

Title: Bogomolov and beyond for Calabi-Yau 3-folds

Date: Oct 08, 2014 02:00 PM

URL: <http://pirsa.org/14100083>

Abstract:

(inequality) ← Chern #s of $(\text{stable})(\text{hol})$ vector bundles
BPS. } coh. sheaves
Bogomolov & beyond → to CY 3-folds

Bogomolov.

1. Bogomolov (type) of thms.

2. Boundedness results (Maruyama, Carlos Simpson) ✓

3. DRY conjecture. (inspiration for going beyond Bogomolov's thm)

hep-th/0604597. M. Douglas, Reinebach, S.T. Yau.

$G(E)$.

hep-th/0604547

M. Douglas, Reineker, S.T. Yau

$G_2(E)$

attractor mechanism for BPS B.H.S. in (4,3) d

DT inv. / Bridgeland-stab:

Douglas P_i

f_i construction

Hartshorne-Serre construction

Deformation theory \rightarrow Stable

$X_3 \begin{cases} E \\ \downarrow \\ \mathbb{P}^2 \end{cases} = \text{reflexive sheaves}$

codim 3 \rightarrow p.t.s

hep-th/0604597

M. Douglas, Reineker, S.T. Yau

$G_2(E)$

attractor mechanism for BPS B.H.S. in (Y, field)

DT inv. / Bridgeland-stab:

Douglas P_i

f. construction: $\left\{ \begin{array}{l} \text{Hartshorne-Serre construction} \\ \text{Deformation theory} \rightarrow \text{Stable} \end{array} \right.$

$X_3 \begin{cases} E \\ \downarrow \\ \mathbb{P}^2 \end{cases} = \text{reflexive sheaves}$
Dickson (interesting thing)

$\text{codim } 3 \rightarrow \text{pts}$
 $M(\text{stable obj}) \cong M_{\text{BPS}} \text{ compact}$

of (stable) (hol) } vector bundles }
BPS. } coh. sheaves }

5. generalization ; split attractor flow.

(Simpson) ✓

for going beyond Bogomolov's theory)

is, Reinecker, S.-T. Yau

moduli for BPS B.H.S. in (4-fold).

of (stable) (hol) vector bundles
BPS. } coh. sheaves

5. generalization; split attractor flow.

(Simpson) ✓

for going beyond Bogomolov's theory

Reinbacher, S.-T. Yau

moduli for BPS B.H.s. in (4-fold).

1. $E_{YM} =$

$$1. \quad E_{YM} = -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{g} \operatorname{Tr}(F_{ij} F^{ij})$$

$$\stackrel{\text{Bog. mod.}}{=} -\frac{1}{8g_{YM}^2} \int d^4x \sqrt{g} \underbrace{\operatorname{Tr}(F_{ij} \pm F_{ij})^2}_{\text{SD/ASD.}} + \frac{1}{8g_{YM}^2} \int d^4x \sqrt{g} \underbrace{\operatorname{Tr}(F_{ij} F_{kl} \epsilon^{ijkl})}_{C_2(E)}$$

$$1. E_{YM} = -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{g} \text{Tr}(F_{ij} F^{ij})$$

$$\stackrel{\text{Bog's analysis}}{=} -\frac{1}{8g_{YM}^2} \int d^4x \sqrt{g} \text{Tr}(F_{ij} \pm F_{ij})^2 \mp \frac{1}{8g_{YM}^2} \int d^4x \sqrt{g} \text{Tr}(F_{ij} F_{kl} \epsilon^{ijkl})$$

SD/ASD. (C₂(E)) instanton #

4-Manifold. HYM-eg.

C-Talbot. M₄/Sp. 4sf → find ASD connections on E.

$$0 < \left(C_2(E) - \frac{r-1}{2r} C_1(E)^2 \right) = \frac{1}{8\pi^2} \int_M \text{Tr}(F^0 \wedge F^0) = \frac{1}{8\pi^2} \left(\|F^0_-\|^2 - \|F^0_+\|^2 \right)$$

M (F₀) F = \frac{1}{rk} (\text{tr} F) \cdot \text{id.}

ASD SD.

$$0 < \left(G_2(E) - \frac{r-1}{2r} G_1(E) \right)^2 =$$

$$\frac{1}{8\pi^2} \int$$

$$\text{Tr}(F^0 \wedge F^0)$$

$$= \frac{1}{8\pi^2} \left(\|F_{-}^0\|^2 - \|F_{+}^0\|^2 \right)$$

local calculation:

diagonalize F .

