

Title: PHYS 781 - Lecture 7

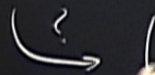
Date: Oct 07, 2014 02:00 PM

URL: <http://pirsa.org/14100052>

Abstract:

Integrable Systems Cont.

$(q_i, p_i) \rightarrow (Q_i, P_i)$ Canonical Trans.



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$\hookrightarrow (\theta_i, J_i)$ s.t. $\dot{\theta}_i = \frac{\partial H}{\partial J_i}$, $\dot{J}_i =$

Integrable Systems Cont.

$(q_i, p_i) \rightarrow (Q_i, P_i)$ Canonical Trans.

$\hookrightarrow (\theta_i, J_i) \quad \text{L.} \quad \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial J_i}, \quad \dot{J}_i = 0.$
 $\theta_i = \dot{\theta}_i$

Integrable Systems Cont.

$(q_i, p_i) \rightarrow (Q_i, P_i)$ Canonical Trans.

$$\hookrightarrow (\theta_i, J_i) \text{ s.t. } \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial J_i}, J_i = 0.$$

Angles Action.

$$\theta_i = \dot{\theta}_i t + \phi_i$$

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For compact ms. $\rightarrow$ bounded

Integrable Systems Cont.

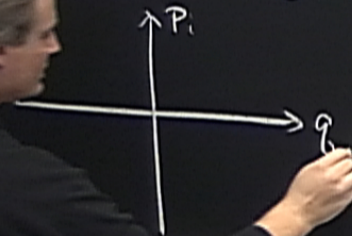
$$(q_i, p_i) \rightarrow (Q_i, P_i) \quad \text{Canonical Trans.}$$

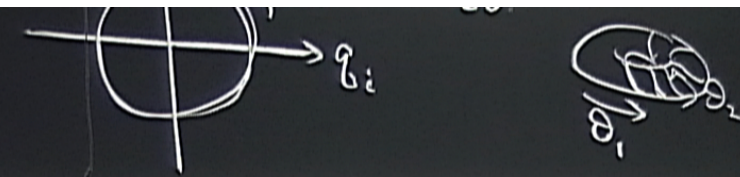
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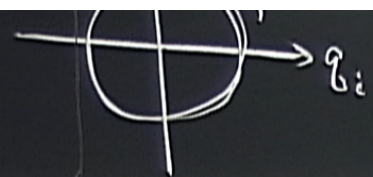
For compact systems $\rightarrow$ bounded
 $\rightarrow$ periodic. $\theta_i \in [0, 2\pi]$





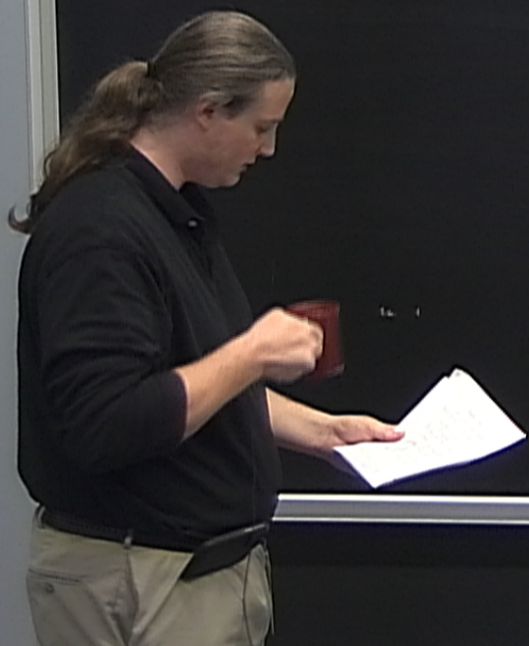
Liouville's Thm. : Cons. of phase space volume.

