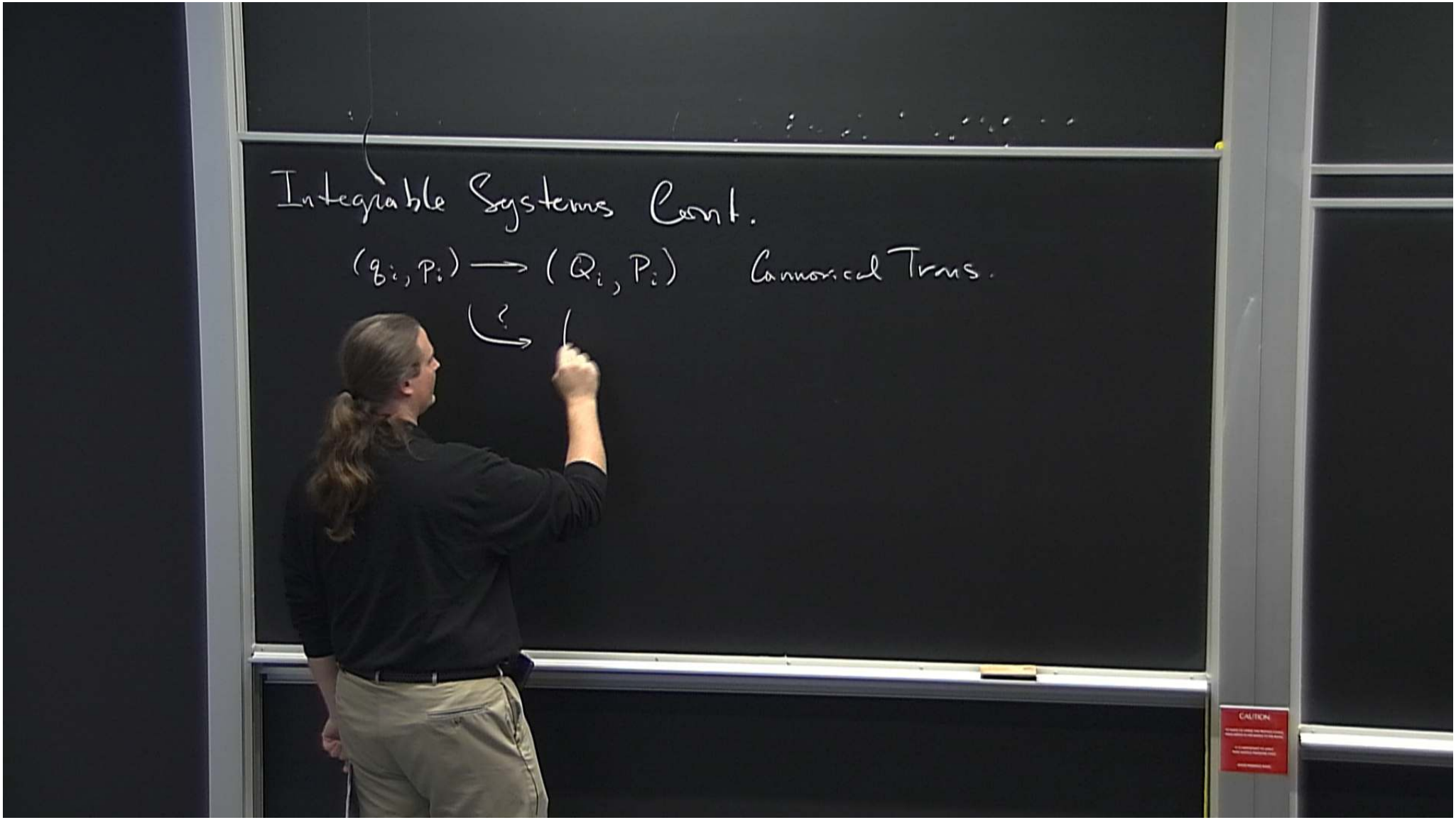
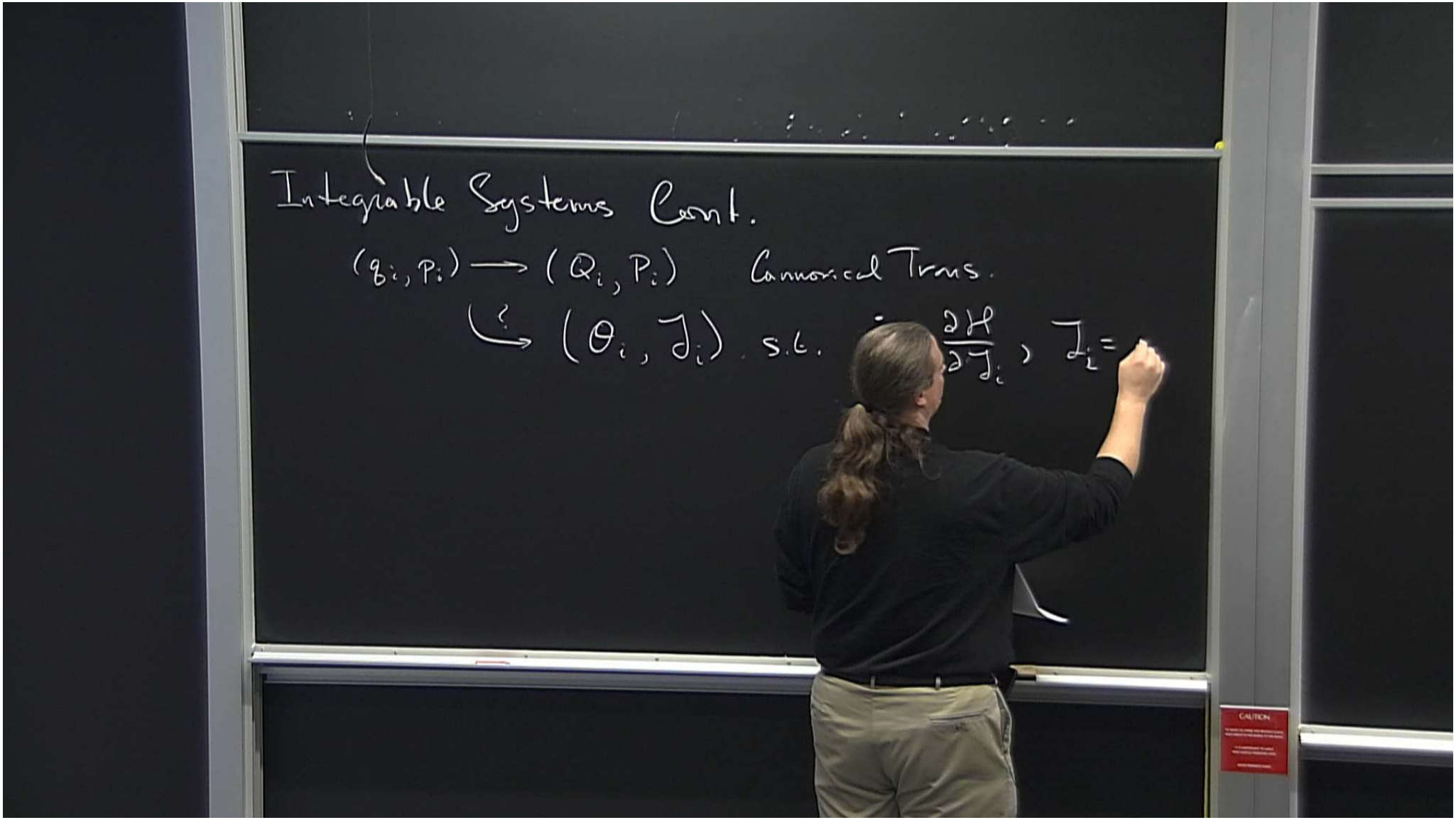
Title: PHYS 781 - Lecture 7

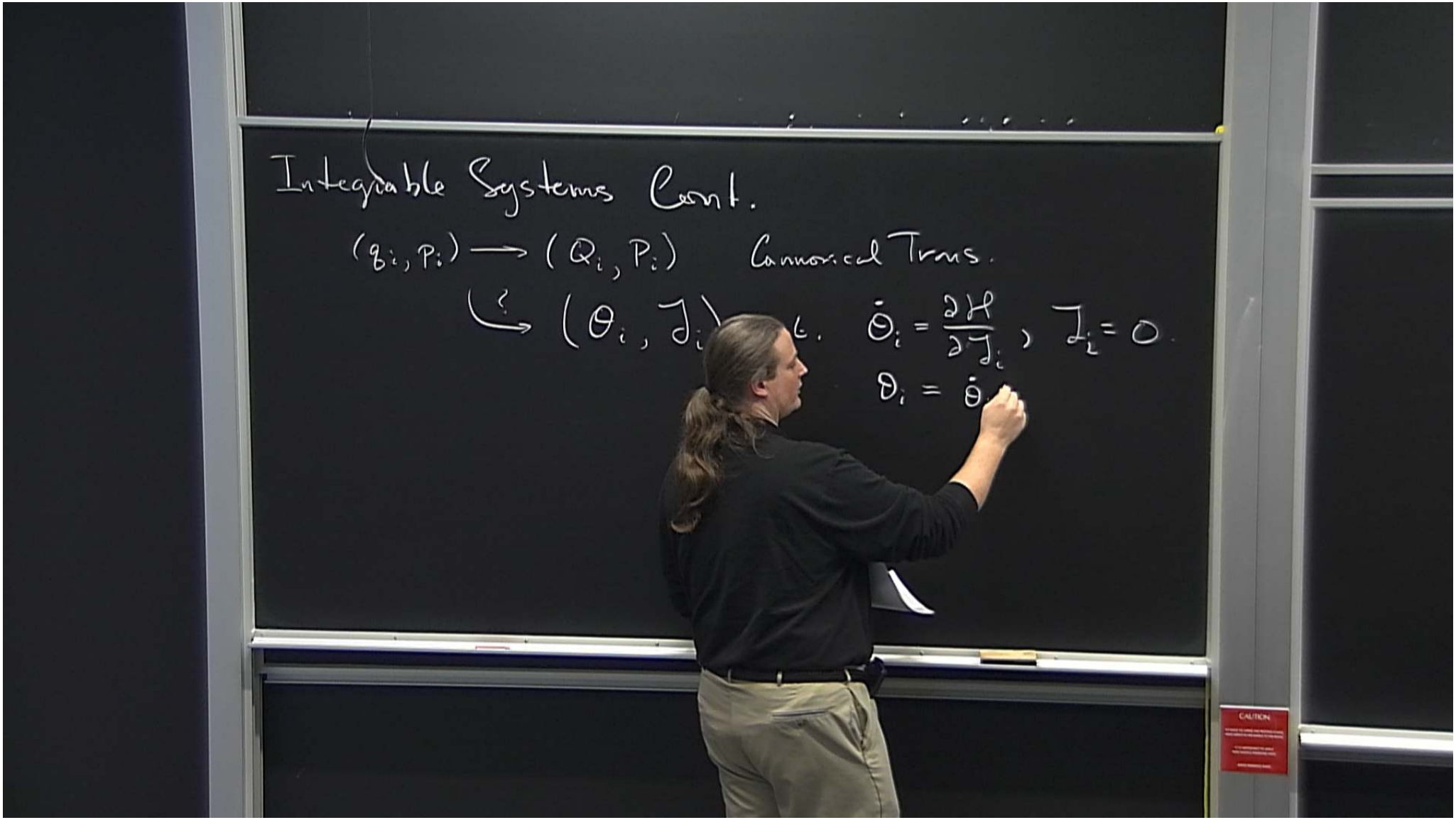
Date: Oct 07, 2014 02:00 PM

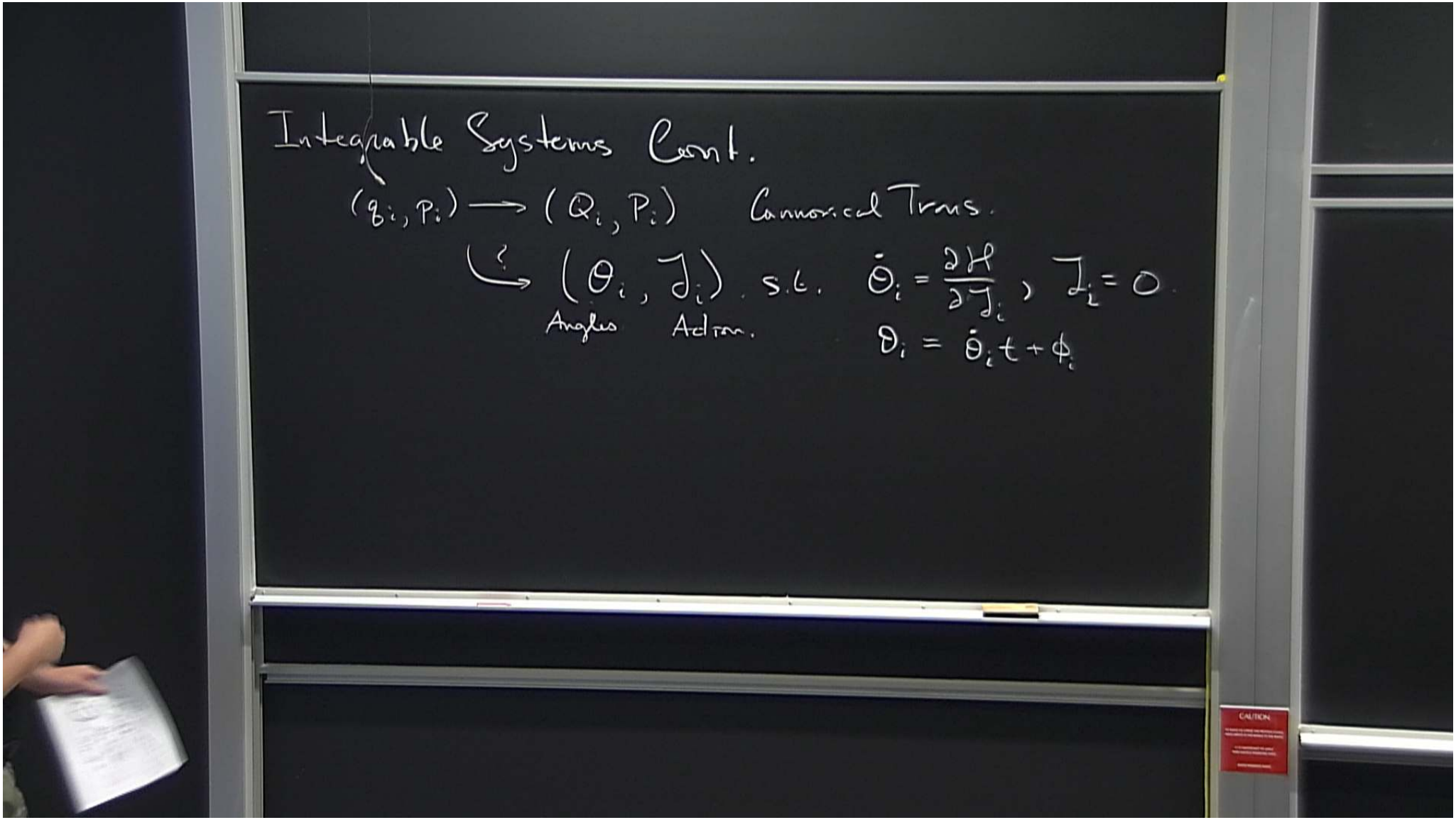
URL: http://pirsa.org/14100052

Abstract:


Integrable Systems Cont.
$(q_i, p_i) \longrightarrow (Q_i, P_i)$ Canonical Trans.
CAUTION


Integrable Systems Cont.
$(q_i, p_i) \longrightarrow (Q_i, P_i)$ Canonical Trans.
$\hookrightarrow (\theta_i, J_i)$ s.t. $\frac{\partial H}{\partial J_i}$, $J_i =$
CAUTION


