Title: (Non)perturbative QCD and Jet Substructure

Date: Oct 14, 2014 01:00 PM

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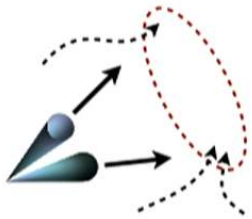
Abstract: With the remarkable performance of the ATLAS and CMS detectors, jets at the LHC can now be characterized not just by their overall direction and energy but also by their substructure. At the same time, there has been substantial progress in predicting the properties of jets from first principles. In this talk, I highlight the ways that theoretical studies of jet substructure have enhanced our understanding of QCD, including examples that blur the boundary between perturbative and nonperturbative physics.

(Non-)Perturbative QCD & Jet Substructure

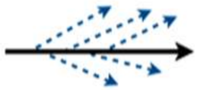
Jesse Thaler

Perimeter Institute Seminar — October 14, 2014

Outline

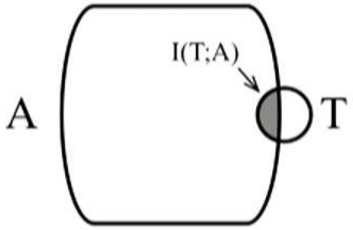


Why Jet Substructure?

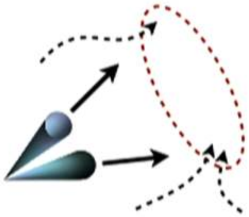


$$\frac{\text{IRC Safe}}{\text{IRC Safe}} = \text{Sudakov Safe}$$

From IRC Safe to Sudakov Safe



Mutual Information with the Truth



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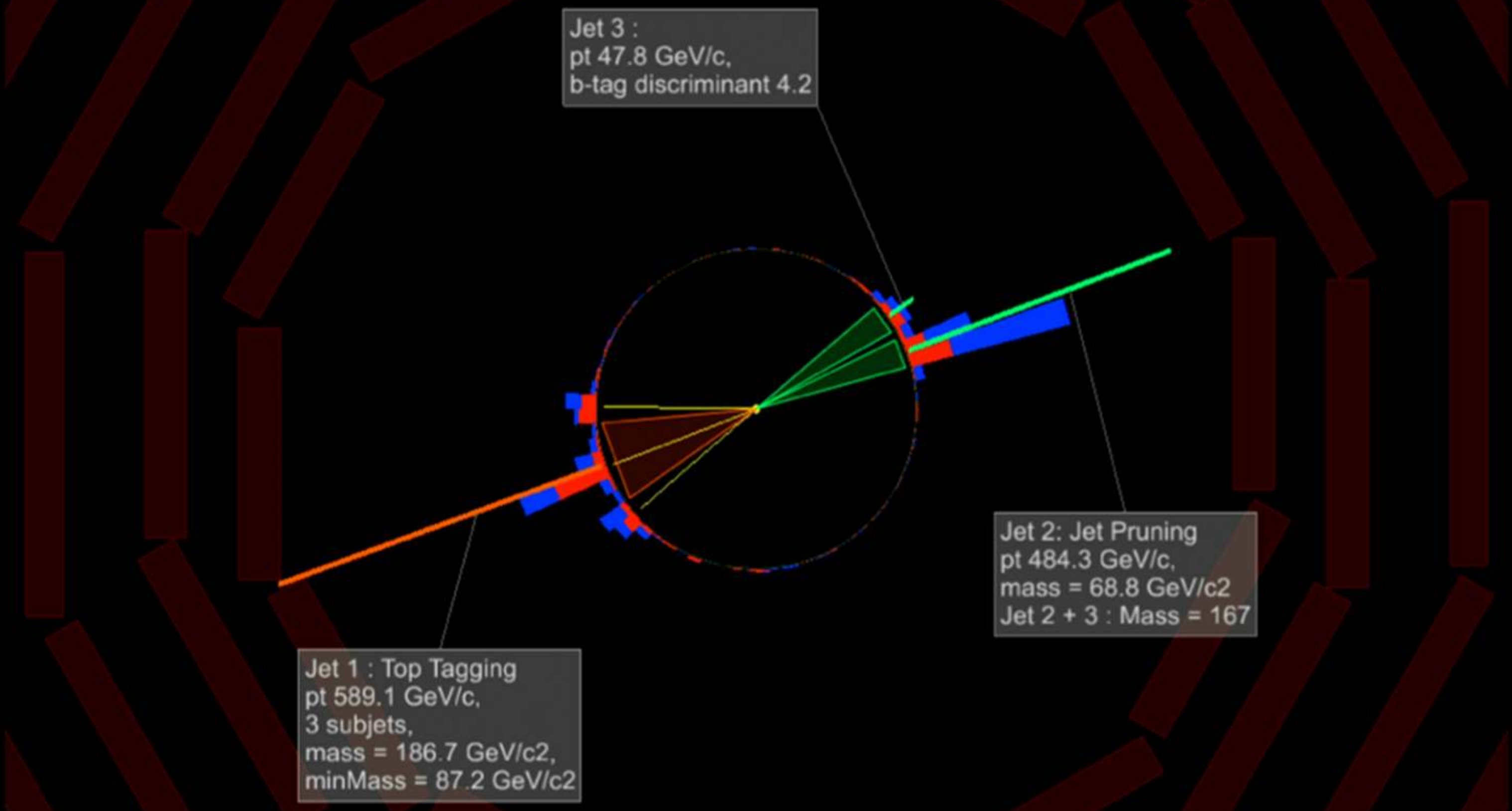
Jets or Jet Substructure?



