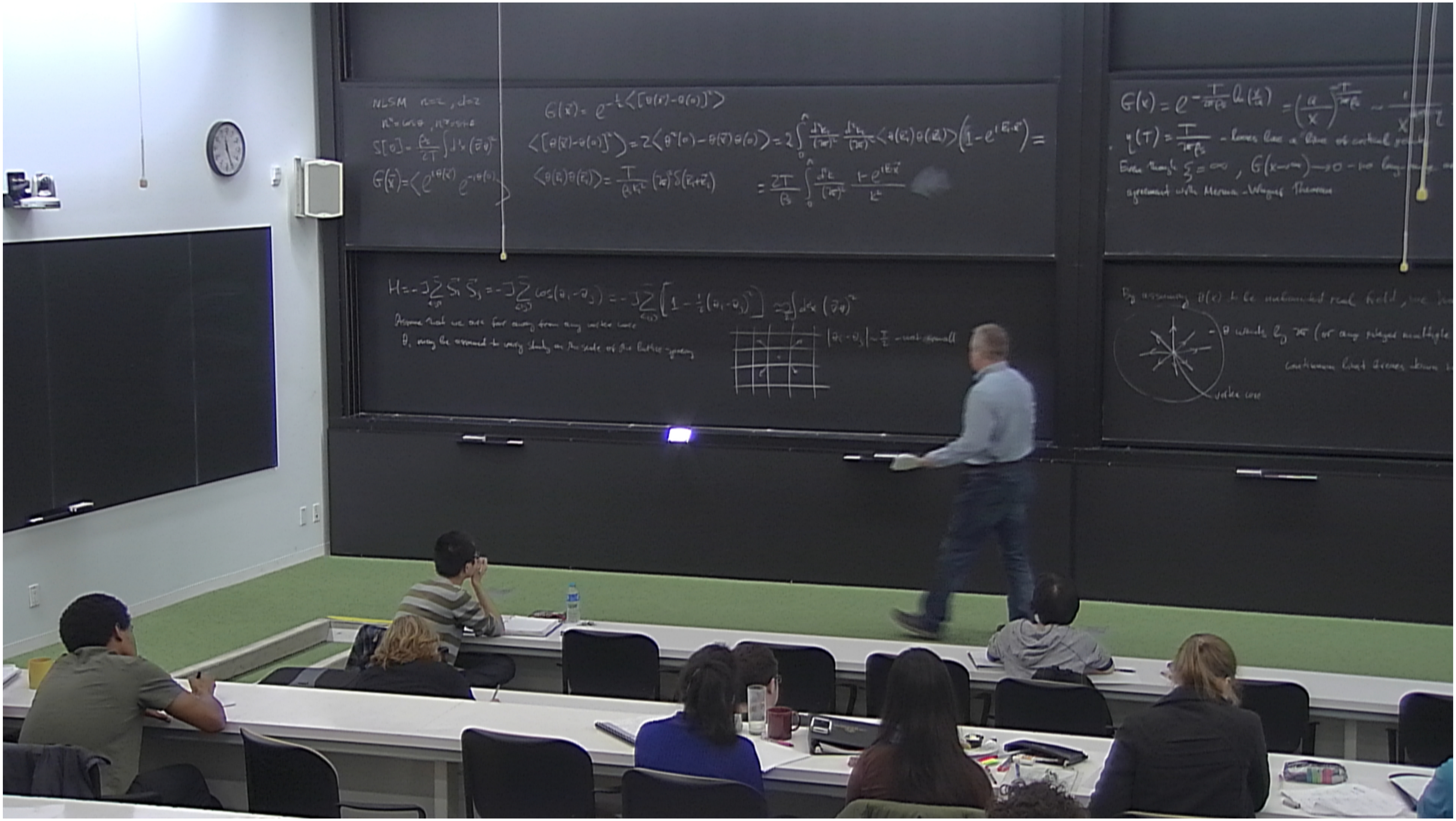

η(T) = T / 2πρs - looks like a line of critical points.
Even though ξ = ∞, G(x→∞) → 0 - no long-range order, in agreement with Mermin-Wagner Theorem
S_i = (cosθ_i, sinθ_i)
S_i S_i = 1
When T >> J, must have a finite correlation length ξ ~ a
By assuming θ(x) to be unbounded real field, we have implicitly neglected the possibility of vortex-like topological defects
- θ winds by 2π (or any integer multiple of 2π) around this point.

