Title: Superconformal Indices, AdS/CFT, and Cyclic Homologies

Date: Oct 16, 2014 11:00 AM

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Abstract: <span>We explain how to obtain the spectrum of operators with protected scaling dimensions in a four-dimensional superconformal field theory from cyclic homology. Additionally, we show that the superconformal index of a quiver gauge theory equals the Euler characteristic of the cyclic homology of the Ginzburg dg algebra associated to the quiver. For quiver gauge theories which are dual to type IIB string theory on the product of an arbitrary smooth Sasaki-Einstein manifold with five-dimensional AdS space, the index is calculated both from the gauge theory and gravity viewpoints. We find complete agreement. Finally we show how to match the spectrum of protected operators on a supergravity compactification involving generalized complex geometry.</span>

# Goal: Prove AdS/CFT

**Less ambitious goal:**

Prove part of AdS/CFT for a subset of protected BPS operators and observables.

**This talk:**

Show that the BPS operators agree under the correspondence.

- Based on joint work with J. Schmude, Y. Tachikawa
- Adv. Theor. Math. Phys. 18 (2014) 129 / arXiv:1207.0573
- arXiv:1305.3547
- and work in progress

# AdS/CFT Cartoon

## Gauge Theory

$\mathbb{R}^{3,1} \times X_6$
$N$ $D3$ branes
$X_6$ Calabi-Yau 6-manifold

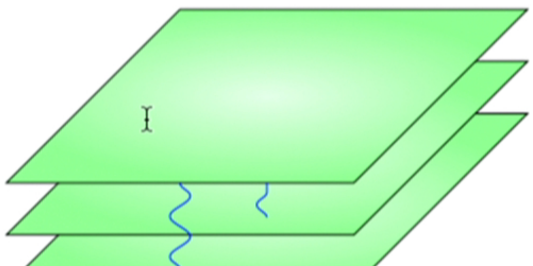


Figure: $N$ D3-branes

## Gravity Theory

$AdS_5 \times L_5$
$N$ units of RR-flux
$L_5$ Sasaki-Einstein 5-manifold

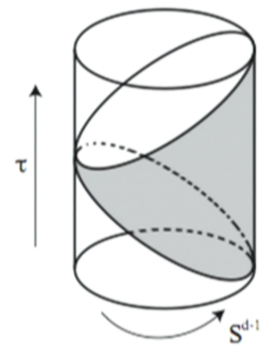



Figure: AdS Space-Time

# Goal: Match Closed String States in the large-N limit

**Gauge Theory**

$\mathbb{R}^{3,1} \times X_6$

**Closed strings:**

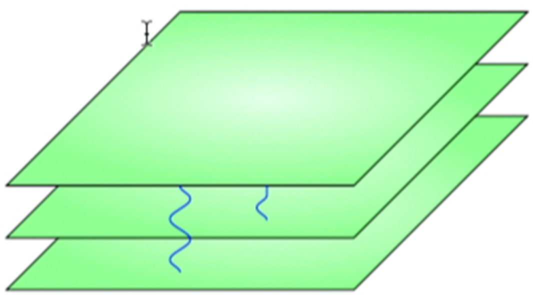
$HC_\bullet(\mathbb{C}Q/\partial W)$

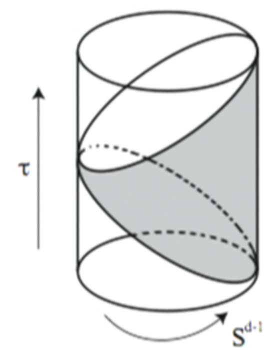
**Gravity Theory**

$AdS_5 \times L_5$

**Closed strings:**

$HP_\bullet(X, \pi = 0)$





# Protected operators in $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM has three adjoint chiral scalar superfields $\Phi^1, \Phi^2, \Phi^3$. Their interactions are described by the superpotential

$$W = \operatorname{Tr} \Phi^1 \left[\Phi^2, \Phi^3\right].$$

Consider an operator of the form

$$\mathcal{O} = T^{z_1 z_2 \ldots z_k} = \operatorname{Tr} \Phi^{z_1} \Phi^{z_2} \ldots \Phi^{z_k}.$$

If $T^{z_1 z_2 \ldots z_k}$ is symmetric in its indices, then the operator is in a short representation of the superconformal algebra. If $T^{z_1 z_2 \ldots z_k}$ is not symmetric, then the operator is a descendant, because the commutators $[\Phi^{z_i}, \Phi^{z_j}]$ are derivatives of the superpotential $W$ [Witten '98].

# Matching protected operators in $\mathcal{N} = 4$ SYM

Under the AdS/CFT dictionary, a scalar excitation $\Phi$ in AdS obeying

$$(\Box_{AdS_5} - m^2)\Phi = 0$$

with asymptotics $\rho^{-\Delta}$ near the boundary of AdS ($\rho \to \infty$) is dual to an operator of scaling dimension

$$m^2 = \Delta(\Delta - d) \to \Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} = m^2}$$

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# Matching protected operators in $\mathcal{N} = 4$ SYM

The operator

$$\mathcal{O} = \mathrm{Tr}\,\Phi^{z_1}\Phi^{z_2}\ldots\Phi^{z_k}$$

has conformal dimension $k$ and is dual to a supergravity state of spin zero and mass

$$m^2 = k(k-4).$$

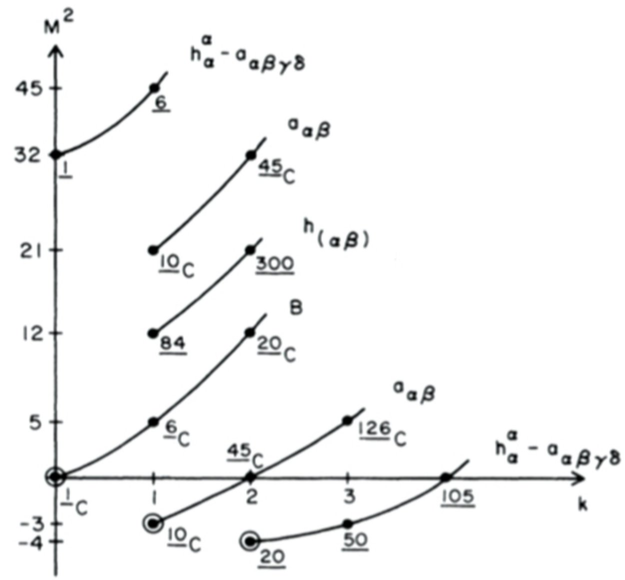


FIG. 2. Mass spectrum of scalars.