Integrable Systems Cont.
$(q_i, p_i) \longrightarrow (Q_i, P_i)$ Canonical Trans.
$\hookrightarrow (\theta_i, J_i)$
$\dot{\theta}_i = \frac{\partial H}{\partial J_i}$, $\dot{J}_i = 0$
$\theta_i = \dot{\theta}$
CAUTION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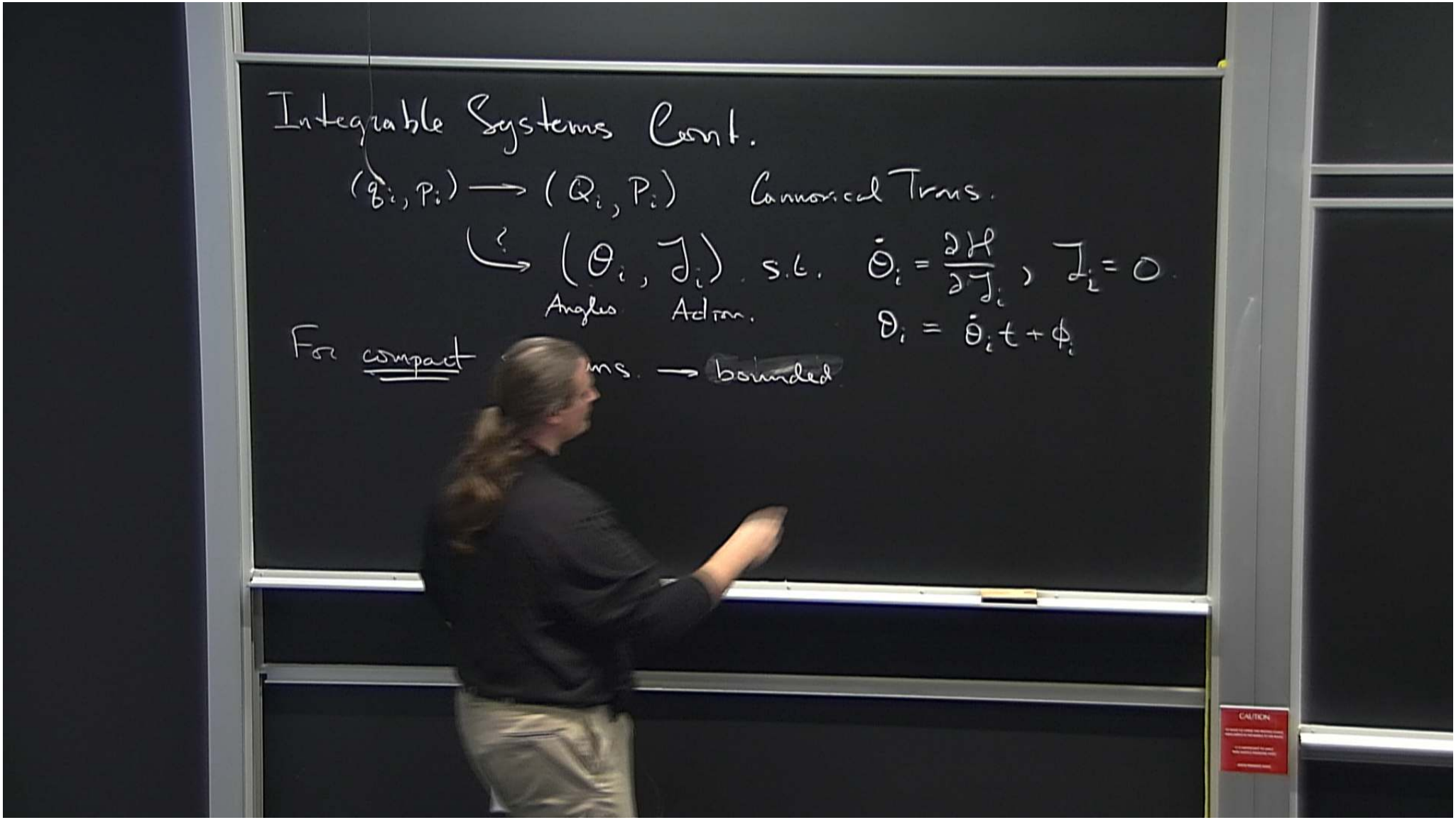
$\hookrightarrow (\theta_i, J_i)$ s.t. $\dot{\theta}_i = \frac{\partial H}{\partial J_i}$, $\dot{J}_i = 0$
Angles Actions.
$\theta_i = \dot{\theta}_i t + \phi_i$
CAUTION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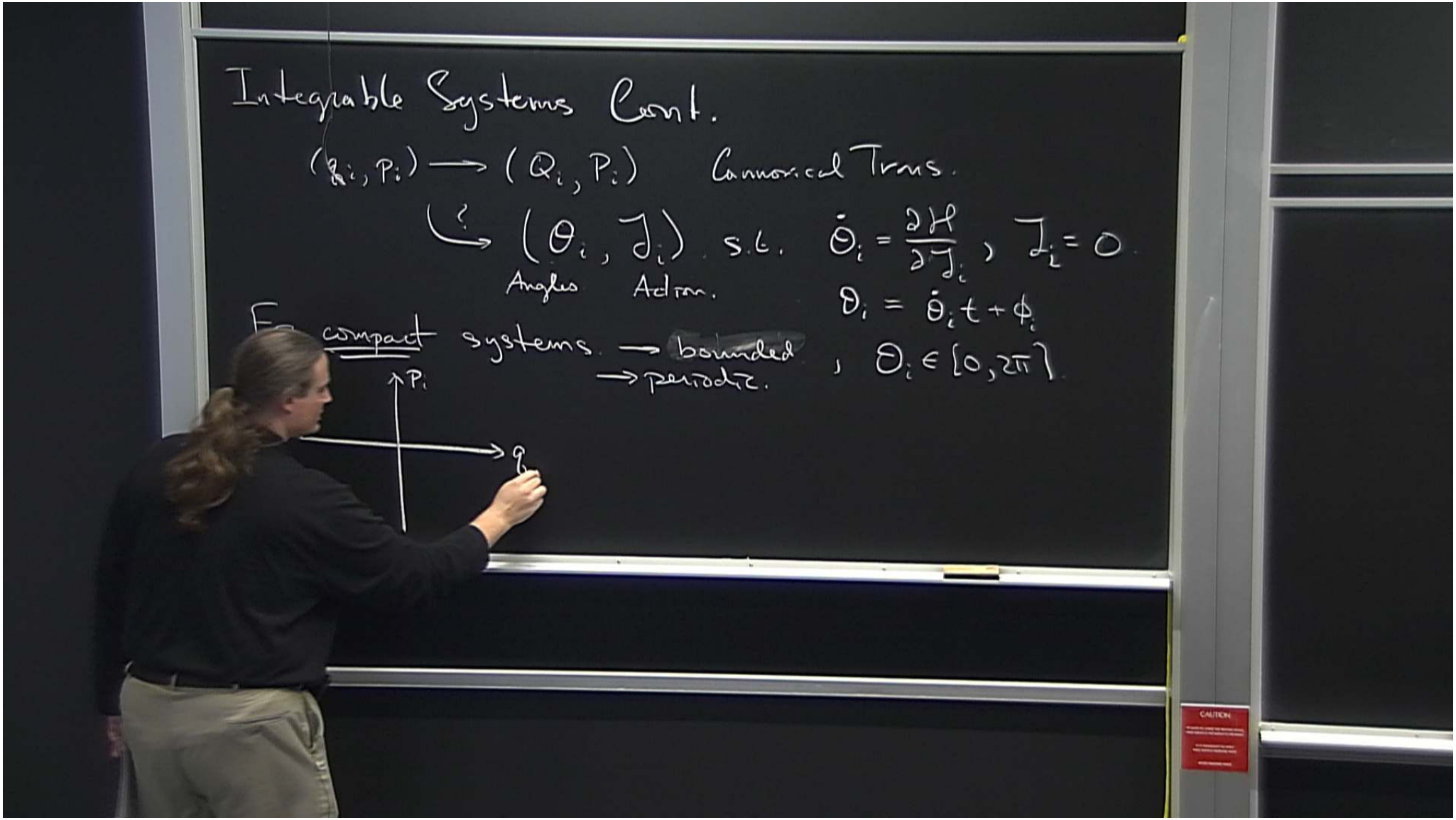
Angles Actions.
$\theta_i = \dot{\theta}_i t + \phi_i$
For compact ms. → bounded
CAUTION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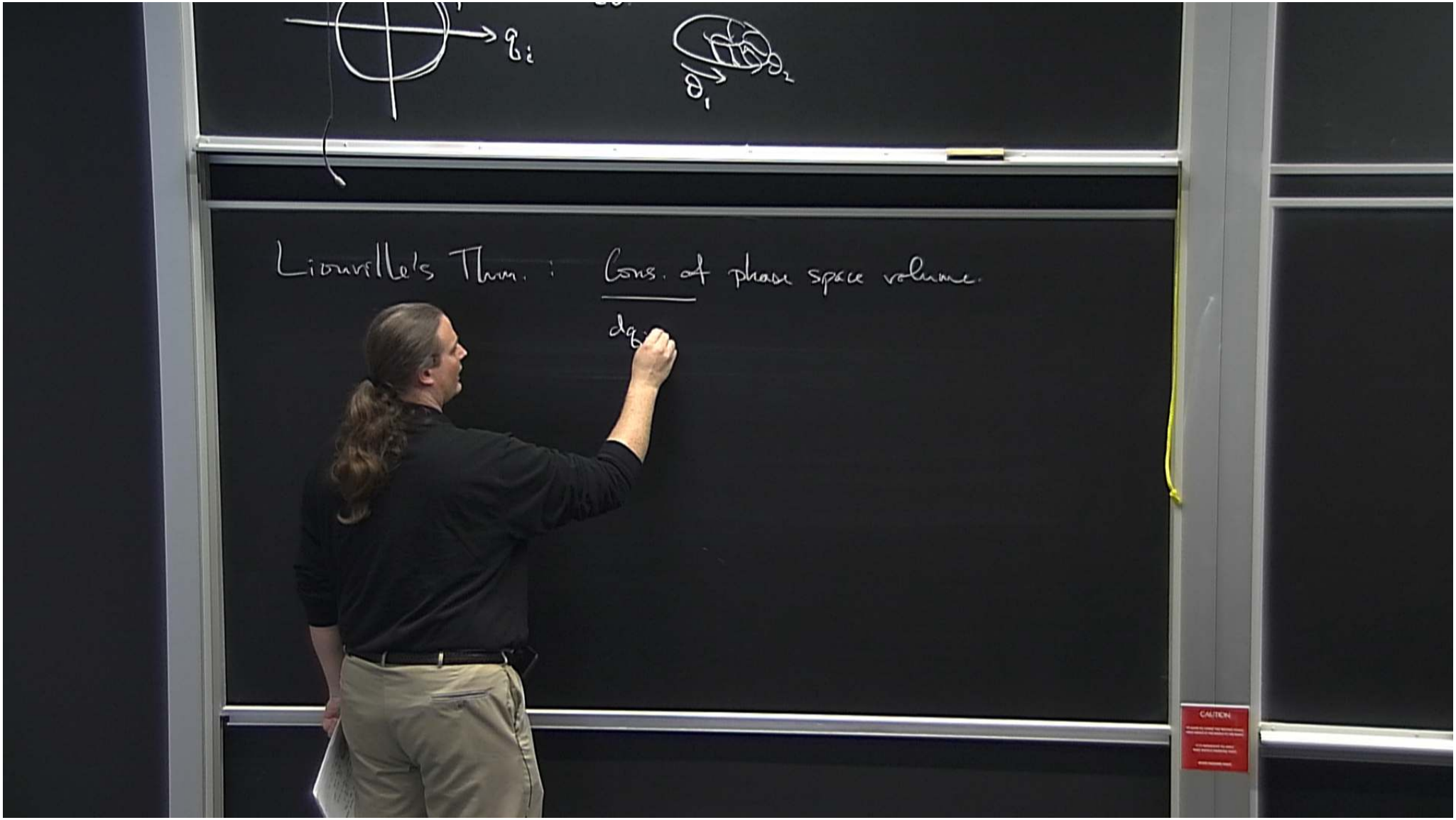
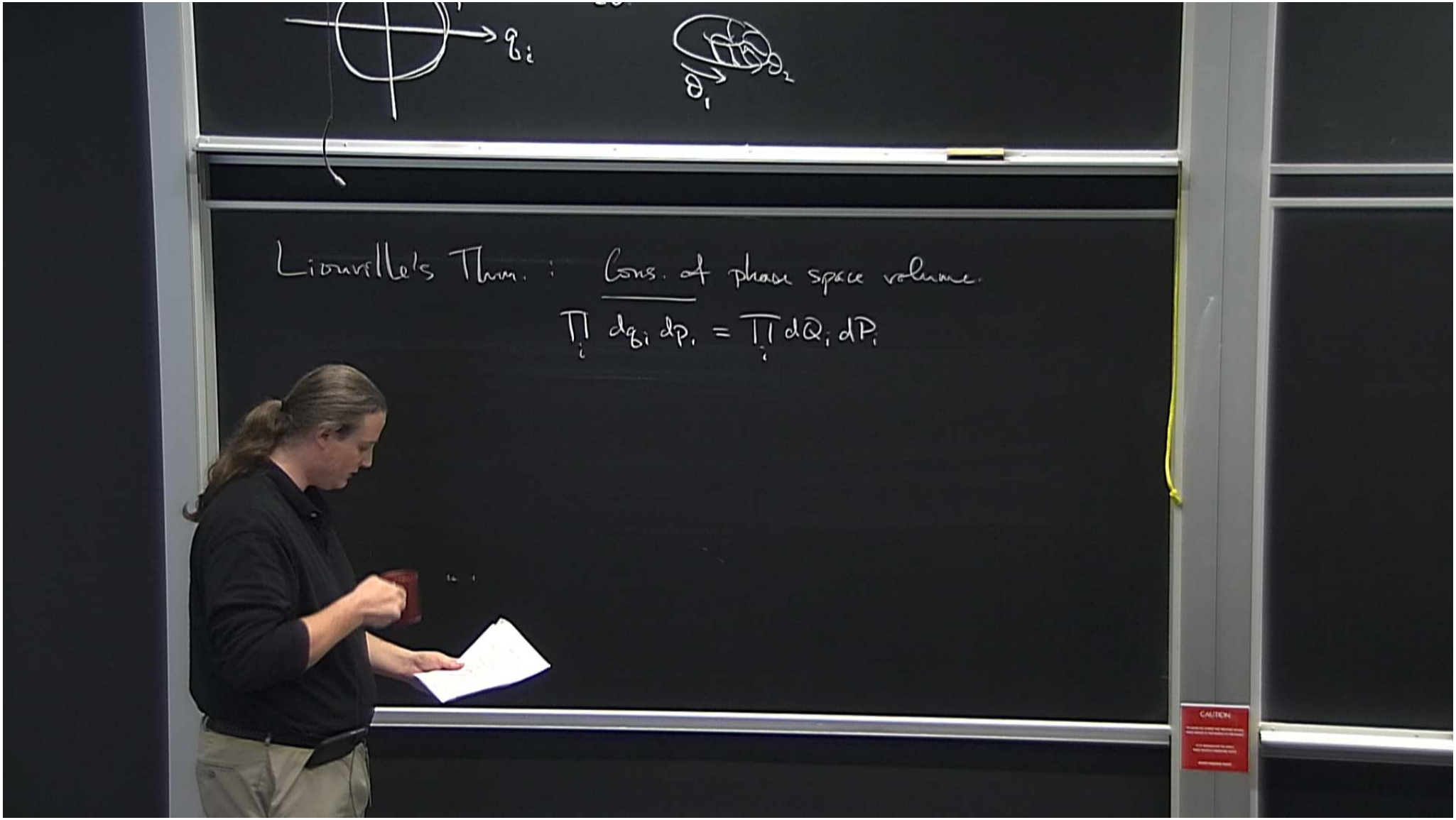
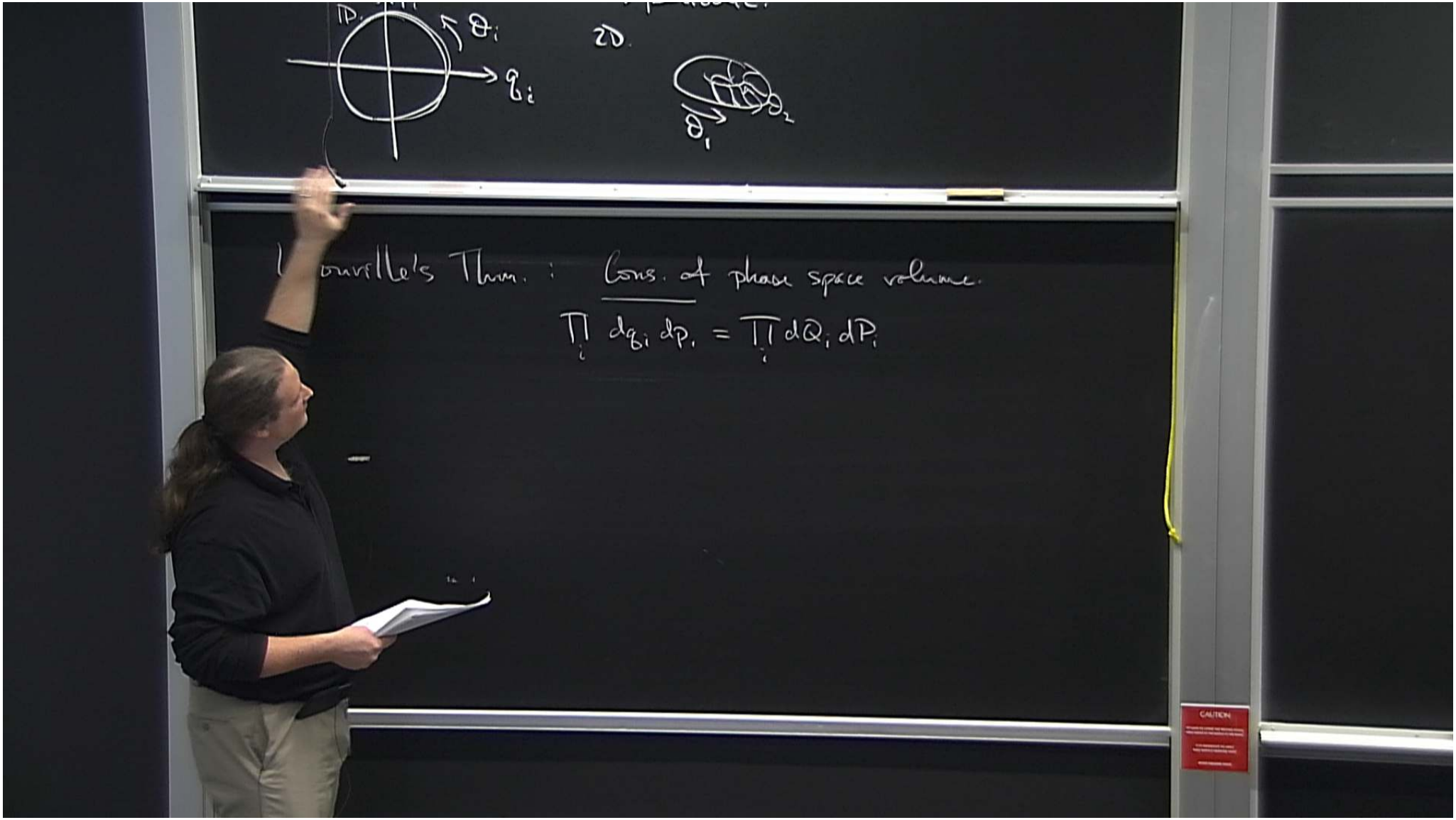
Angles Actions.
$\theta_i = \dot{\theta}_i t + \phi_i$
compact systems → bounded, $\theta_i \in [0, 2\pi]$
→ periodic.
P_i
q_i


q_i
p_2
p_1
Liouville's Thm.: Cons. of phase space volume.
dq_i
CAUTION


q_i
p_i
Liouville's Thm.: Cons. of phase space volume.
$\prod_i dq_i\, dp_i = \prod_i dQ_i\, dP_i$


p_i
q_i
2D
θ_2
θ_1
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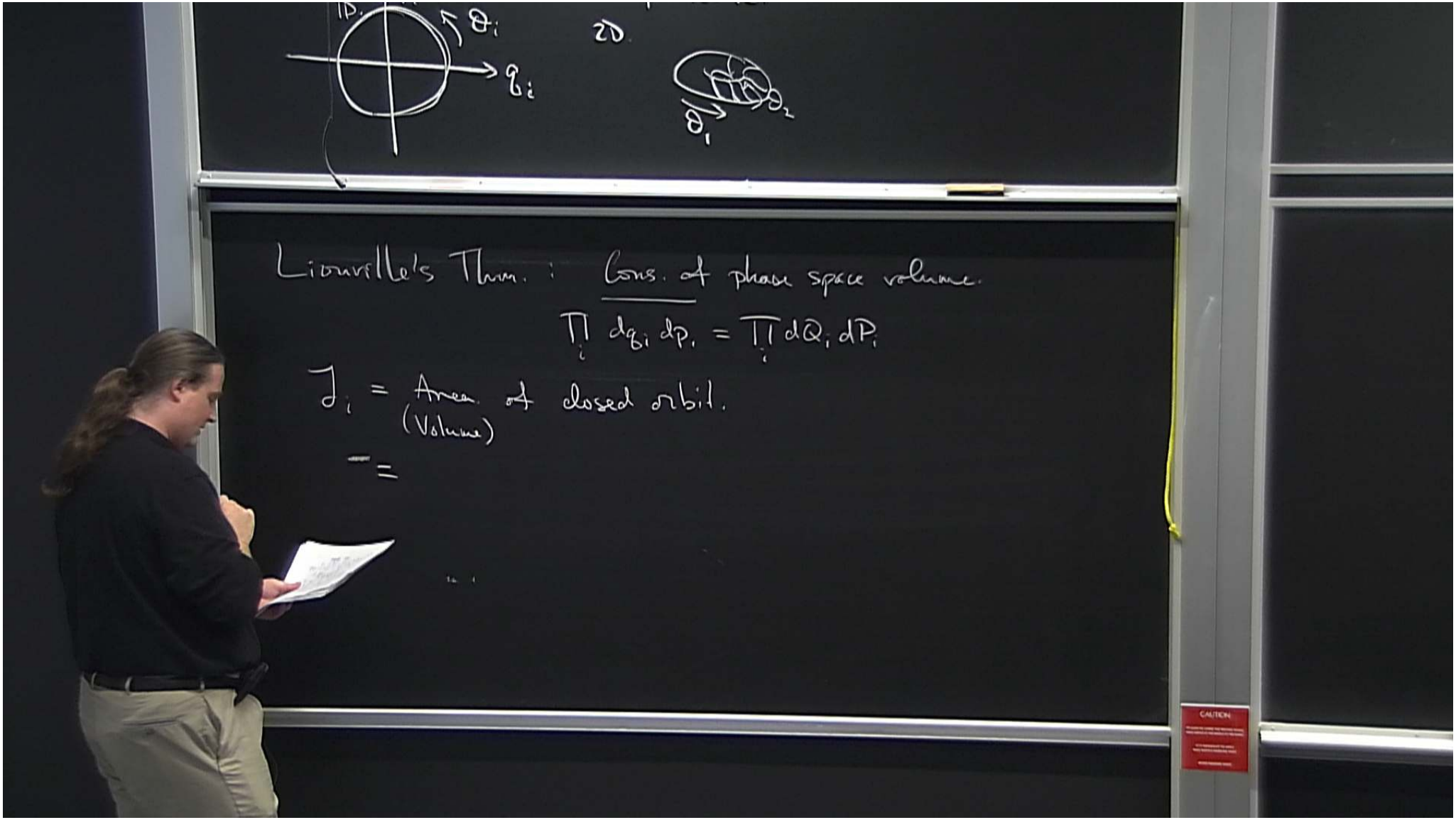
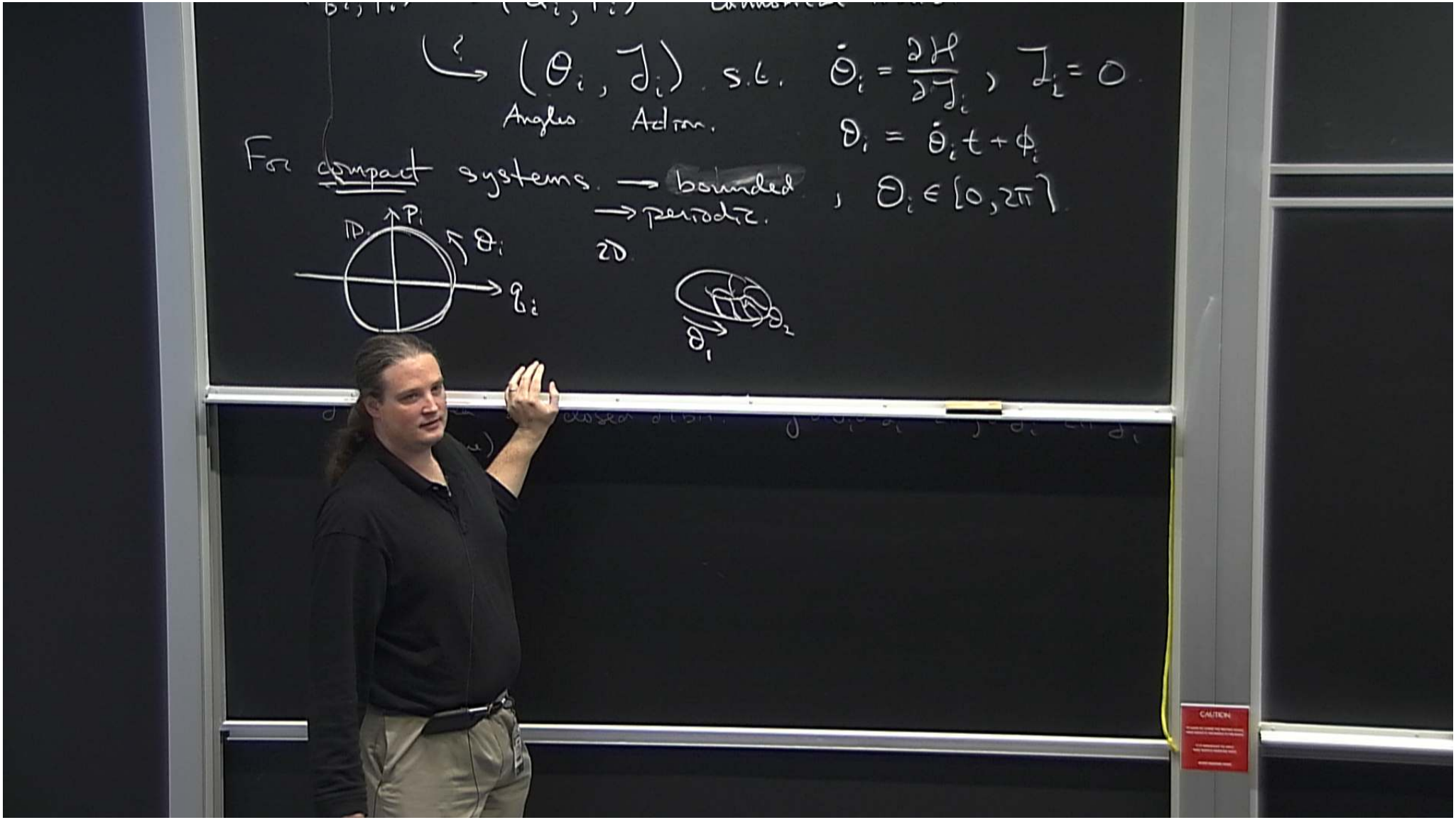
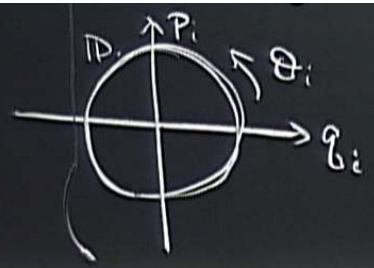
CAUTION


p_i
θ_i
q_i
2D
θ_1
θ_2
Liouville's Thm.: Cons. of phase space volume.
$\prod_i dq_i\, dp_i = \prod_i dQ_i\, dP_i$
J_i = Area (Volume) of closed orbit.
=


$\hookrightarrow (\theta_i, J_i)$ s.t. $\dot{\theta}_i = \frac{\partial H}{\partial J_i}$, $\dot{J}_i = 0$
Angles Actions
$\theta_i = \dot{\theta}_i t + \phi_i$
For compact systems → bounded, $\theta_i \in [0, 2\pi]$
→ periodic
p_i
θ_i
q_i
2D
θ_1
θ_2



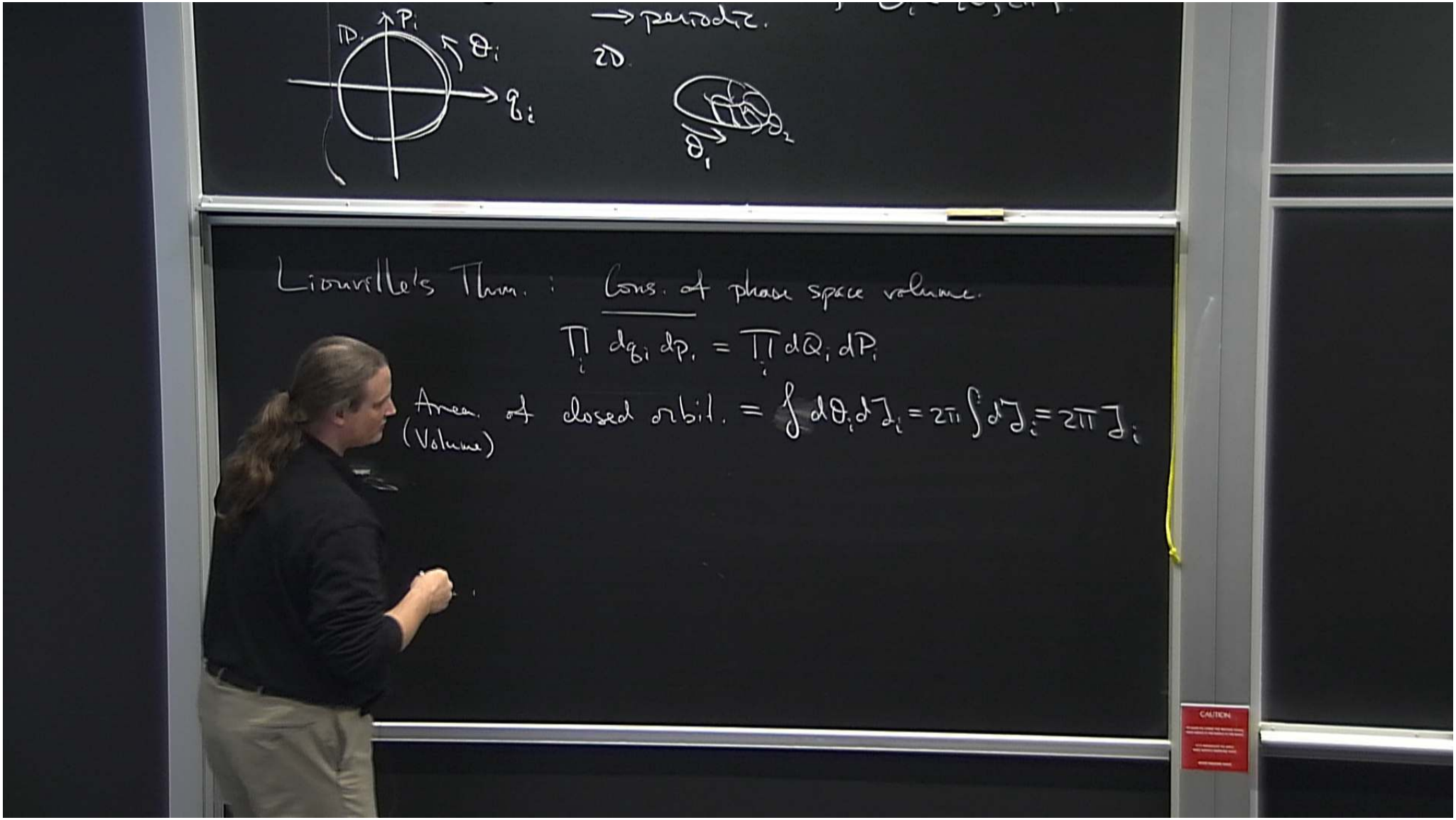
→ periodic.

