Title: Bulk Entanglement Spectrum: From Topological States to Quantum Criticality

Date: Oct 14, 2014 03:30 PM

URL: http://pirsa.org/14100075

Abstract: <span>A quantum phase transition is usually achieved by tuning physical parameters in a Hamiltonian at zero temperature. Here, we demonstrate that the ground state of a topological phase itself encodes critical properties of its transition to a trivial phase. To extract this information, we introduce a partition of the system into two subsystems both of which extend throughout the bulk in all directions. The resulting bulk entanglement spectrum (BES) has a low-lying part that resembles the excitation spectrum of a bulk Hamiltonian, which allows us to probe a topological phase transition from a single wavefunction by tuning either the geometry of the partition or the entanglement temperature. As an example, this remarkable correspondence between topological phase transition and entanglement criticality is rigorously established for integer quantum Hall states. We also implement BES using tensor networks, derive the universality classes of topological phase transitions from the spin-1 chain Haldane phase, and demonstrate that the AKLT wavefunction (and its generalizations) remarkably contains critical six-vertex (and in general eight-vertex) models within it. </span>

# Outline

**Part I: Bulk Entanglement Spectrum (BES)**
with Liang Fu, PRL 113, 106801 (2014)

Topological State $\ni$ Topological Phase Transition

BES of Chern insulator

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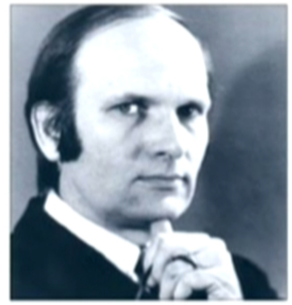
**Part II: Analytical Implementation of BES**
with Liang Fu and Xiao-Liang Qi, PRB 90, 085137 (2014)

BES via Tensor Networks

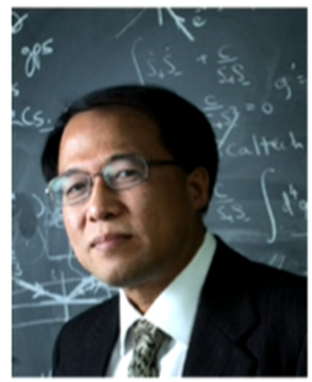
BES of Spin-1 chain, AKLT $\ni$ Critical six-vertex models

# Motivation: Ground State Wavefunction









photocredit: cerncourier.com (top), nndb.com (left), caltech.edu (mid), newswire.com (right)

# Topological Ground States

No local order parameter

Information more subtly encoded: invariants, ground state degeneracy, **entanglement**

# Main Point and Technique

Many topologically nontrivial ground states contain information about a topological phase transition from nontrivial to trivial.

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Information extracted using new extensive partition for entanglement spectrum