$$J = i \lambda^2 \wedge d\bar{z}^2$$

$$\left. \begin{aligned} & (\text{Tr } F)^2 + r \text{Tr}(F^2) \\ & \left. \right\} \lambda^2 \geq 0. \end{aligned}$$

$$\dim X = n.$$

Calculations: $\sum p_i \cdot s_i^2$

$$\left\langle \left(C_2(E) - \frac{r-1}{2r} C_1(E)^2 \right)^2 \right\rangle = \frac{1}{\text{StH}^2} \int_M \text{Tr}(F^0 \wedge F^0) = \frac{1}{\text{StH}^2} \left(\|F_{-}^0\|^2 - \|F_{+}^0\|^2 \right)$$

local calculation:

$$F^0 = F - \frac{1}{rk} (\text{tr} F) \cdot \text{Id.}$$

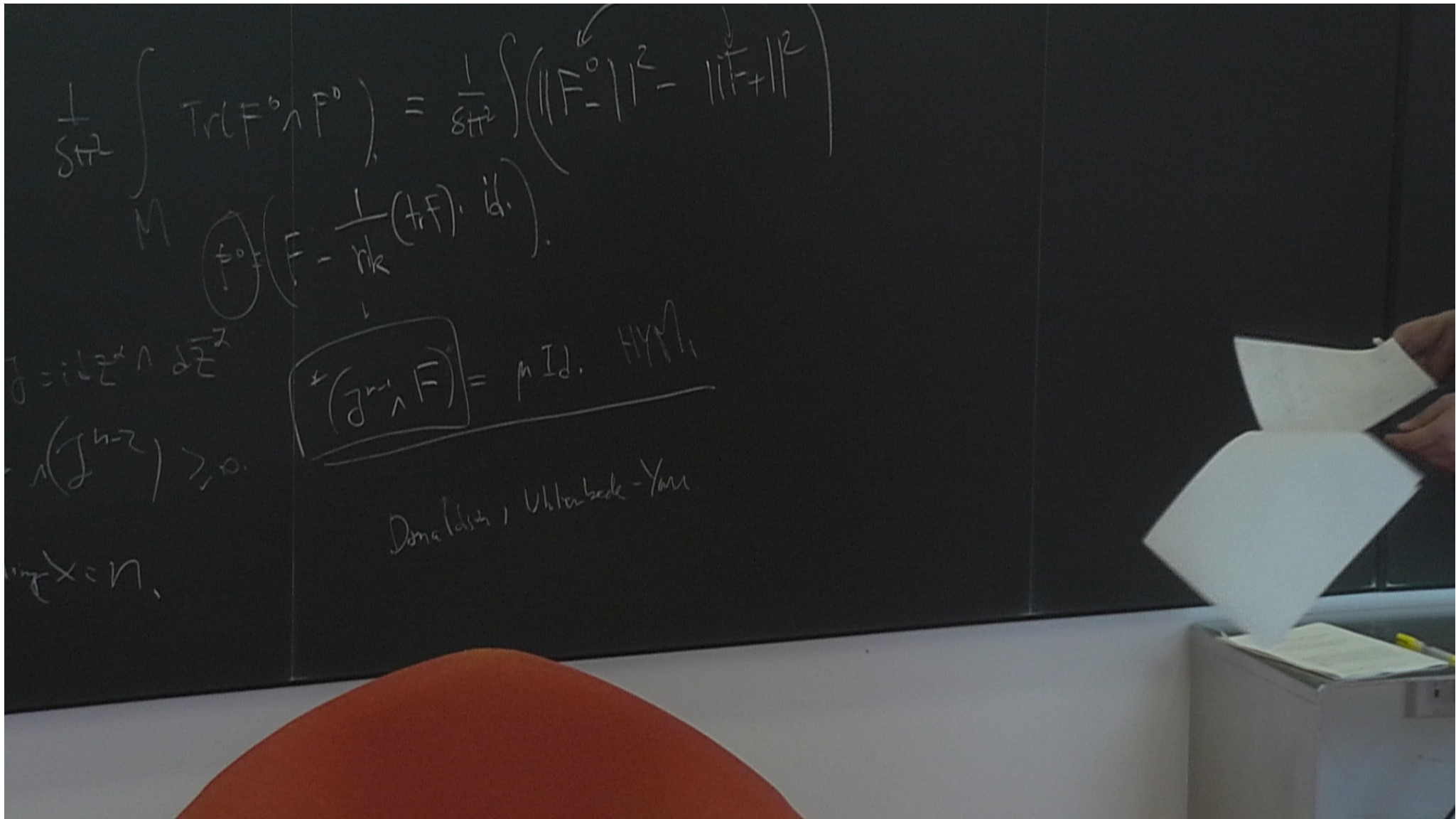
diagonalize F .