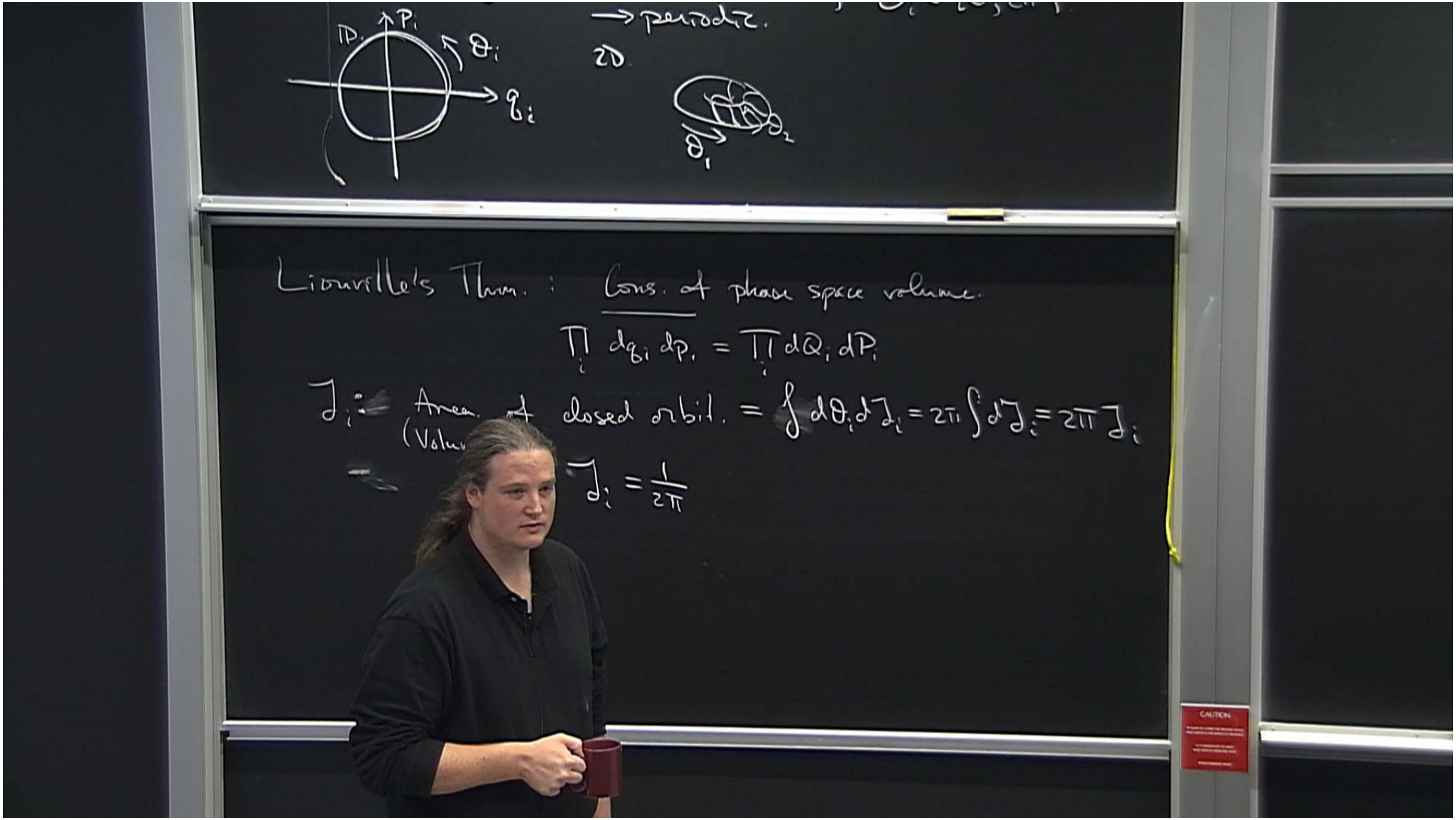
2D

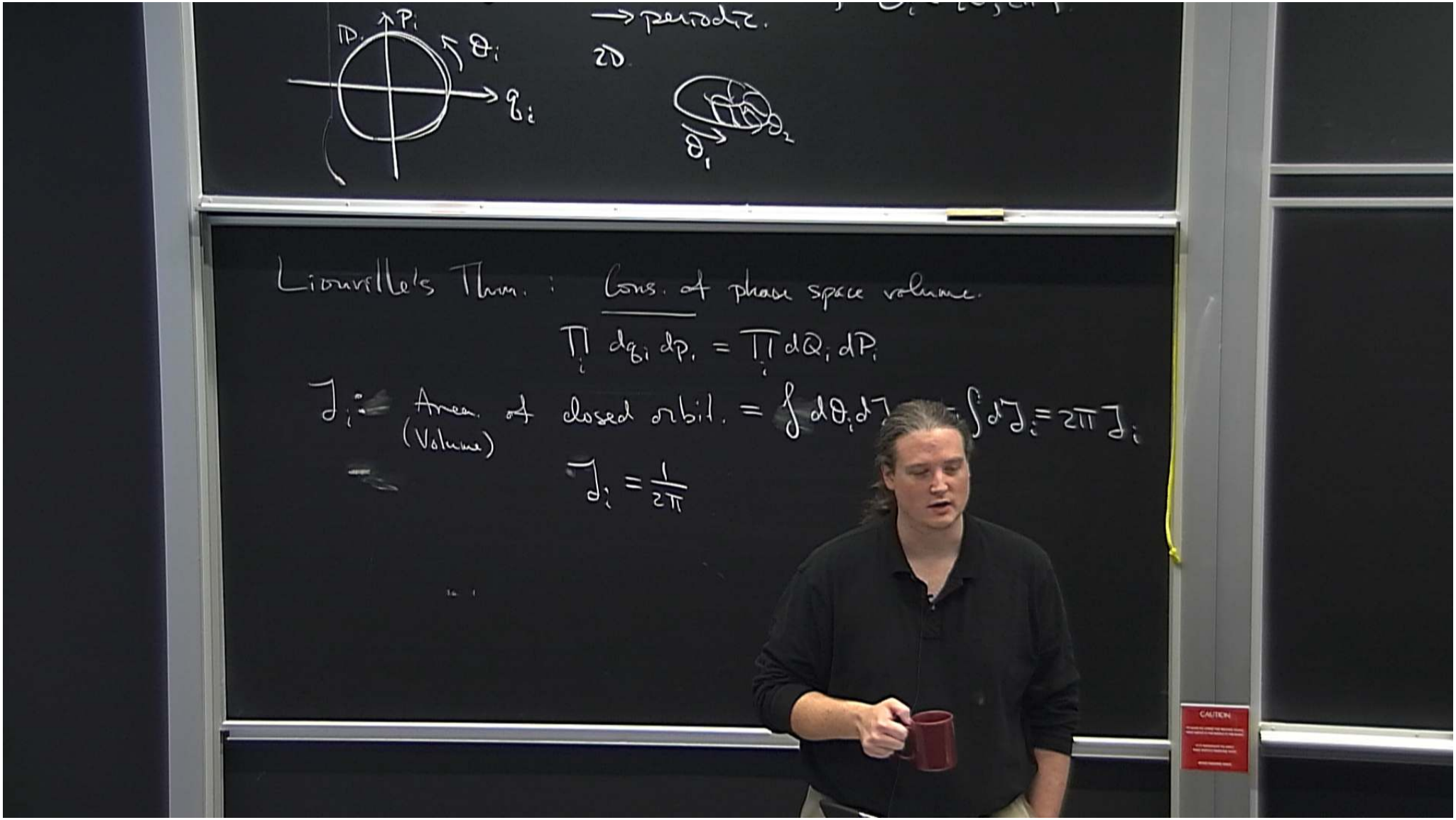
Liouville's Thm.: Cons. of phase space volume.

$$\prod_i dq_i\, dp_i = \prod_i dQ_i\, dP_i$$

J_i := Area (Volume) of closed orbit. $= \oint d\theta_i\, dJ_i = 2\pi \int dJ_i = 2\pi J_i$


p_i
Q_i
q_i
→ periodic.
2D
θ_2
θ_1
Liouville's Thm. : Cons. of phase space volume.
$\prod_i dq_i\, dp_i = \prod_i dQ_i\, dP_i$
Area (Volume) of closed orbit. $= \oint d\theta_i\, dJ_i = 2\pi \int dJ_i = 2\pi J_i$


p_i
q_i
θ_i
→ periodic.
2D
θ_1
θ_2
Liouville's Thm.: Cons. of phase space volume.
$\prod_i dq_i\, dp_i = \prod_i dQ_i\, dP_i$
J_i = Area (Volu… of closed orbit. $= \oint d\theta_i\, dJ_i = 2\pi \int dJ_i = 2\pi J_i$
$J_i = \frac{1}{2\pi}$
CAUTION


p_i
P_i
θ_i
q_i
→ periodic.
2D
θ_1
θ_2
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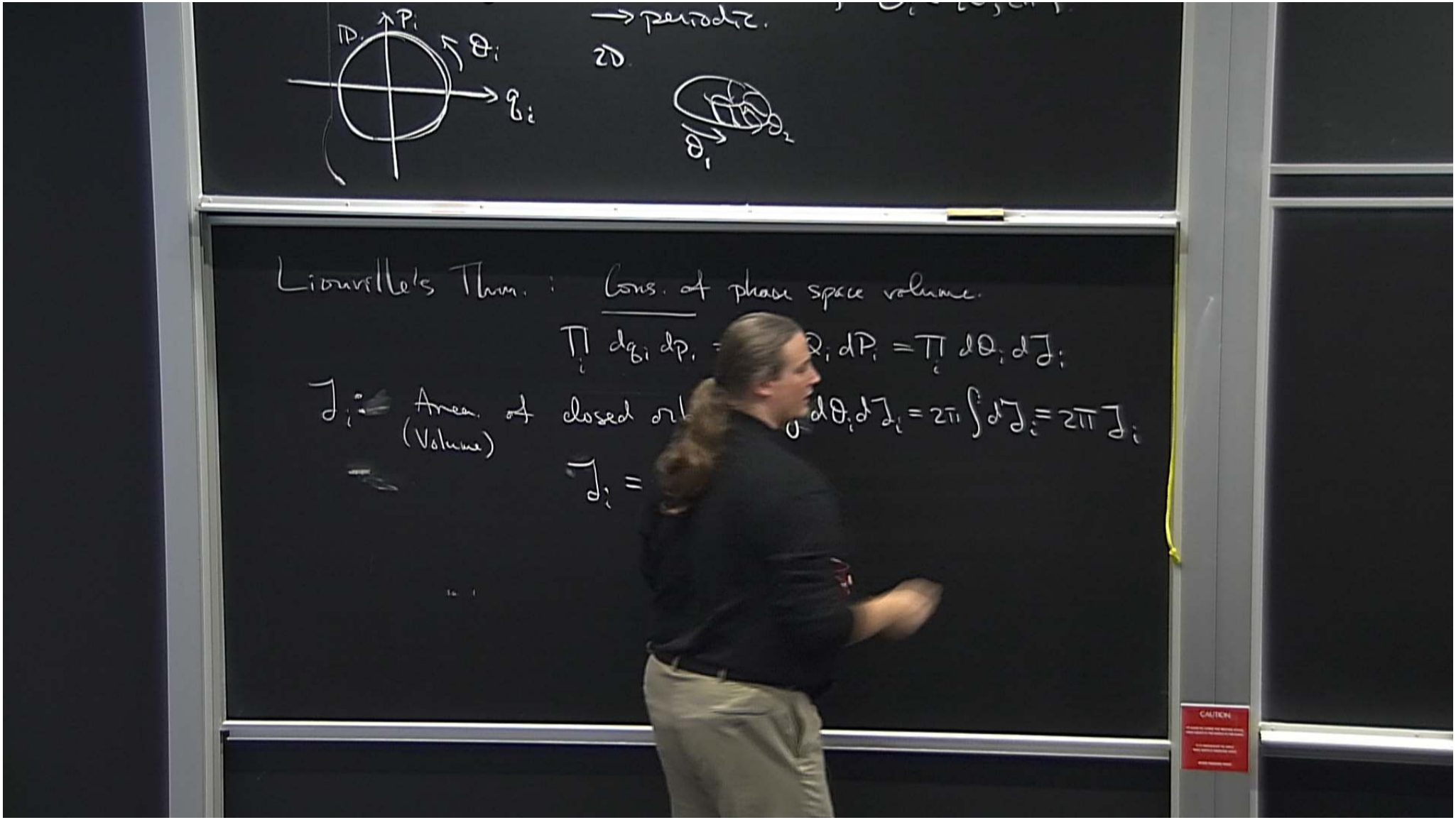
p_i
q_i
θ_i
θ_1
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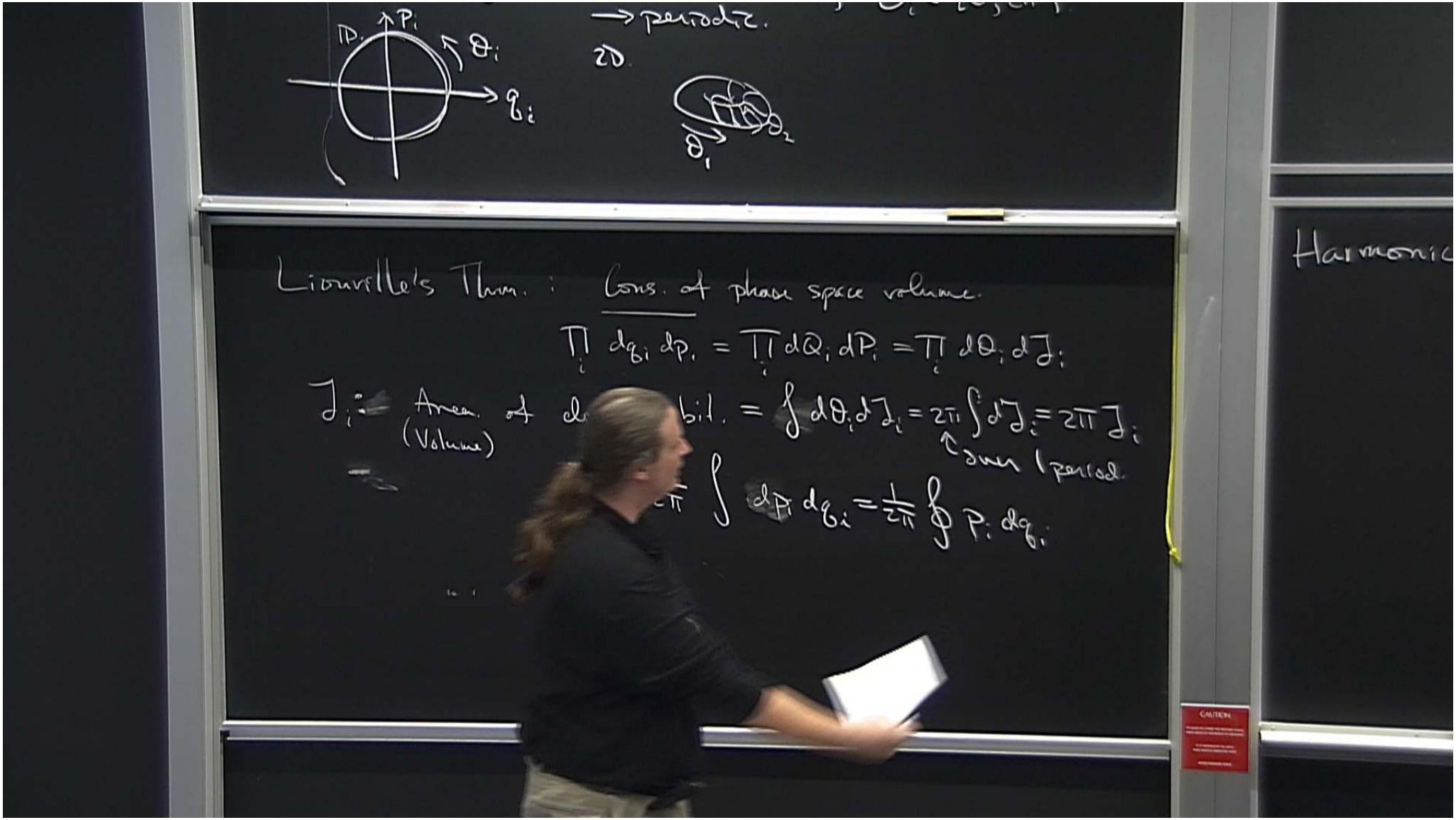
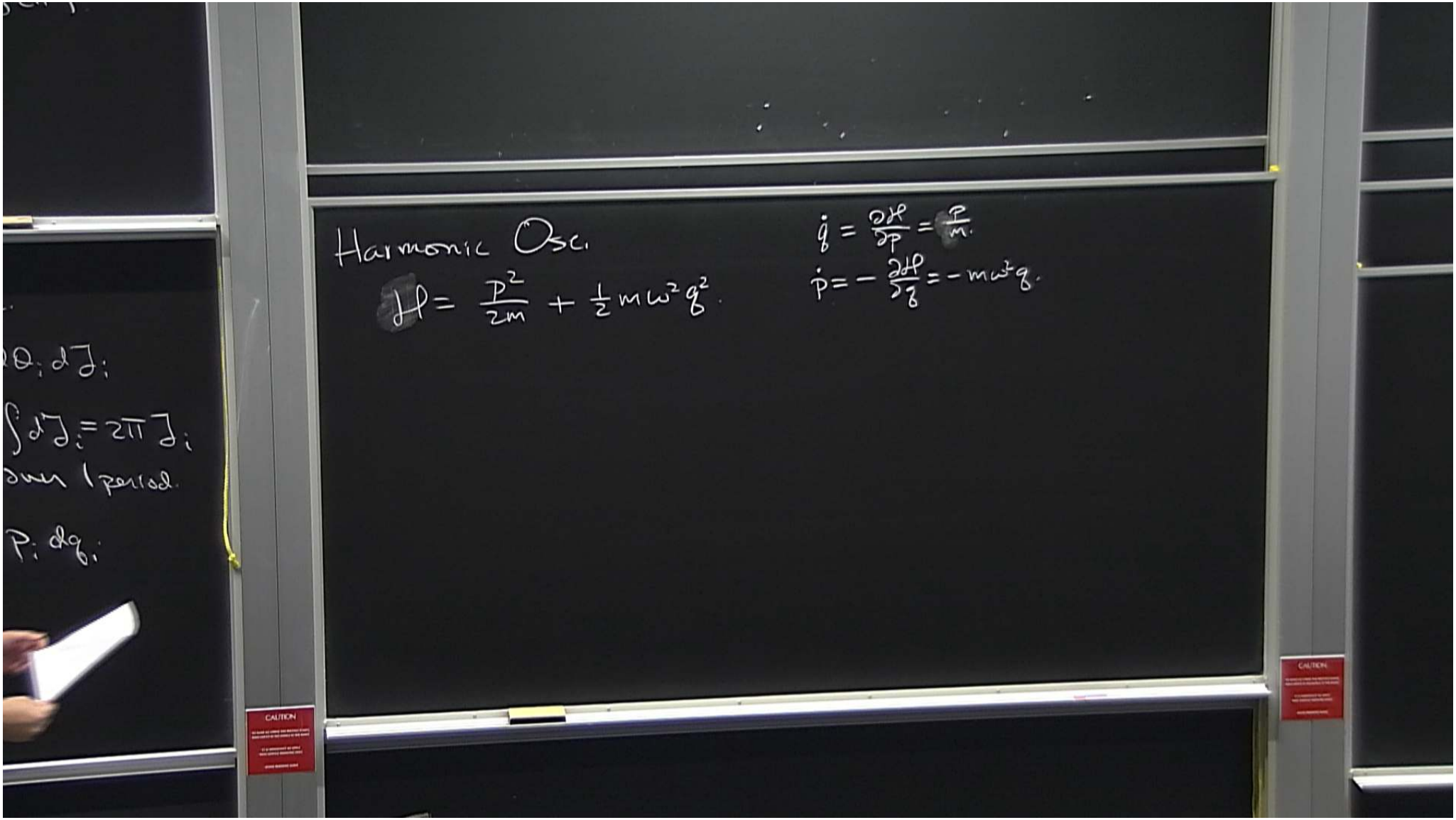
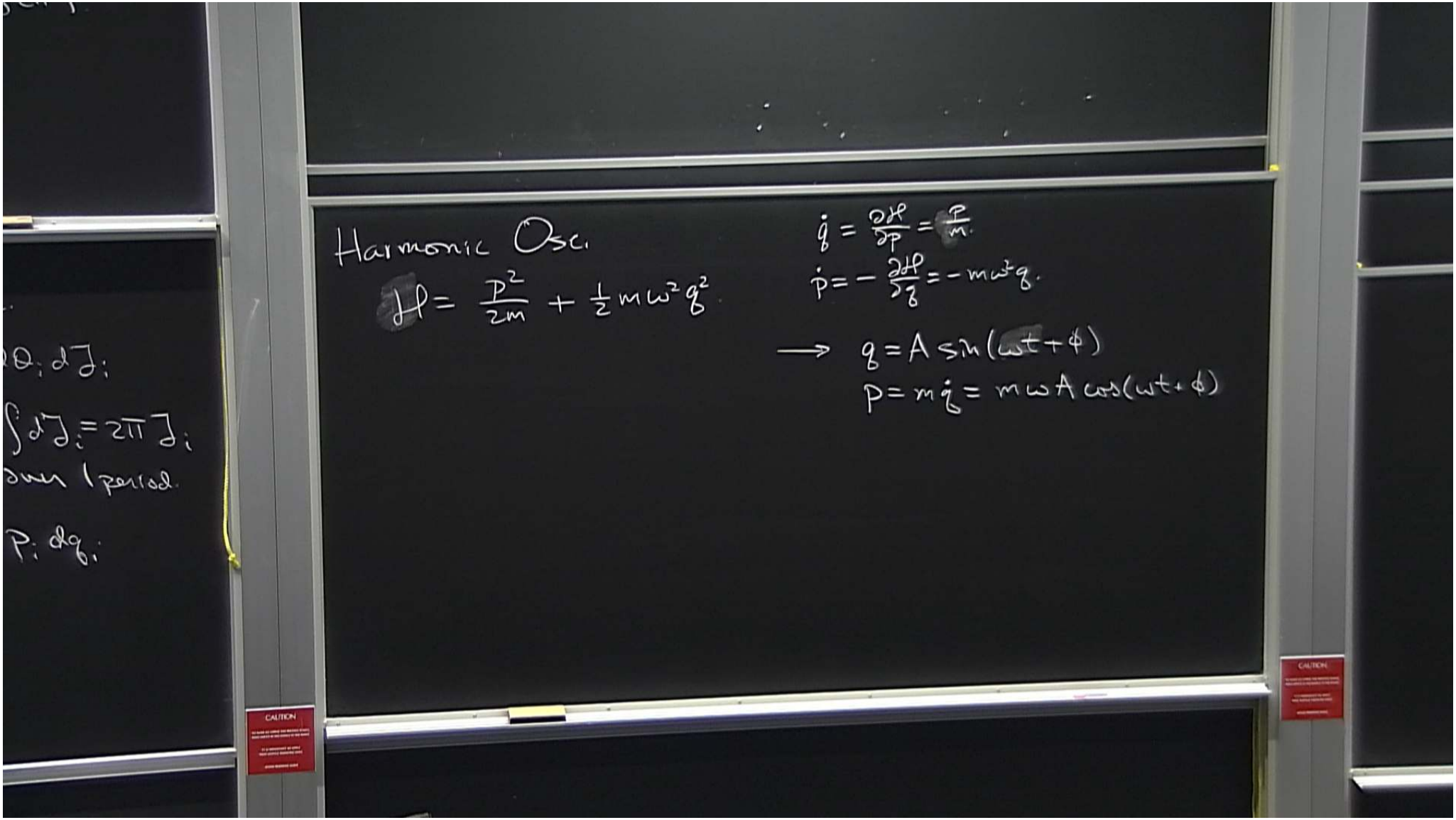
periodic.
2D
Liouville's Thm.: Cons. of phase space volume.
$\prod_i dq_i dp_i = \prod_i dQ_i dP_i = \prod_i d\theta_i dJ_i$
J_i = Area (Volume) of closed orb
$\int_0^{2\pi} d\theta_i dJ_i = 2\pi \int dJ_i = 2\pi J_i$
$\overline{J}_i =$


periodic.
2D
Harmonic
Liouville's Thm.: Cons. of phase space volume.
$\prod_i dq_i\,dp_i = \prod_i dQ_i\,dP_i = \prod_i d\theta_i\,dJ_i$
J_i = Area (Volume) of
$= \oint d\theta_i\,dJ_i = 2\pi \int dJ_i = 2\pi J_i$
over 1 period.
$\int dp_i\,dq_i = \frac{1}{2\pi} \oint p_i\,dq_i$