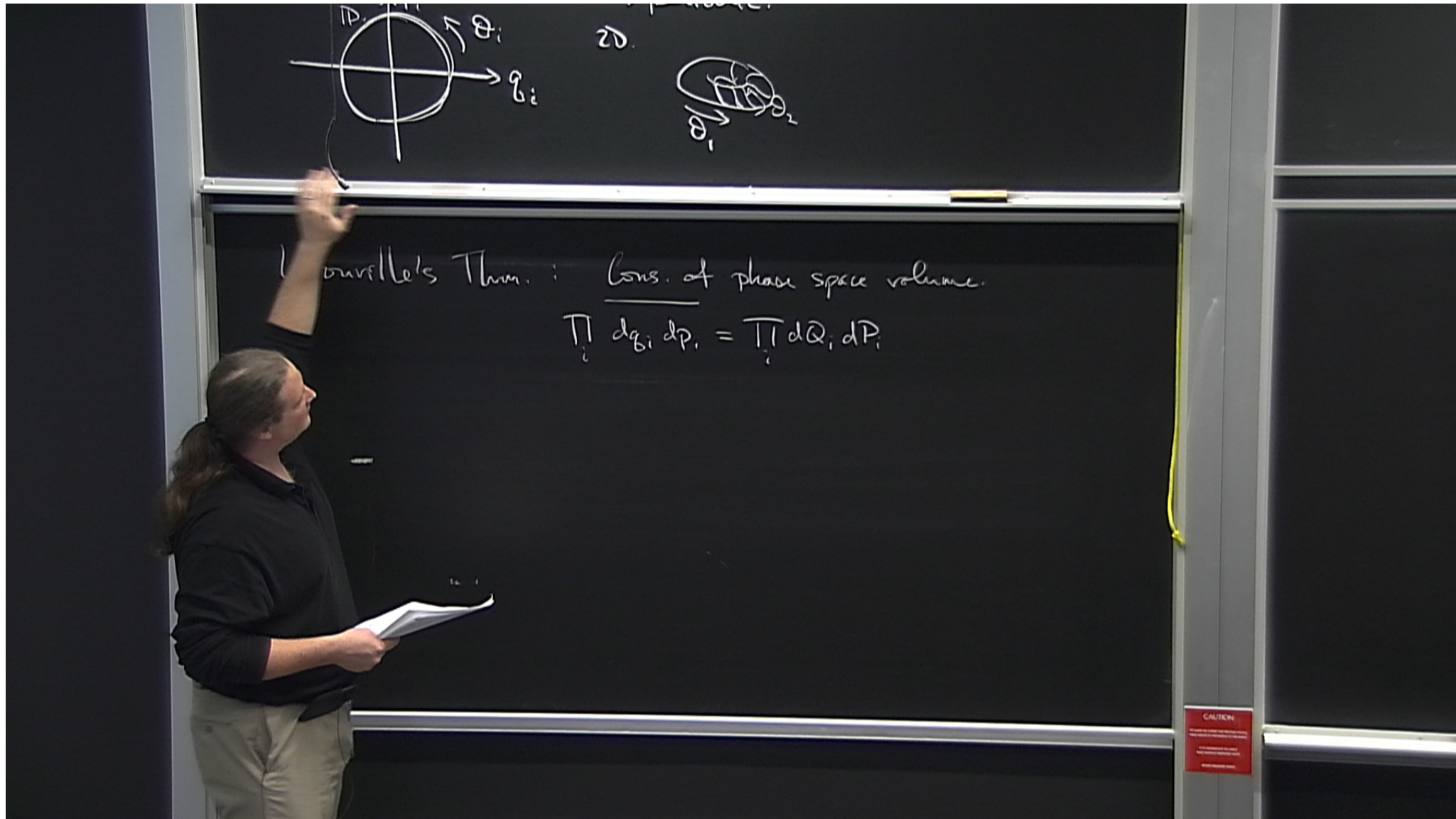
dq_i

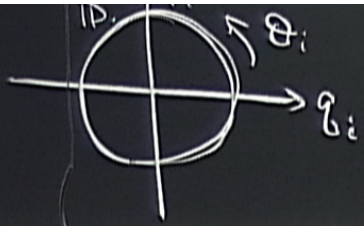


Liouville's Thm. : Cons. of phase space volume.

$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i$$





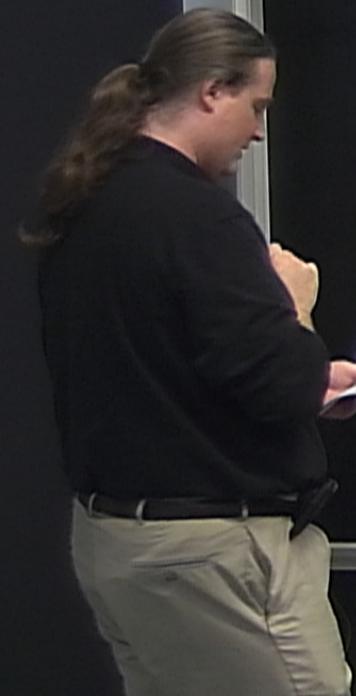


Liouville's Thm. : Cons. of phase space volume.

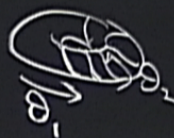
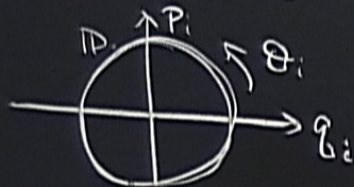
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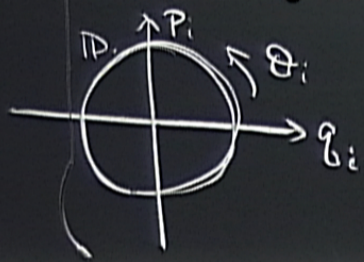
$J_i = \text{Area of closed orbit.}$
(Volume)

$=$

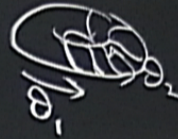


(θ_i, J_i) s.t. $\dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial J_i}$, $J_i = 0$
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 For compact systems $\rightarrow$ bounded, $\theta_i \in [0, 2\pi]$
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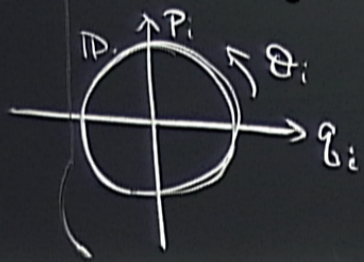
→ periodic.
2D



Liouville's Thm. : Cons. of phase space volume.