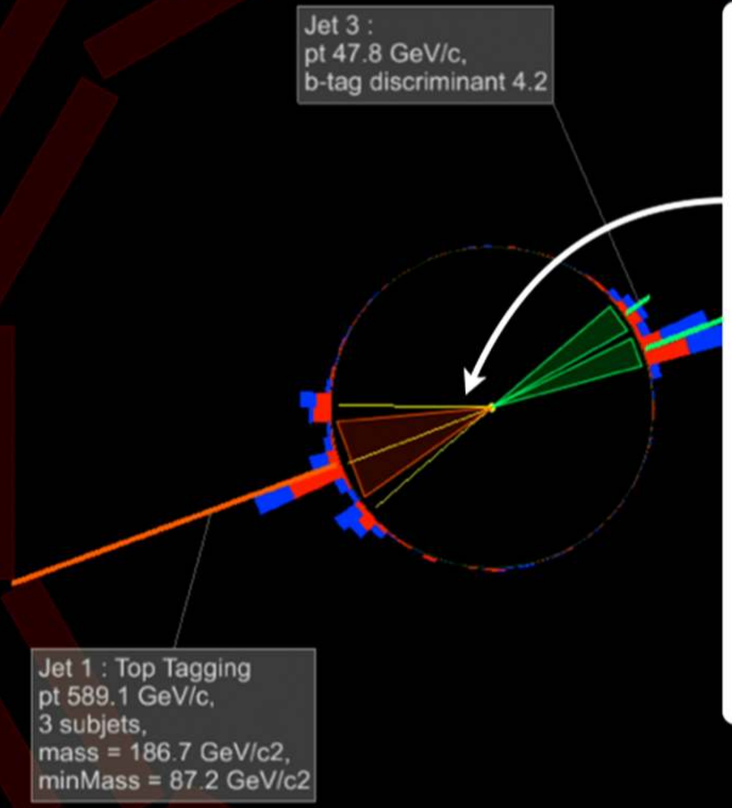
Jet 3 :
pt 47.8 GeV/c,
b-tag discriminant 4.2

Jet 2: Jet Pruning
pt 484.3 GeV/c,
mass = 68.8 GeV/c2
Jet 2 + 3 : Mass = 167

Jet 1 : Top Tagging
pt 589.1 GeV/c,
3 subjets,
mass = 186.7 GeV/c2,
minMass = 87.2 GeV/c2

[CMS EXO-11-006, CMS JME-13-007]
[Using JHU/CMSTopTagger: Kaplan, Rehermann, Schwartz, Tweedie, 0806.0848]
[Using Pruning: Ellis, Vermilion, Walsh, 0903.5081]

Jets or Jet Substructure?