Ideal goal: study some interesting phase transitions

# Entanglement Spectrum

Ground state $|\Psi\rangle$


$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \equiv e^{-H_A}$$

Entanglement Hamiltonian $H_A$ with entanglement spectrum $\{\epsilon_i\}$

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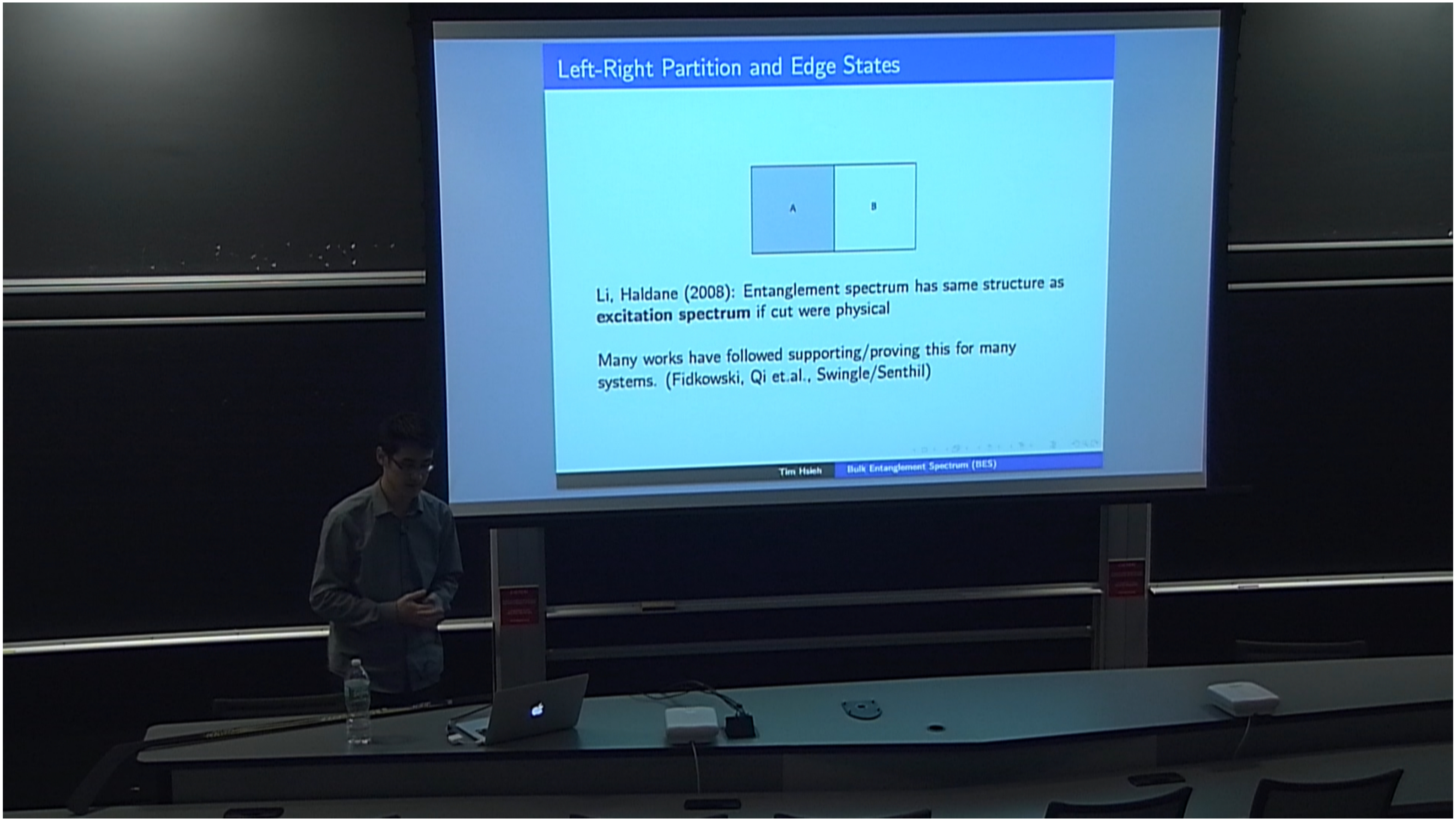
Schmidt decomposition

$$|\Psi\rangle = \sum_i e^{-\frac{\epsilon_i}{2}} |\psi_i\rangle_A \otimes |\tilde{\psi}_i\rangle_B$$

# Left-Right Partition and Edge States


Li, Haldane (2008): Entanglement spectrum has same structure as **excitation spectrum** if cut were physical


Left-Right Partition and Edge States
A
B
Li, Haldane (2008): Entanglement spectrum has same structure as **excitation spectrum** if cut were physical
Many works have followed supporting/proving this for many systems. (Fidkowski, Qi et.al., Swingle/Senthil)
Tim Hsieh
Bulk Entanglement Spectrum (BES)

# Extensive Partition and Bulk Entanglement

Probe bulk?

| | | |
|---|---|---|
| A | B | A |
| B | A | B |
| A | B | A |

Extensive in all directions

# Extensive Partition and Bulk Entanglement

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Entanglement Hamiltonian $H_A$ is **bulk** Hamiltonian, with BES and bulk ground state defined on $A$

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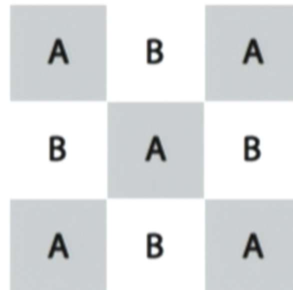


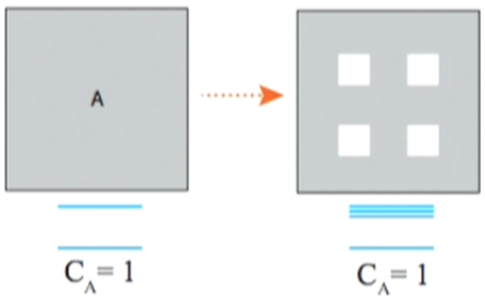

Extensive in all directions

Entanglement Hamiltonian $H_A$ is **bulk** Hamiltonian, with BES and bulk ground state defined on $A$

What do we learn about bulk? Phase transition!