$$J = i \partial \bar{\partial} \wedge d\bar{z}^2$$

$$\left\{ \begin{array}{l} (\text{Tr} F)^2 + r \text{Tr}(F^2) \\ \wedge (J^{n-2}) \geq 0 \end{array} \right.$$

$$\left(\int \wedge F \right) = \mu \text{Id.} \quad \text{HYM}$$

$\dim X = n$.



$C_2(E)$ instanton #

SD connections on E .

$$\frac{1}{8\pi^2} \int_M \text{Tr}(F^0 \wedge F^0) = \frac{1}{8\pi^2} \left(\overset{\text{ASD}}{\|F^0\|^2} - \overset{\text{SD}}{\|F^+\|^2} \right)$$

$$F^0 = \left(F - \frac{1}{\text{rk}} (\text{tr} F) \cdot \text{id.} \right)$$

$\int_M \text{tr} F^0 \wedge F^0$

$\int_M \text{tr} F^0 \wedge F^0 \geq 0$

$n = 4$

$$\int_M \text{tr} (F^0 \wedge F^0) = \mu \text{Id.}$$

conf. inv. hermitic σ -model
 $(2,0)$ -SUGRA

Donaldson, Uhlenbeck-Yau

flavor of "bivector geometry"

Surv

$d^4x / 8 \text{Tr}(F_{\mu\nu} F^{\mu\nu})$

$(C_2(E))$ instanton #

of 4 manifolds, SW.
on E .

ASD

$$\int \text{tr} F^0 = \frac{1}{8\pi^2} \int (\|F^0\|^2 - 1)$$

$$F = \frac{1}{rk} (\text{tr} F) \cdot \text{id}$$

$$\frac{1}{8\pi^2} \left(\|F^0\|^2 - \|F^+\|^2 \right)$$
 ASD \swarrow SD

$$F = \mu Id.$$

conf. inv. heterotic σ -model
 (2,0)-theory

$$\mu(E) = \frac{c_1(E)}{rk(E)}$$
 "slope"

moduli, Uhlenbeck-Yau

#

SD
 \downarrow
 $\|F+1\|^2$

surf. inv.
 heterotic σ -model
 (2,0)-theory

$\mu(E) = \frac{c_1(E)}{rk(E)}$

$\mu(F) > \mu(E)$

$\Delta(E) > 0$ stable
 < 0 unstable

~~DT inv.~~
~~Be...~~

$vk=2$ Bogomolov:

$0 \rightarrow \underbrace{Q_x(D)}_{\text{line bundle}} \rightarrow E \rightarrow Q_x(D') \otimes \mathbb{I}_Z \rightarrow 0$

Dirac Surface, Divisor (divisor)

#

SD

$\frac{1}{|F+|R}$

surf. inv.
 heterotic σ -model
 (2,0)-theory

$\mu(E) = \frac{c_1(E)}{rk(E)}$

$\mu(F) > \mu(E)$

$\Delta(E) > 0$ stable

$\Delta(E) < 0$ unstable

~~DT inv.~~

~~Re...~~

$vk=2$

Bogomolov

$0 \rightarrow \underbrace{Q_x(D)}_{\text{line bundle}} \rightarrow E \rightarrow \underbrace{Q_x(D) \otimes \mathcal{I}_Z}_{\text{Dirac Surface, Divisor (divisor)}} \rightarrow 0$