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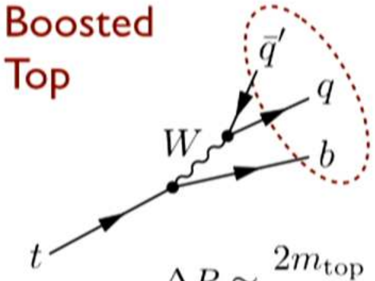
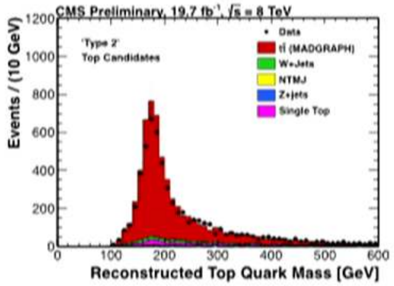
Boosted Top

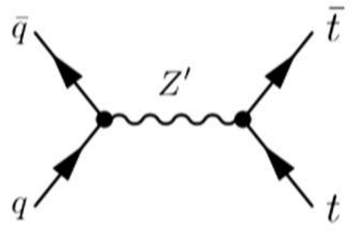


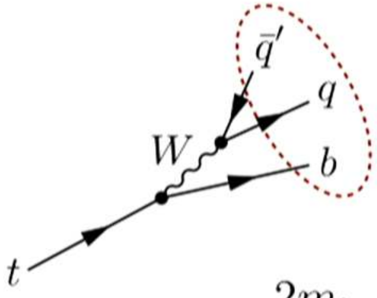
$$\Delta R \simeq \frac{2m_{\rm top}}{p_T}$$



[CMS EXO-11-006, CMS JME-13-007]
[Using JHU/CMSTopTagger: Kaplan, Rehermann, Schwartz, Tweedie, 0806.0848]
[Using Pruning: Ellis, Vermilion, Walsh, 0903.5081]

High Energy: Boosted Regime is Inevitable





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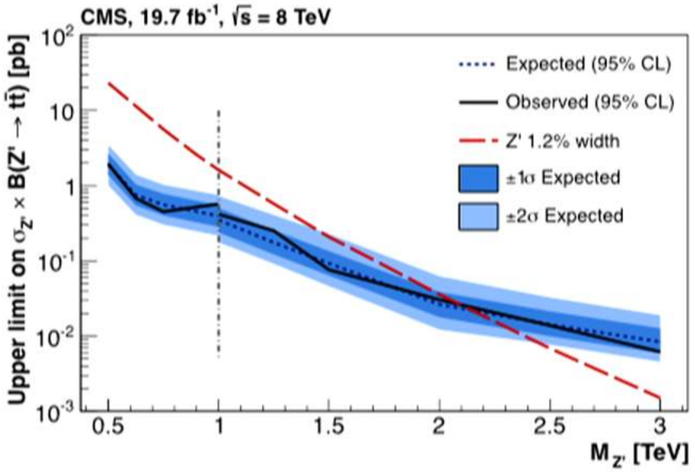
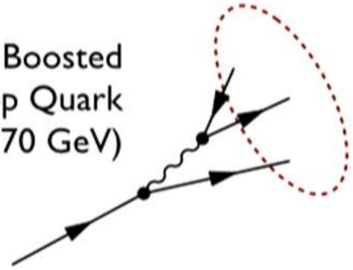


[CMS B2G-13-001]

High Luminosity: Pileup is Inevitable

Boosted
Top Quark
($m_{top} \approx 170$ GeV)