Figure: From Kim-Romans-van Nieuwenhuizen [Phys.Rev. D32 (1985) 389]

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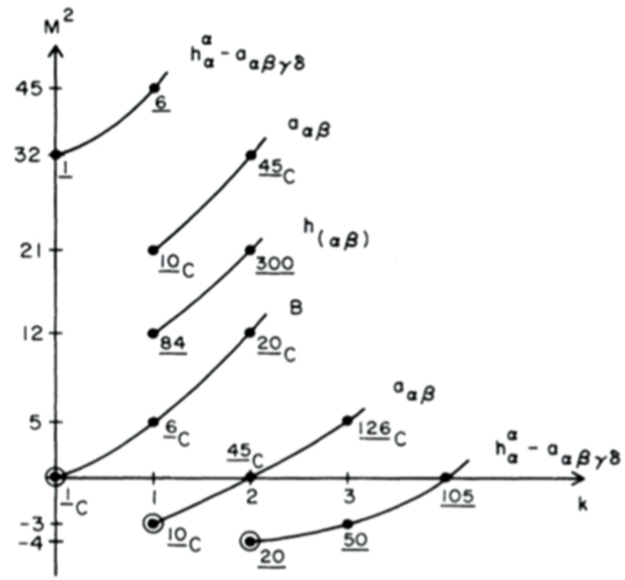


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# Goal: Test AdS/CFT by small deformations

$\mathcal{N} = 4$ SYM has superpotential

$$W = \mathrm{Tr}\,(XYZ - XZY)\,.$$

What happens when we deform it by giving a mass to one of the scalars

$$W = \mathrm{Tr}\left(XYZ - XZY + mZ^2\right)$$

or deform the coupling constants?

$$W = \mathrm{Tr}\left(qXYZ - q^{-1}XZY\right)$$

Can we still match the spectrum of protected operators?

## Quivers and Representations

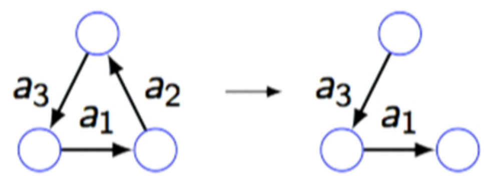
- A *quiver* $Q = (V, A, h, t : A \to V)$ is a collection of vertices $V$ and arrows $A$ between the vertices of the quiver.
- Arrows are directed edges with the head and tail of an arrow $a \in A$ given by maps $h(a)$ and $t(a)$ respectively. An arrow *starts* at the vertex $t(a)$ and *terminates* at the vertex $h(a)$.
- A *representation* $X$ of a quiver is an assignment of $\mathbb{C}$-vector spaces $X_v$ to every vertex $v \in V$ and a $\mathbb{C}-$linear map $\phi_a : X_{t(a)} \to X_{h(a)}$ to every arrow $a \in A$.
- The *dimension vector* $\mathbf{n} \in \mathbb{N}^{|V|}$ of a representation $X$ is a vector with an entry for each vertex $v \in V$ equal to the dimension of the vector space $X_v$.

# Superpotential algebras

- The vector space $\mathbb{C}Q_{cyc} = \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$ has a basis of cycles.
- Define a map $\frac{\partial}{\partial x_j} : \mathbb{C}Q_{cyc} \to \mathbb{C}Q$ acting on a cyclic word $\Phi = x_{i_1} x_{i_2} \ldots x_{i_r}$

$$\frac{\partial \Phi}{\partial x_j} = \sum_{s | i_s = j} x_{i_s+1} x_{i_s+2} \ldots x_{i_r} x_{i_1} x_{i_2} \ldots x_{i_s-1}$$

- Example: $\Phi = a_3 a_2 a_1$, $\frac{\partial \Phi}{\partial a_2} = a_1 a_3$



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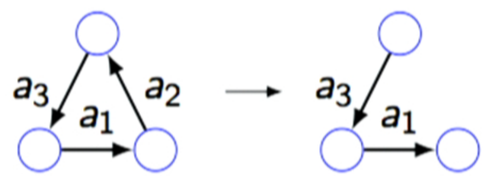

# Superpotential algebras II

## Definition

Given by a quiver $Q$ and a superpotential $W \in \mathbb{C}Q_{cyc}$, the superpotential algebra $\mathcal{A}$ associated to $(Q, W)$ is

$$\mathcal{A} = \mathbb{C}\langle x_1, \ldots x_N \rangle / ((\partial W / \partial x_i))_{i=1,\ldots N}$$

where $x_i$ are the arrows of the quiver.

# Quivers from geometry

## Quivers from geometry

Given a local CY $X$, a completely general method to determine $(Q, W)$ given by Aspinwall and Katz.

$$D^b(\mathrm{Coh}X) \cong D^b(\mathrm{mod}-\mathcal{A})$$

For $X = \mathbb{C}^2/\Gamma \times \mathbb{C}$, $\Gamma \subset SU(2)$, $(Q, W)$ first determined by Douglas and Moore. $Q$ is a decorated affine ADE diagram.