$|Z|$ # of points

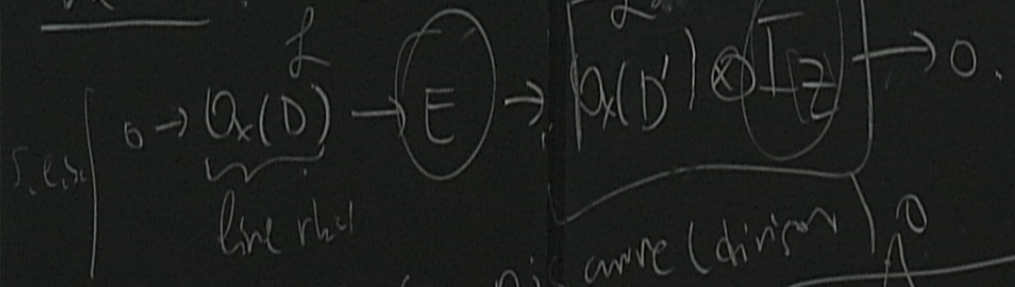
$\Delta(E) < 0$ stable

$\Delta(E) > 0$ unstable

DT inv.

$|z|$ # of points

$\nu_k = 2$ Bogomolov



$|z| = \#$ of DO's

Physics: $D - D', D^2$

$(D - D')$ bound state on same alg. surface.

$$\Delta(E) = C_1^2(E) - 4C_2(E) = (D - D')^2 - 4|Z|$$

Non zero # of pits in surface

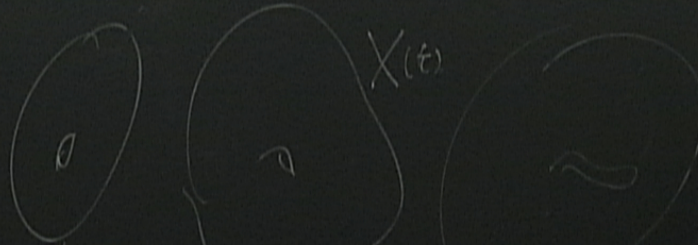
$$\frac{C_1(E)}{rk(E)}$$

3. DRY Configuration

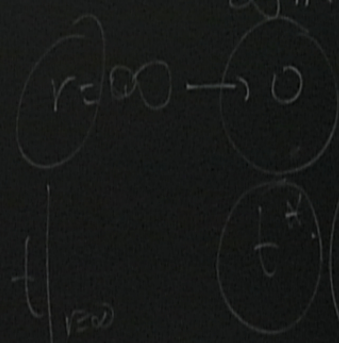
3 cycles u, X
 H^3

attractor mechanism

DB bruel. \rightarrow



\mathbb{R}^3 , $r = |x_i|^2$
 H



horizon.
 (determined by charges)
 BPS solutions

$|Z|$ # of points

$|Z| = \#$ of DO's

Physics: $D-D', D2$

$(D1-D2)$ bound state

DT inv.

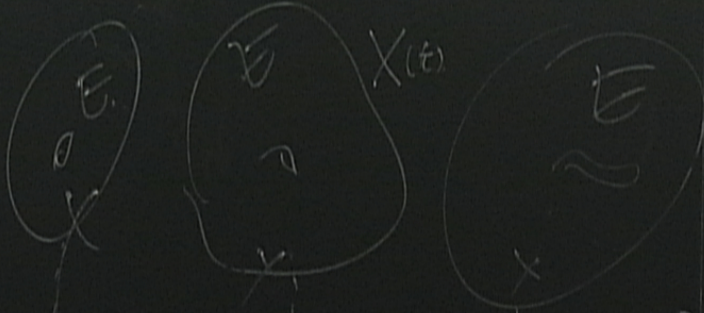
$\rightarrow \rho(D') \otimes \mathbb{I}_Z \rightarrow 0.$

3. DRY Compactification

for 3 cycles u, X :

attractor mechanism

D3 branes



Hodge Jump:

$$H^3(X, \mathbb{R}) = (H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3})$$

$$\Gamma = (p_0, \vec{p}; \vec{q}, q_0)$$

$$\dim(M_{\text{cptx}}) = h^{2,1}$$

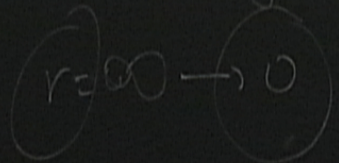
$$P_{\text{att}}(E) = \text{Re}(\bar{C}, \Omega^{3,0})$$

f points

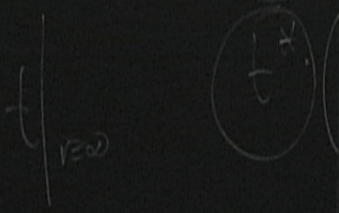
$$|z| = \# \text{ of D0's}$$

Physics: $D-D', D2$

(D0-D2) bound state



horizon



(determined by charges)