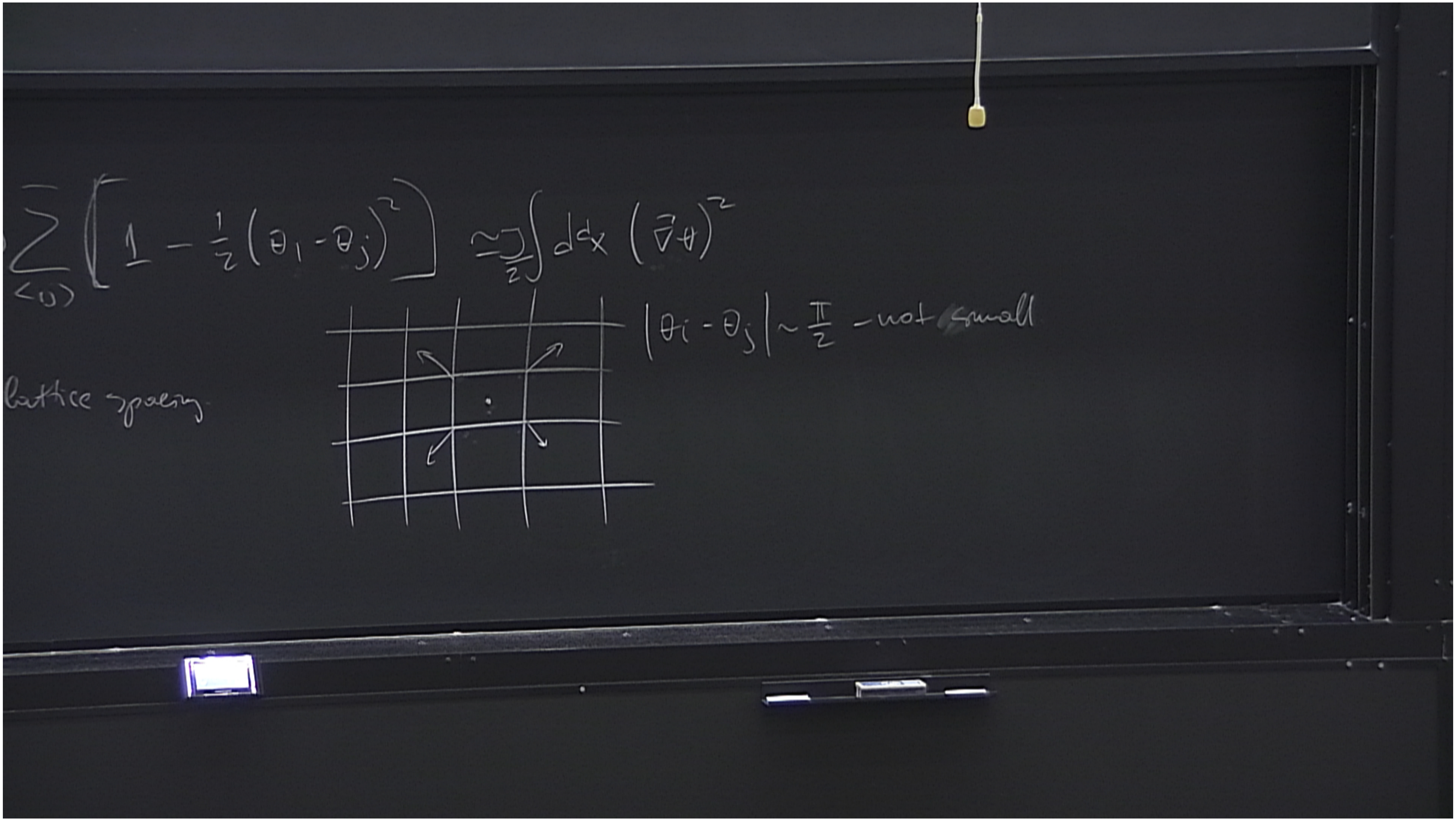

Even though ξ = ∞ , G(x→∞) → 0 – no long-range order, in agreement with Mermin-Wagner Theorem
S_i S_i = 1
When T >> J, must have a finite correlation length ξ ~ a
By assuming θ(x) to be unbounded real field, we have implicitly neglected the possibility of vortex-like topological defects
θ winds by 2π (or any integer multiple of 2π) around this point.
continuum limit breaks down near the vortex core
vortex core

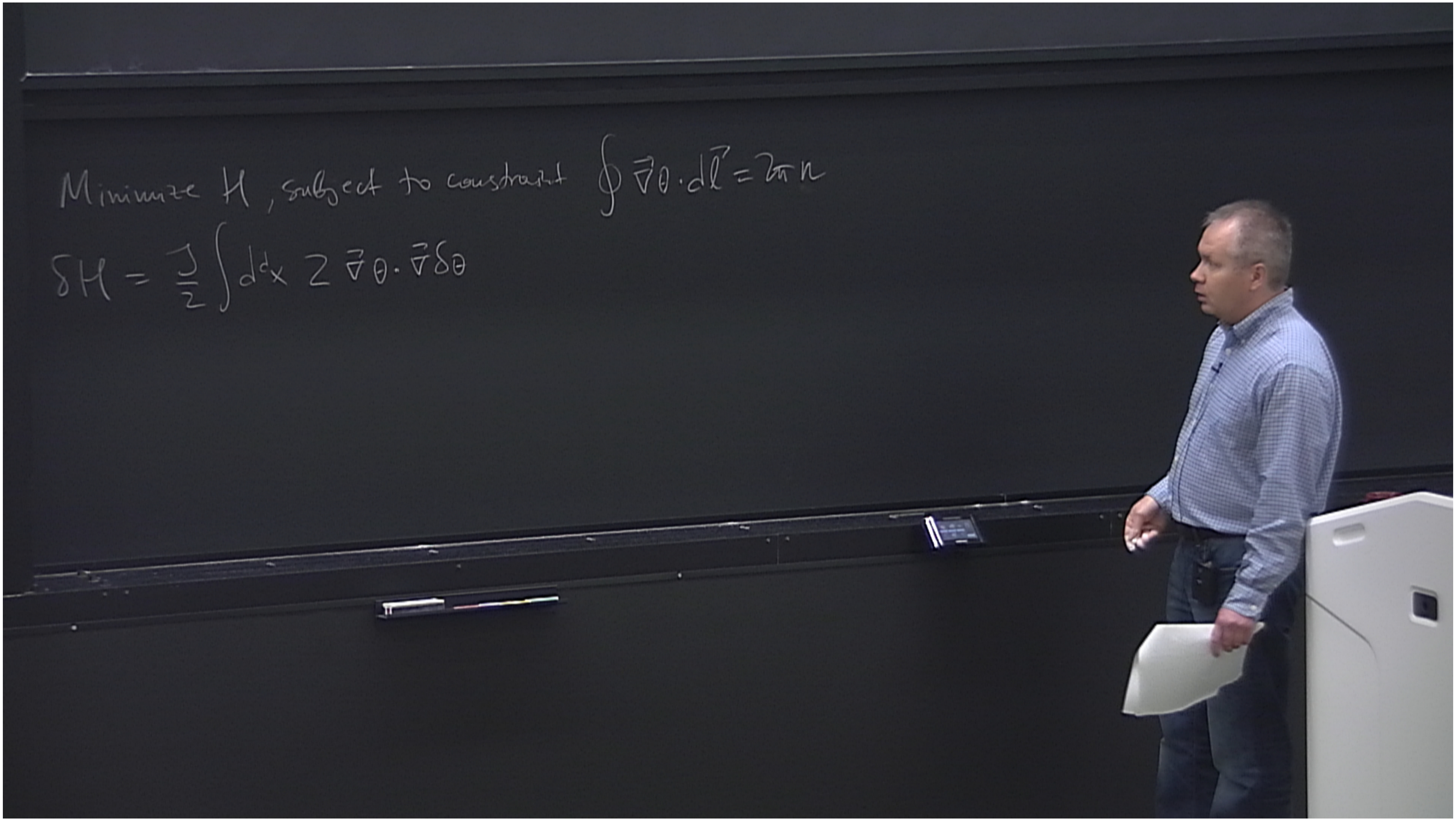
$$H = -J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) \simeq -J\sum_{\langle ij\rangle}\left[1 - \frac{1}{2}(\theta_i - \theta_j)^2\right]$$