# Tuning the Partition

Begin with a topologically nontrivial ground state $|\Psi\rangle$.



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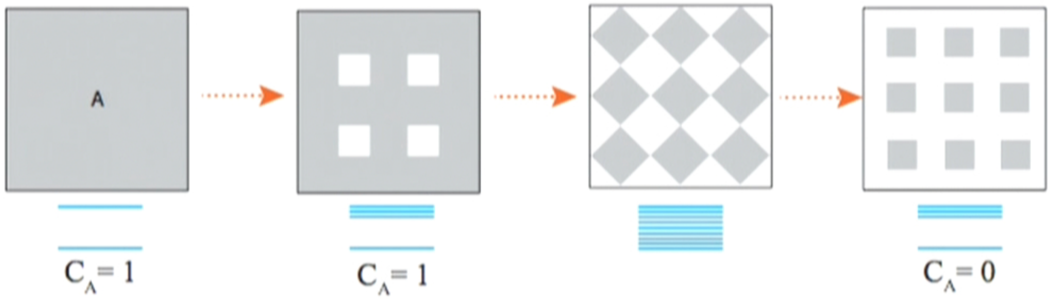

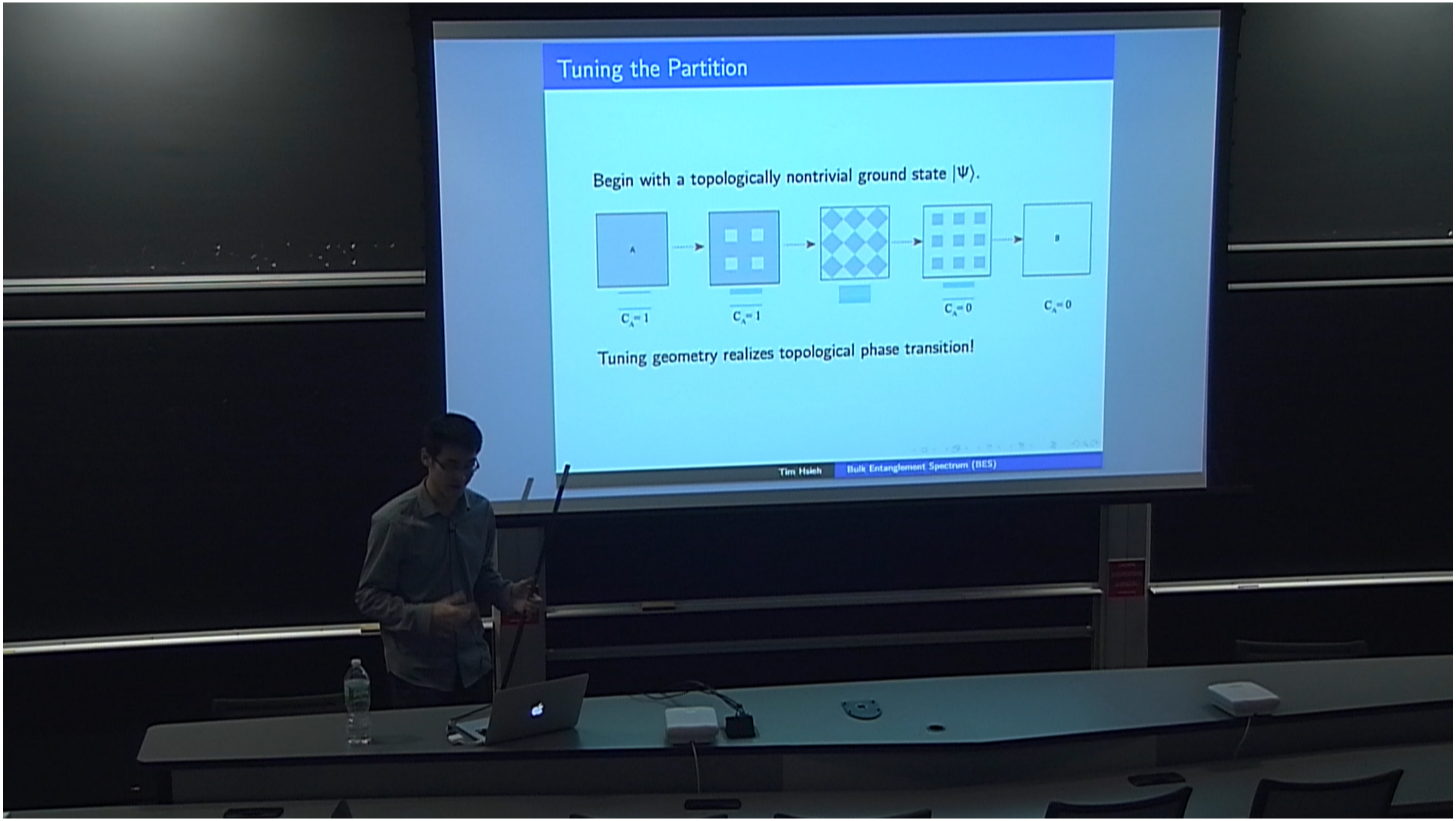
Tuning the Partition
Begin with a topologically nontrivial ground state |Ψ⟩.
A
B
C_A = 1
C_A = 1
C_A = 0
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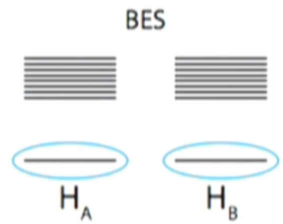
Tuning geometry realizes topological phase transition!
Tim Hsieh
Bulk Entanglement Spectrum (BES)