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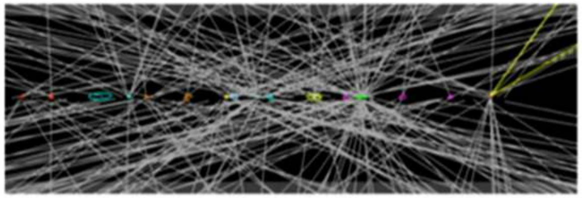
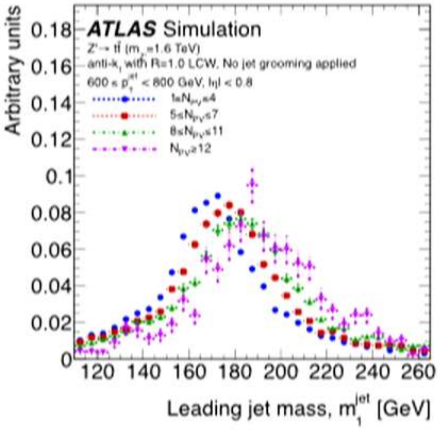


Secondary Collision Debris



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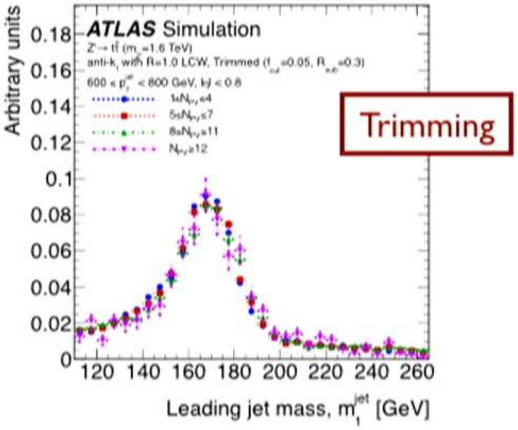
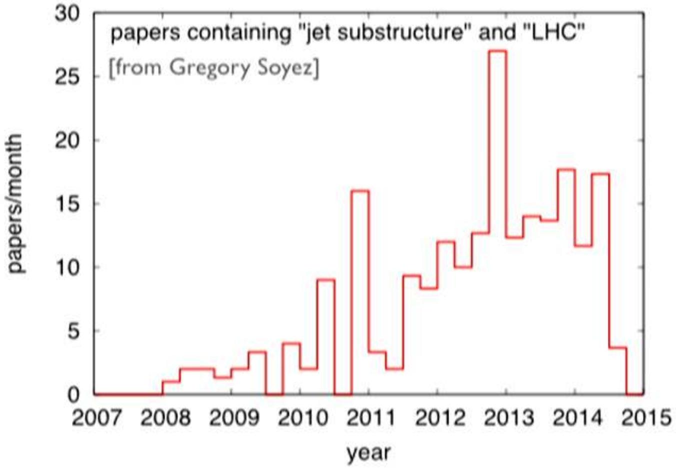


Trimming

[ATLAS PERF-2012-02]
[Krohn, JDT, Wang, 0912.1342]

High Stakes: Cleverness is Inevitable



Mass Drop, p_T Balance, Y-splitter, Filtering, Trimming, Pruning, Soft Drop, **Angularities**, Planar Flow, N-subjettiness, Angular Structure Functions, Jet Charge, Jet Pull, Energy Correlation Functions, Dipolarity, p_T^D, Zernike Coefficients, Fox-Wolfram Moments, JHU/CMSTopTagger, HEPTopTagger, Template Method, Shower Deconstruction, Jets Without Jets, Subjet Counting, Wavelets, Q-Jets, Telescoping Jets, Jet Reclustering, SoftKiller, PUPPI, Constituent Subtraction, etc.