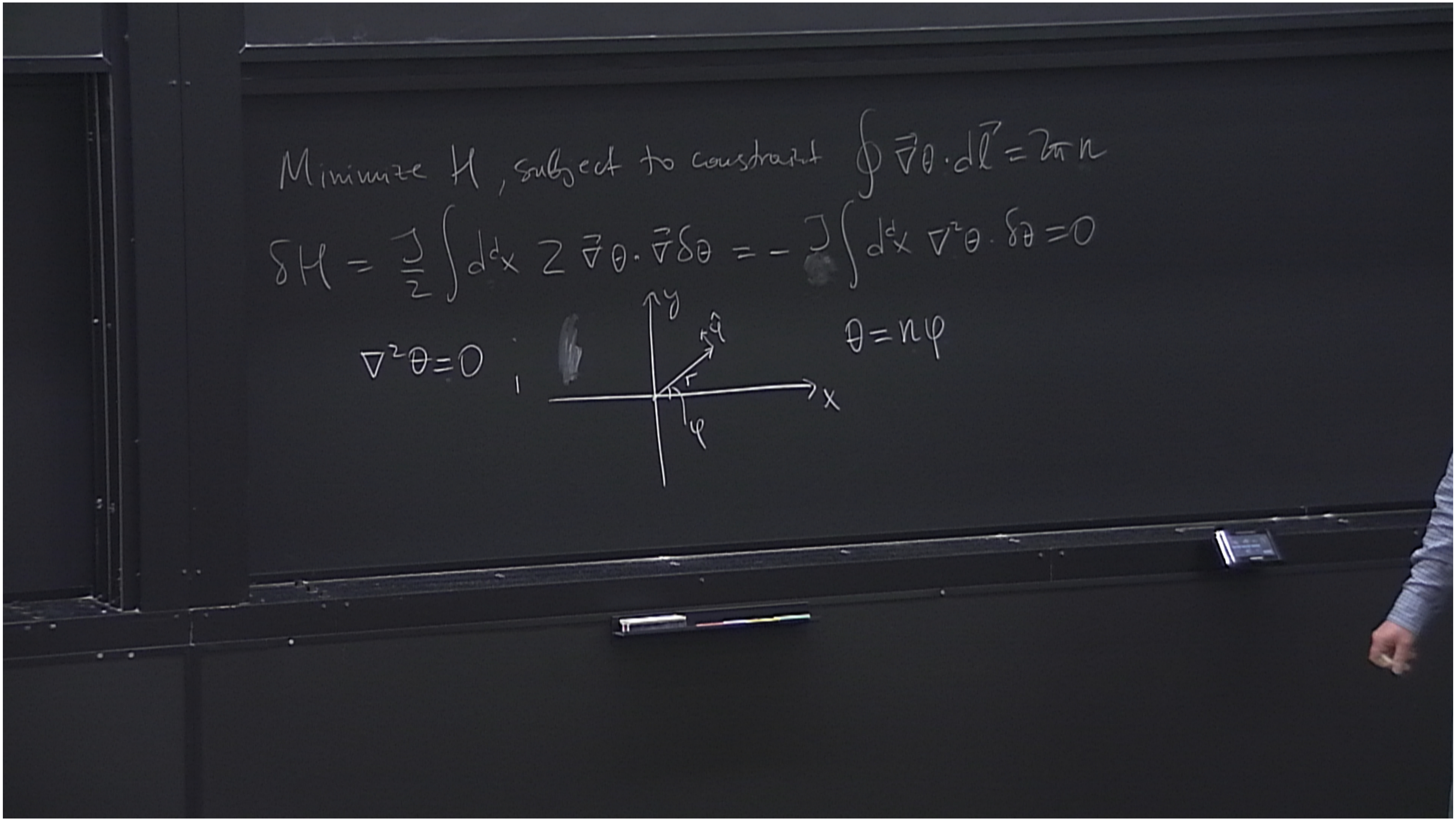
Assume that we are far away from any vortex core