# A Conjecture




$$|\Psi\rangle = \sum_i e^{-\frac{\epsilon_i}{2}} |\psi_i\rangle_A \otimes |\tilde{\psi}_i\rangle_B$$

Assume BES is gapped. The largest-weight state $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$ in the Schmidt decomposition possesses the **same topological order** as the original ground state $|\Psi\rangle$.

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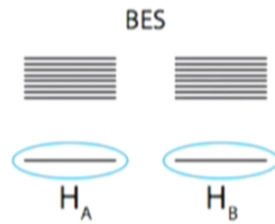


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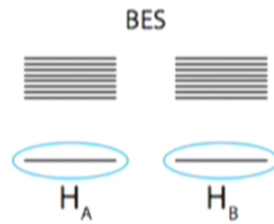


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Assume the original topological state is irreducible: it is not product state of two identically topologically ordered states

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Assume for sake of contradiction that BES at symmetric partition is gapped

# Application to Chern Insulator (Free Fermions)

Free fermion system $\rightarrow$ entanglement Hamiltonian also free fermion (Peschel)

Rigorously establish that for symmetric, extensive partitions of ground states with odd Chern number, BES is gapless

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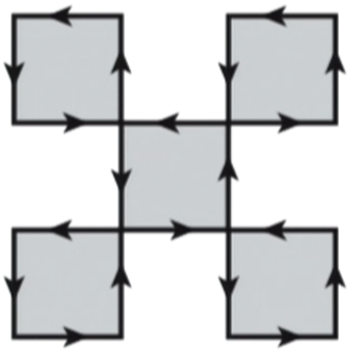
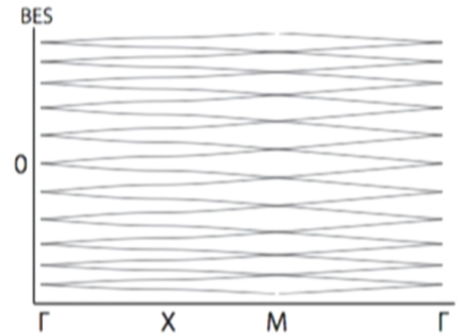
Rigorously establish that for symmetric, extensive partitions of ground states with odd Chern number, BES is gapless

Proof based on Chern number as obstruction against continuous choice of phase in Brillouin zone (see Supplementary Material of TH and L. Fu, PRL 113, 106801 (2014))