$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i$$

$$J_i \stackrel{\text{Area. of closed orbit.}}{\text{(Volume)}} = \int d\theta_i dJ_i = 2\pi \int dJ_i = 2\pi J_i$$



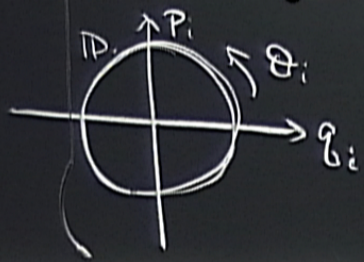
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$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i$$

Area. of closed orbit. = $\int d\theta_i dJ_i = 2\pi \int dJ_i = 2\pi J_i$
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→ periodic.
2D.

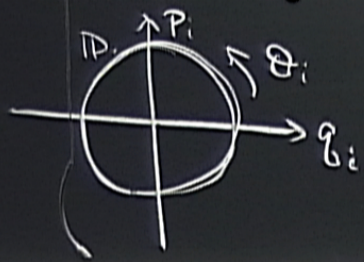


Liouville's Thm. : Cons. of phase space volume.

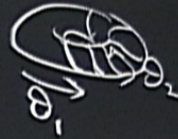
$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i$$

J_i := Area of closed orbit. = $\int d\theta_i dJ_i = 2\pi \int dJ_i = 2\pi J_i$
(Volume)

$$J_i = \frac{1}{2\pi}$$



→ periodic.
2D



Liouville's Thm. : Cons. of phase space volume.

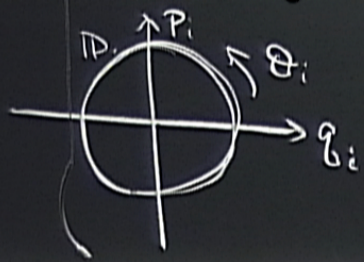
$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i$$

$$J_i \equiv \text{Area. of closed orbit.} = \int d\theta_i d\Gamma = \int dJ_i = 2\pi J_i$$

(Volume)

$$J_i = \frac{1}{2\pi}$$





→ periodic.
2D



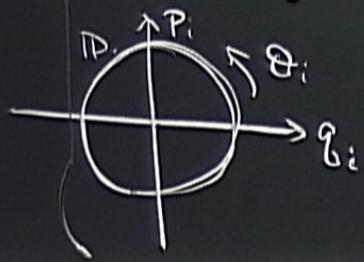
Liouville's Thm. : Cons. of phase space volume.

$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i = \prod_i dQ_i dJ_i$$

J_i = Area of closed orbit
(Volume)

$$J_i =$$

$$\int dQ_i dJ_i = 2\pi \int dJ_i = 2\pi J_i$$



→ periodic.
2D.



Liouville's Thm. : Cons. of phase space volume.

$$\prod_i dq_i dp_i = \prod_i dQ_i dP_i = \prod_i dQ_i dJ_i$$

$$J_i \equiv \text{Area. of d.o.f. bit.} = \int d\theta_i dJ_i = \int_{\text{over 1 period}} dJ_i = 2\pi J_i$$

$$\frac{1}{2\pi} \int dp_i dq_i = \frac{1}{2\pi} \oint p_i dq_i$$

Harmonic

Harmonic Osc.

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q$$

$Q; dJ;$

$\oint dJ = 2\pi J;$
over 1 period.

$P; dq;$

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$$\rightarrow q = A \sin(\omega t + \phi)$$

$$p = m\dot{q} = m\omega A \cos(\omega t + \phi)$$

$Q; dJ;$

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Harmonic Osc.

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\begin{aligned} J &= \frac{1}{2\pi} \oint p dq \\ &= \frac{1}{2\pi} m \omega \oint \sqrt{A^2 - q^2} dq \\ &= \end{aligned}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

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$$\begin{aligned} p &= m \dot{q} = m \omega A \cos(\omega t + \phi) \\ &= m \omega \sqrt{A^2 - q^2} \end{aligned}$$

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$$= \frac{1}{2\pi} m \omega \oint dq$$

$$= \frac{1}{2} m \omega A^2$$

$$= E$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

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Harmonic Osc.

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$$= m\omega \sqrt{A^2 - q^2}$$

$$J = \frac{1}{2\pi} \oint p dq$$

$$= \frac{1}{2\pi} m\omega \int_{-A}^A \sqrt{A^2 - q^2} dq$$

$$= \frac{1}{2} m\omega \pi A^2$$

$$= E$$

$$\Rightarrow J = \frac{E}{h}$$

$Q; dJ;$

$\oint dJ = 2\pi J;$
over 1 period.

$P; dq;$

Harmonic Osc.