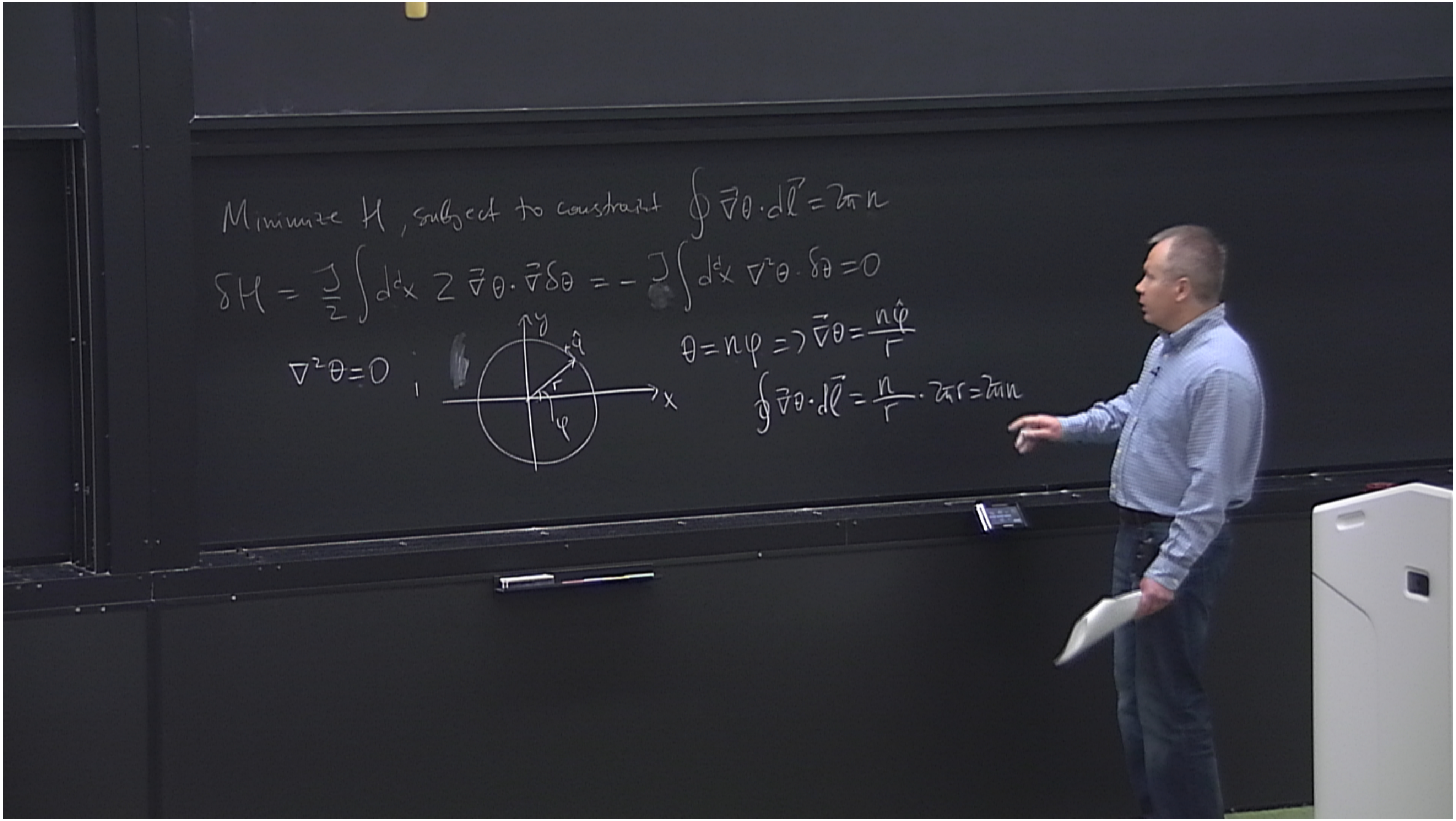
θ_i may be assumed to vary slowly on the scale of the lattice spacing