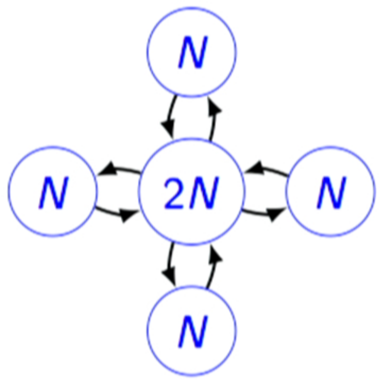


Figure: $\widehat{D}_4$ Quiver

## Quiver Gauge Theories

A *quiver gauge theory* is specified by a quiver and superpotential in the following manner:

- The gauge group
$$G = \prod_{v \in V} U(n_v)$$
is a product of unitary groups $U(n_v)$.
- Arrows $a \in A$ represent chiral superfields $\Phi_a$ transforming in the fundamental representation of $U(n_{h(a)})$ and in the anti-fundamental representation of $U(n_{t(a)})$.
- The superpotential
$$W = \sum_{l = a_1 a_2 \ldots a_k \in L} \lambda_l \operatorname{Tr}\left[\Phi_{a_1}\Phi_{a_2}\ldots\Phi_{a_k}\right]$$
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# Superconformal Field Theories from Quiver Gauge Theories

At a conformal fixed point in the infrared, we expect the NSVZ 1-loop exact beta functions of the gauge groups $SU(n_v)$ and couplings $\lambda_l$ to vanish. These constraints are

$$\hat{\beta}_{1/g_v^2} = 0 \quad 2n_v + \sum_{e \in Q_1} (R(e) - 1) n_{t(e)} + \sum_{e \in Q_1} (R(e) - 1) n_{h(e)} = 0 \tag{0.1}$$

$$\hat{\beta}_{\lambda_l} = 0 \quad -2 + \sum_{e \in \text{loop } l} R(e) = 0. \tag{0.2}$$

The last condition implies that at a superconformal fixed point, every term in the superpotential has total $R$-charge 2.

# The Superconformal Algebra

The 4D superconformal algebra combines both the conformal algebra and $\mathcal{N} = 1$ supersymmetry algebra. The conformal algebra consists of Lorentz generators $M_{\mu\nu}$, momenta $P_{\mu}$, special conformal generators $K_{\mu}$ and a dilatation $D$.

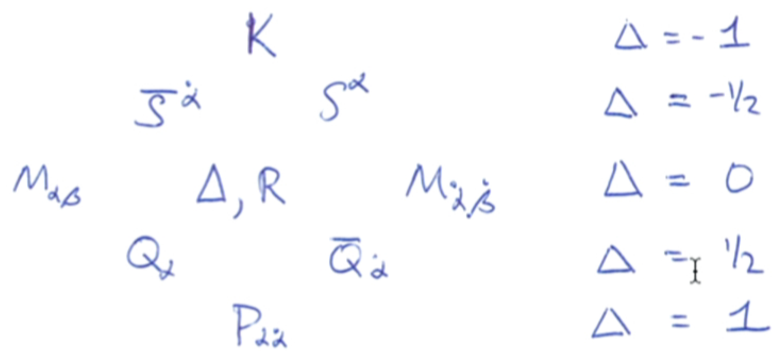


Figure: Generators of the Superconformal Algebra

# The Superconformal Index

**The SCI is a 4D analog of the Witten index in quantum mechanics**

Defined as

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F e^{-\beta\delta} e^{-\mu_i \mathcal{M}_i}$$

- The trace is over the Hilbert space of states on $S^3$
- $\mathcal{Q}$ is one of the Poincare supercharges
- $\mathcal{Q}^\dagger$ is the conjugate conformal supercharge
- $\delta \equiv \frac{1}{2}\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}$
- $\mathcal{M}_i$ are $\mathcal{Q}-$closed conserved charges

# Operators contributing to the index

Key commutation relations:

$$\{Q_\alpha, Q^{\dagger\beta}\} = E + 2M_\alpha^\beta + \frac{3}{2}r$$
$$\{\overline{Q}_{\dot\alpha}, \overline{Q}^{\dagger\dot\beta}\} = E + 2\overline{M}_{\dot\alpha}^{\dot\beta} - \frac{3}{2}r$$

Operators for which $\overline{\mathcal{Q}}^{\dot\alpha}\mathcal{O} = 0$ are called chiral primaries.
Operators contributing to the (right-handed) index have $\delta = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$. Choosing $\mathcal{Q} = \overline{Q}_{\dot{-}}$, operators contributing to the index satisfy

$$E - 2j_2 - \frac{3}{2}r = 0. \qquad (0.3)$$

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# The 4D Letter Index

| Letter | $(j_1, j_2)$ | $\mathcal{I}$ |
|---|---|---|
| $\phi$ | $(0,0)$ | $t^{3r}$ |
| $\overline{\psi}_2$ | $(0,1/2)$ | $-t^{3(2-r)}$ |
| $\partial_{\pm -}$ | $(\pm 1/2, 1/2)$ | $t^3 y^{\pm 1}$ |

| Letter | $(j_1, j_2)$ | $\mathcal{I}$ |
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| $\lambda_1$ | $(1/2,0)$ | $-t^3 y$ |
| $\lambda_2$ | $(-1/2,0)$ | $-t^3 y^{-1}$ |
| $\overline{f}_{22}$ | $(0,1)$ | $t^6$ |
| $\partial_{\pm -}$ | $(\pm 1/2, 1/2)$ | $t^3 y^{\pm 1}$ |

- Fields contributing to the index, from a chiral multiplet (left) and from a vector multiplet (right) [1]
- Consider letters modulo the equation of motion $\partial_{22}\lambda_1 = \partial_{12}\lambda_2$

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Table: Fields contributing to the index, from a chiral multiplet (left) and from a vector multiplet (right), after the cancellation of $W_\alpha$ and the spacetime derivatives $\partial_\mu$ are taken into account.

Ginzburg's DG algebra is a free differential-graded algebra

$$\mathfrak{D} = \mathbb{C}\langle x_1, \ldots, x_n, \theta_1, \ldots, \theta_n, t_1, \ldots, t_m \rangle$$

where $\phi, \overline{\psi}_2, \overline{f}_{22}$ correspond to $x, \theta, t$ respectively.