Harmonic Osc.
$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$.
$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$.
$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q$.
$dQ_i dJ_i$
$\int dJ_i = 2\pi J_i$
own period.
$P_i dq_i$
CAUTION
CAUTION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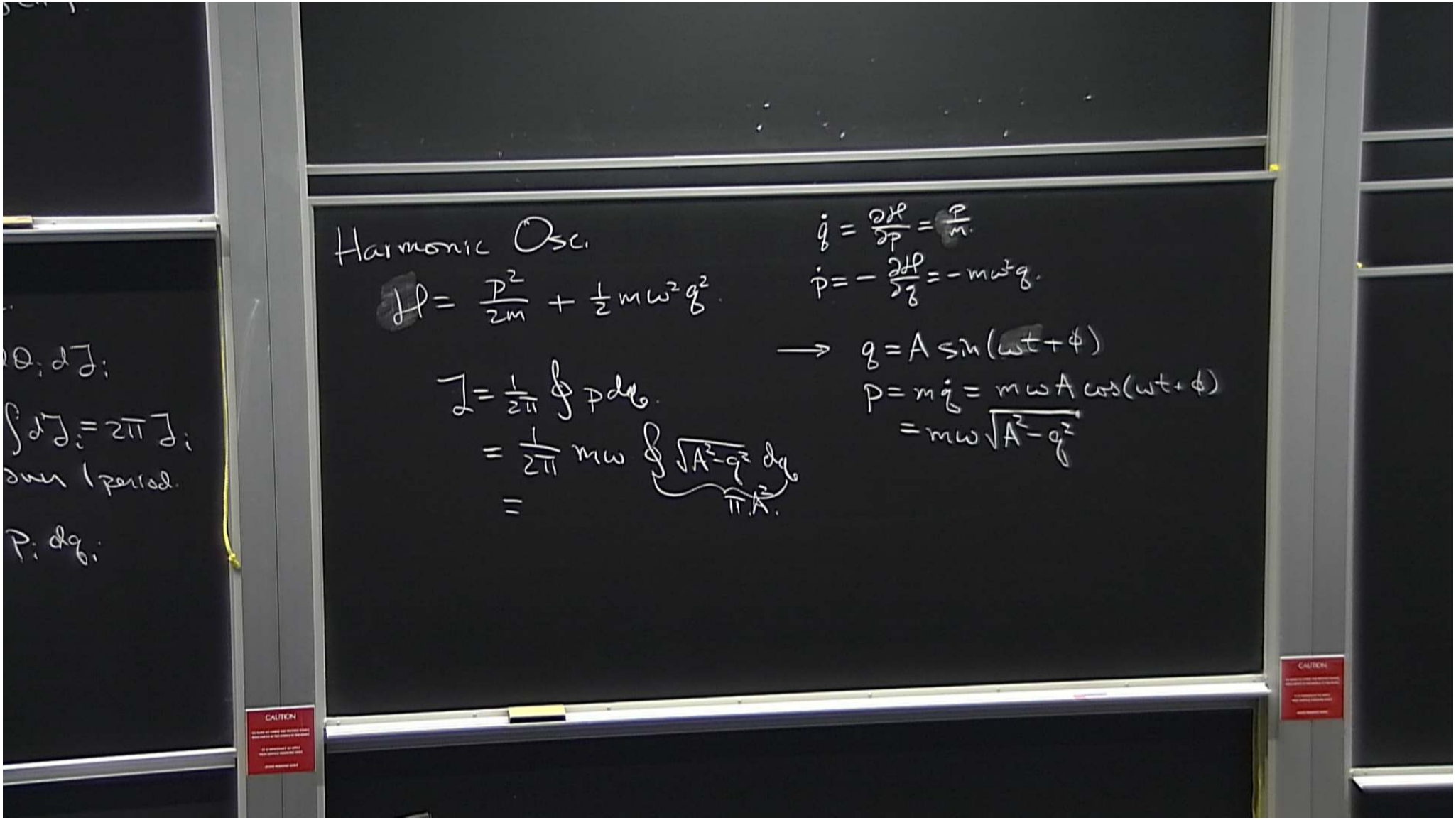
$\longrightarrow q = A\sin(\omega t + \phi)$
$p = m\dot{q} = m\omega A\cos(\omega t + \phi)$
$\oint dJ_i = 2\pi J_i$
$P_i\, dq_i$
CAUTION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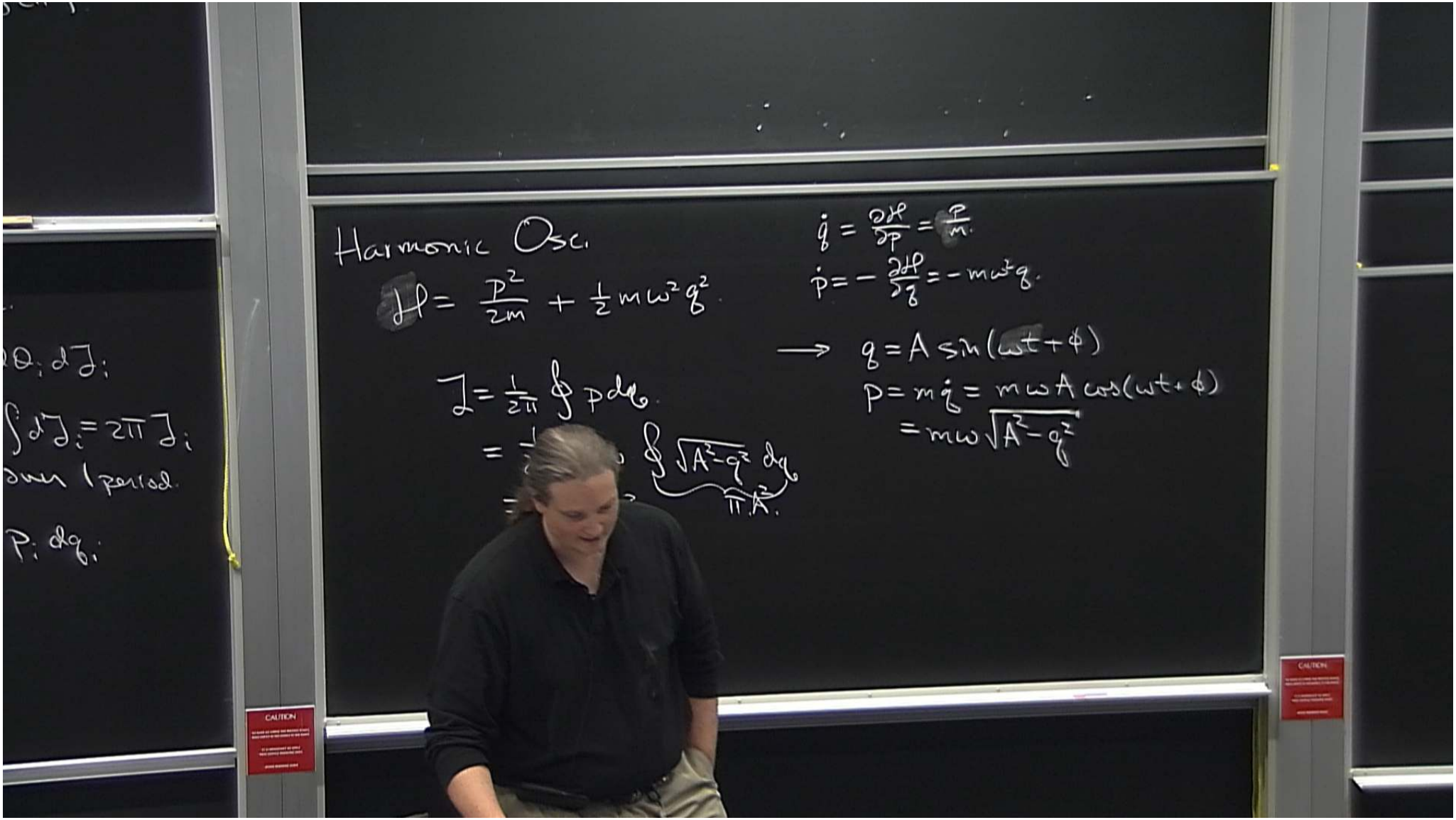
$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$
$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q$
$\longrightarrow q = A\sin(\omega t + \phi)$
$p = m\dot{q} = m\omega A\cos(\omega t + \phi)$
$= m\omega\sqrt{A^2 - q^2}$
$J = \frac{1}{2\pi}\oint p\,dq$
$= \frac{1}{2\pi} m\omega \oint \sqrt{A^2 - q^2}\,dq$
πA^2
$= $
$\oint dJ_i = 2\pi J_i$
$P_i\,dq_i$
CAUTION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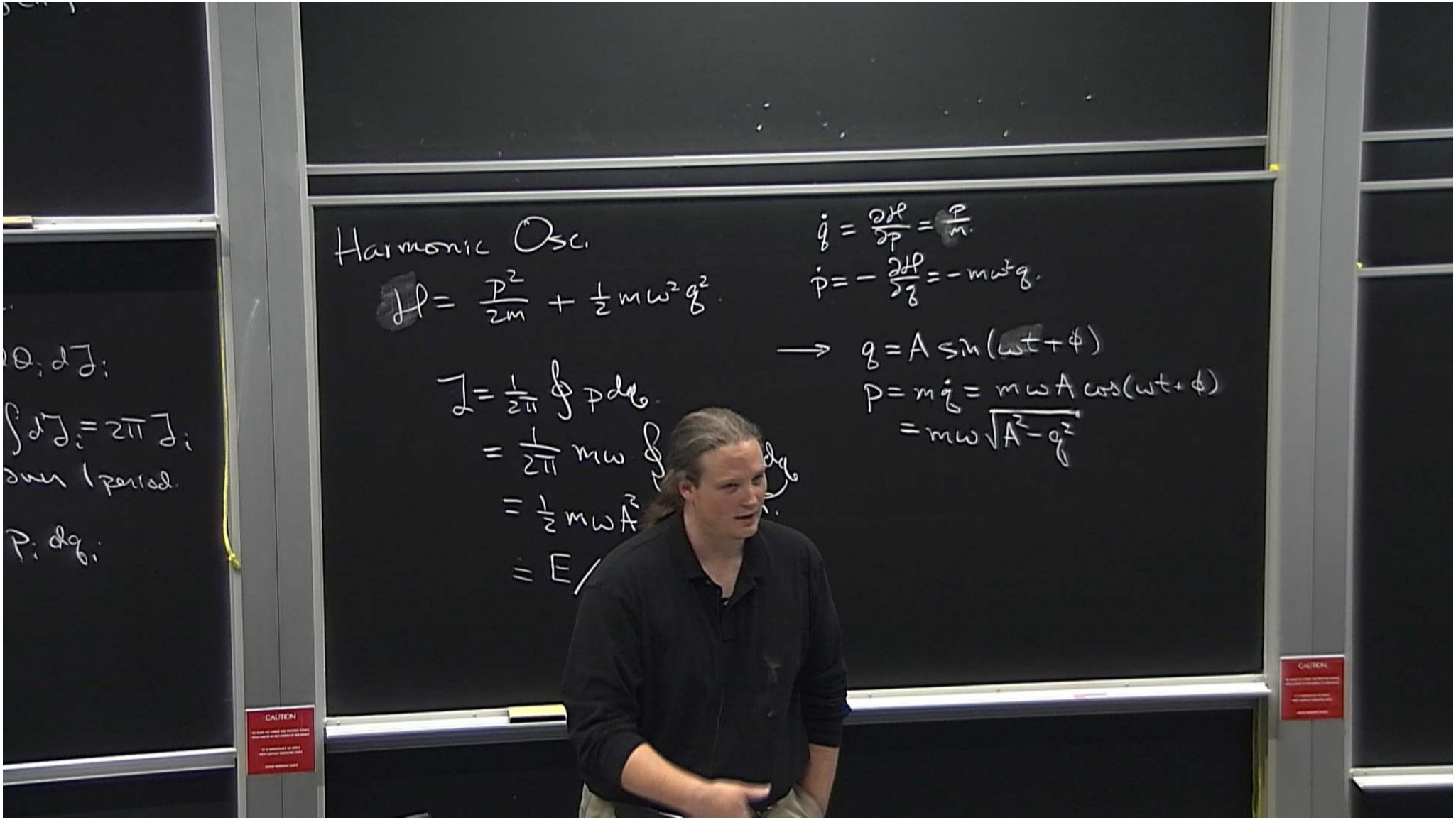
πA^2
CAUTION


Harmonic Osc.
$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$
$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$
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$\longrightarrow q = A\sin(\omega t + \phi)$
$p = m\dot{q} = m\omega A\cos(\omega t + \phi)$
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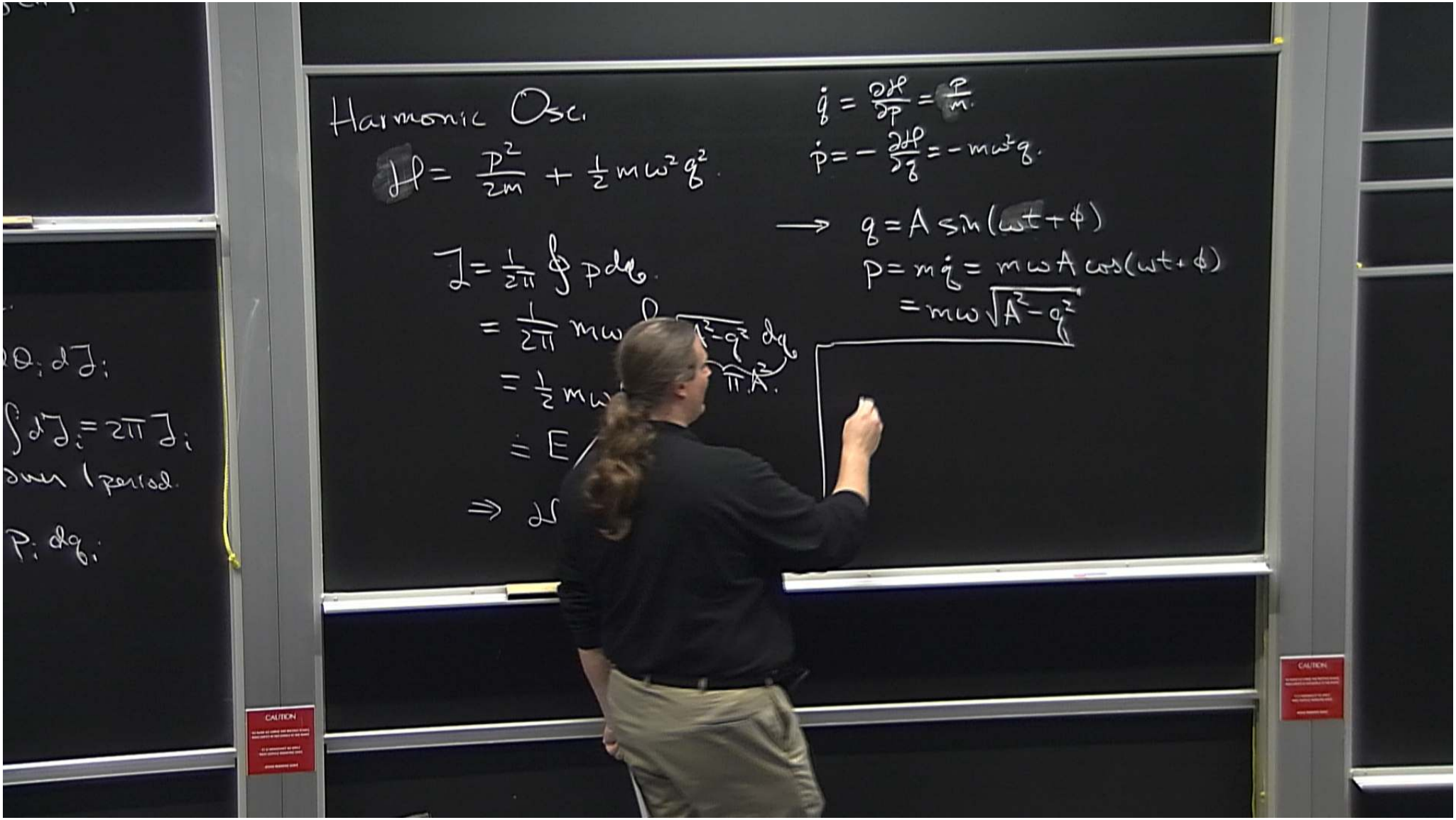
$J = \frac{1}{2\pi}\oint p\,dq$
$= \frac{1}{2\pi} m\omega \oint$
$= \frac{1}{2} m\omega A^2$
$= E/$
CAUTION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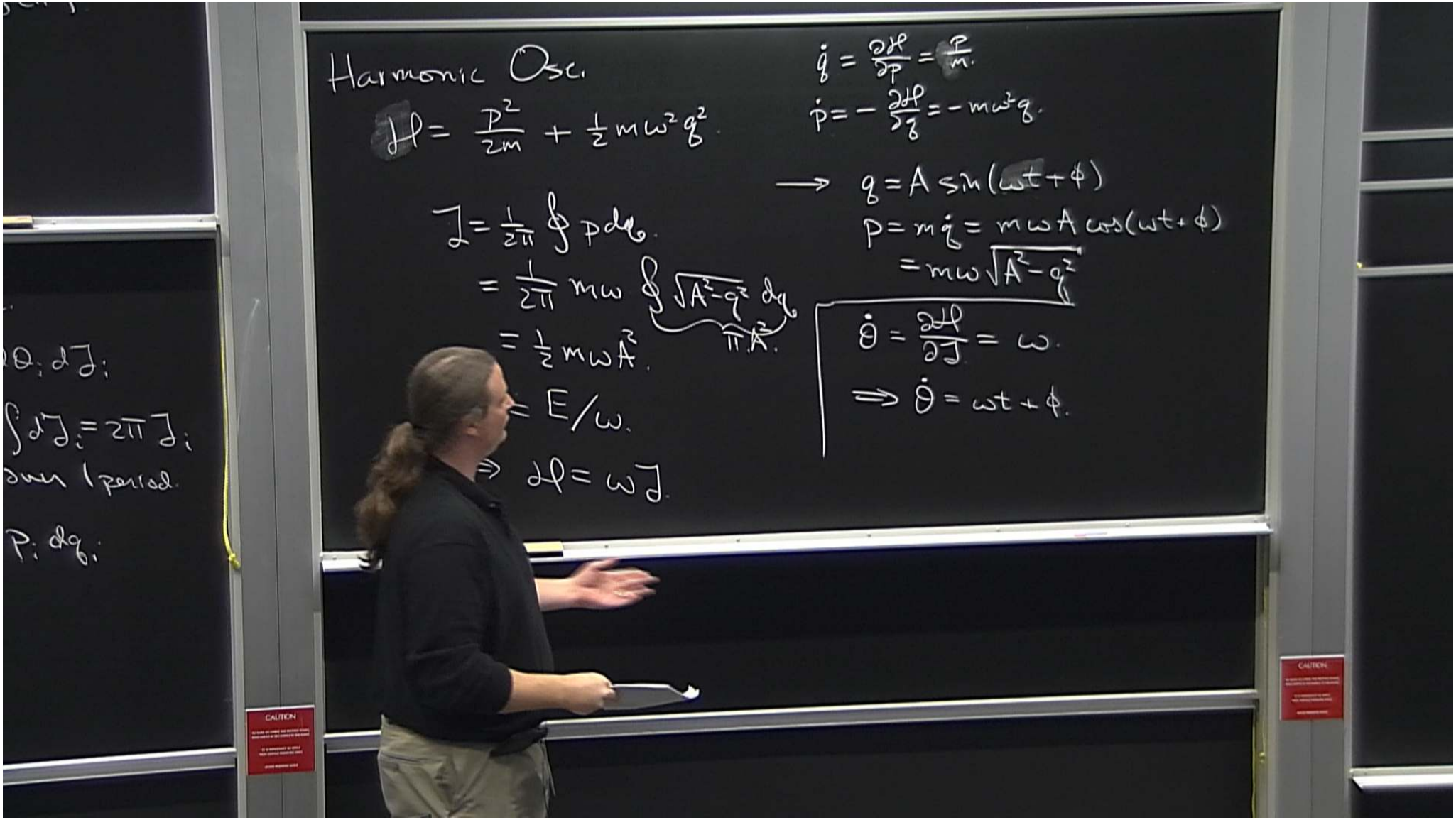
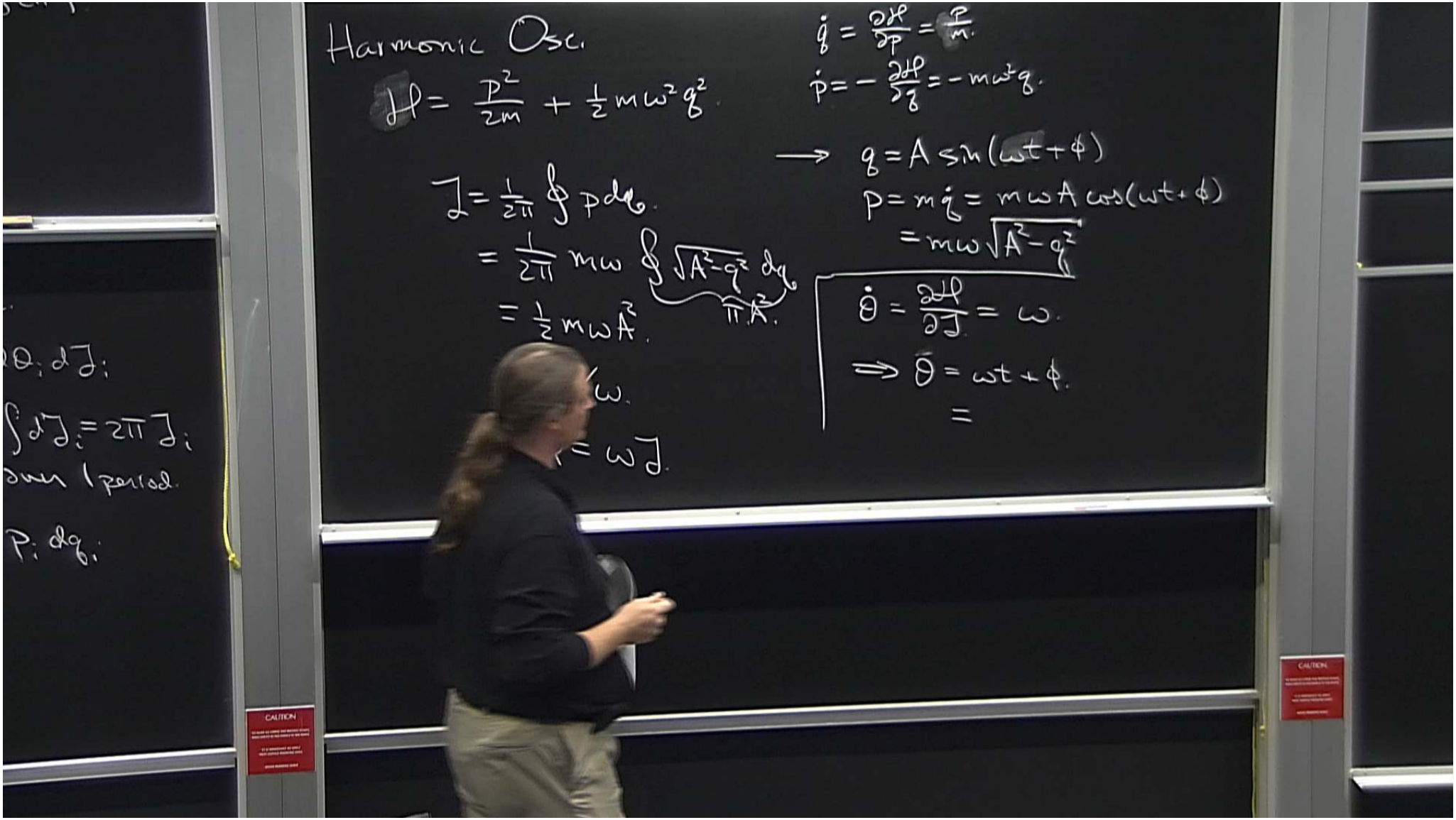
$= m\omega\sqrt{A^2 - q^2}$
$J = \frac{1}{2\pi}\oint p\,dq$
$= \frac{1}{2\pi} m\omega$
$= \frac{1}{2} m\omega$
$= E$
$\Rightarrow \Delta$
πA^2
$dQ_i\, dJ_i$
$\int dJ_i = 2\pi J_i$
own / period.
$P_i\, dq_i$