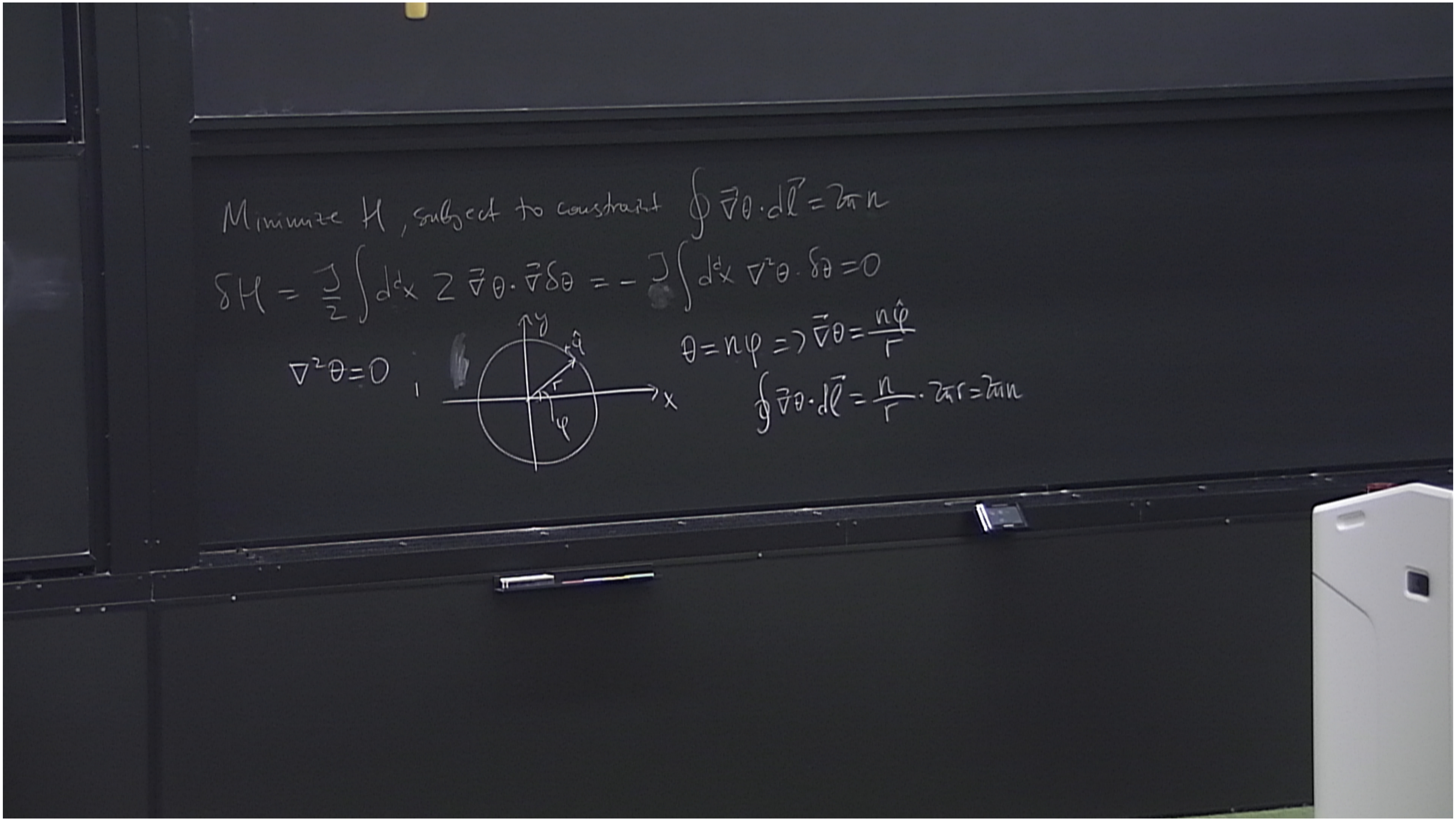


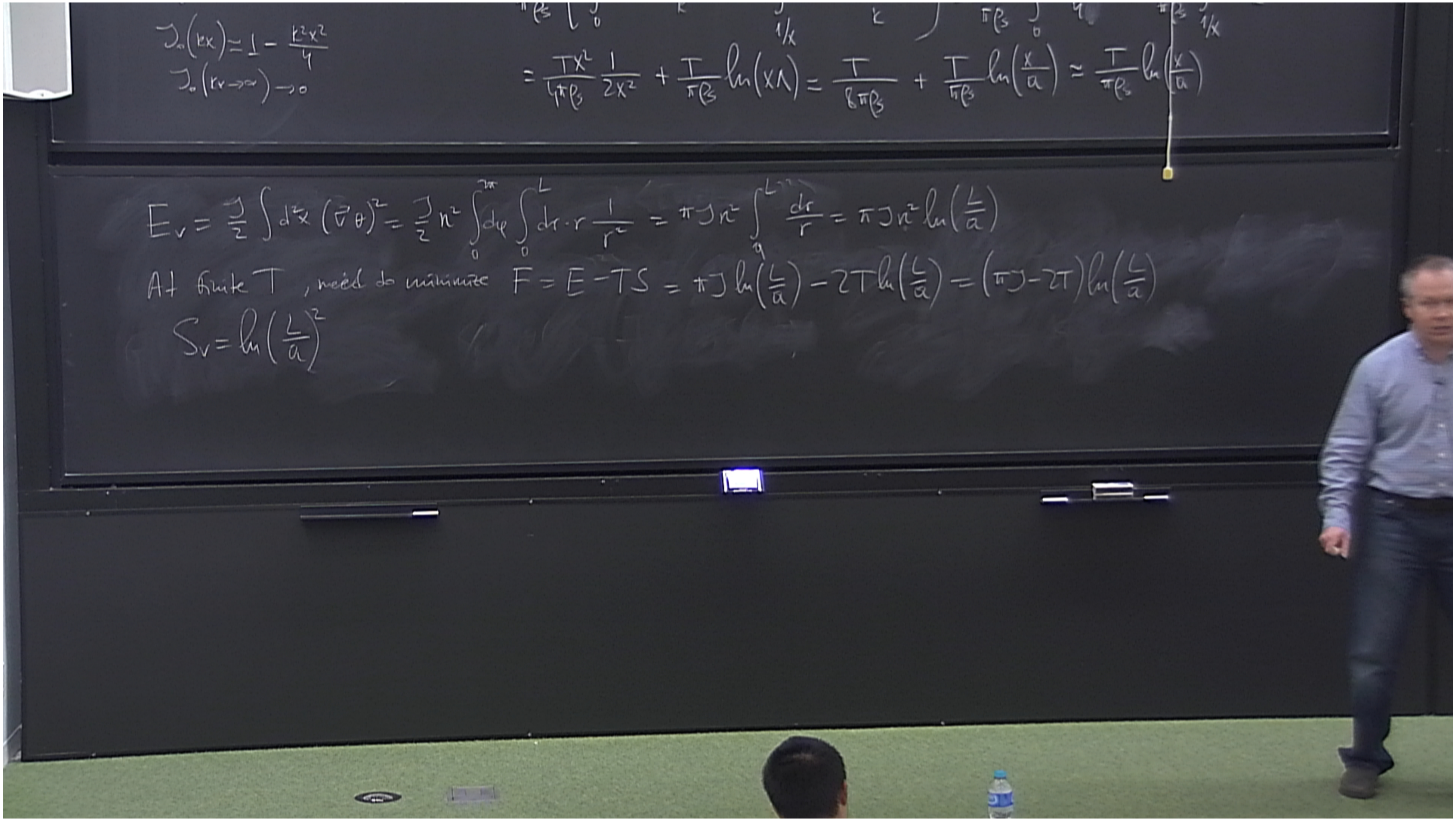

$\sum_{\langle ij \rangle} \left[1 - \frac{1}{2}(\theta_i - \theta_j)^2 \right] \simeq \frac{J}{2} \int d^d x \, (\vec{\nabla}\theta)^2$
$|\theta_i - \theta_j| \sim \frac{\pi}{2}$ - not small
lattice spacing