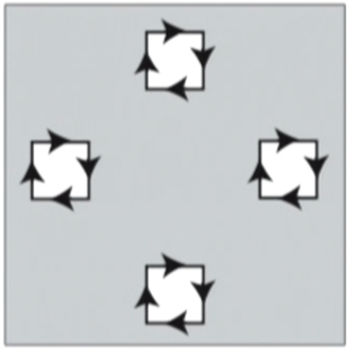
# BES of Chern Insulator



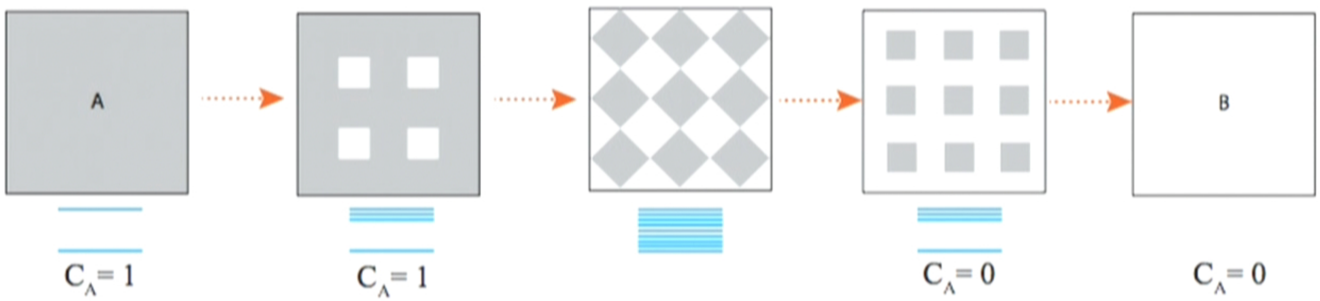



At symmetric partition, entanglement edge modes percolate a la Chalker-Coddington network model.

# BES of Chern Insulator



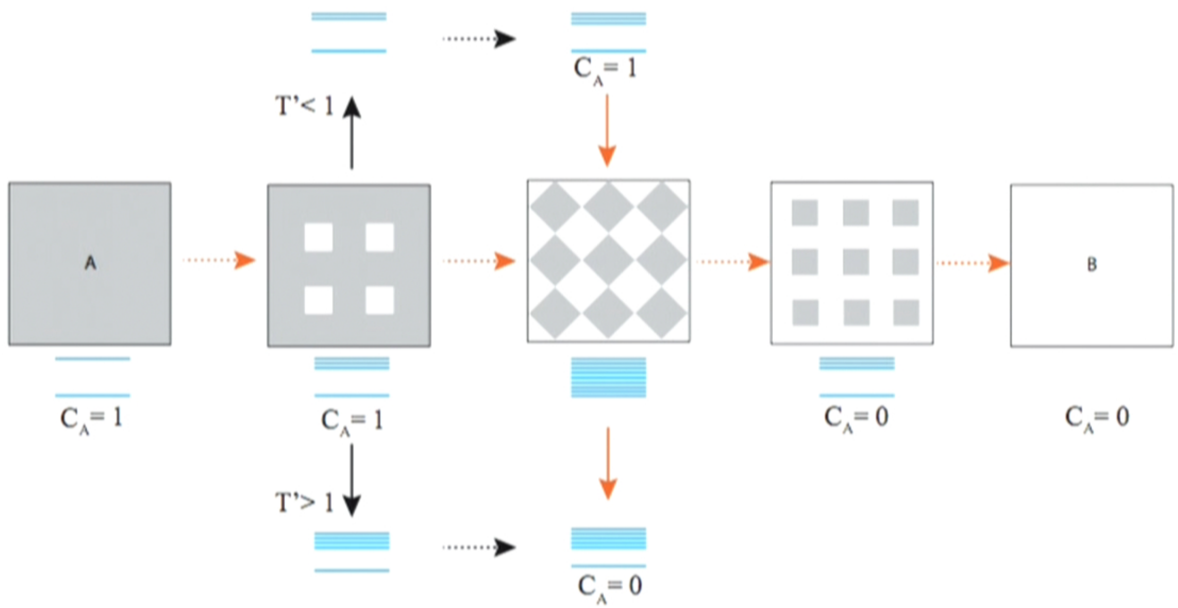
# Phase Transition via Entanglement Temperature




Fix the Hilbert space: $A_s \equiv A$ at symmetric partition.

# Recap

Realize topological phase transition by either tuning geometry of partition or entanglement temperature