Harmonic Osc.
$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$
$J = \frac{1}{2\pi}\oint p\,dq$
$= \frac{1}{2\pi} m\omega \oint \sqrt{A^2 - q^2}\,dq$
$= \frac{1}{2} m\omega A^2$
$= E/\omega$
$\Rightarrow H = \omega J$
$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$
$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q$
$\longrightarrow q = A\sin(\omega t + \phi)$
$p = m\dot{q} = m\omega A\cos(\omega t + \phi)$
$= m\omega\sqrt{A^2 - q^2}$
$\dot{\theta} = \frac{\partial H}{\partial J} = \omega$
$\Rightarrow \theta = \omega t + \phi$
CAUTION
CAUTION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$J = \frac{1}{2\pi}\oint p\,dq$
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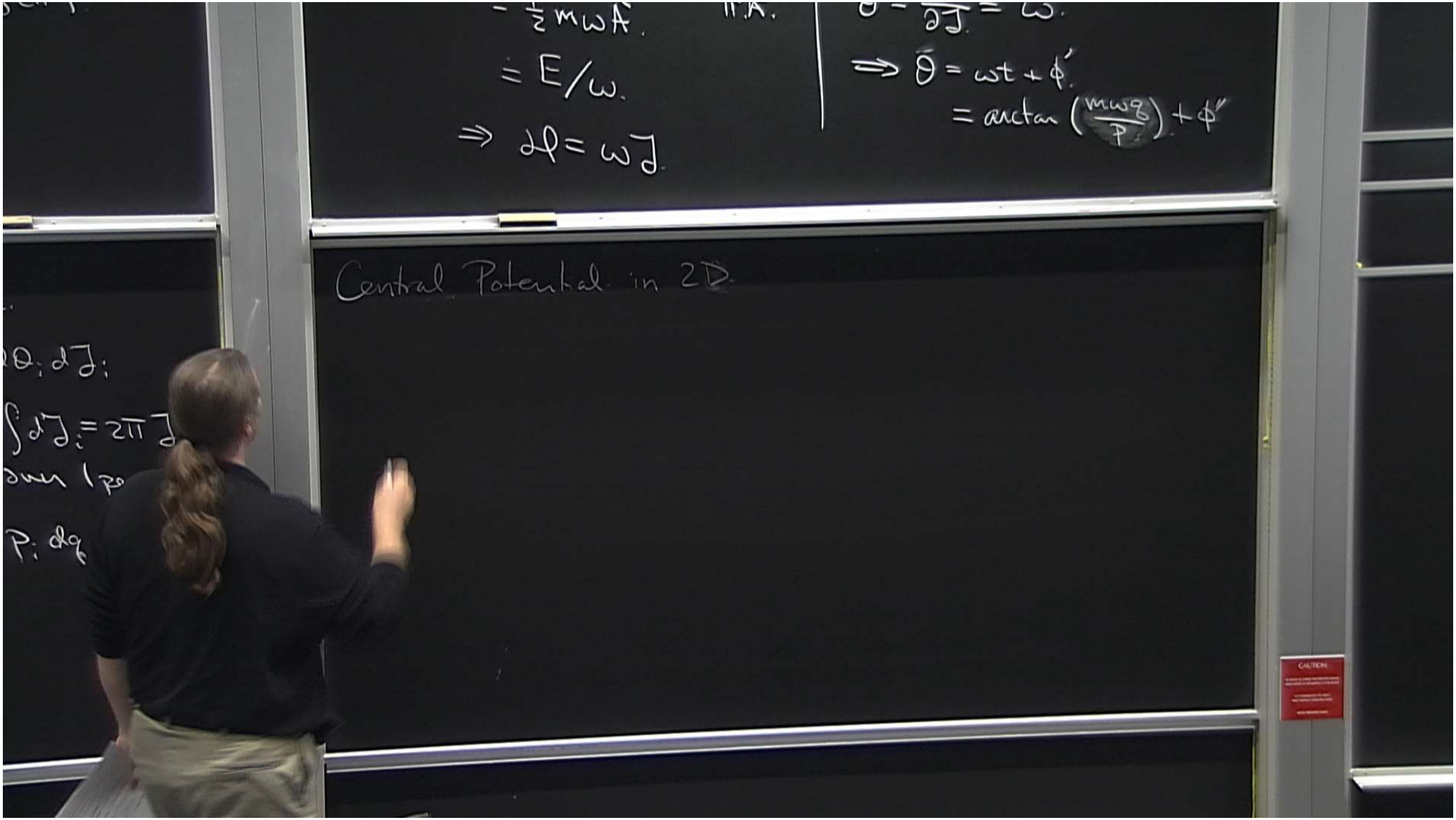
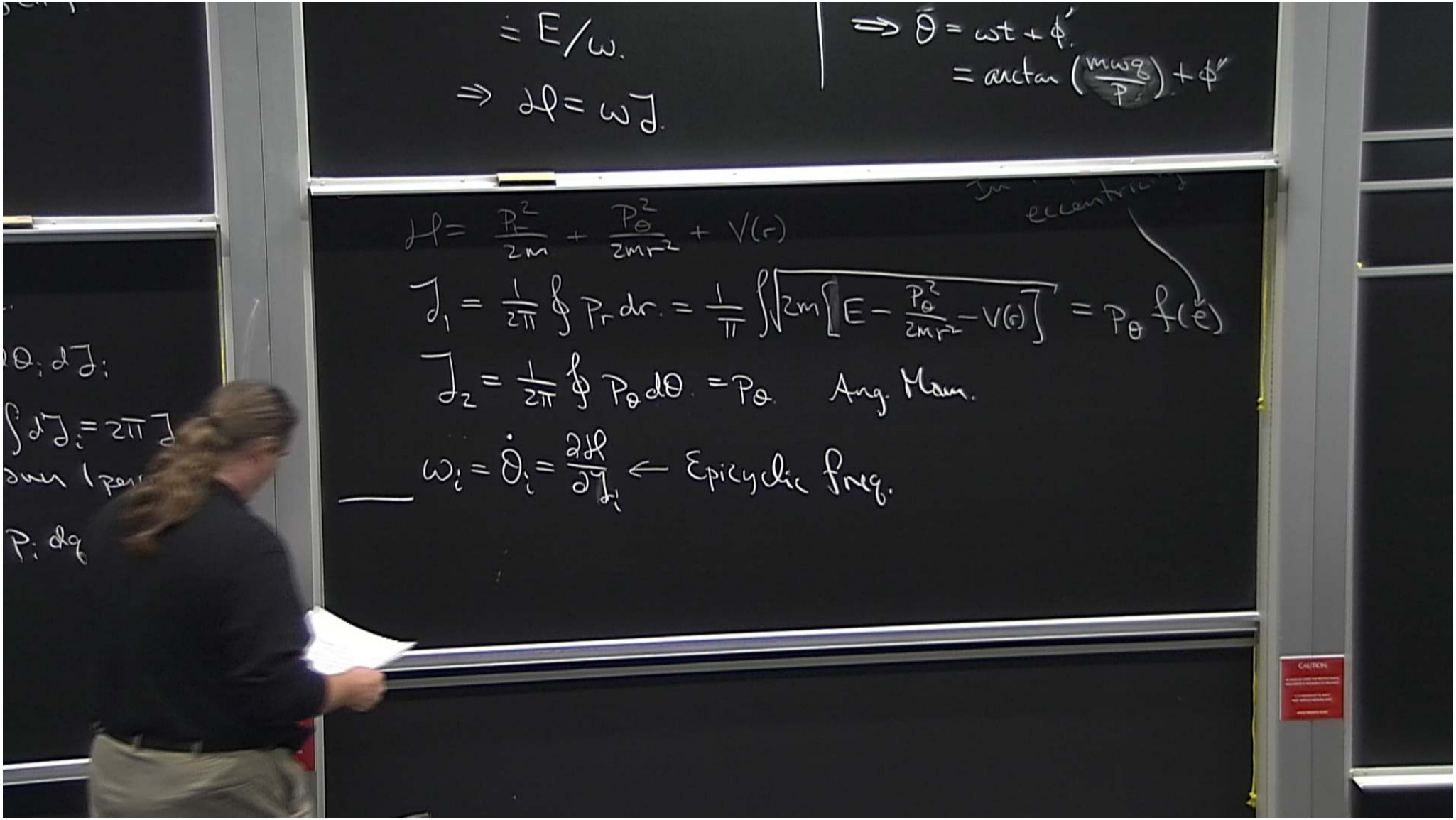
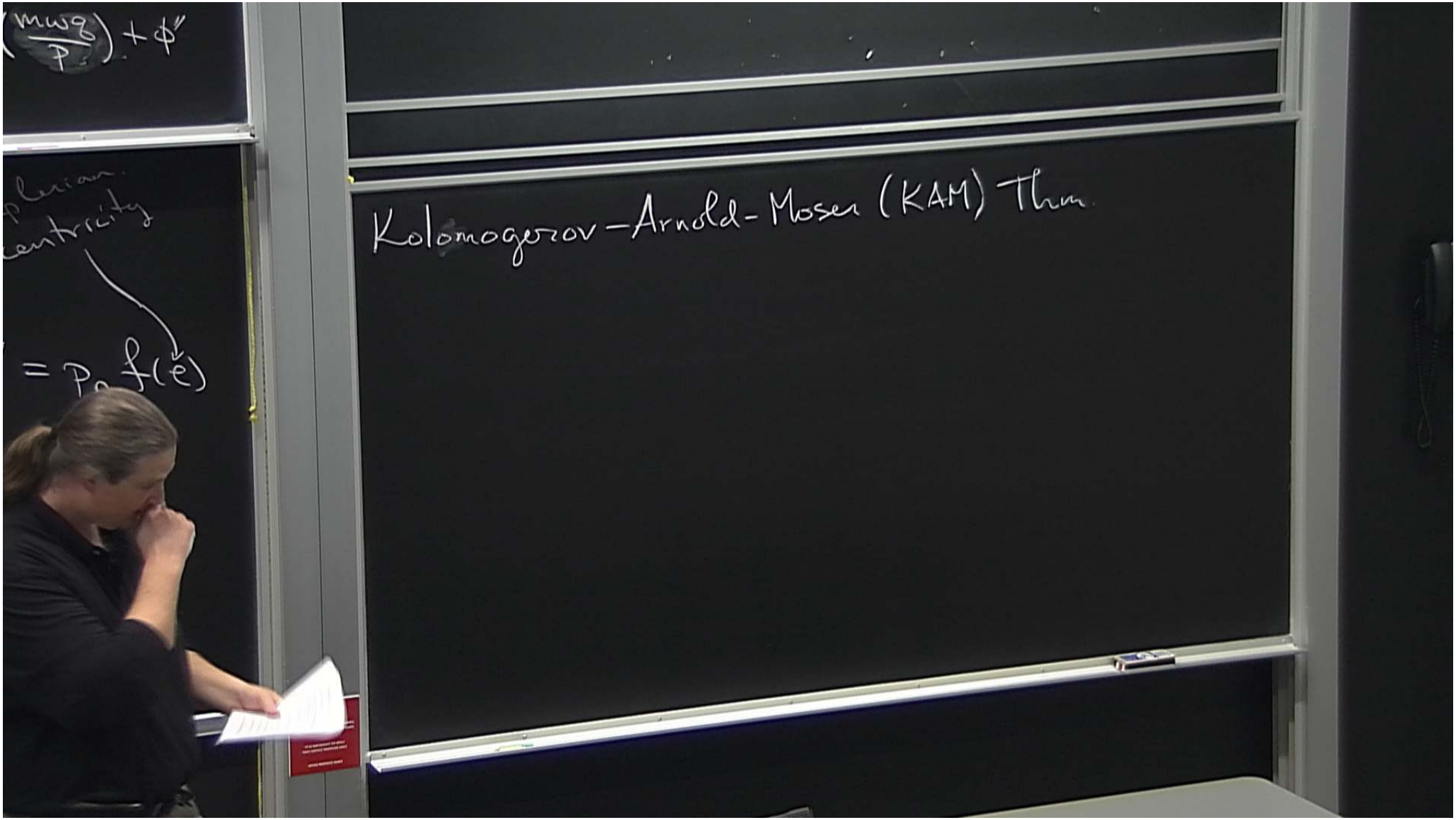
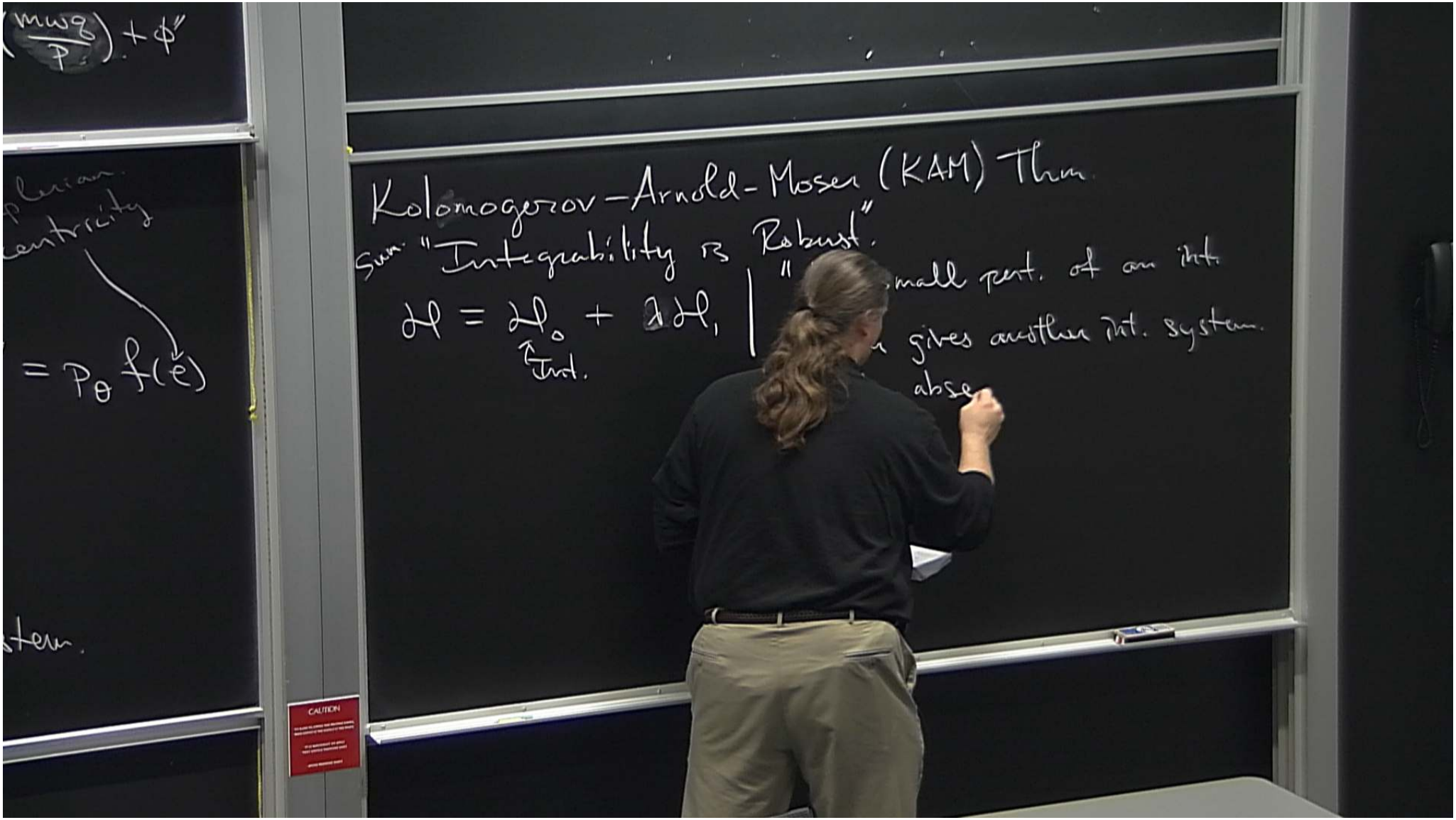
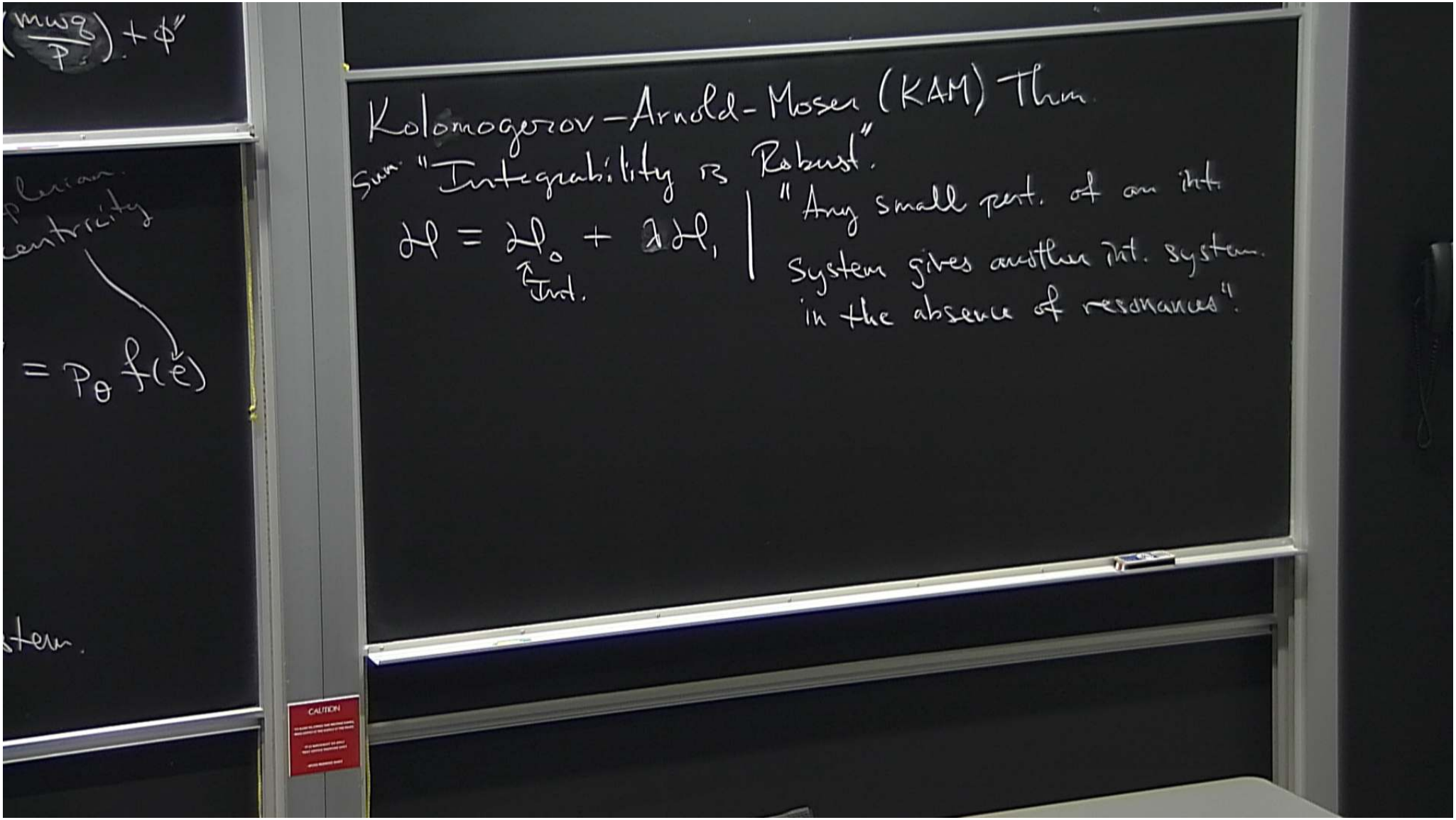
πA^2
$= \frac{1}{2}m\omega A^2$
$\dot{\theta} = \frac{\partial H}{\partial J} = \omega$
$\Rightarrow \theta = \omega t + \phi$
$=$
CAUTION
CAUTION