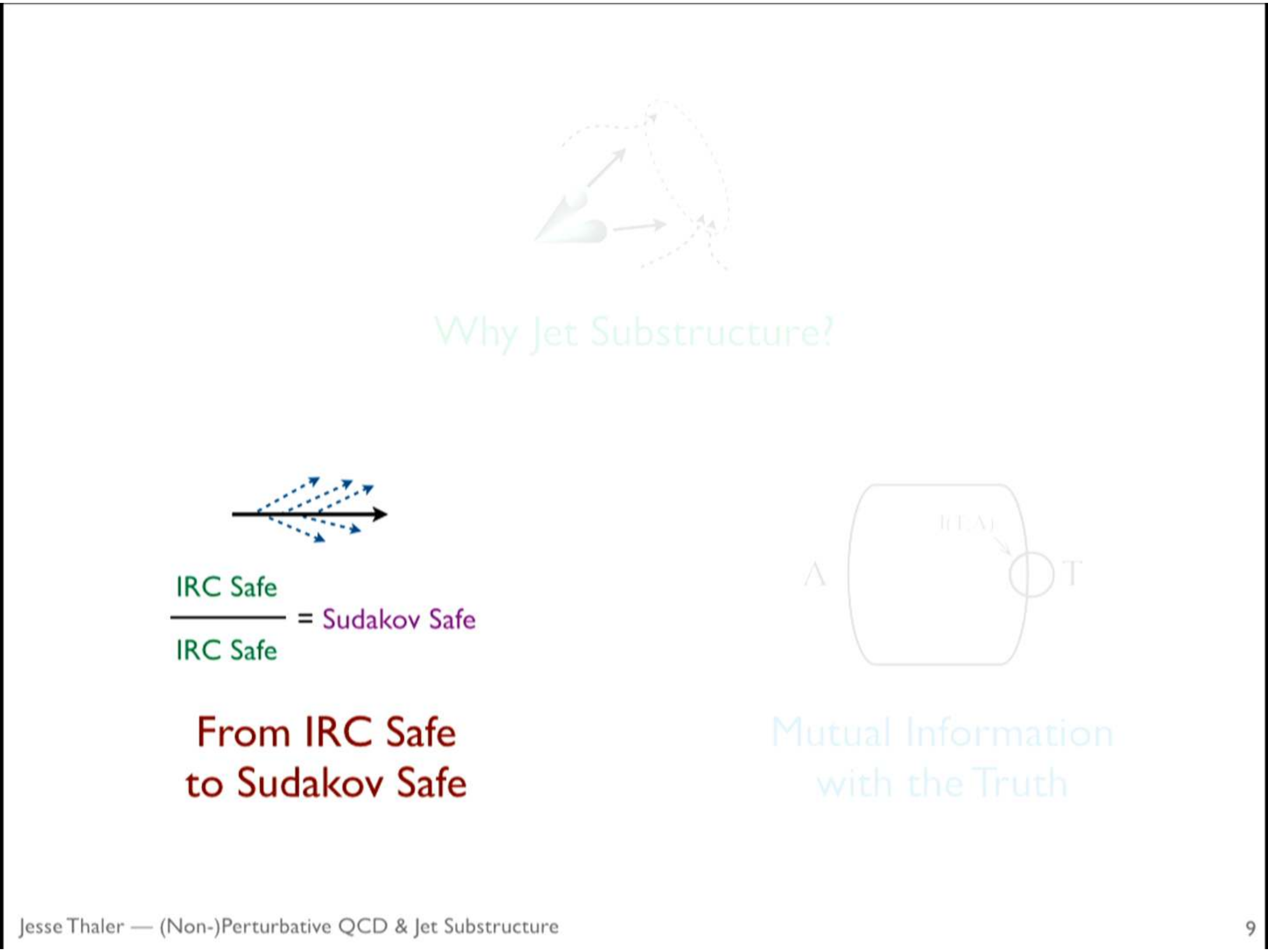
Core Principles of Jet Substructure:

Prong-like Behavior
Radiation Patterns
Flavor Tagging
Pileup Mitigation

Rest of this talk:

Jet substructure as motivation to delve into some subtleties of QCD

(More examples in backup)



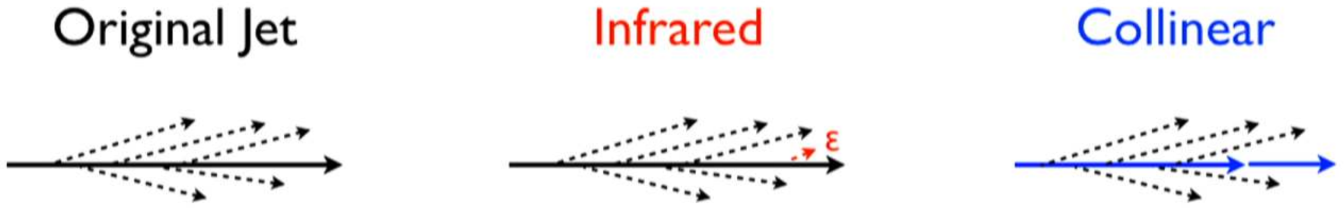
Why Jet Substructure?

$$\frac{\text{IRC Safe}}{\text{IRC Safe}} = \text{Sudakov Safe}$$

From IRC Safe
to Sudakov Safe

Mutual Information
with the Truth

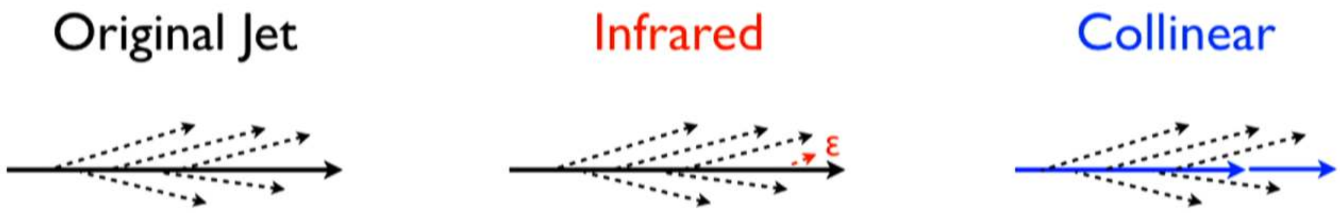
Infrared/Collinear Safety



IRC Safe Observable: Insensitive to IR or C emissions

		IRC Safe	IRC Unsafe
Lore:	Calculable in pQCD?	✓	✗
	Controlled Λ_{QCD} Effects?	✓	✗

Infrared/Collinear Safety



IRC Safe Observable: Insensitive to IR or C emissions

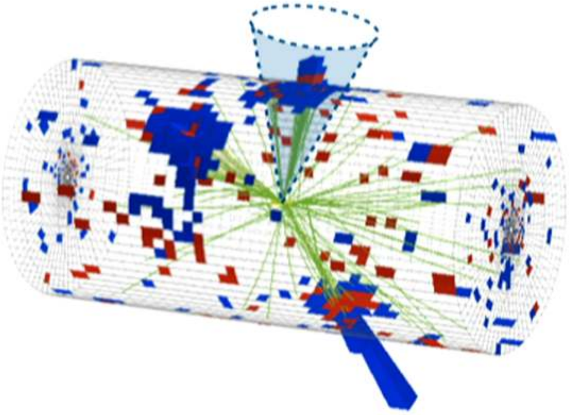
New Lore:

	IRC Safe	IRC Unsafe
Calculable in pQCD?	✓	$\boxed{✓*}$
Controlled Λ_{QCD} Effects?	✓*	✓*

Examples

Jet p_T: $$\sum_{i \in \mathrm{jet}} p_{T,i}$$ IRC Safe

$p_T{}^D$: [CMS HIG-11-027] $$\sum_{i \in \mathrm{jet}} \frac{p_{T,i}^2}{p_{T\mathrm{jet}}^2}$$ IR Safe
C Unsafe