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# SUSY Transformations

## The differential $\mathcal{Q}$ on Ginzburg's DG algebra

$$\mathcal{Q}\phi_e = 0,$$
$$\mathcal{Q}\bar{\psi}_{e,2} = \partial W(\phi_e)/\partial\phi_e,$$
$$\mathcal{Q}\bar{f}_{v,22} = \sum_{h(e)=v} \phi_e\bar{\psi}_{e,2} - \sum_{t(e)=v} \bar{\psi}_{e,2}\phi_e.$$

Let $[\mathfrak{D},\mathfrak{D}]$ be a $\mathbb{C}$-linear space spanned by commutators. The basis of $\mathfrak{D}_{\rm cyc} = \mathfrak{D}/(\mathbb{C}+[\mathfrak{D},\mathfrak{D}])$ corresponds to the set of closed path of $\hat{Q}$, or equivalently, the single-trace operators formed from $\phi_e$, $\bar{\psi}_{e,2}$ and $\bar{f}_{v,22}$.

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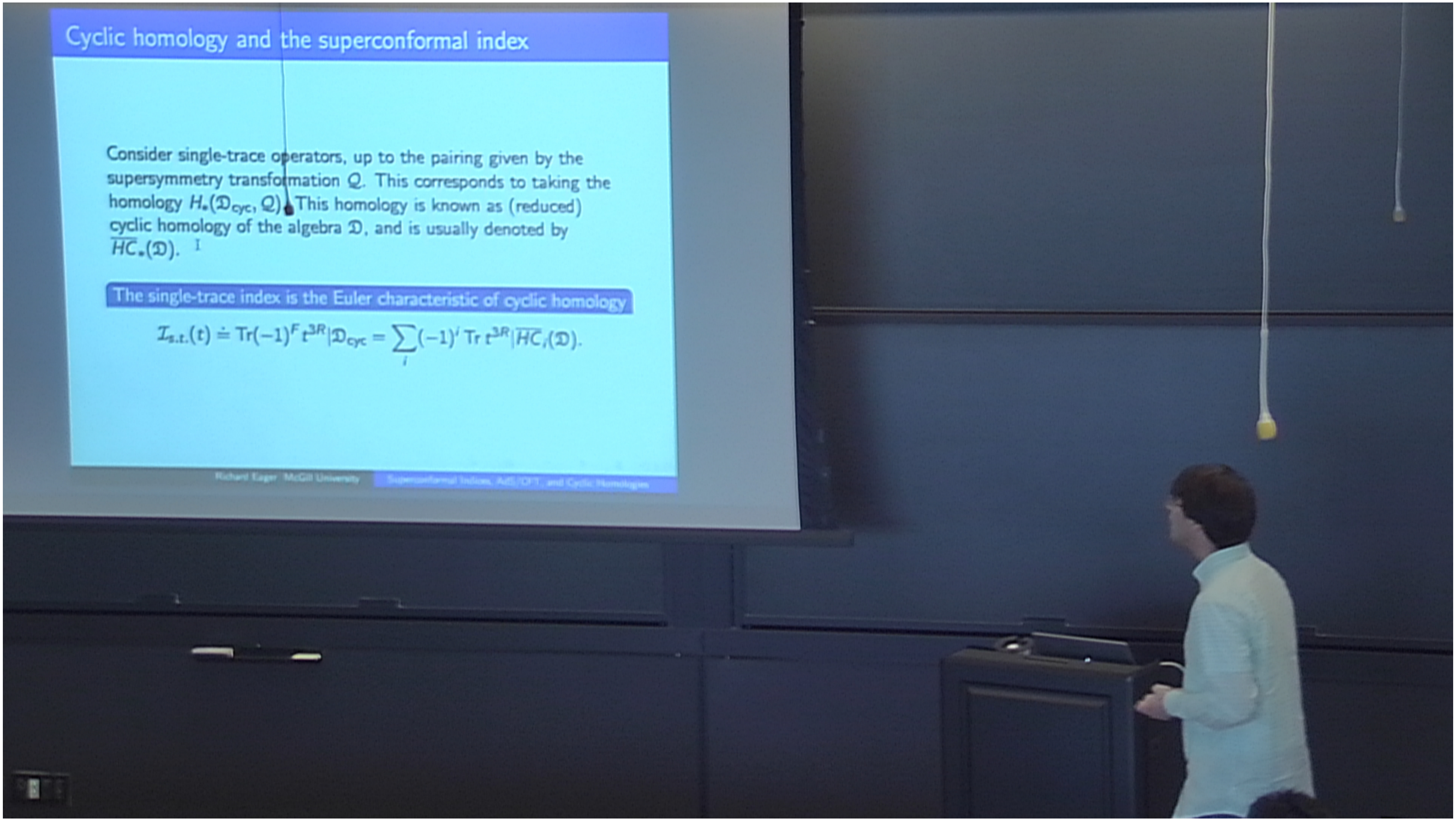
Cyclic homology and the superconformal index
Consider single-trace operators, up to the pairing given by the supersymmetry transformation $\mathcal{Q}$. This corresponds to taking the homology $H_*(\mathfrak{D}_{\text{cyc}}, \mathcal{Q})$. This homology is known as (reduced) cyclic homology of the algebra $\mathfrak{D}$, and is usually denoted by $\overline{HC}_*(\mathfrak{D})$.
The single-trace index is the Euler characteristic of cyclic homology
$\mathcal{I}_{\text{s.t.}}(t) \doteq \operatorname{Tr}(-1)^F t^{3R}|\mathfrak{D}_{\text{cyc}} = \sum_i (-1)^i \operatorname{Tr} t^{3R}|\overline{HC}_i(\mathfrak{D})$.
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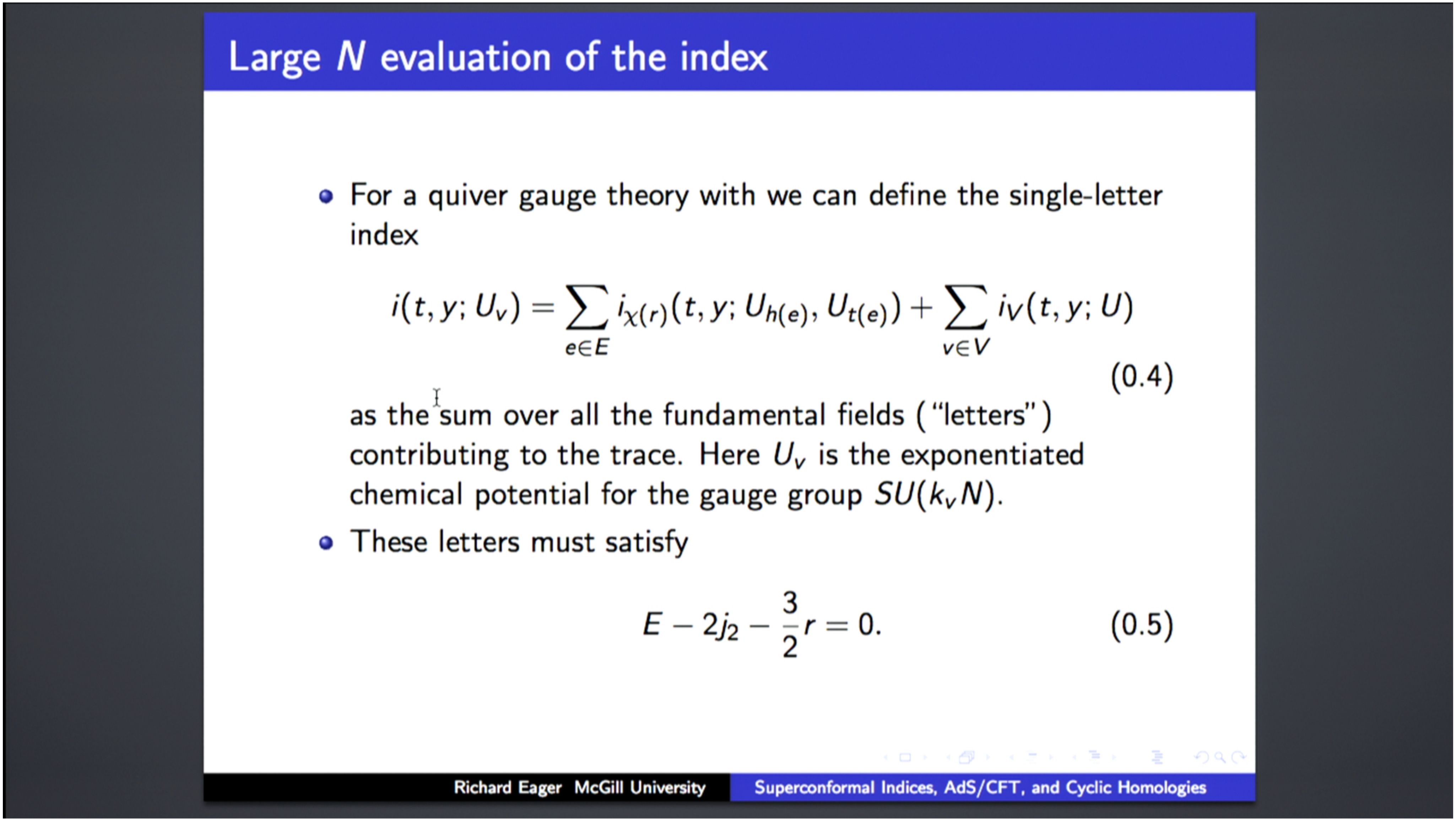
# Large-$N$ superconformal index