Derived from a single ground state only

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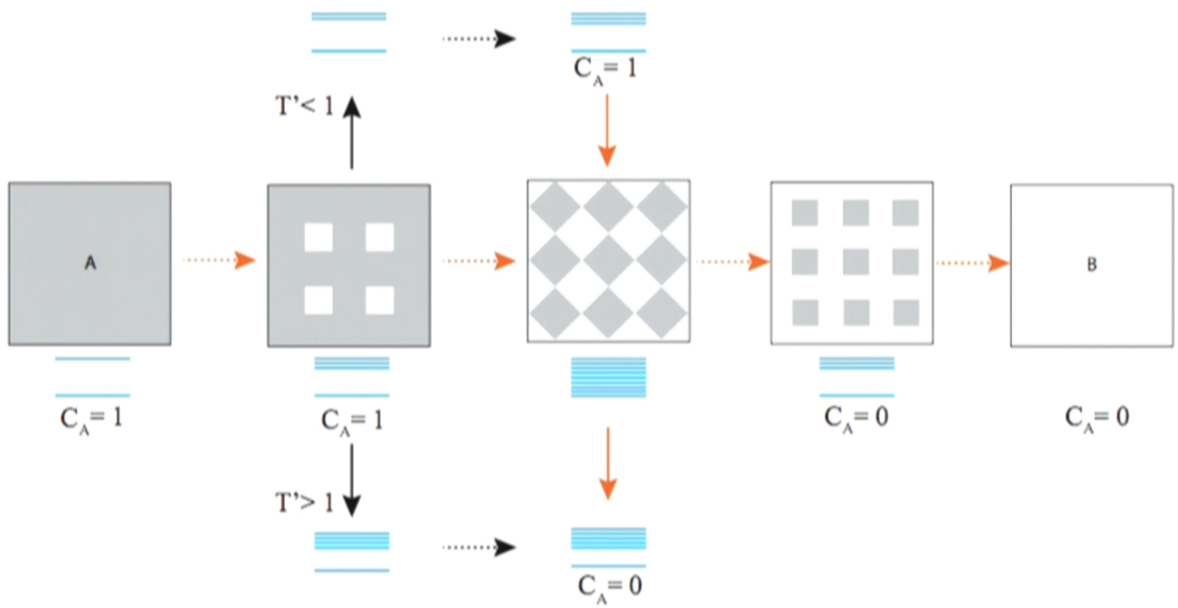


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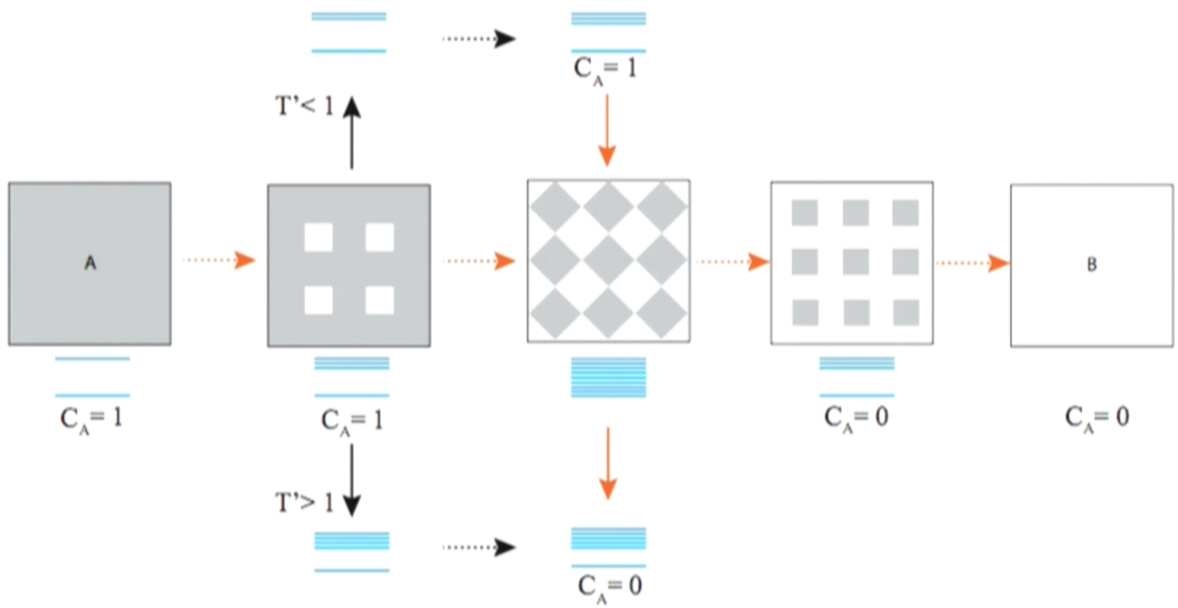


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# BES via Tensor Networks

TH, L. Fu, and X-L. Qi PRB 90, 085137 (2014)

W-J Rao, X. Wan, and G-M. Zhang PRB 90, 075151 (2014)

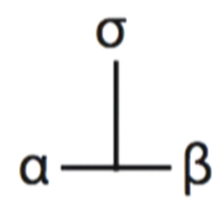
R. Santos (arXiv:1408.1716)

Analytical approach, appealing graphical perspective

Classical partition function of a quantum **critical** system

# Matrix Product States (MPS)

Tensor $T^{\sigma}_{\alpha\beta}$




Matrix product state

$$\rho_A = \sum_{\sigma_B} \langle \sigma_B | \psi \rangle \langle \psi | \sigma_B \rangle$$