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

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$$\rightarrow q = A \sin(\omega t + \phi)$$

$$p = m\dot{q} = m\omega A \cos(\omega t + \phi) \\ = m\omega \sqrt{A^2 - q^2}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial J} = \omega$$

$$\Rightarrow \theta = \omega t + \phi$$

$$J = \frac{1}{2\pi} \oint p dq$$

$$= \frac{1}{2\pi} m\omega \oint \sqrt{A^2 - q^2} dq$$

$$= \frac{1}{2} m\omega A^2$$

$$= E/\omega$$

$$\Rightarrow \mathcal{H} = \omega J$$

Harmonic Osc.

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

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$$\Rightarrow \theta = \omega t + \phi$$

$$J = \frac{1}{2\pi} \oint p dq$$

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$$= \frac{1}{2} m\omega A^2$$

$$\omega$$

$$\omega J$$

$Q; dJ;$

$\oint dJ = 2\pi J;$
over 1 period.

$P; dq;$

$$= \frac{1}{2} m \omega A^2$$

$$= E / \omega$$

$$\Rightarrow d\phi = \omega dJ$$

11.A.

$$\theta - \frac{\partial J}{\partial J} = \omega$$

$$\Rightarrow \theta = \omega t + \phi'$$

$$= \arctan\left(\frac{m\omega g}{p}\right) + \phi''$$

Central Potential in 2D

$\theta, dJ:$

$$\int dJ = 2\pi J$$

over $1p$

$p: dq$

$$= E/\omega.$$

$$\Rightarrow \mathcal{L} = \omega \mathcal{J}.$$

$$\Rightarrow \bar{\theta} = \omega t + \phi'$$

$$= \arctan\left(\frac{m\omega r}{p}\right) + \phi'$$

$$\mathcal{H} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$$

$$\mathcal{J}_1 = \frac{1}{2\pi} \oint P_r dr = \frac{1}{\pi} \int \sqrt{2m \left[E - \frac{P_\theta^2}{2mr^2} - V(r) \right]} = P_\theta f(e)$$

$$\mathcal{J}_2 = \frac{1}{2\pi} \oint P_\theta d\theta = P_\theta \quad \text{Ang. Mom.}$$

$$\omega_i = \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial \mathcal{J}_i} \leftarrow \text{Epicyclic Freq.}$$

$$\left(\frac{m\omega g}{p}\right) + \phi''$$

planar
centricity

$$= p_0 f(\vec{e})$$

Kolmogorov-Arnold-Moser (KAM) Thm.

$$\left(\frac{m\omega g}{p}\right) + \phi''$$

planar
centricity

$$= p_0 f(\epsilon)$$

stem.

Kolmogorov-Arnold-Moser (KAM) Thm.
Sum: "Integrability is Robust".

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1$$

↑
Int.

"small pert. of an int.
gives another int. system.
absen

CAUTION
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WHILE THE LECTURE IS IN PROGRESS
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PLEASE ASK THE LECTURER

$$\left(\frac{m\omega g}{p}\right) + \phi''$$

planar
centricity

$$= p_0 f(\vec{e})$$

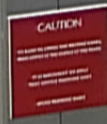
stem

Kolmogorov-Arnold-Moser (KAM) Thm.
Sum: "Integrability is Robust".

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1$$

$\uparrow$
Int.

"Any small pert. of an int. system gives another int. system in the absence of resonances".



$$\left(\frac{m\omega g}{P}\right) + \phi''$$

planar
centricity

$$= P_0 f(\vec{e})$$

stem.

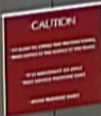
Kolmogorov-Arnold-Moser (KAM) Thm.
Sum: "Integrability is Robust".