Minimize H, subject to constraint ∮ ∇⃗θ · dl⃗ = 2πn
δH = J/2 ∫ d^dx 2 ∇⃗θ · ∇⃗δθ


Minimize H, subject to constraint $\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$
$\delta H = \frac{J}{2}\int d^dx \; 2\,\vec{\nabla}\theta\cdot\vec{\nabla}\delta\theta = -J\int d^dx \; \nabla^2\theta \cdot \delta\theta = 0$
$\nabla^2\theta = 0$;
y
$\hat{\varphi}$
r
x
φ
$\theta = n\varphi$


Minimize H, subject to constraint $\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$
$\delta H = \frac{J}{2}\int d^dx \, 2\, \vec{\nabla}\theta \cdot \vec{\nabla}\delta\theta = -J\int d^dx \, \nabla^2\theta \cdot \delta\theta = 0$
$\nabla^2\theta = 0$
$\theta = n\varphi \Rightarrow \vec{\nabla}\theta = \frac{n\hat{\varphi}}{r}$
$\oint \vec{\nabla}\theta \cdot d\vec{l} = \frac{n}{r} \cdot 2\pi r = 2\pi n$
y
x
r
φ
$\hat{\varphi}$


Minimize H, subject to constraint $\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$
$\delta H = \frac{J}{2}\int d^dx \; 2\, \vec{\nabla}\theta \cdot \vec{\nabla}\delta\theta = -J\int d^dx \; \nabla^2\theta \cdot \delta\theta = 0$
$\nabla^2\theta = 0$
y
$\hat{\varphi}$
r
φ
x
$\theta = n\varphi \Rightarrow \vec{\nabla}\theta = \frac{n\hat{\varphi}}{r}$
$\oint \vec{\nabla}\theta \cdot d\vec{l} = \frac{n}{r} \cdot 2\pi r = 2\pi n$



$J_0(kx) \simeq 1 - \frac{k^2x^2}{4}$
$J_0(kr \to \infty) \to 0$
$= \frac{Tx^2}{4\pi\rho_s}\frac{1}{2x^2} + \frac{T}{\pi\rho_s}\ln(x\Lambda) = \frac{T}{8\pi\rho_s} + \frac{T}{\pi\rho_s}\ln\left(\frac{x}{a}\right) \simeq \frac{T}{\pi\rho_s}\ln\left(\frac{x}{a}\right)$
$E_v = \frac{J}{2}\int d^2x\,(\vec\nabla\theta)^2 = \frac{J}{2}n^2\int_0^{2\pi}d\varphi\int_0^L dr\cdot r\frac{1}{r^2} = \pi J n^2\int_a^L\frac{dr}{r} = \pi J n^2\ln\left(\frac{L}{a}\right)$
At finite T, need to minimize $F = E - TS = \pi J\ln\left(\frac{L}{a}\right) - 2T\ln\left(\frac{L}{a}\right) = (\pi J - 2T)\ln\left(\frac{L}{a}\right)$
$S_v = \ln\left(\frac{L}{a}\right)^2$

$J_0(r\to\infty)\to 0$

$$= \frac{T x^2}{4\pi\beta_s}\frac{1}{2x^2} + \frac{T}{\pi\beta_s}\ln(x\Lambda) = \frac{T}{8\pi\beta_s} + \frac{T}{\pi\beta_s}\ln\left(\frac{x}{a}\right) \simeq \frac{T}{\pi\beta_s}\ln\left(\frac{x}{a}\right)$$