The large-N superconformal index was first computed as a large-N matrix integral by mathematicians [P. Etingof, V. Ginzburg] and independently by physicists [A. Gadde, L. Rastelli, S. S. Razamat, W. Yan].

The index can also simply computed as the Euler characteristic of a free dg-algebra [P. Etingof, V. Ginzburg].


Large $N$ evaluation of the index
For a quiver gauge theory with we can define the single-letter index
$$i(t,y;U_v) = \sum_{e\in E} i_{\chi(r)}(t,y;U_{h(e)},U_{t(e)}) + \sum_{v\in V} i_V(t,y;U) \quad (0.4)$$
as the sum over all the fundamental fields ("letters") contributing to the trace. Here $U_v$ is the exponentiated chemical potential for the gauge group $SU(k_v N)$.
These letters must satisfy
$$E - 2j_2 - \frac{3}{2}r = 0. \quad (0.5)$$
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# Chiral and vector multiplet contributions

The single-letter index of a chiral multiplet with R-charge $r$ is

$$i_{\chi(r)}(t,y;U) = i_{\phi(r)}(t,y;U) + i_{\bar{\psi}(r)}(t,y;U), \tag{0.6}$$

where

$$i_{\phi(r)}(t,y;U) = \frac{t^{3r}\chi_R(U)}{(1-t^3y)(1-t^3y^{-1})} \tag{0.7}$$

$$i_{\bar{\psi}(r)}(t,y;U) = -\frac{t^{3(2-r)}\chi_{\overline{R}}(U)}{(1-t^3y)(1-t^3y^{-1})}. \tag{0.8}$$

Similarly the single-letter index of a vector multiplet is

$$i_V(t,y;U) = \frac{2t^6 - t^3(y+\frac{1}{y})}{(1-t^3y)(1-t^3y^{-1})}\chi_{\text{adj}}(U). \tag{0.9}$$

## Chiral and vector multiplet contributions

The single-letter index of a chiral multiplet with R-charge $r$ is

$$i_{\chi(r)}(t,y;U) = i_{\phi(r)}(t,y;U) + i_{\bar{\psi}(r)}(t,y;U), \qquad (0.6)$$

where

$$i_{\phi(r)}(t,y;U) = \frac{t^{3r}\chi_R(U)}{(1-t^3y)(1-t^3y^{-1})} \qquad (0.7)$$

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# The Large-$N$ Matrix Integral

- Take the plethystic exponential of the single letter index
- Project to the gauge-invariant operators by integrating over $U$:

$$\mathcal{I}(t, y) = \int \prod_{v} [dU_v] PE[i(t, y; U_v)]. \qquad (0.10)$$

- Here $PE$ is the plethystic exponential defined by

$$f(t) = \sum_{n \geq 1} a_n t^n \mapsto PE[f(t)] = \prod_{n \geq 1} \frac{1}{(1 - t^n)^{a_n}}. \qquad (0.11)$$