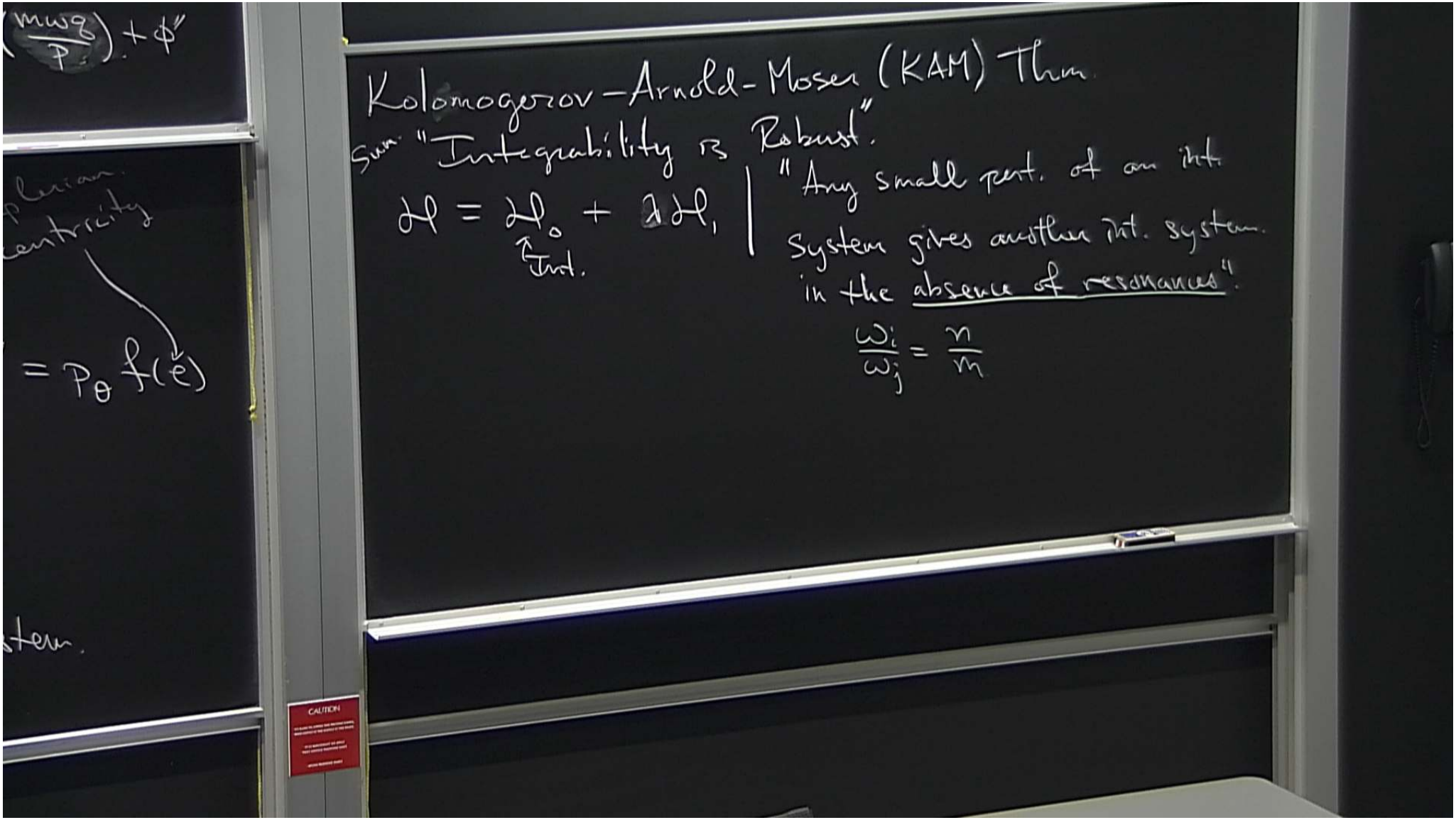

$= E/\omega$.
$\Rightarrow d\phi = \omega\, dJ$.
$\Rightarrow \theta = \omega t + \phi'$.
$= \arctan\left(\frac{m\omega q}{p}\right) + \phi''$
Central Potential in 2D.

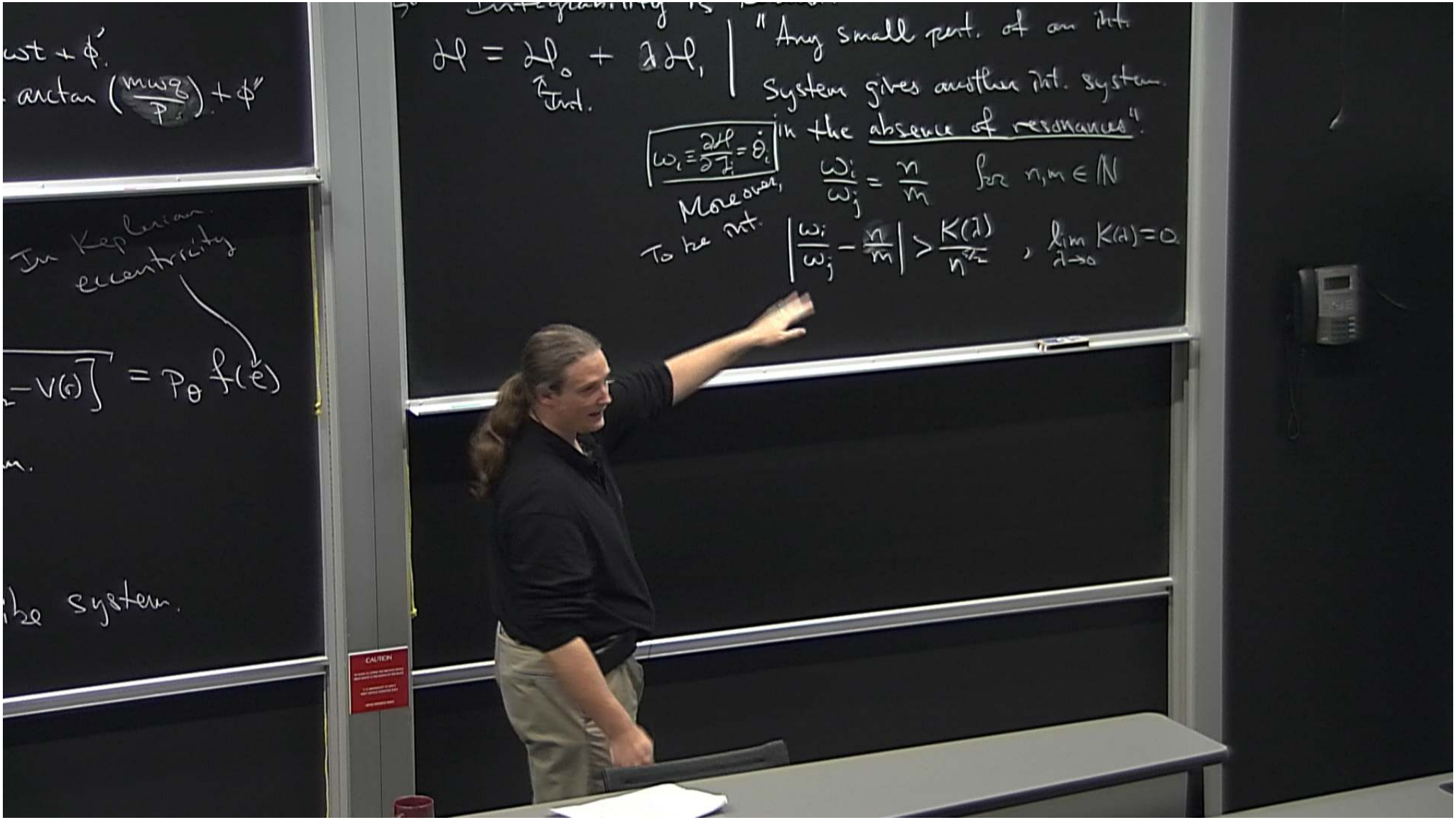

$= E/\omega$.
$\Rightarrow \mathcal{H} = \omega J$.
$\Rightarrow \theta = \omega t + \phi'$
$= \arctan\left(\frac{m\omega q}{p}\right) + \phi''$
$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$
eccentricity
$J_1 = \frac{1}{2\pi}\oint p_r\, dr = \frac{1}{\pi}\int\sqrt{2m\left[E - \frac{p_\theta^2}{2mr^2} - V(r)\right]} = p_\theta f(e)$
$J_2 = \frac{1}{2\pi}\oint p_\theta\, d\theta = p_\theta$ Ang. Mom.
$\omega_i = \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial J_i}$ ← Epicyclic freq.
$\theta_i\, dJ_i$
$\int dJ_i = 2\pi J$
$p_i\, dq$

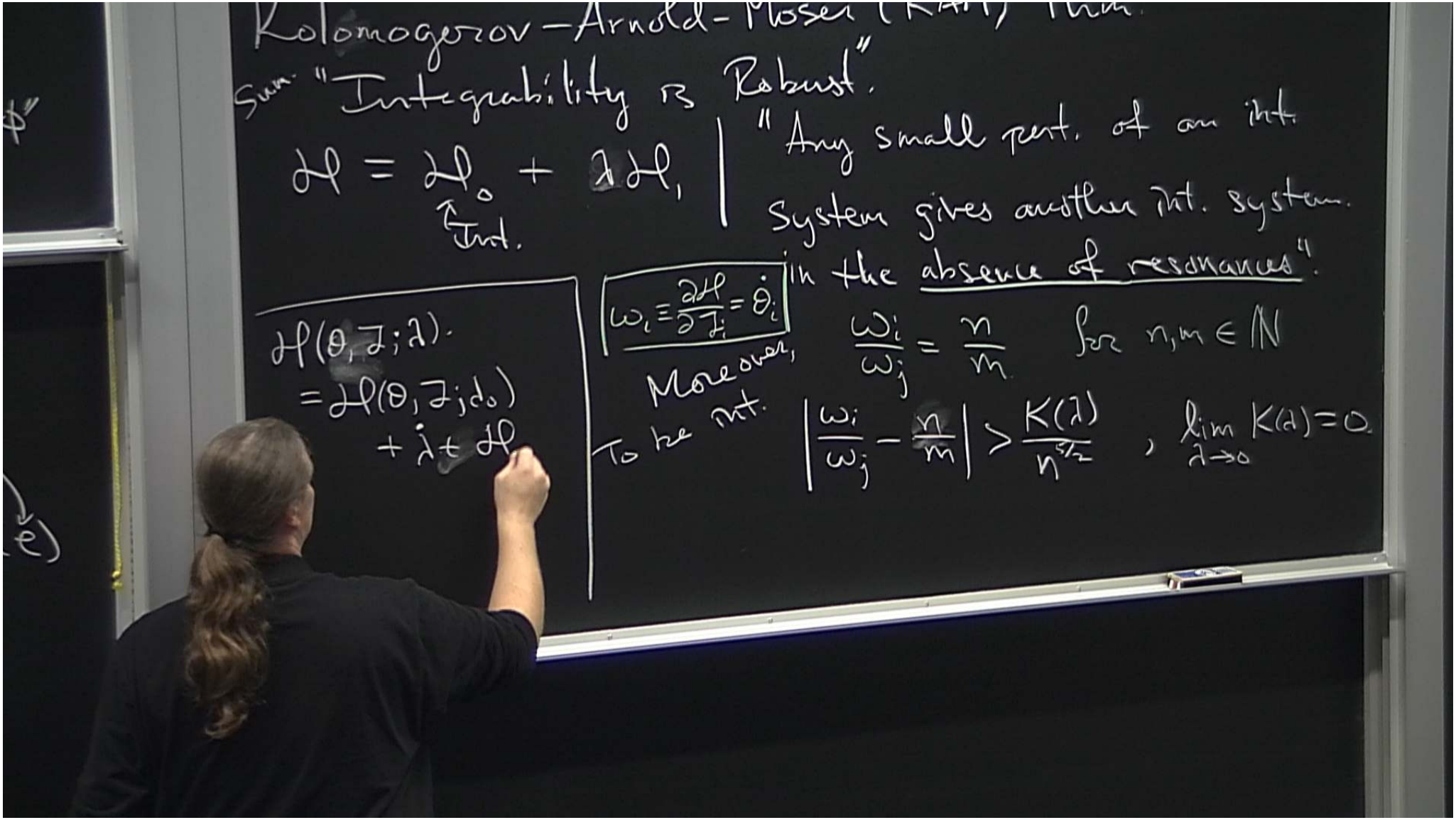

Kolomogerov–Arnold–Moser (KAM) Thm

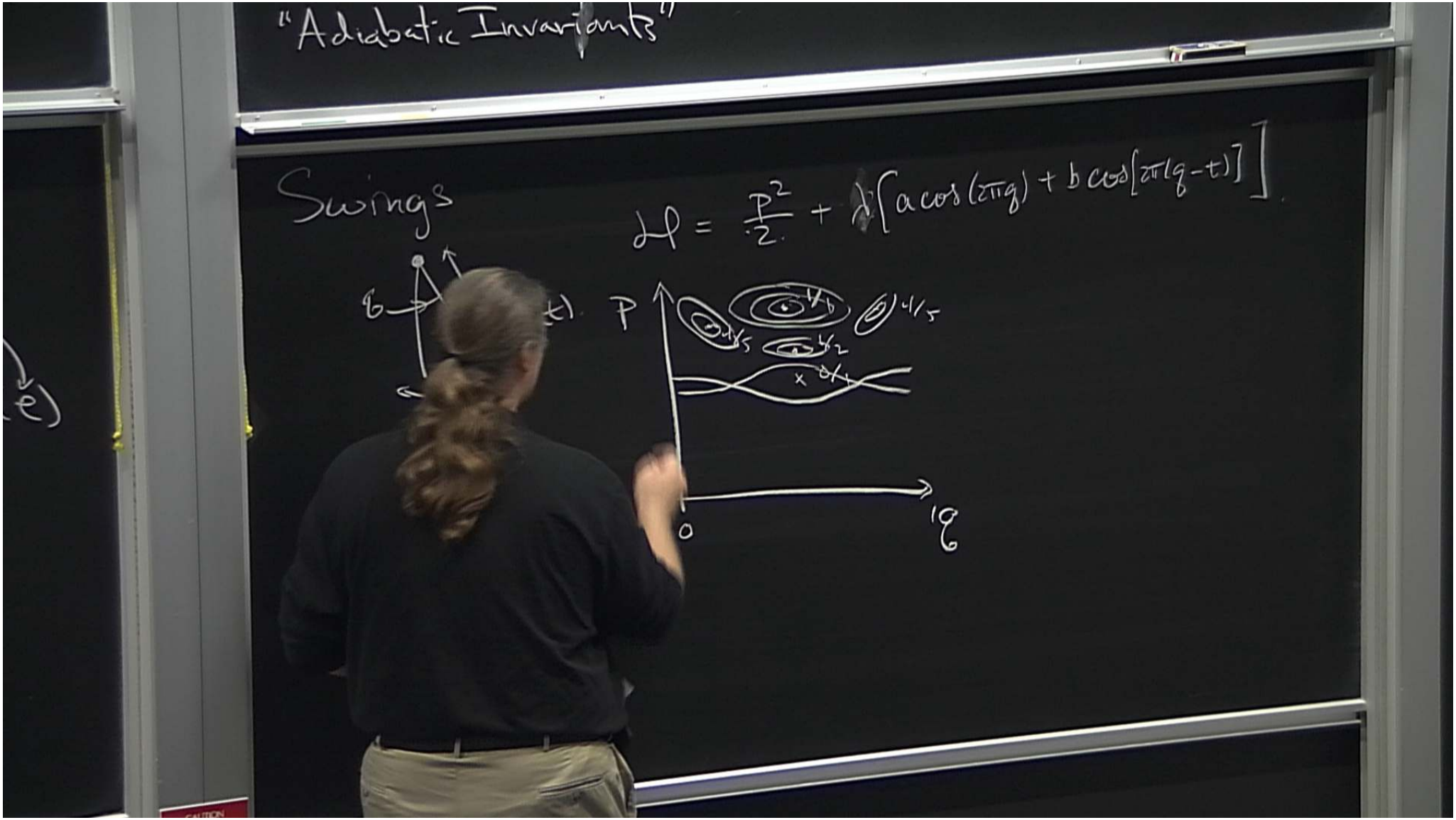

Kolomogorov–Arnold–Moser (KAM) Thm
Sum: "Integrability is Robust".
$H = H_0 + \lambda H_1$
Int.
small pert. of an int.
gives another int. system.
abse
$= P_\theta f(e)$
stem.
CAUTION

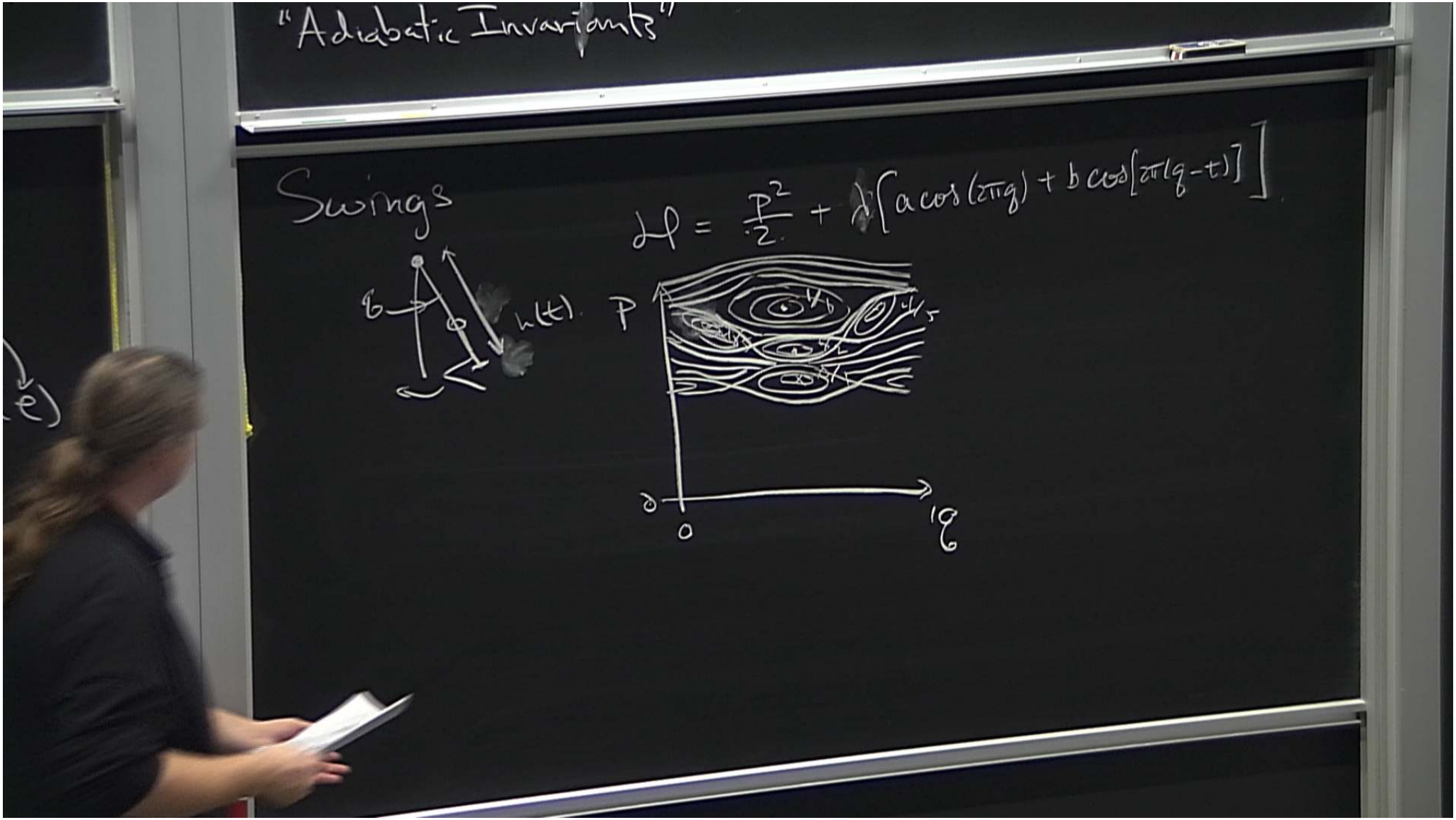

Kolomogerov–Arnold–Moser (KAM) Thm.
Sum: "Integrability is Robust".
$H = H_0 + \lambda H_1$
Int.
"Any small pert. of an int. system gives another int. system in the absence of resonances".
eccentricity
$= P_\theta f(e)$
CAUTION