BPS solutions

3. DRY Compactification

for 3 cycles u, X :

attractor mechanism

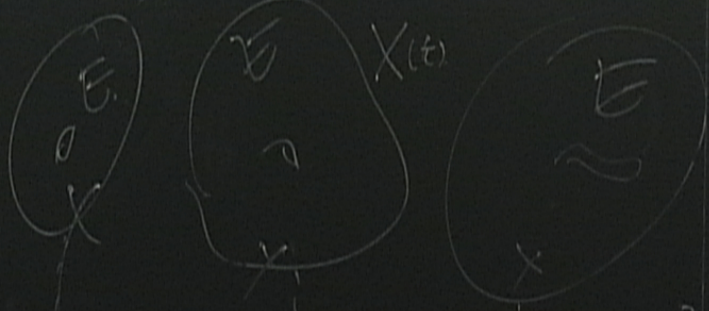
D3 branes

Hodge Jump:

$$H^3(X, \mathbb{R}) = (H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3})$$

$$\Gamma = (\underbrace{p_0, \vec{p}}_{\text{charges}}, \underbrace{\vec{q}, q_0}_{\text{charges}})$$

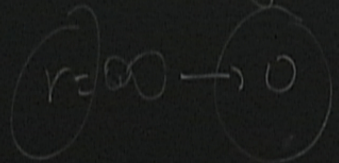
$$\dim(M_{\text{cplx}}) = h^{2,1}$$



horizon

$$\mathbb{R}^3, r = \sum |x_i|^2$$

horizon



t^x (determined by charges)

BPS solutions

$$P_{\text{att}}(X) = \text{Re}(\bar{C} \Omega^{3,0})$$

rel. components \rightarrow B-branes

$$X \xrightarrow{\text{minor}} X$$

$$P_{\text{att}}(E) = \text{Re}(\bar{C} \hat{\Omega}_X)$$

f points

$$|z| = \# \text{ of D0's}$$

physics: $D-D', D2$

(D0-D2) bound state

χ :

ge. desc.

$$(X, \mathbb{Z}) = (1-3^0 \oplus \dots \oplus \dots)$$

$$= (\rho_0, \vec{p}; \vec{q}, \rho_0)$$

$$\dim(M_{\text{cplx}}) = h^{2,1}$$

$$P_{\text{att}}(\chi) = \text{Re}(\bar{C} \Omega^{3,0})$$

helic. counterparts \rightarrow B-branes

$$X \xrightarrow{\text{mirror}} X$$

$$\mathcal{G} = 1 + (\beta + i\gamma) + \frac{1}{2}(\beta + i\gamma)^2 + \frac{1}{6}(\beta + i\gamma)^3$$

\leftarrow

ge. desc.

$(X, \mathcal{P}) = (1-3^0 \oplus \dots \oplus \dots)$

$(P_0, \vec{P}; \vec{q}, q_0)$

$\dim(M_{\text{cpt}} X) = h^{2,1}$

$P_{\text{att}}(\mathcal{V}_3) = \text{Re}(\bar{C}, \Omega^{3,0})$

rel. components \rightarrow B-branches

$X \xrightarrow{\text{mirror}} X$

$\mathcal{G} = 1 + (B+iJ) + \frac{1}{2}(B+iJ)^2 + \frac{1}{6}(B+iJ)^3$

$\mathcal{P} = \int_X ch(E) \sqrt{Td(X)}$ Mukai vector

$g_1(E) = 0$

$g_2(E) \sim H^2$

$g_3(E) \sim H^3 \left(\frac{c_2}{1+c_2} \right)$

$1+c_2 \sim 2c_2$

$\left| \frac{g_3(E)}{h(E)} \right| \sim \left| \frac{g_2(E)}{h(E)} \right|$

$\hat{H}CX$

$$\dim(M_{\text{opt}}) = h^{2,1}$$

$$g(E) = 0$$

$$G_2(E) \sim H^2$$

$\hat{H} \subset X$

$$P_{\text{opt}}(\gamma) = \text{Re}(\bar{C} \Omega^{3,0})$$

$$G_3(E) \sim H^3 \left(\frac{g}{1+g^2} \right)$$

rel. counterparts \rightarrow B-branes

$X \xrightarrow{\text{minor}} X_1$

$$1+g^2 \gg 2g \quad \uparrow \quad \frac{3}{2}$$

$$\left| \frac{G_3(E)}{r(E)} \right| \sim \left| \frac{G_2(E)}{r(E)} \right| \left(\frac{3}{2} \right)^2$$