- In the large $N$ limit the matrix integral is evaluated using the saddle-point method [Kinney et. al. '05,Gadde et. al. '10]. The result is that the superconformal index for $SU(N)$ gauge group is

$$\mathcal{I}(x) = \prod_k \frac{e^{-\frac{1}{k}\operatorname{Tr} i(x^k)}}{\det(1 - i(x^k))}. \tag{0.12}$$

- Here, $i(x) \equiv i(t,y)$ is a $n_v \times n_v$ matrix given by

$$i(t,y) = \sum_v i_V(t,y) E_{v,v} + \sum_e i_{\phi(r)}(t,y) E_{h(e),t(e)} + \sum_e i_{\bar{\psi}(r)}(t,y) E_{t(e)} \tag{0.13}$$

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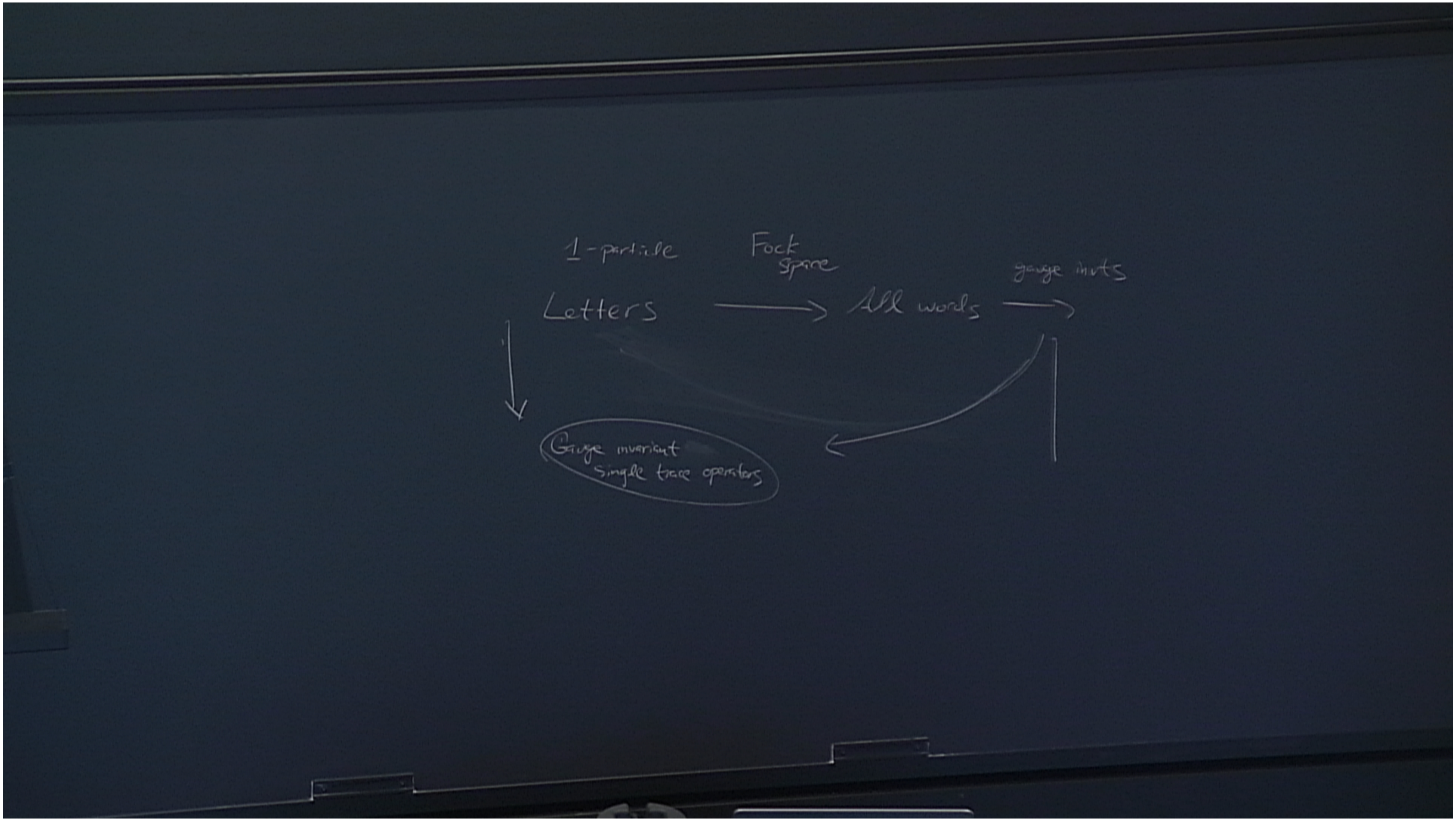
# The Single-Trace Index

The single-trace superconformal index can be extracted from the multi-trace index using the plethystic logarithm

$$\mathcal{I}_{s.t.} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{I}(x^n) \tag{0.14}$$

$$= -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log[\det(1 - i(x^n))] - \operatorname{Tr} i(x). \tag{0.15}$$