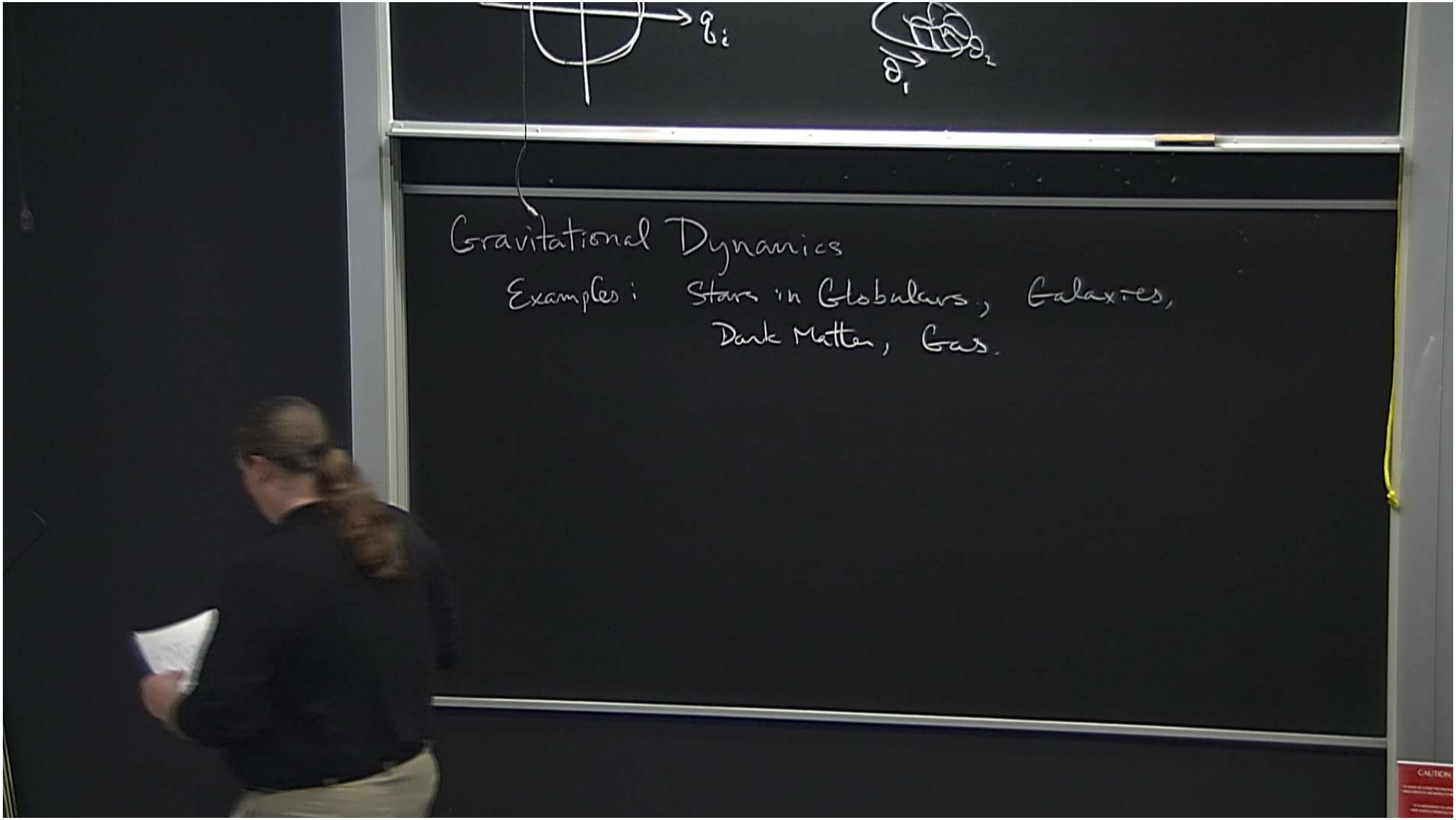

Kolomogorov-Arnold-Moser (KAM) Thm
Sum: "Integrability is Robust".
$H = H_0 + \lambda H_1$
Int.
"Any small pert. of an int. system gives another int. system in the absence of resonances".
$\frac{\omega_i}{\omega_j} = \frac{n}{m}$
$= P_0 f(e)$
CAUTION

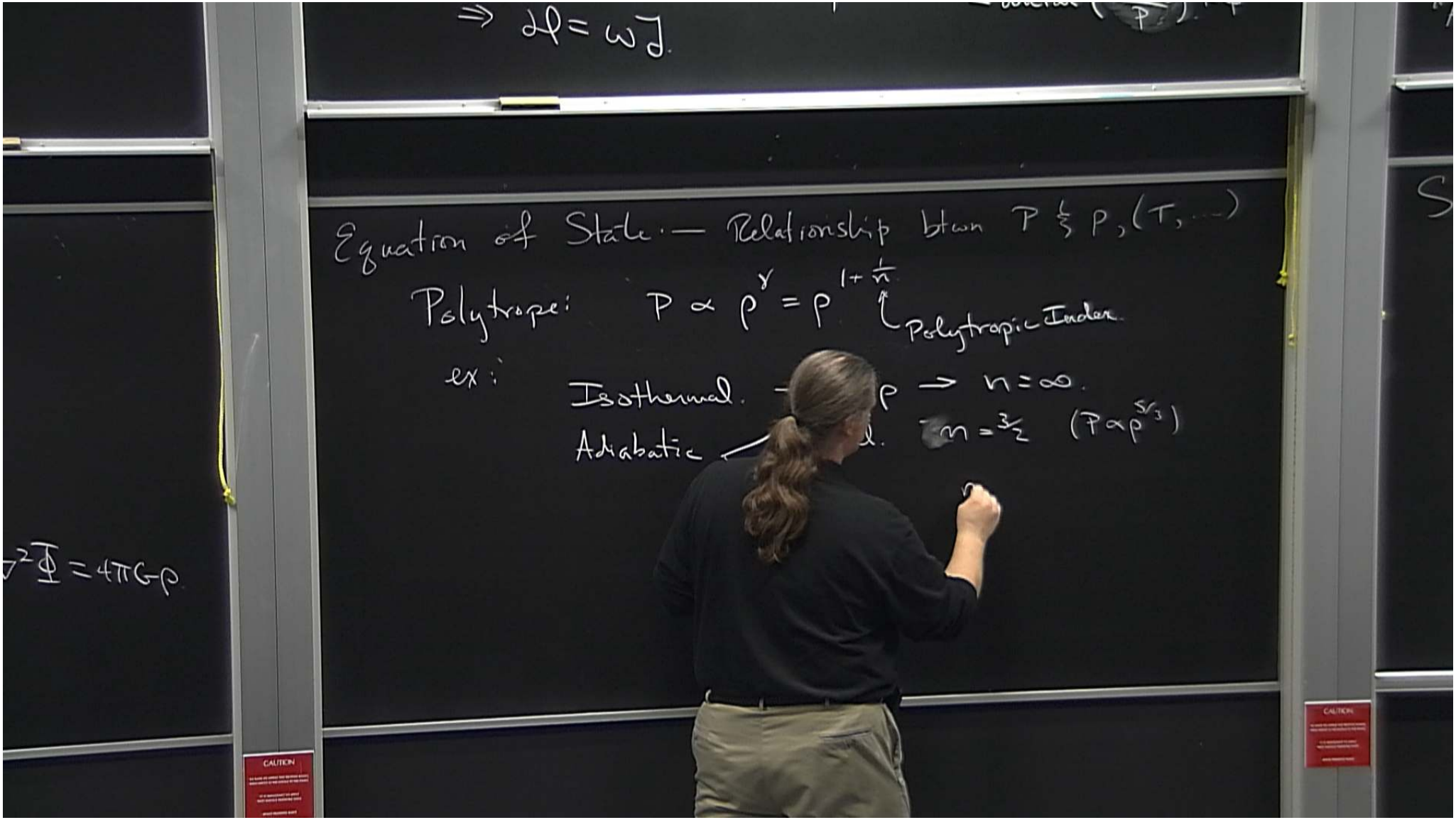

$\omega t + \phi'$.
$\arctan\left(\frac{m\omega_0}{p_\theta}\right) + \phi''$
In Keplerian.
eccentricity
$-V(r)] = p_\theta f(e)$
the system.
$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1$
Int.
"Any small pert. of an int. System gives another int. system. in the absence of resonances".
$\omega_i = \frac{\partial \mathcal{H}}{\partial I_i} = \dot{\theta}_i$
Moreover, $\frac{\omega_i}{\omega_j} = \frac{n}{m}$ for $n, m \in \mathbb{N}$
To be int. $\left|\frac{\omega_i}{\omega_j} - \frac{n}{m}\right| > \frac{K(\lambda)}{n^{5/2}}$, $\lim_{\lambda \to 0} K(\lambda) = 0$
CAUTION

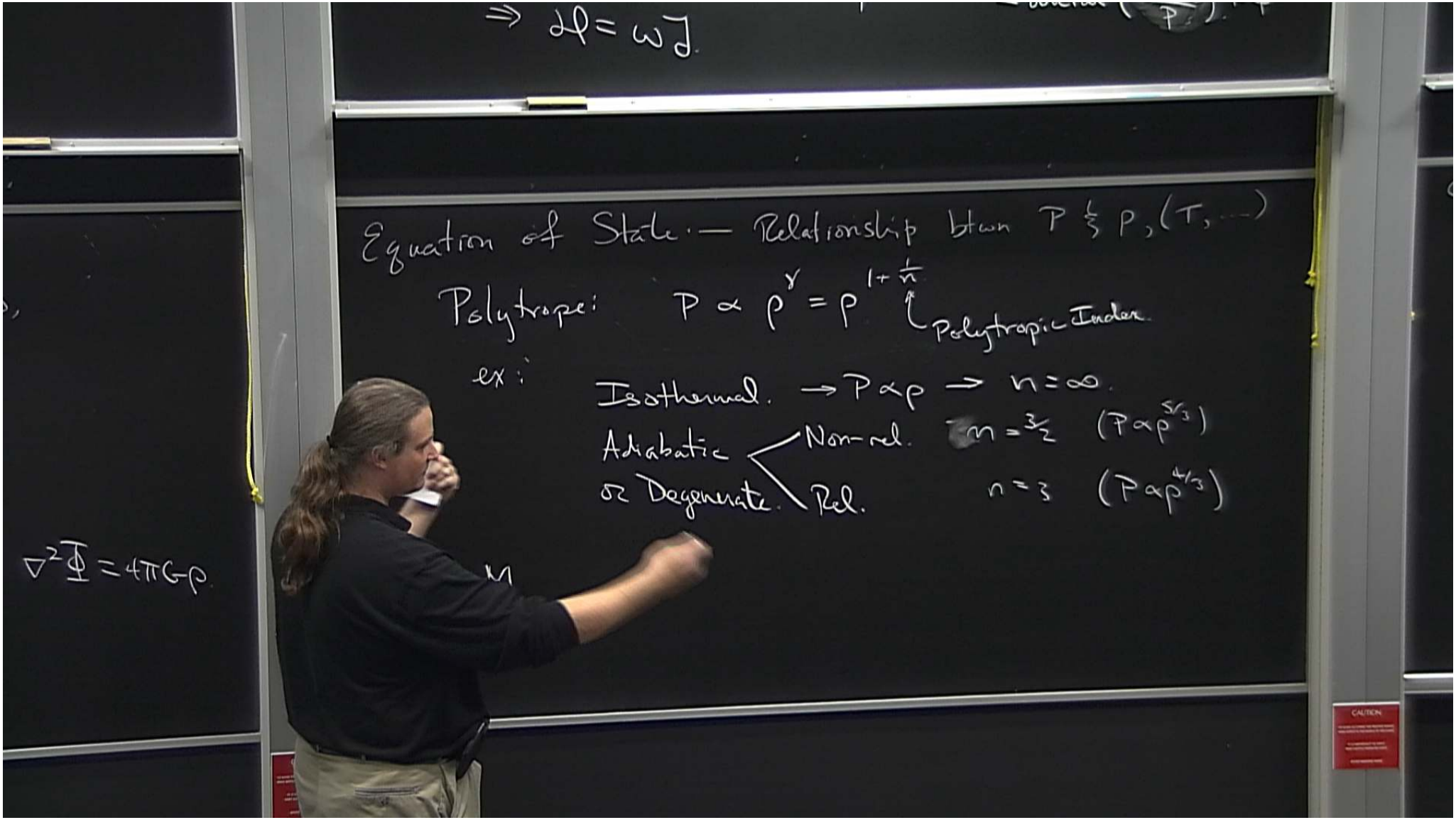

Kolmogorov–Arnold–Moser (KAM) Thm.
Sum: "Integrability is Robust".
$H = H_0 + \lambda H_1$
Int.
"Any small pert. of an int. system gives another int. system in the absence of resonances".
$H(\theta, I; \lambda)$
$= H(\theta, I; d_0)$
$+ \lambda t H_1$
$\omega_i = \frac{\partial H}{\partial I_i} = \dot{\theta}_i$
Moreover,
To be int.
$\frac{\omega_i}{\omega_j} = \frac{n}{m}$ for $n, m \in \mathbb{N}$
$\left| \frac{\omega_i}{\omega_j} - \frac{n}{m} \right| > \frac{K(\lambda)}{n^{5/2}}$, $\lim_{\lambda \to 0} K(\lambda) = 0$


"Adiabatic Invariants"
Swings
$\mathcal{H} = \frac{p^2}{2} + \lambda[a\cos(2\pi q) + b\cos[2\pi(q-t)]]$
p
q
0
1


"Adiabatic Invariants"
Swings
$\mathcal{H} = \frac{p^2}{2} + \lambda\left[a\cos(2\pi q) + b\cos[2\pi(q-t)]\right]$
L(t)
p
0
q


q_i
q_2
q_1
Gravitational Dynamics
Examples: Stars in Globulars, Galaxies,
Dark Matter, Gas.


$\Rightarrow \nabla\Phi = \omega J$.
Equation of State — Relationship btwn P & ρ, $(T, \ldots)$
Polytrope: $P \propto \rho^{\gamma} = \rho^{1+\frac{1}{n}}$
Polytropic Index
ex:
Isothermal. $\rho \rightarrow n = \infty$.
Adiabatic $n = \frac{3}{2}$ $(P \propto \rho^{5/3})$
$\nabla^2\Phi = 4\pi G\rho$
CAUTION
CAUTION


$\Rightarrow \mathcal{H} = \omega \mathcal{J}$.
Equation of State — Relationship btwn P & ρ, $(T, \ldots)$
Polytrope: $P \propto \rho^{\gamma} = \rho^{1+\frac{1}{n}}$
Polytropic Index
ex:
Isothermal. $\rightarrow P \propto \rho \rightarrow n = \infty$.
Adiabatic or Degenerate. < Non-rel. $n = \frac{3}{2}$ $(P \propto \rho^{5/3})$
Rel. $n = 3$ $(P \propto \rho^{4/3})$
$\nabla^2 \Phi = 4\pi G \rho$

$\Rightarrow \mathcal{H} = \omega \mathcal{J}$.

Equation of State — Relationship btwn P & ρ, $(T, \ldots)$

Polytrope: $P \propto \rho^{\gamma} = \rho^{1+\frac{1}{n}}$ ← Polytropic Index

ex:

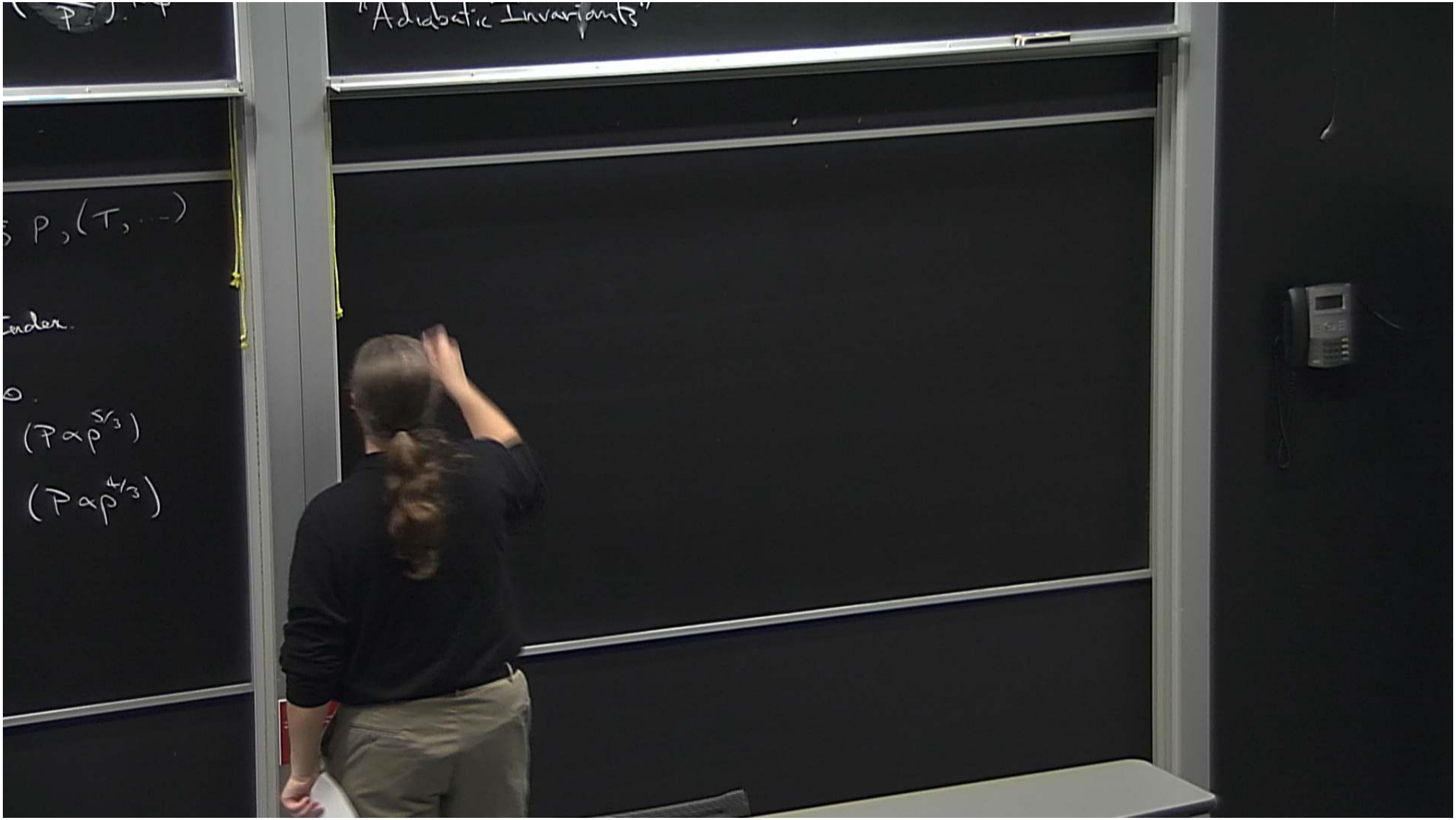
Isothermal. $\rightarrow P \propto \rho \rightarrow n = \infty$.

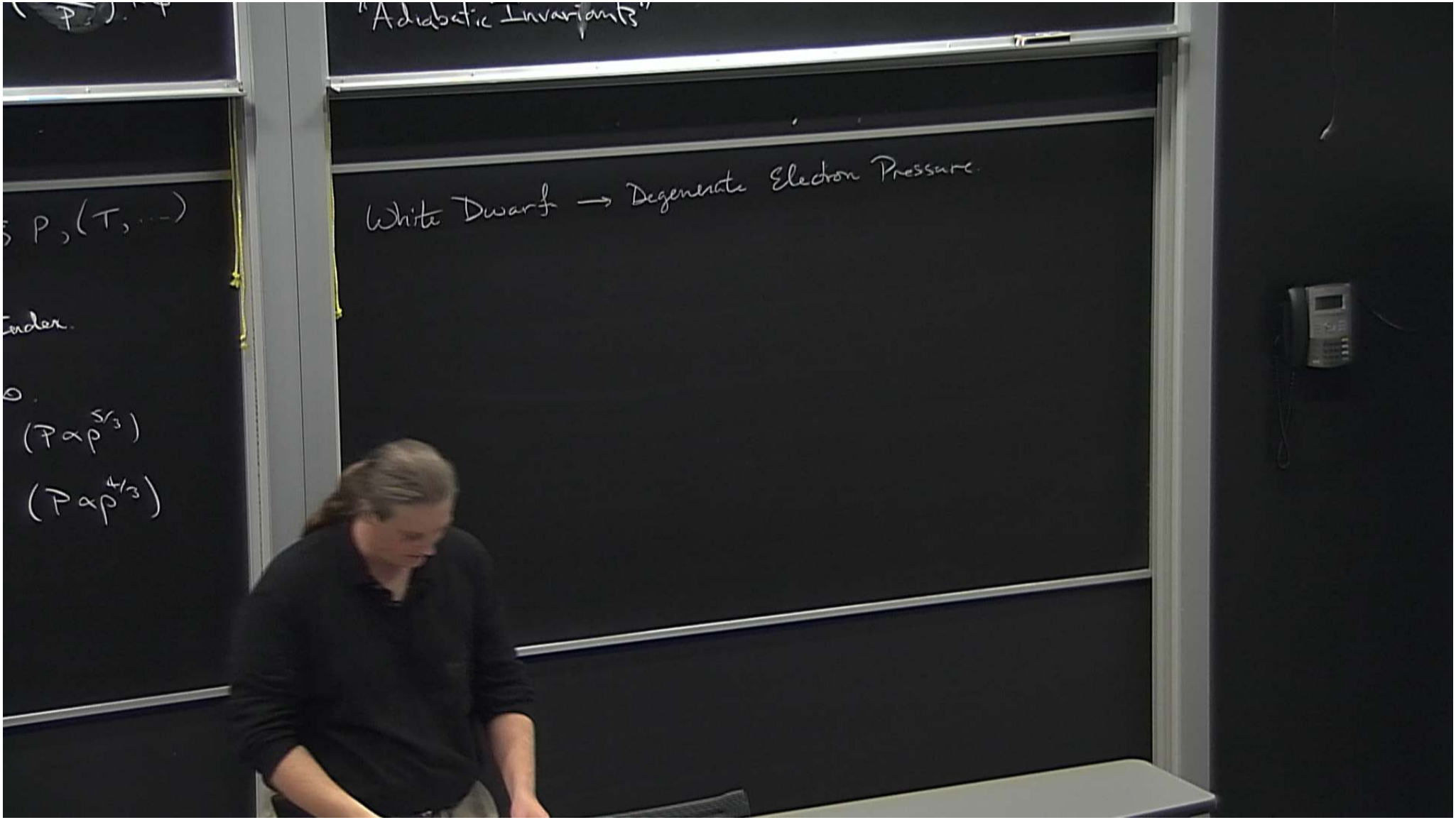
Adiabatic < Non-rel. $n = 3/2$ $(P \propto \rho^{5/3})$