# Matrix Product States (MPS)

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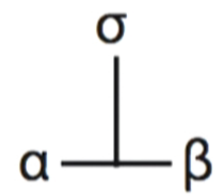


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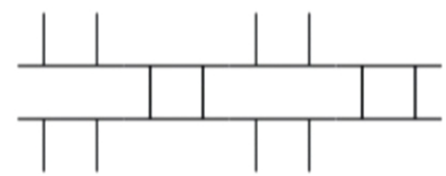
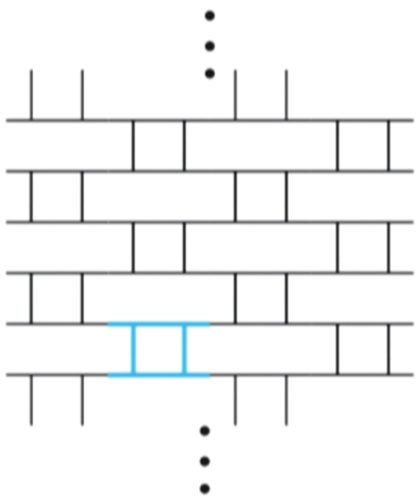


# Partition Function of Entanglement Hamiltonian

$$Z = tr(e^{-\beta H_A}) = tr(\rho_A^\beta)$$



Allows us to change perspective from physical to virtual degrees of freedom

Will want to evaluate building block

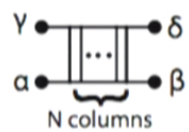
# Spin-1 Chain

Haldane phase, with AKLT MPS as representative

$$T^{+} = \sqrt{\frac{2}{3}}\sigma^{+},\ T^{0} = -\frac{1}{\sqrt{3}}\sigma^{z},\ T^{-} = -\sqrt{\frac{2}{3}}\sigma^{-}$$

Physical: spin-1. Virtual: spin-1/2.

Trace out every $N$ spins.




$$B^{N}_{\gamma\delta,\alpha\beta} = \left(1 - \left(-\frac{1}{3}\right)^{N}\right) I + \left(-\frac{1}{3}\right)^{N} P,$$

P is projection onto singlet.

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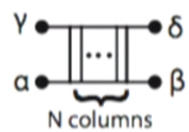


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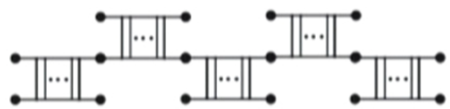
P is projection onto singlet.
Odd $N$: FM interaction; Even $N$: AFM interaction

# Large-$N$ Limit

$$B^N = \left(1 - \left(-\frac{1}{3}\right)^N\right) I + \left(-\frac{1}{3}\right)^N P$$

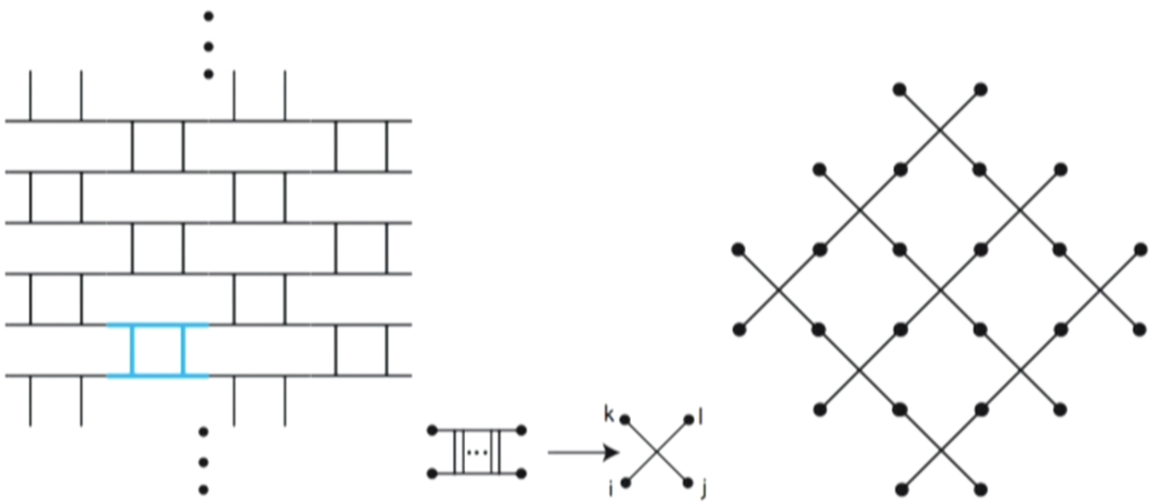
$$B^\infty \approx I + \left(-\frac{1}{3}\right)^N P \approx e^{(-\frac{1}{3})^N P}$$