1-particle
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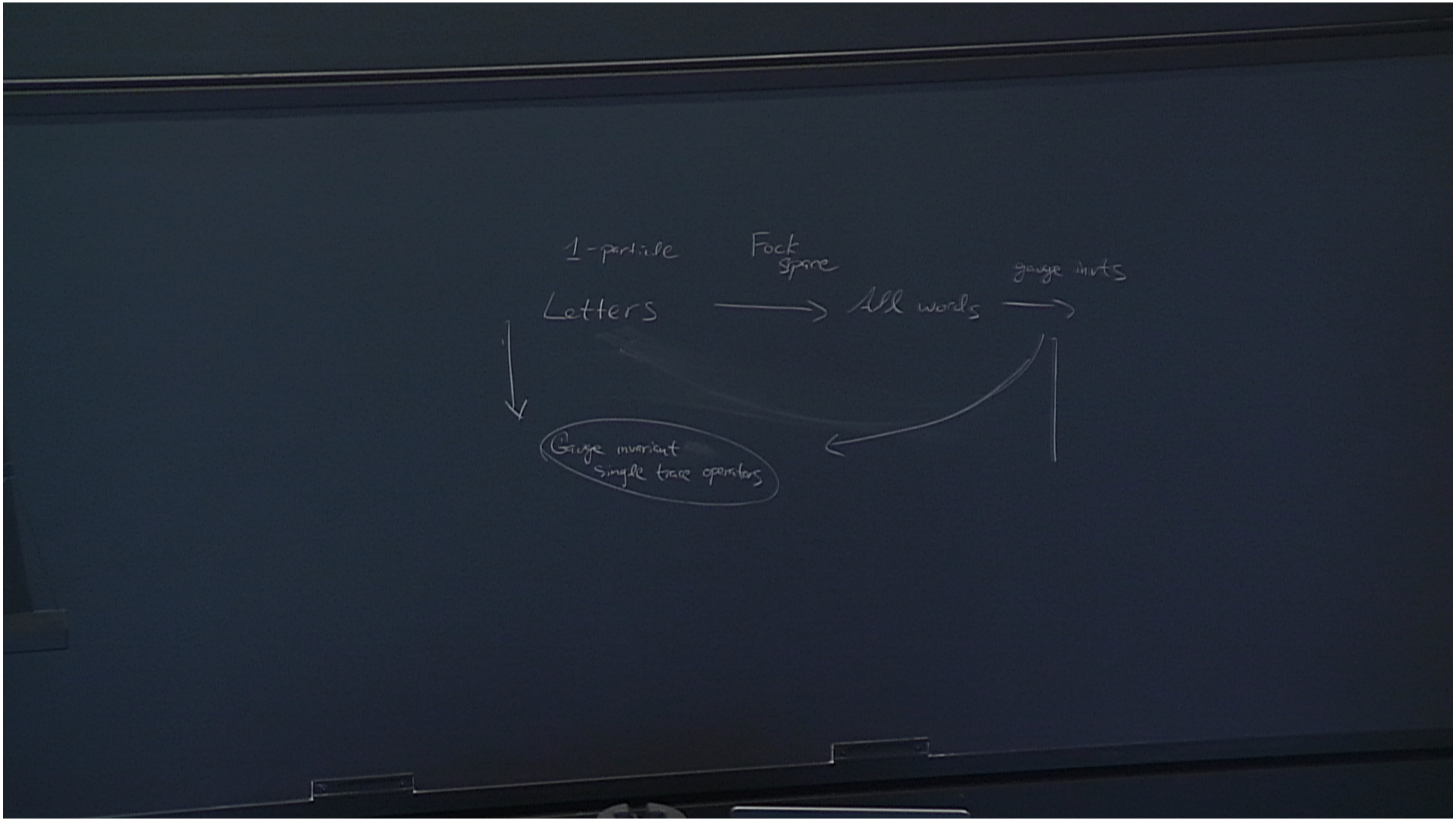
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# Operators in $\mathcal{N} = 4$ Super Yang-Mills

For $X = \mathbb{C}^3, L^5 = S^5$. The corresponding gauge theory is $\mathcal{N} = 4$ SYM, whose superpotential algebra is

$$\mathcal{A} = \mathbb{C}\langle x, y, z\rangle/(xy - yx, yz - zy, zx - xz) \cong \mathbb{C}[x, y, z]$$

| | $1$ | $t^2$ | $t^4$ | $t^6$ | $t^8$ | $t^{10}$ | $t^{12}$ | ... |
|---|---|---|---|---|---|---|---|---|
| $HC_0$ | 1 | 3 | 6 | 10 | 15 | 21 | 28 | ... |
| $HC_1$ | 0 | 0 | 3 | 8 | 15 | 24 | 35 | ... |
| $HC_2$ | 0 | 0 | 0 | 1 | 3 | 6 | 10 | ... |
| $\mathcal{I}(t)$ | 1 | 3 | 3 | 3 | 3 | 3 | 3 | ... |

Table: Cyclic homology group dimensions for $\mathcal{N} = 4$ SYM

Elements $\mathcal{O} \in HC_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ are of the form

$$\mathcal{O} = \mathrm{Tr}\, x^i y^j z^k, \qquad i, j, k \in \mathbb{N}_{\geq 0}$$

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# The $\beta$-deformation

The $\beta$-deformation of $\mathcal{N} = 4$ super Yang-Mills theory is a quiver gauge theory with potential $W = qxyz - q^{-1}xzy$ where $q = e^{i\beta}$. The F-term relations are

$$xy = q^{-2}yx$$
$$yz = q^{-2}zy$$
$$zx = q^{-2}xz$$

The cyclic homology groups were computed by Nuss and Van den Bergh.

# Chiral Primaries in the $\beta$-deformation

Consider an operator $\mathcal{O} = \mathrm{Tr}\, l_1 l_2 \ldots l_n$, where $l_i$ is one of the letters $x, y$, or $z$. Suppose that $l_1$ is an $x$. The F-term conditions imply that

$$\mathcal{O} = \mathrm{Tr}\, l_1 l_2 \ldots l_{n-1} l_n = q^{2(|z|-|y|)}\, \mathrm{Tr}\, l_n l_1 l_2 \ldots l_{n-1},$$

where $|x|$, $|y|$, and $|z|$ are the total number of $x$'s, $y$'s, and $z$'s in the operator $\mathcal{O}$. Thus the single-trace chiral primaries have charges $(k,0,0),(0,k,0),(0,k,0),(k,k,k)$ [D. Berenstein, V. Jejjala, R. G. Leigh]. [2] For $q$ a $k$-th root of unity, the cyclic homology groups jump.

[2]For $G = SU(N)$ there are additional chiral primaries $\mathrm{Tr}\, xy, \mathrm{Tr}\, xz$ and $\mathrm{Tr}\, yz$. This agrees with the perturbative one-loop spectrum of chiral operators found in [D. Z. Freedman, U. Gursoy].