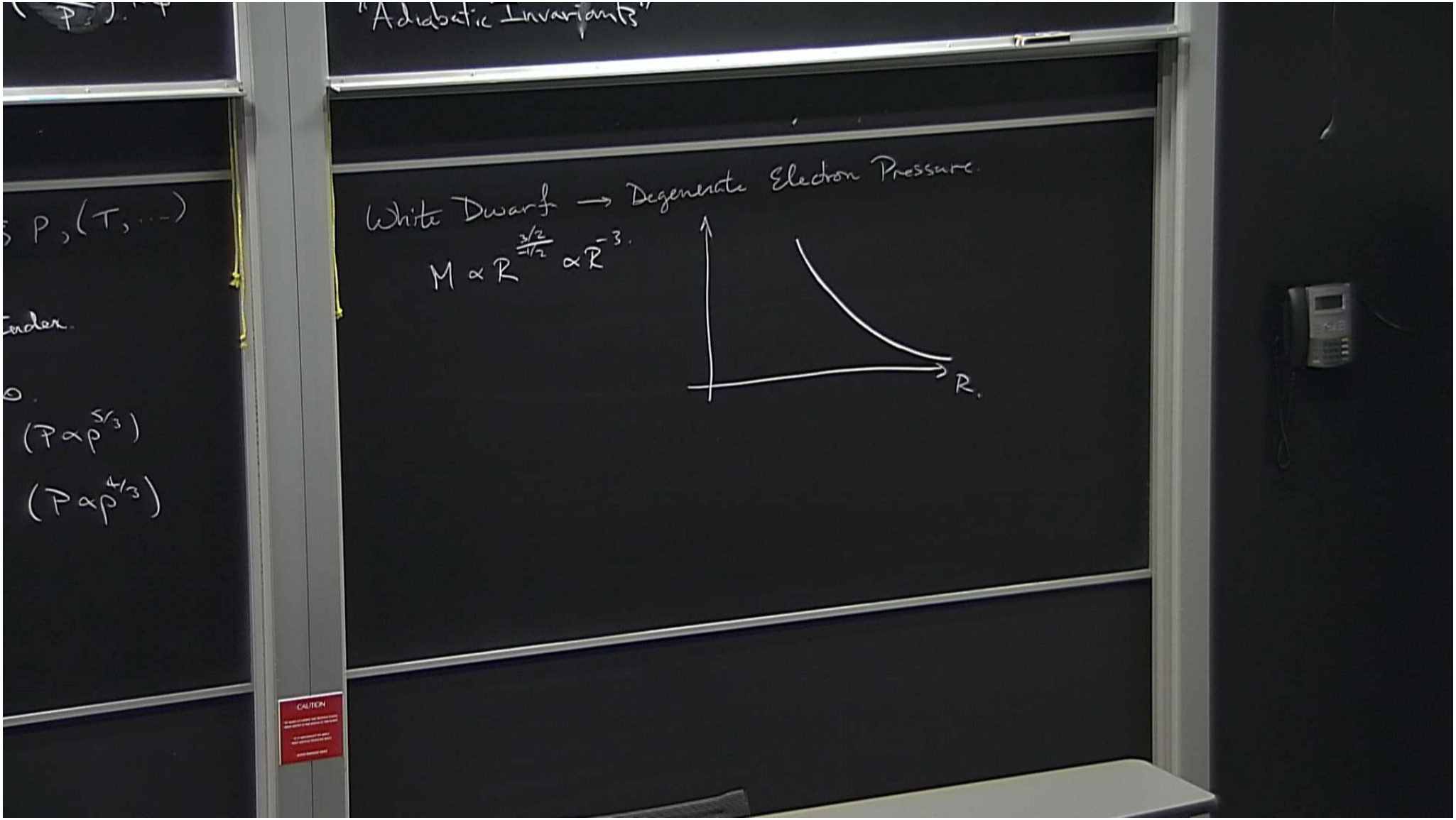
or Degenerate. < Rel. $n = 3$ $(P \propto \rho^{4/3})$

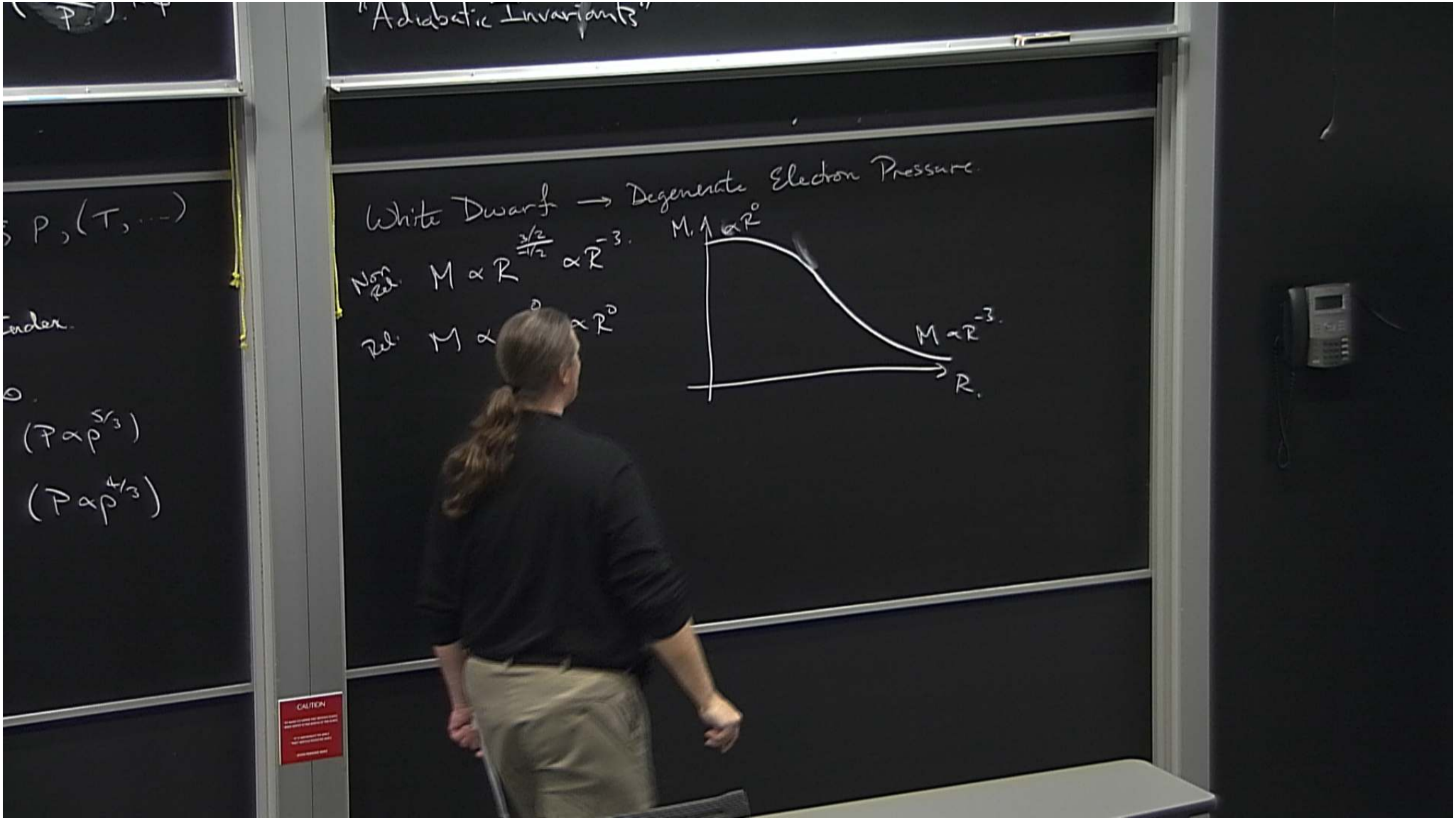
$$M \propto R^{\frac{3-n}{1-n}}$$

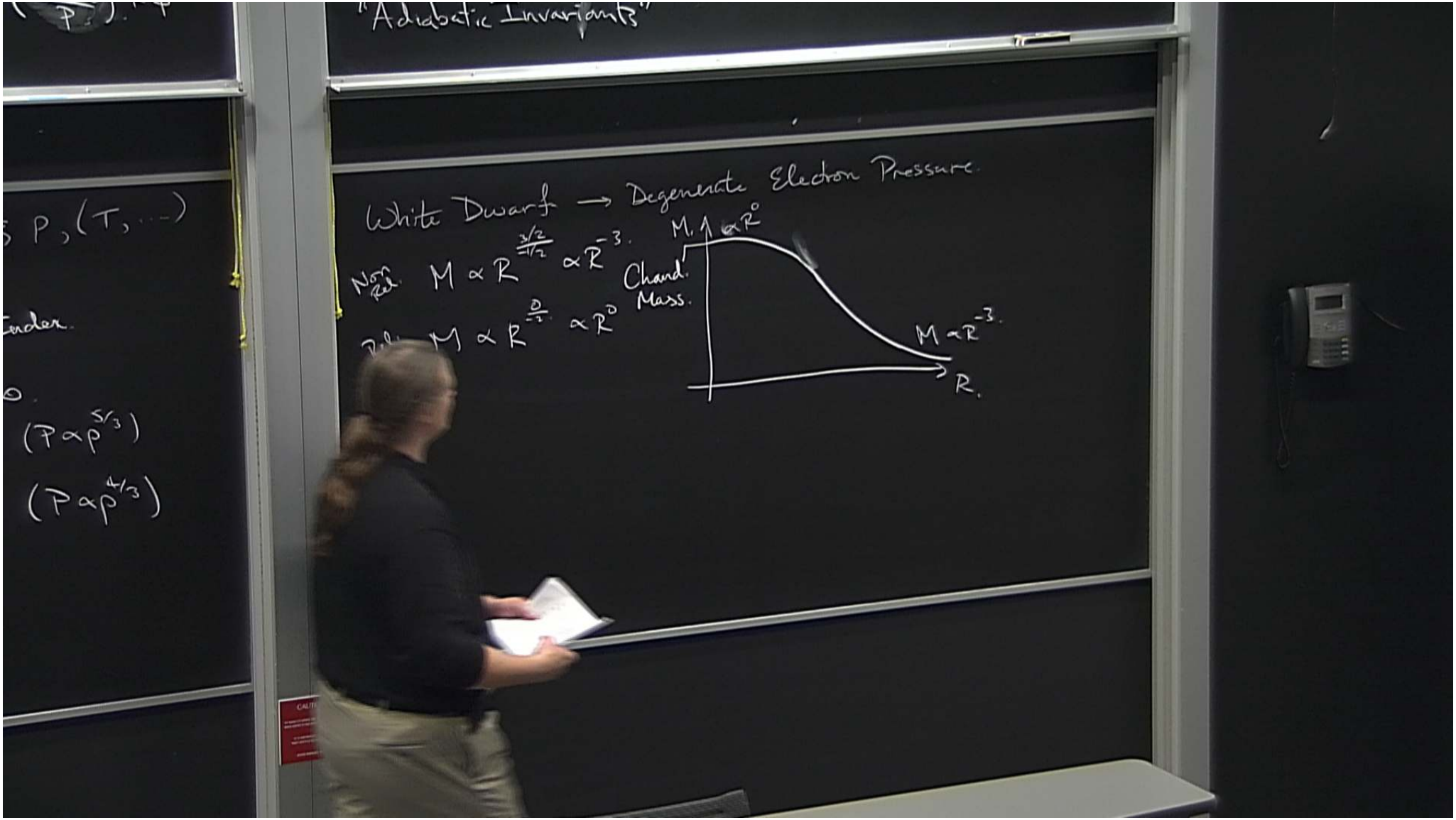
$\nabla^2 \Phi = 4\pi G \rho$

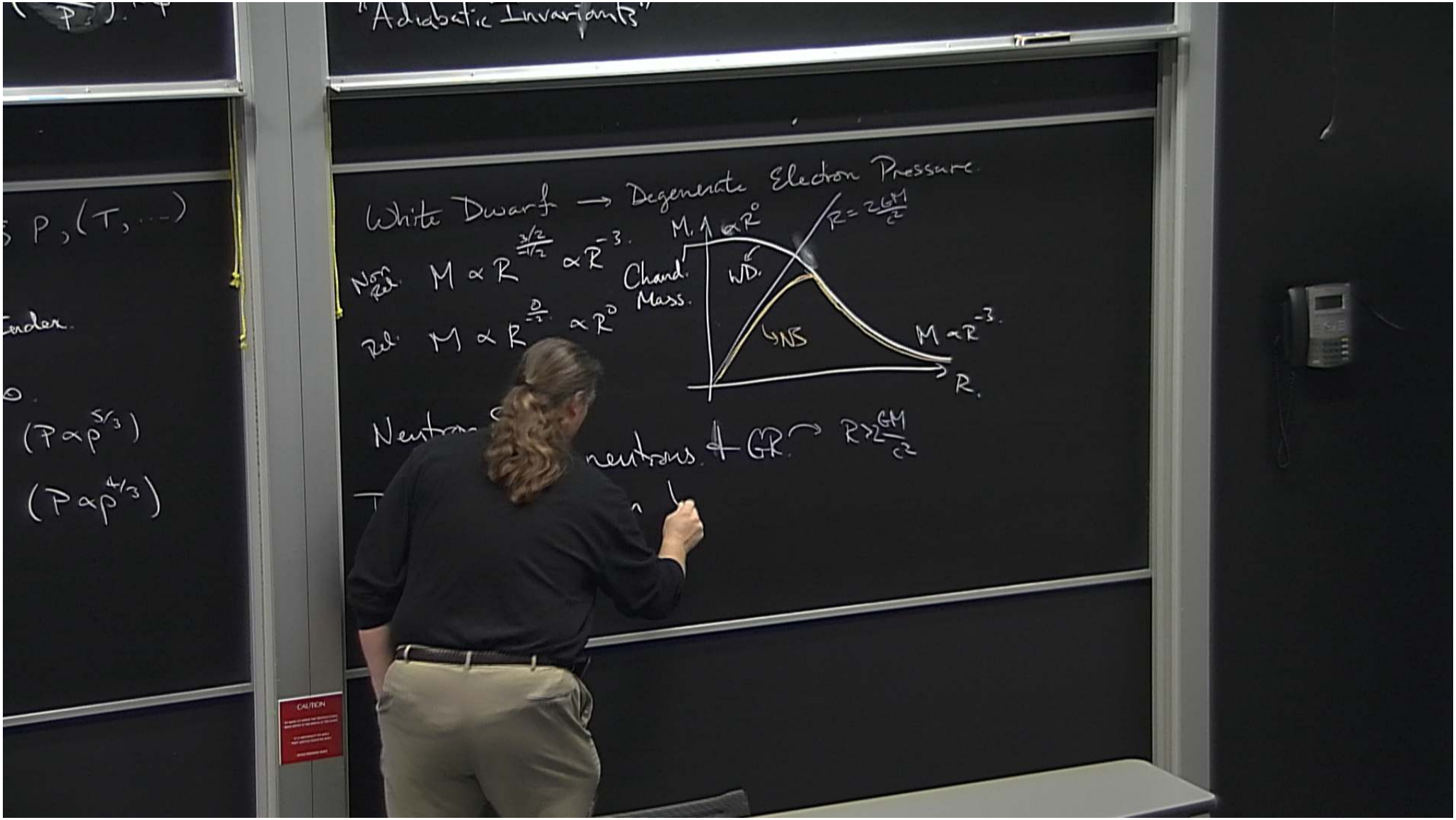

"Adiabatic Invariants"
$P_3(T_3, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$

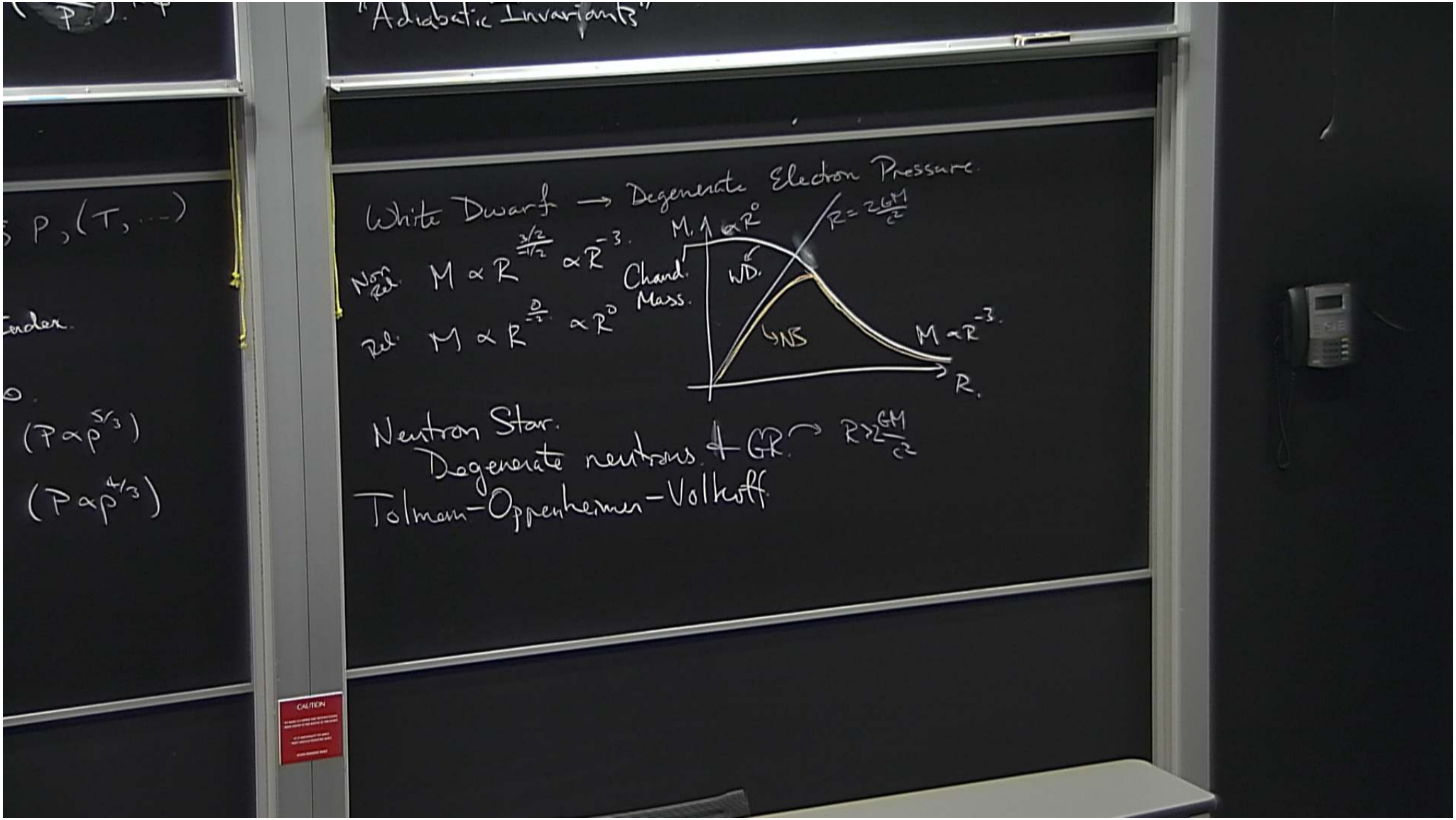

"Adiabatic Invariants"
White Dwarf → Degenerate Electron Pressure.
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$


"Adiabatic Invariants"
White Dwarf → Degenerate Electron Pressure.
$M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$.
R.
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$
CAUTION


"Adiabatic Invariants"
White Dwarf → Degenerate Electron Pressure.
Non Rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$.
Rel. $M \propto R^{0}$
$M \propto R^{0}$
$M \propto R^{-3}$.
R.
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$
CAUTION


"Adiabatic Invariants"
White Dwarf → Degenerate Electron Pressure.
Non Rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$.
$M \propto R^{\frac{0}{-1}} \propto R^{0}$
$M \propto R^{0}$
Chand. Mass.
$M \propto R^{-3}$.
R.
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$


"Adiabatic Invariants"
White Dwarf → Degenerate Electron Pressure
Non Rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$.
Rel. $M \propto R^{\frac{0}{-1}} \propto R^{0}$
Chand. Mass.
WD.
NS
$M \propto R^{-3}$.
R.
$R = \frac{2GM}{c^2}$
neutrons + GR → $R > 2\frac{GM}{c^2}$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$
CAUTION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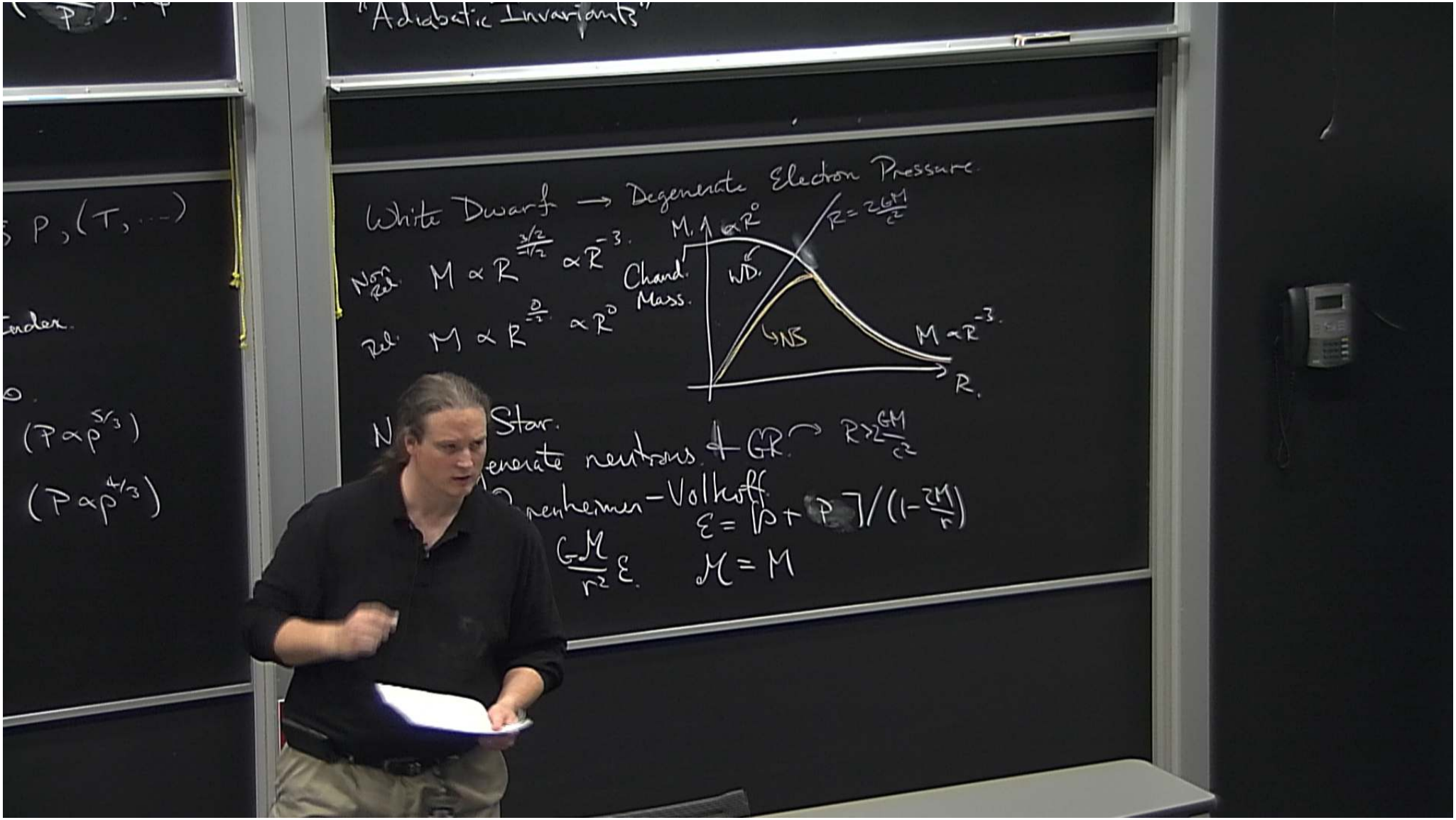
Rel. $M \propto R^{\frac{0}{-1}} \propto R^{0}$
Chand. Mass.
$M \propto R^{0}$
$R = \frac{2GM}{c^2}$
WD.
NS
$M \propto R^{-3}$.
R.
Neutron Star.
Degenerate neutrons + GR → $R > 2\frac{GM}{c^2}$
Tolman-Oppenheimer-Volkoff
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$
CAUTION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White Dwarf → Degenerate Electron Pressure.
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Rel. $M \propto R^{\frac{0}{-1}} \propto R^{0}$
Chand. Mass.
M.
WD.
NS
$M \propto R^{-3}$.
R.
$R = \frac{2GM}{c^2}$
Star.
generate neutrons + GR → $R > 2\frac{GM}{c^2}$
renheimer-Volkoff.
$\mathcal{E} = [\rho + P]/(1 - \frac{2M}{r})$
$\frac{GM}{r^2}\mathcal{E}$.
$\mathcal{M} = M$
$P_3(T, \ldots)$
$(P \propto \rho^{5/3})$
$(P \propto \rho^{4/3})$