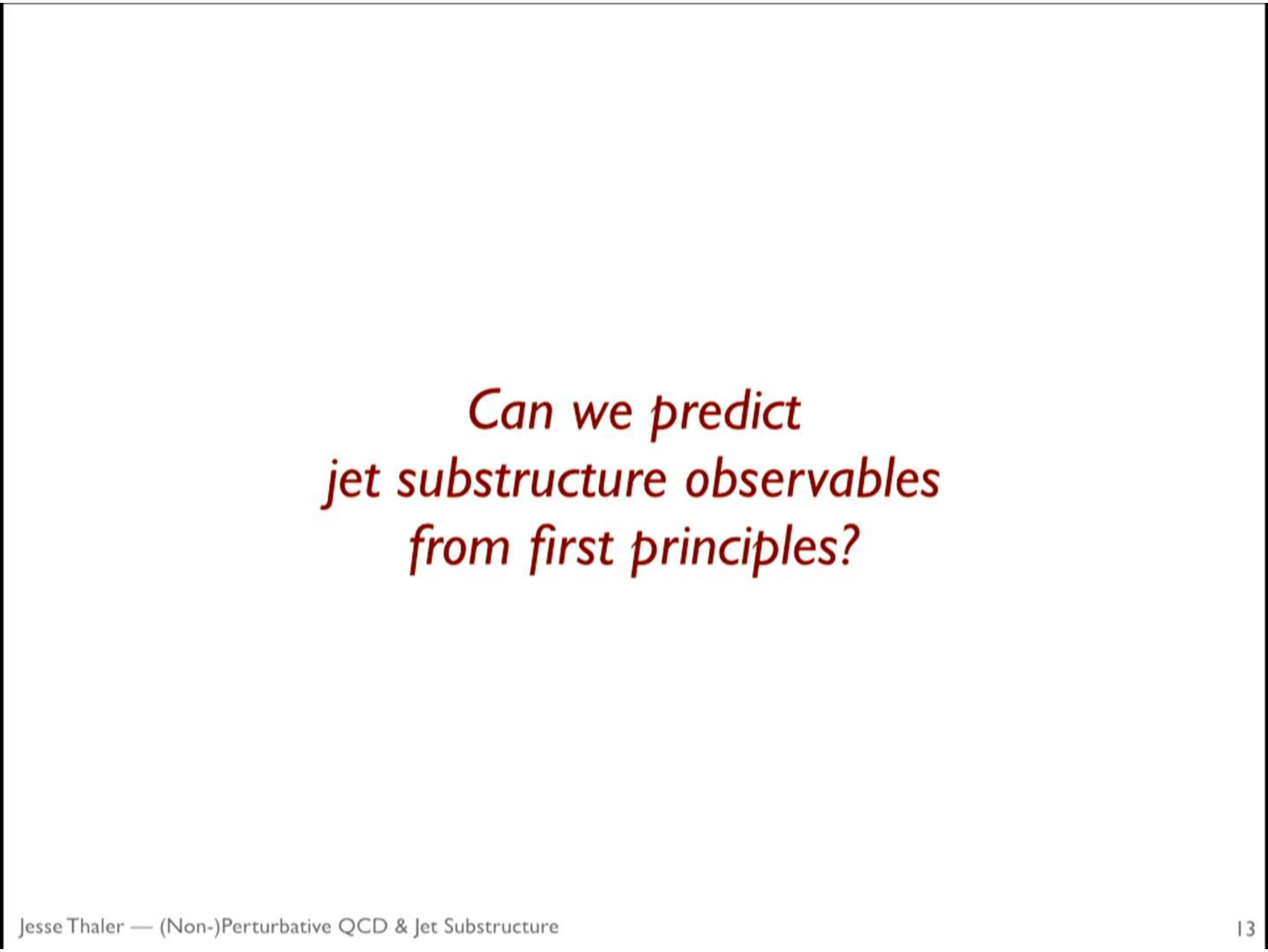


Multiplicity: $$\sum_{i \in \mathrm{jet}} 1$$ IRC Unsafe

Jet Mass: $$\sum_{i,j \in \mathrm{jet}} p_i \cdot p_j$$ IRC Safe

N-subjettiness: $$\sum_{i \in \mathrm{jet}} p_{T,i} \min \left\{ \Delta R_{i,1}, \Delta R_{i,2}, \ldots, \Delta R_{i,N} \right\}^\beta$$ IRC Safe