# Operators in the $\beta$-deformation

Cyclic homology gives a prediction for the spectrum of protected operators in the $\beta$-deformation. The corresponding gravity solution was found by Lunin and Maldacena.

| | $1$ | $t^2$ | $t^4$ | $t^6$ | $t^8$ | $t^{10}$ | $t^{12}$ | ... |
|---|---|---|---|---|---|---|---|---|
| $HC_0$ | 1 | 3 | 3 | 4 | 3 | 3 | 4 | ... |
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# Massive Deformation

After adding a mass deformation $\Delta\mathcal{L} = \mathrm{Tr}\, mz^2$, to $\mathcal{N} = 4$ super Yang-Mills, the superpotential is $W = xyz - xzy + mz^2$. Since $z$ is massive, it can be integrated out of the Lagrangian using its equations of motion. The result is $W = \frac{1}{m}[x, y]^2$. Both superpotential algebras are Morita equivalent and have the same $\mathcal{Q}$ cohomology groups. The F-term relations are

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# Massive Deformation II

The $Q$-cohomology for the massive deformation is

| | $1$ | $t^{3/2}$ | $t^3$ | $t^{9/2}$ | $t^6$ | $t^{15/2}$ | $t^9$ | $\ldots$ |
|---|---|---|---|---|---|---|---|---|
| $HC_0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\ldots$ |
| $HC_1$ | 0 | 0 | 0 | 2 | 3 | 4 | 5 | $\ldots$ |
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Table: Cyclic homology group dimensions for the massive deformation

We will compare these protected operators to the short representations in the KK-spectrum of the exact SUGRA solution found by Pilch and Warner.

## Pilch-Warner Solution

A new critical point of $\mathcal{N} = 8$ gauge supergravity on $AdS_5$ was discovered by Khavaev-Pilch-Warner. Pilch and Warner found the full type IIB supergravity solution.

$$ds_{10}^2 = \Delta^{-1} ds_{AdS_5}^2 + L^2 \Delta^1 ds_5^2(\rho, \chi)$$

$$ds_5^2(\rho, \chi) = (dx^I Q_{IJ}^{-1} dx^J) + \frac{\sinh^2 \chi}{\xi^2} (x^I J_{IJ} dx^J)^2$$

$\rho$ and $\chi$ are critical points of the supergravity potential. For the Pilch-Warner critical point $\rho = 2^{1/6}, \chi = \frac{1}{2} \log 3$. The warp-factor is

$$\Delta = \Omega^{-2}$$

where $\Omega^2 = \xi \cosh \chi$.

## Glueball spectrum

The KK-spectrum of glue balls is found by finding solutions of the warped-Laplacian

$$\mathcal{L} \equiv \frac{\Delta^{-1}}{\sqrt{-g_5}} \partial_\alpha \left( \sqrt{-g_5} \Delta^{-1} g^{\alpha\beta} \partial_\beta \right)$$

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# Further applications of cyclic homology

For CY-3 algebras

$$HC_j(\mathcal{A}) = 0 \text{ for } j > 2$$

This corresponds to the AdS dual theory having no particles of spin higher than 2.

$$HC_2(\mathcal{A}) = Z(\mathcal{A})$$

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# Final Remarks

## Summary

We have shown how to compare the protected fields on both sides of the AdS/CFT correspondence at large-$N$.

- Further extension to finite $N$ is possible, although the cyclic homology groups become much harder to compute.

## Further directions

Use all fields that contribute to the SCI. This corresponds to proving AdS/CFT under a holomorphic twist of the both the gauge and gravity theories [K. Costello].

- Other twists are also interesting [C. Beem, L. Rastelli, B. C. van Rees].
- Extensions to M-theory compactifications [R.E., J. Schmude].

# Toric Cohomology

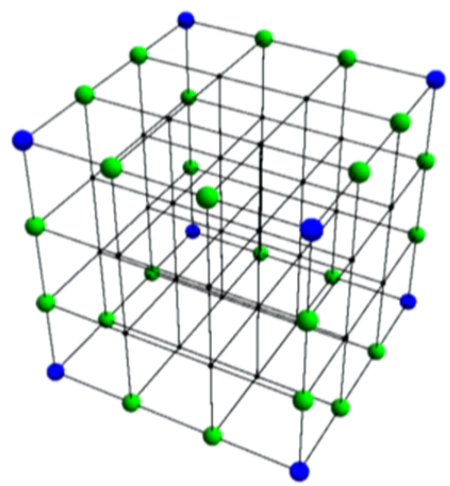
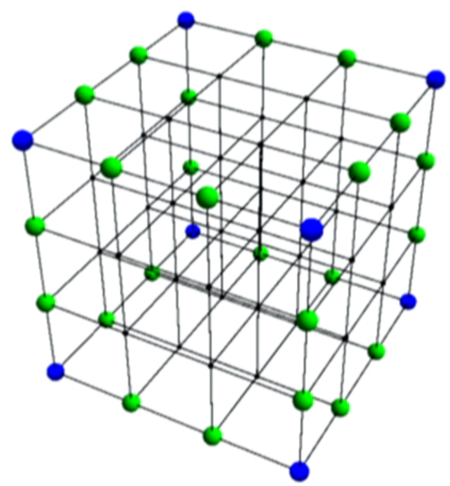


Figure: Contributions to $M^{\circ(1)}$ and $M^{\circ(2)}$ for $Q^{1,1,1}$ with $l = 3$ are colored green and blue respectively.

| $l$ | $M^{\circ(1)}$ | $M^{\circ(2)}$ | $M^{\circ(3)}$ | $M^{\circ(4)}$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 8 | 0 | 0 | 0 |
| 2 | 8 | 12 | 6 | 1 |
| 3 | 8 | 24 | 24 | 8 |
| 4 | 8 | 36 | 54 | 27 |
| 5 | 8 | 48 | 96 | 64 |

Table: Toric data for $Q^{1,1,1}$.

# Toric Cohomology



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