Using $e^A e^B \approx e^{A+B}$ for small $A, B$,

$$\begin{aligned} Z &\approx tr(e^{-\beta \tilde{H}_A}) \\ \tilde{H}_A &\equiv -\left(-\frac{1}{3}\right)^N \sum_i P_{i,i+1} \rightarrow \sum_i s_i \cdot s_{i+1} \end{aligned}$$

# Quantum to Classical Mapping



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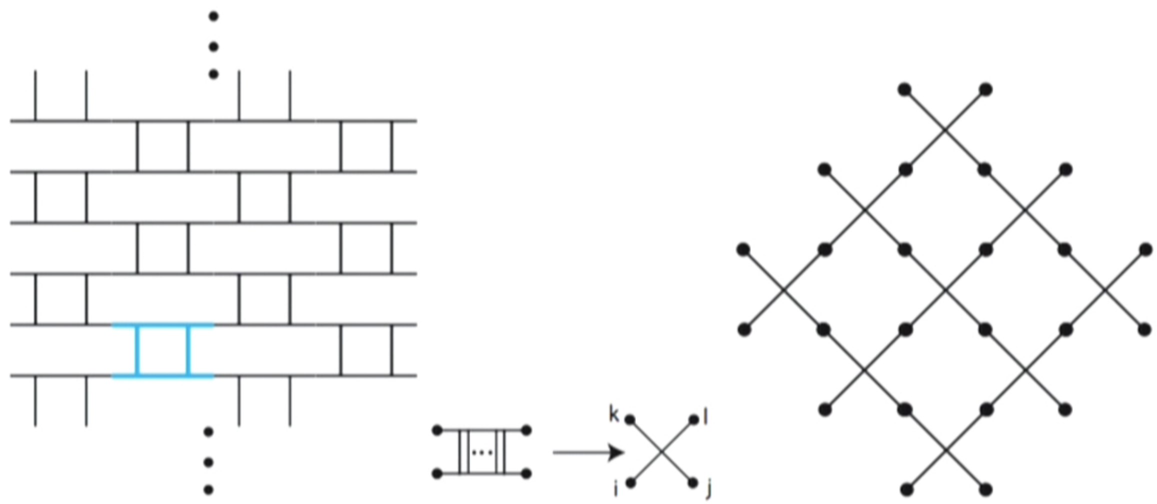


$B^N$ for even $N$ maps to 'six-vertex' model with vertex

$$V^{ij}_{kl} = \delta^i_k \delta^j_l + \lambda \delta^i_j \delta^k_l$$

$\rightarrow$ 4-state Potts model $\rightarrow$ continuum theory level-1 SU(2) WZW (Affleck 1985) (thanks to Senthil)

# Which Phase Transition?

Affleck, Haldane (1987)

**Spontaneous** dimerization of Haldane chain described by level-**2** SU(2) WZW

In general, dimerization of Haldane chain (**without demanding translation invariance** of Hamiltonian) is described by level-**1** SU(2) WZW

# Summary: A rhyme/recipe

*Take topological state*
*Carve extensive partition*
*Get entanglement spectrum*
*And voila: phase transition!*

Phase transition by tuning geometry of partition or entanglement temperature

Interesting directions:

Entanglement topological invariants (J. Borchmann, A. Farrell, S. Matsuura, T. Pereg-Barnea arXiv:1407.5980)

'Entanglement Chern number' (T. Fukui, Y. Hatsugai arXiv:1408.3471)

# Future Directions

Tensor network approach to analyze phase transitions from symmetry protected topological states (SPT)

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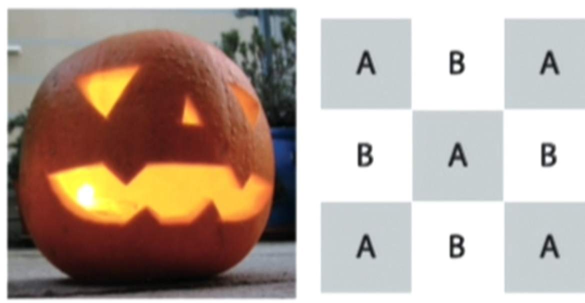
2D$\rightarrow$ 3D a bit messier- Monte Carlo, tensor renormalization?

Extensive partition as renormalization of topological phase




BES: trick or treat?