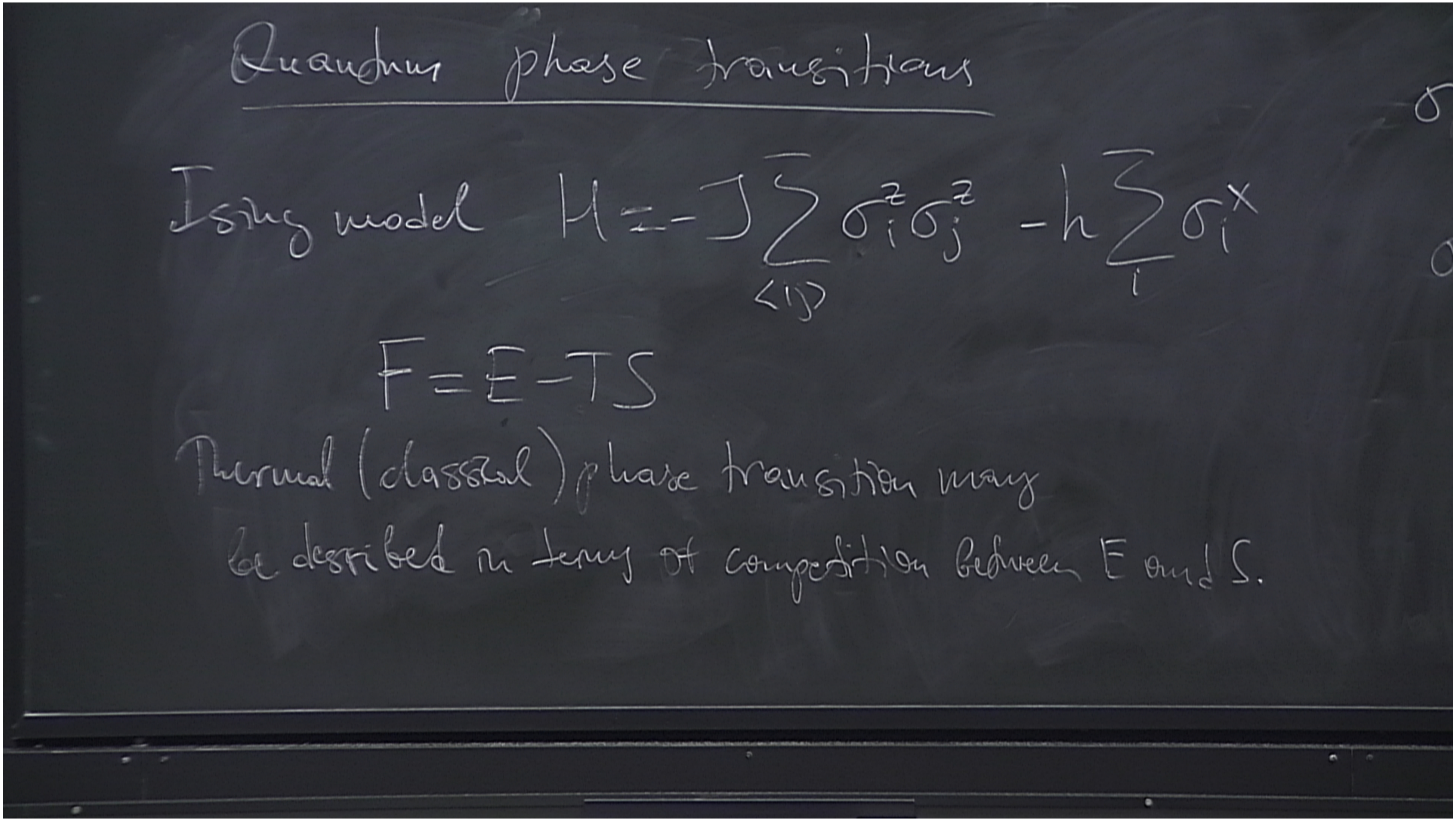
$$E_v = \frac{J}{2}\int d^2x\,(\vec\nabla\theta)^2 = \frac{J}{2}n^2\int_0^{2\pi}d\varphi\int_0^L dr\cdot r\,\frac{1}{r^2} = \pi J n^2\int_a^L\frac{dr}{r} = \pi J n^2\ln\left(\frac{L}{a}\right)$$

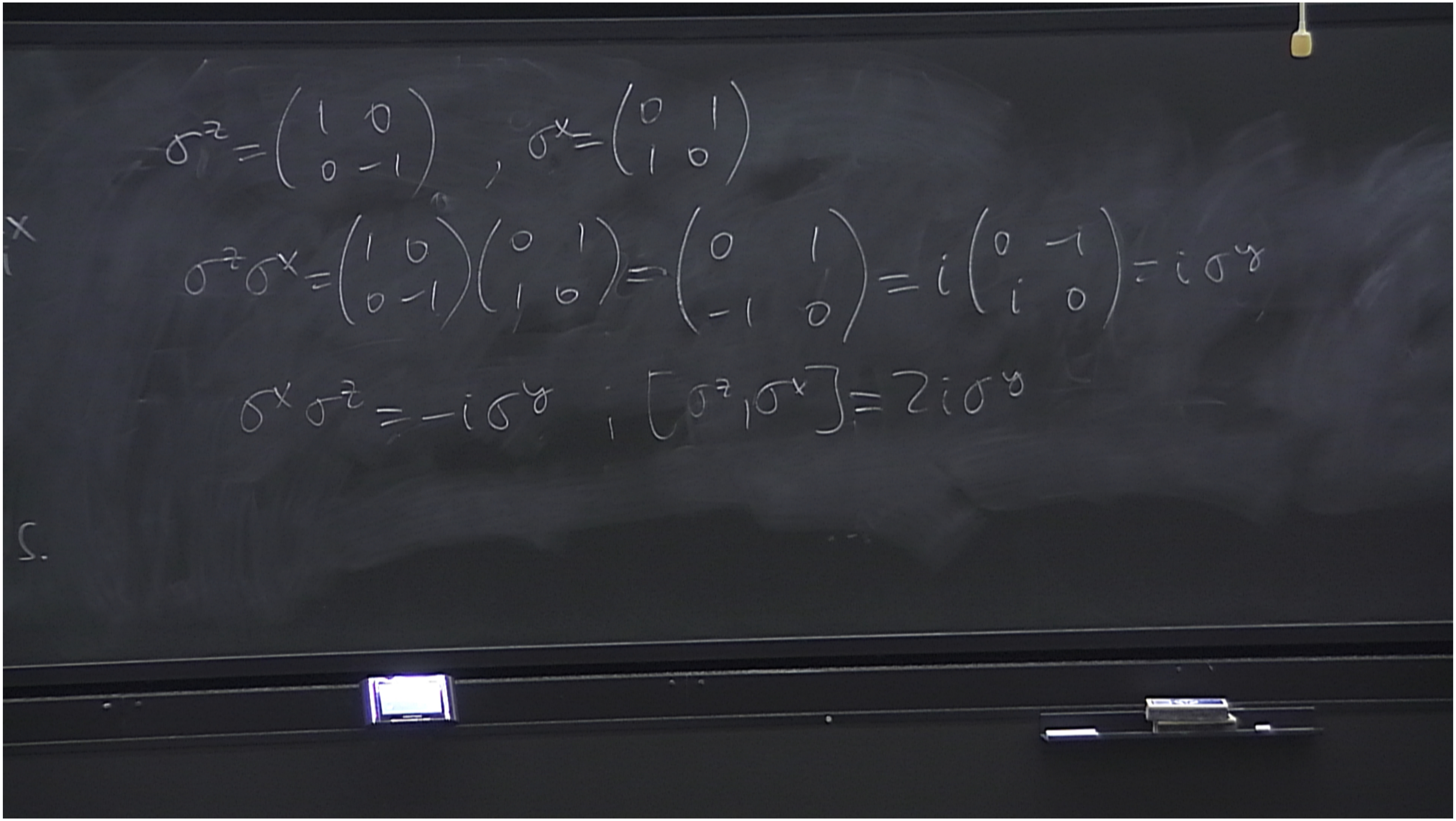
At finite T, need to minimize $F = E - TS = \pi J\ln\left(\frac{L}{a}\right) - 2T\ln\left(\frac{L}{a}\right) = (\pi J - 2T)\ln\left(\frac{L}{a}\right)$