$$P_{\text{opt}}(E) = \text{Re}(\bar{C} \hat{\Omega}_X)$$

D0 charges \rightarrow D2 charges

$$\hat{\Omega} = \left[e^{\frac{B+iF}{2}} \right]$$

Very many examples: $\left| \frac{G_3(E)}{r(E)} \right| \sim \left| \frac{G_2(E)}{r(E)} \right| \frac{HRR_1}{4\pi}$

- Moore)

$$(B+iJ)^2 + \frac{1}{6}(B+iJ)^3.$$

Mukai vector

4. does this agree w/ PRY'S
Crijetone (S)

$$\hat{H}^2$$

$$\hat{H}^3 \left(\frac{50}{1+y^2} \right)$$

$\hat{H}^2 CX$

CICY 3folds (monod)

ELLIPTIC 3folds: $E \xrightarrow{\text{Spectral cover}} B_2$

funny case

X is self-mirror

Schoen's CY 3fold

2ζ

$\frac{3}{2}$

\hat{H}^2

DRY → holds for CY 3fold
stable objects (reflexive sheaf)

$$\hat{H}^2$$

$$\hat{H}^3 \left(\frac{50}{1+y^2} \right)$$

$\hat{H}^2 CX$

CICY 3folds (monod)

ELLIPTIC 3folds: $E \xrightarrow{\text{Spectral}}$
 \downarrow cover
 B_2

funny case

X is self-mirror

Schoen's CY 3fold

$2\mathbb{Z}$

\hat{H}^2

$\left(\frac{3}{2} \right)$

(E)

DRY \rightarrow holds for CY 3fold
stable objects (reflexive sheaf)

existence failed

$(r, C_i), i=1,2,3.$

① $\frac{1}{2r^2}(\Delta(E) - \frac{r^2}{12} G_2(T_x)) = \tilde{H}^2$ *positive*

② $\frac{1}{6r^2}(C_1^3(E) + 3r(\underline{r ch_3(E) - ch_2(E) G_1(E)}))$

$\{ \# \cdot r \cdot \tilde{H}^3 \}$

\mathcal{F}

Bogomolov
 \downarrow
 Bogomolov

1. Bogomolov
2. Borel

Bertalan, Maciej, Toda

3. PR
 h_{ep-1}
 $G_2(E)$
 DT

(r, C_i), i=1,2,3.

$$\textcircled{1} \quad \frac{1}{2r^2} \left(\Delta(E) - \frac{r^2}{12} G_2(T_x) \right) = \overset{\sim 2}{H} \quad \text{positive}$$

$$\textcircled{2} \quad \frac{1}{6r^2} \left(C_1^3(E) + 3r \left(r \text{ch}_3(E) - \text{ch}_2(E) C_1(E) \right) \right)$$

$$\prec \# \cdot r \cdot \overset{\sim 3}{H}$$

$\forall J$, there is such a bundle E .

Bertalan, Macca

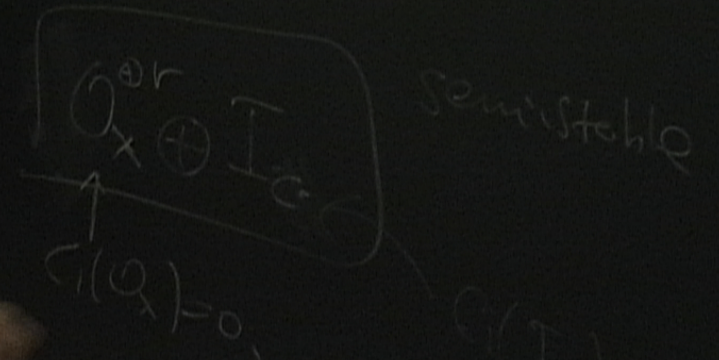
(r, c_i), i=1,2,3.

$$\textcircled{1} \quad \frac{1}{2r^2} \left(\Delta(E) - \frac{r^2}{12} c_2(T_X) \right) = \frac{\sim 2}{H} \quad \text{positive}$$

$$\textcircled{2} \quad \frac{1}{6r^2} \left(c_1^3(E) + 3r \left(r \text{ch}_3(E) - \text{ch}_2(E) c_1(E) \right) \right)$$

$$\prec \# \cdot r \cdot \frac{\sim 3}{H}$$

$\forall J$, there is only a bundle E .