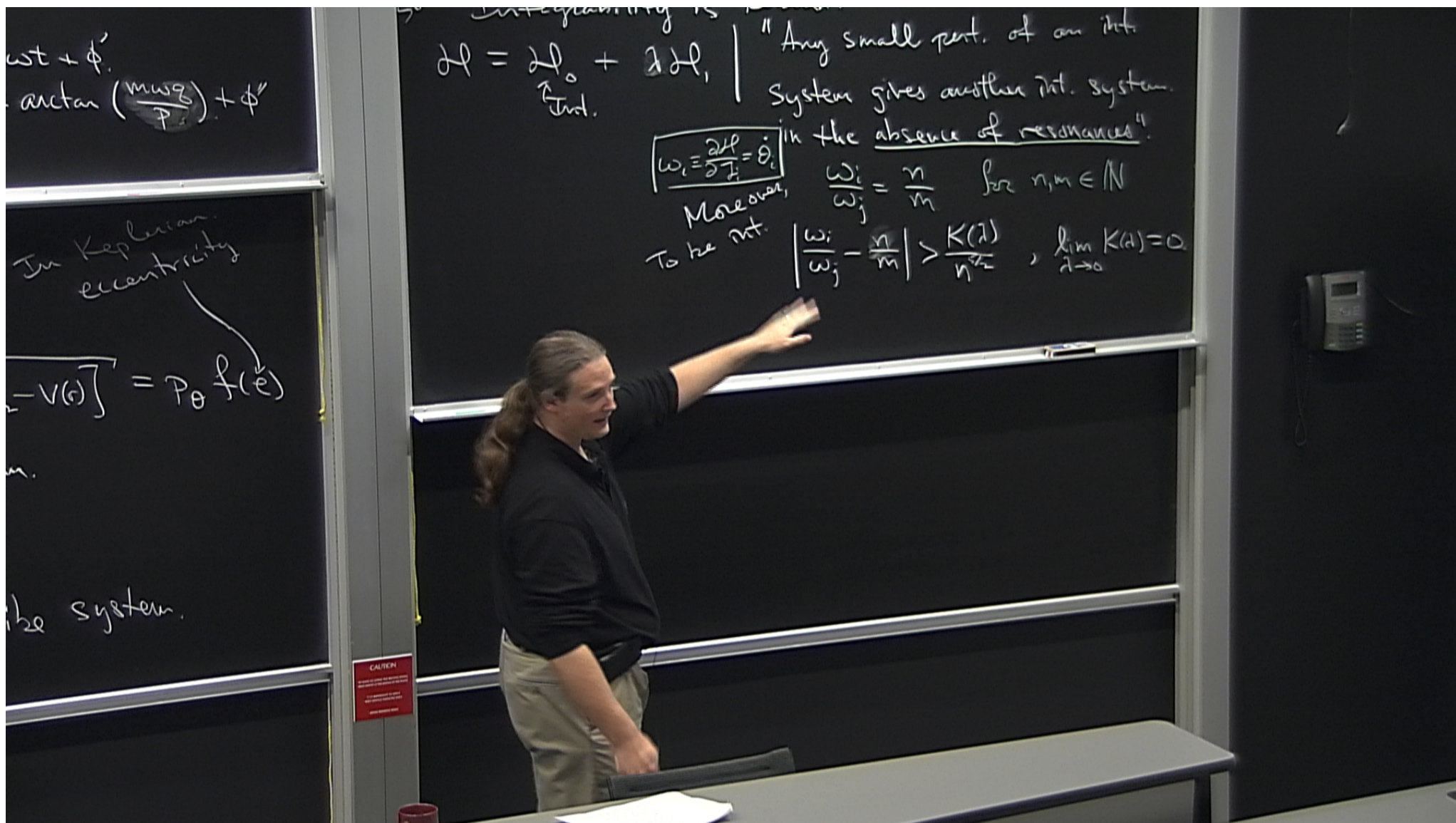
$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1$$

↑
Int.

"Any small pert. of an int. system gives another int. system in the absence of resonances".

$$\frac{\omega_i}{\omega_j} = \frac{n}{m}$$





Kolmogorov-Arnold-Moser (KAM) Theorem.
 Sum: "Integrability is Robust".

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1,$$

$\uparrow$
Int.

"Any small pert. of an int. system gives another int. system. in the absence of resonances".

$$\begin{aligned} \mathcal{H}(\theta, I; \lambda) \\ = \mathcal{H}(\theta, I; \lambda_0) \\ + \lambda \mathcal{H}_1 \end{aligned}$$

$$\boxed{\omega_i \equiv \frac{\partial \mathcal{H}}{\partial I_i} = \dot{\theta}_i}$$

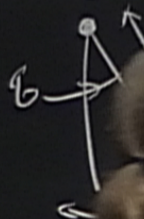
Moreover,
To be int.

$$\frac{\omega_i}{\omega_j} = \frac{n}{m} \quad \text{for } n, m \in \mathbb{N}$$

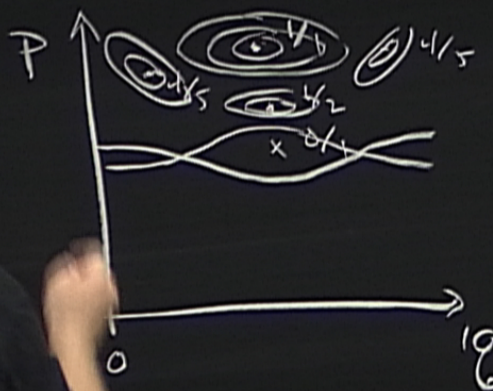
$$\left| \frac{\omega_i}{\omega_j} - \frac{n}{m} \right| > \frac{K(\lambda)}{n^{s/2}}, \quad \lim_{\lambda \rightarrow 0} K(\lambda) = 0.$$

"Adiabatic Invariants"

Swings

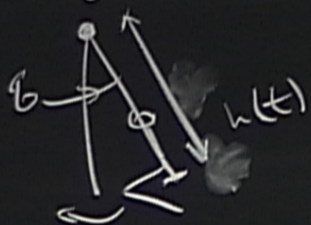


$$\mathcal{L} = \frac{P^2}{2} + \lambda [a \cos(2\pi q) + b \cos[2\pi(q-t)]]$$



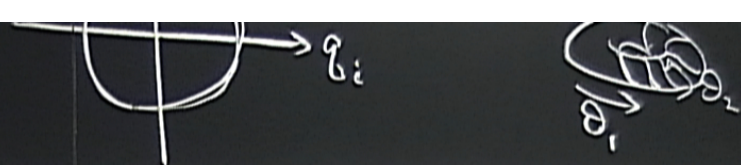
"Adiabatic Invariants"

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$$\mathcal{L} = \frac{p^2}{2} + \lambda [a \cos(2\pi q) + b \cos[2\pi(q-t)]]$$

A phase space plot with q on the horizontal axis and p on the vertical axis. The origin is marked with 0 . The plot shows several closed, nested loops representing periodic orbits, and a separatrix that passes through saddle points. The loops are labeled with numbers 1 through 5.