$S_v = \ln\left(\frac{L}{a}\right)^2$

When $T > T_{KT} = \frac{\pi J}{2}$, vortices will appear in the system

$T < T_{KT}$ the presence of vortices may be ignored $G(x)\sim\frac{1}{x^{\eta(T)}}$; $\eta(T) = \frac{T}{2\pi J}$

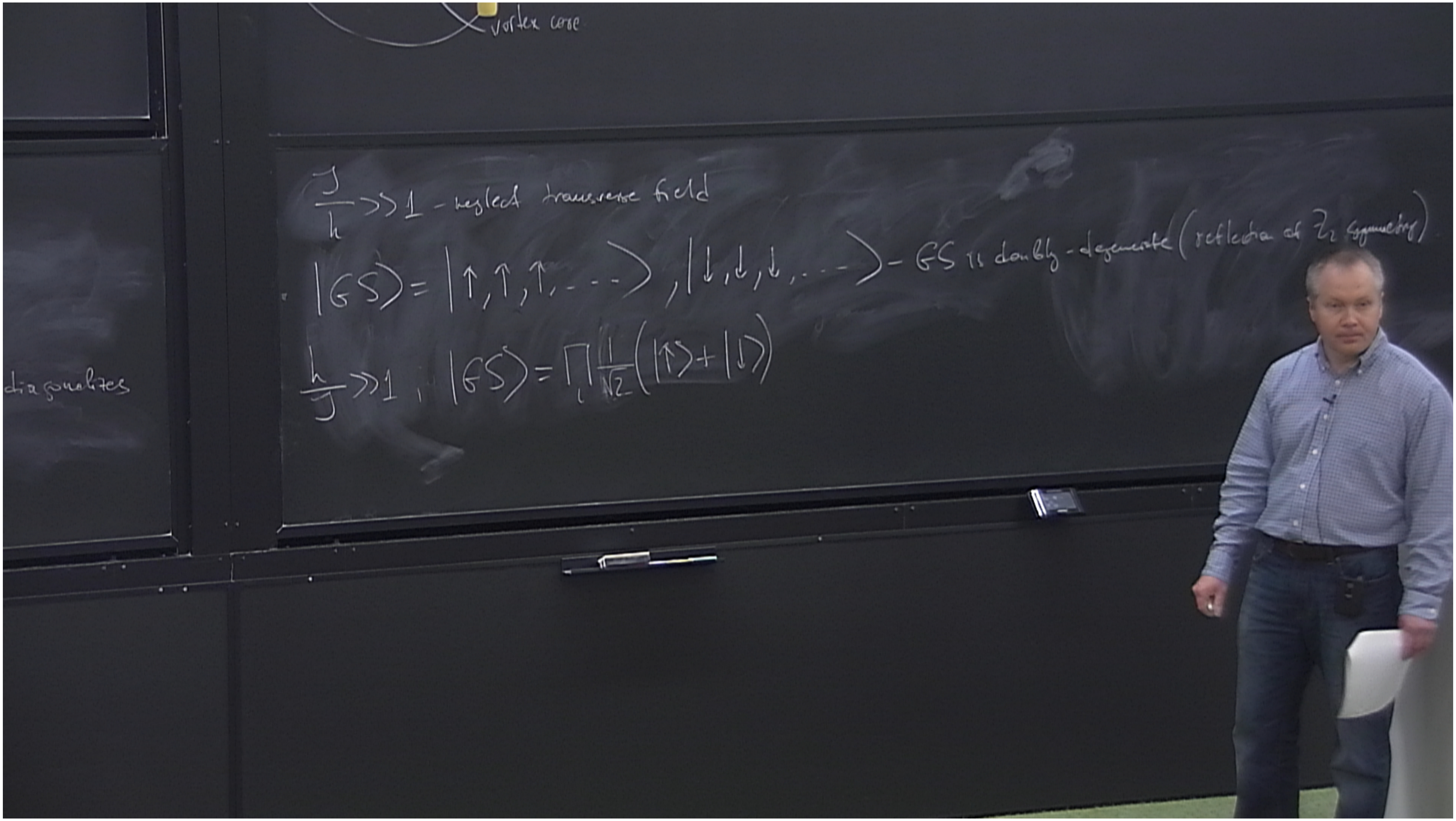

Quantum phase transitions
Ising model $H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$
$F = E - TS$
Thermal (classial) phase transition may
be described in terms of competition between E and S.


$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\sigma^z \sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma^y$
$\sigma^x \sigma^z = -i\sigma^y \quad ; \quad [\sigma^z, \sigma^x] = 2i\sigma^y$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z \sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma^y$$

$\sigma^x \sigma^z = -i\sigma^y \; ; \; [\sigma^z, \sigma^x] = 2i\sigma^y$ – no choice of basis that diagonalizes both terms in H


vortex core
J/h >> 1 - neglect transverse field
|GS⟩ = |↑,↑,↑,...⟩ , |↓,↓,↓,...⟩ - GS is doubly-degenerate (reflection of Z2 symmetry)
h/J >> 1 , |GS⟩ = ∏ 1/√2 (|↑⟩ + |↓⟩)
diagonalizes