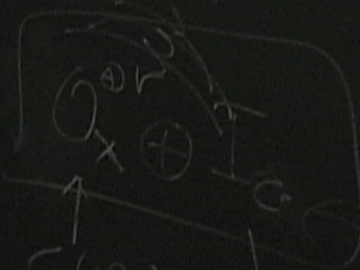
Bertalan, Mac

Bertalan, Mar 2016

$$\# \cdot r \cdot H^3$$

$\forall \mathcal{F}$

there is such a bundle E .

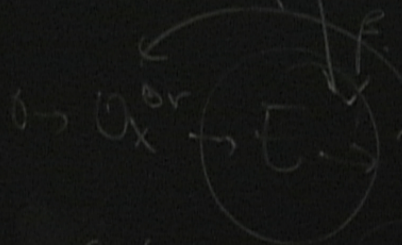


semistable

$$c_1(E) = 0$$

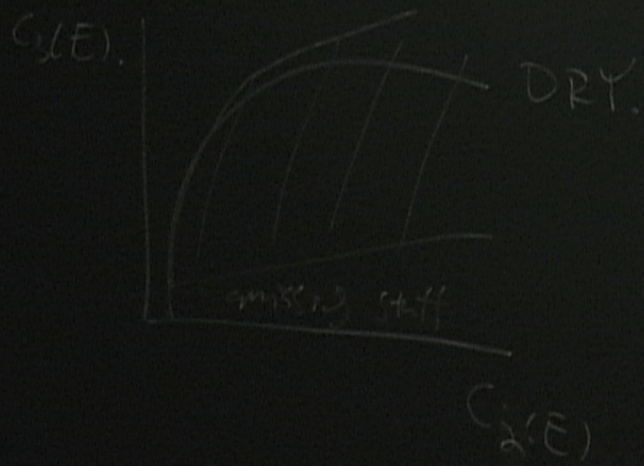
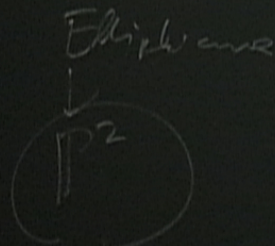
$$c_2(E) = 0$$

1405.5676
B. Wu, S.-T. Yang



$$c_2(E) = n[H]^2, \quad c_3(E) = 27[H]^3$$

\mathbb{P}^3 is a curve
in \mathbb{P}^3



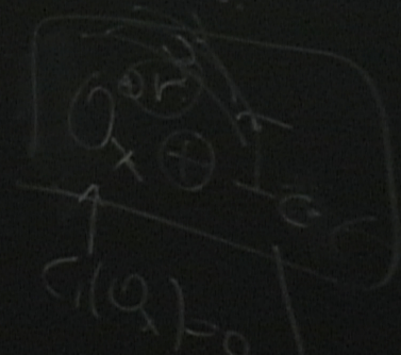
(r, C₁, ...)

① $\frac{1}{2r^2} (\Delta(E) - \frac{r^2}{12} G_2)$

② $\frac{1}{6r^2} (C_1^3(E) + 3r(r \text{ ch}_3))$

≠ r · H³

∇ ∫, this is not



$(r, C_i), i=1,2$
 ① $\frac{1}{2r^2} \left(L(E) - \frac{r^2}{12} G(T_x) \right)$
 ② $\frac{1}{6r^2} \left(C_1^3(E) + 3r (r \cdot ch_3(E)) \right)$
 $\# r \cdot H^3$
 the result =
 $C_1(Q_2) = 0$

The chalkboard contains the following content:

- Diagram (Left):** An ellipse with a point E at the top. A vertical dashed line descends from E to a point WMS on the ellipse. A point $P2$ is marked on the left side of the ellipse. Below the ellipse, there is a graph with a curve and the label $C_2(E)$. The text "DRY" is written in the center of the diagram.
- Equations (Right):**

$$\frac{1}{2r^2} \left(\Delta(E) - \frac{r^2}{12} G_2(T_x) \right)$$

$$\frac{1}{6r^2} \left(r^3 - 1 + 3r(r \operatorname{ch}_3(E)) \right)$$
- Equation (Bottom Left):**

$$\frac{1}{V_j} = \langle P_{1j}, P_{1j} \rangle \operatorname{Im}(z \bar{z}_j) > 0$$

Below this equation, the coordinates (E_j, E_j) are written.