Gravitational Dynamics

Examples: Stars in Globulars, Galaxies,
Dark Matter, Gas.

$$\Rightarrow \Delta\phi = \omega J$$

Equation of State — Relationship btwn P & ρ , $(T, \dots)$

Polytrope: $P \propto \rho^\gamma = \rho^{1 + \frac{1}{n}}$ Polytropic Index

ex:

Isothermal. $\rho \rightarrow n = \infty$

Adiabatic $\rho \rightarrow n = \frac{5}{2}$ ($P \propto \rho^{\frac{5}{3}}$)

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\Rightarrow \Delta\phi = \omega J$$

Equation of State — Relationship btwn P & ρ , $(T, \dots)$

Polytrope: $P \propto \rho^\gamma = \rho^{1 + \frac{1}{n}}$ ↑ Polytropic Index

ex:

Isothermal. $\rightarrow P \propto \rho \rightarrow n = \infty$

Adiabatic $\left\{ \begin{array}{l} \text{Non-rel. } n = \frac{5}{2} \quad (P \propto \rho^{5/3}) \\ \text{or Degenerate } \text{Rel. } n = 3 \quad (P \propto \rho^{4/3}) \end{array} \right.$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\Rightarrow \mathcal{H} = \omega J.$$

Equation of State. — Relationship btwn P & ρ , ($T, \dots$)

Polytrope: $P \propto \rho^\gamma = \rho^{1 + \frac{1}{n}}$ ↑ Polytropic Index.

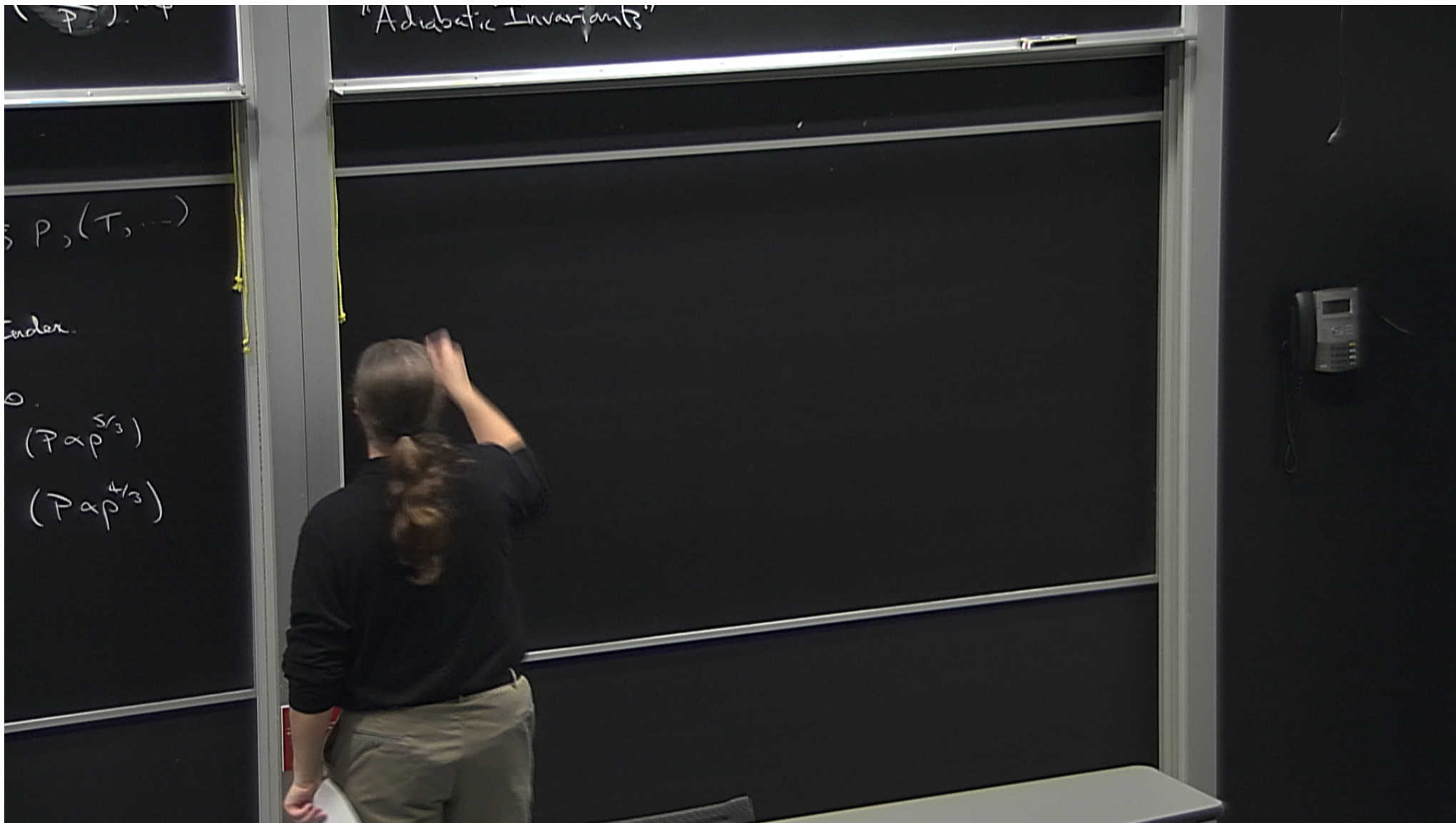
ex:

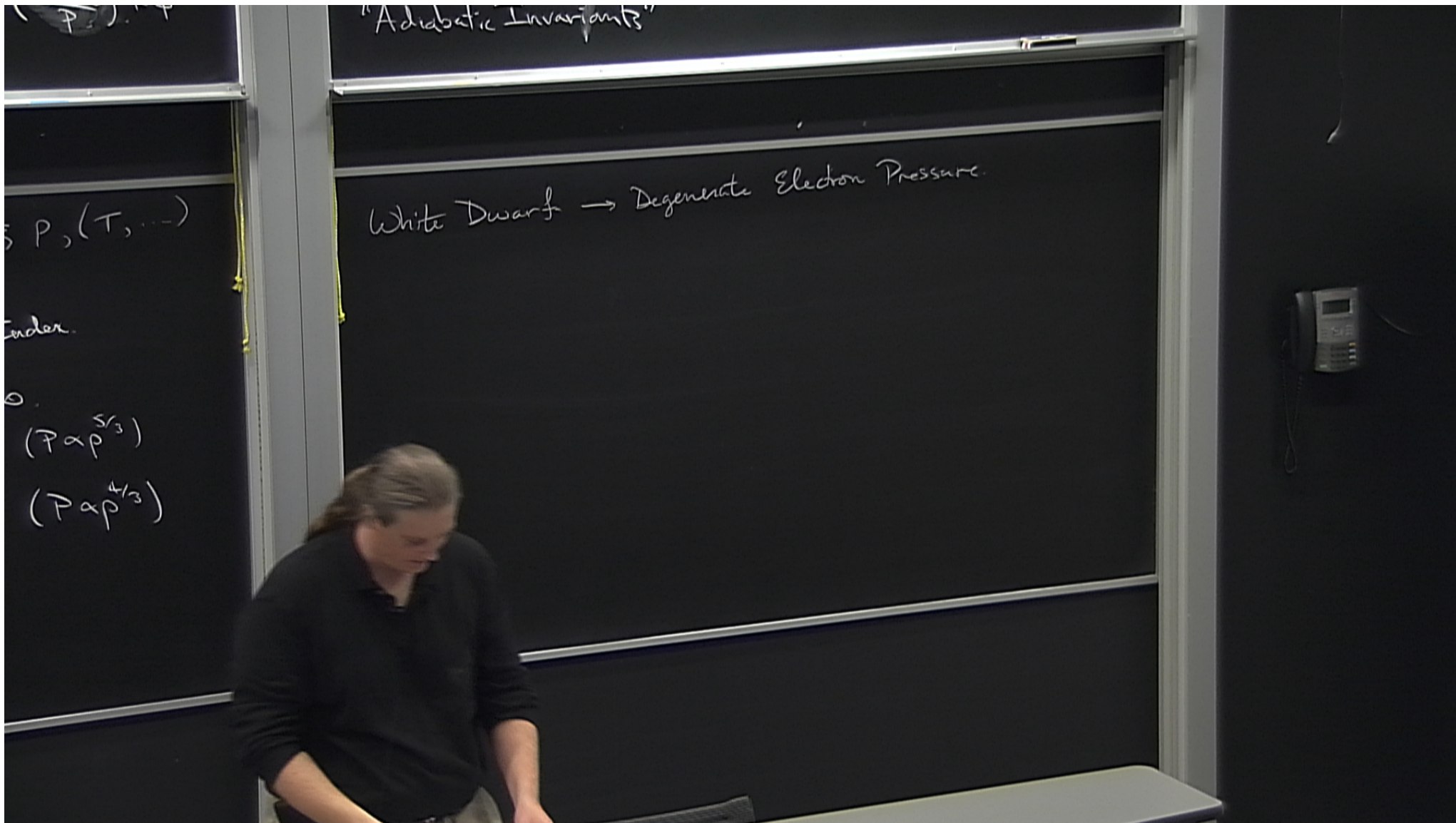
Isothermal. $\rightarrow P \propto \rho \rightarrow n = \infty$.

Adiabatic $\left\{ \begin{array}{l} \text{Non-rel. } n = \frac{5}{2} \quad (P \propto \rho^{5/2}) \\ \text{or Degenerate. Rel. } n = 3 \quad (P \propto \rho^{4/3}) \end{array} \right.$

$$\boxed{M \propto R^{\frac{3-n}{1-n}}}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

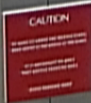
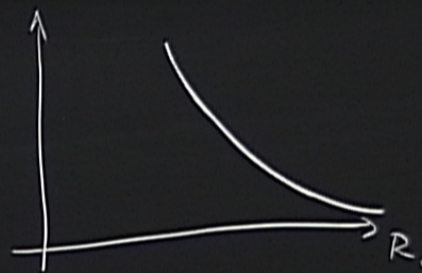




"Adiabatic Invariants"

White Dwarf $\rightarrow$ Degenerate Electron Pressure.

$$M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}.$$

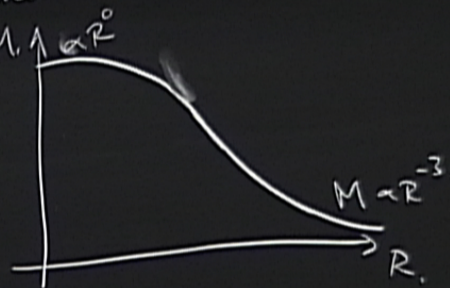


"Adiabatic Invariants"

White Dwarf $\rightarrow$ Degenerate Electron Pressure.

Non-rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$. $M \uparrow \propto R^0$

Rel. $M \propto R^0 \propto R^0$



$P_s(T, \dots)$

Index.

$(P \propto \rho^{5/3})$

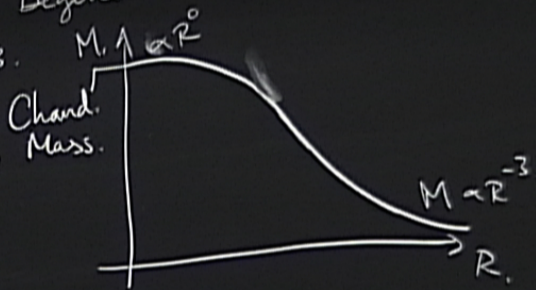
$(P \propto \rho^{4/3})$

"Adiabatic Invariants"

White Dwarf $\rightarrow$ Degenerate Electron Pressure.

Non-rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$

Rel. $M \propto R^{\frac{0}{-1/2}} \propto R^0$



$P_0(T, \dots)$

Index.

$(P \propto \rho^{5/3})$

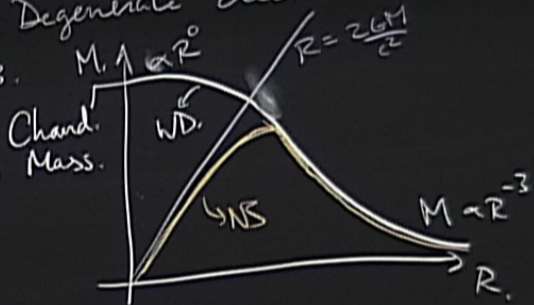
$(P \propto \rho^{4/3})$

"Adiabatic Invariants"

White Dwarf $\rightarrow$ Degenerate Electron Pressure.

Non-rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$

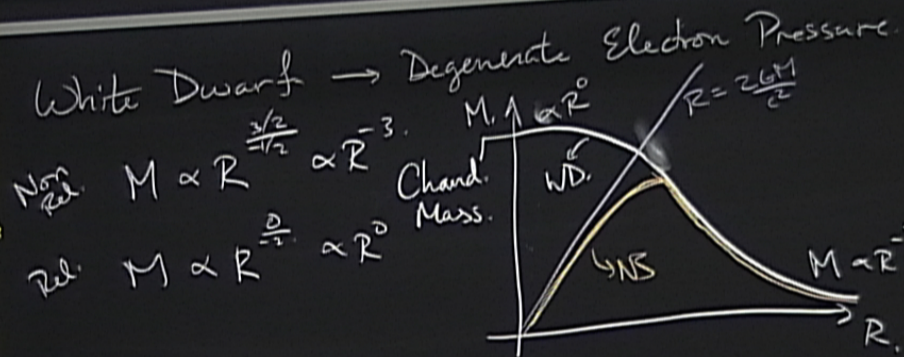
Rel. $M \propto R^{\frac{0}{-1/2}} \propto R^0$



Neutron $\rightarrow$ neutrons + GR $\rightarrow R > \frac{2GM}{c^2}$

T

"Adiabatic Invariants"



Neutron Star.
 Degenerate neutrons + GR $\rightarrow R > \frac{2GM}{c^2}$
 Tolman-Oppenheimer-Volkoff

$P_2(T, \dots)$

Index.

$(P \propto \rho^{5/3})$

$(P \propto \rho^{4/3})$

"Adiabatic Invariants"

White Dwarf $\rightarrow$ Degenerate Electron Pressure.

Non-rel. $M \propto R^{\frac{3/2}{-1/2}} \propto R^{-3}$

Rel. $M \propto R^{\frac{0}{-1}} \propto R^0$

Star. generate neutrons + GR $\rightarrow R > \frac{2GM}{c^2}$

neutronium-Volkoff

$$\epsilon = \left(\rho + \frac{P}{c^2} \right) / \left(1 - \frac{2M}{r} \right)$$

$$\frac{GM}{r^2} \epsilon$$

$$\mathcal{M} = M$$