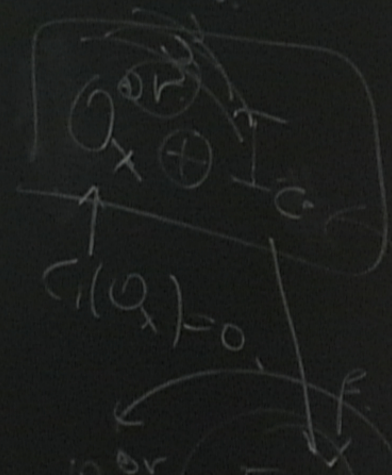
$$2r^2(L(E)) - \frac{1}{12} c_2(T_X) = H \quad \text{argento}$$

$$\frac{1}{6r^2} (c_1^3(E) + 3r(c_1 c_2(E) - c_3(E)) c_1(E))$$

$\{ \# r, \tilde{H} \}$

$E_1 \rightarrow E_2$

\exists , there is such a bundle E .



semistable

$$c_1(E) = 0$$

$$c_1(T_X) = 0 \Rightarrow c_1(E) = 0 \text{ is a } \dots$$

Bertalan, Macaul, Toda

$$c_2(E) \sim c_1(E)$$

$$\text{Pic}(X) = \mathbb{Z}$$

hyperplane
quintic

Bogomolny.

1. Bogomolov (type) of \dots
2. Boundedness results
3. DPV conjecture

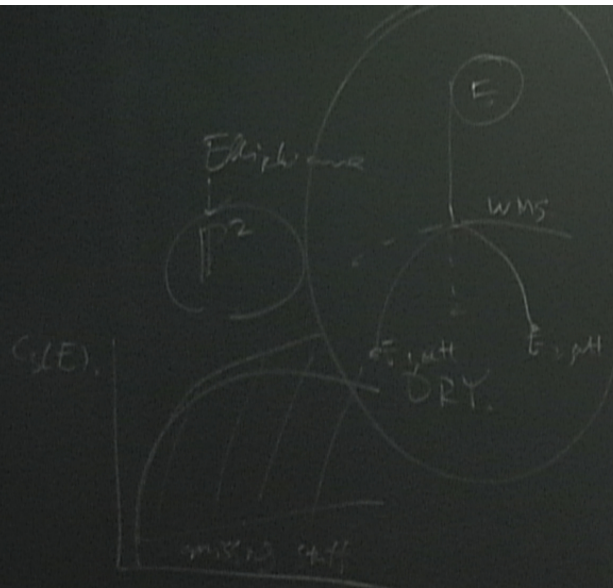
hep-th/060454

$$c_1(E)$$

DT inv. / Bridgeland

Doughlas P.

construction



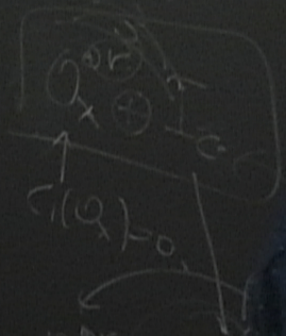
(r, C_i), i=1,2,3.

$$\textcircled{1} \quad \frac{1}{2r^2} \left(\Delta(E) - \frac{r^2}{12} G(T_x) \right) = \sim 2 \quad \text{H}$$

$$\textcircled{2} \quad \frac{1}{6r^2} \left(C_1^3(E) + 3r(r \text{ch}_3(E) - \text{ch}_3(E)) C_1(E) \right)$$

$$\# \cdot r \cdot \text{H}^3$$

$\partial \bar{D}$, theoretical = b



$\rightarrow Q_x \text{ or } E \rightarrow$

$$C(E) = n \text{H}$$

positive

Pentagon, Macaulay

$$\text{ch}_2(E) \sim \text{ch}_1(E)$$

$$\text{Pic}(X) = \mathbb{Z}$$

hyper-surface
quintic

$$\frac{1}{V_1} = \langle T_i, P_i \rangle \int_{\mathbb{R}^3} \bar{z}_i > 0$$

(E_1, E_2)

1405.5676
B. Wu, S.T. Yau