[JDT, Van Tilburg, 1011.2268, 1108.2701]



Can we predict
jet substructure observables
from first principles?

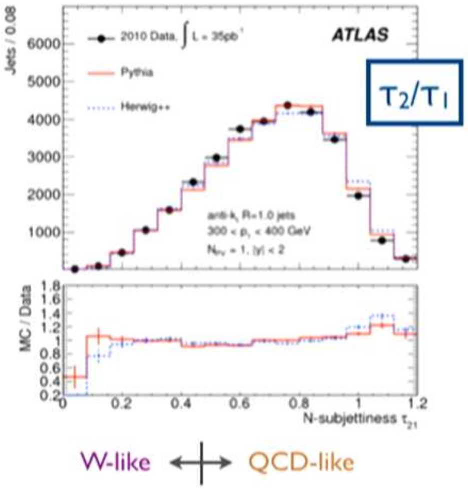
Ratio Observables?

IRC Safe ⇒ Useful Ratio

$$\tau_N \Rightarrow \frac{\tau_N}{\tau_{N-1}}$$

Ubiquitous in jet substructure

[ATLAS, 1203.4606]



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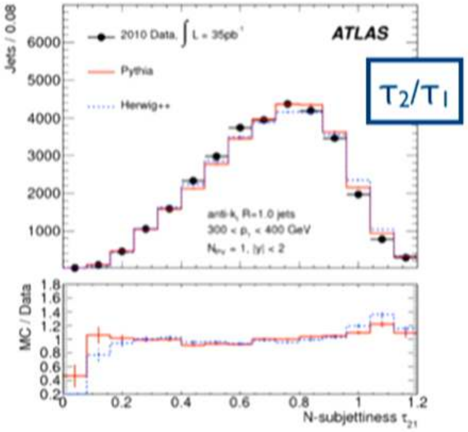


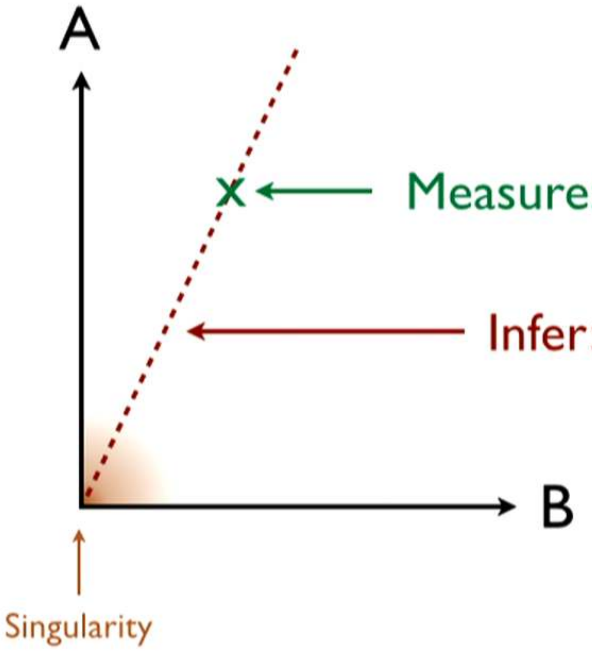
W-like ↔ QCD-like

$$\frac{\text{IRC Safe Numerator}}{\text{IRC Safe Denominator}} = \text{IRC Unsafe Ratio}$$

[Soyez, Salam, Kim, Dutta, Cacciari, 1211.2811]

WHAT?!

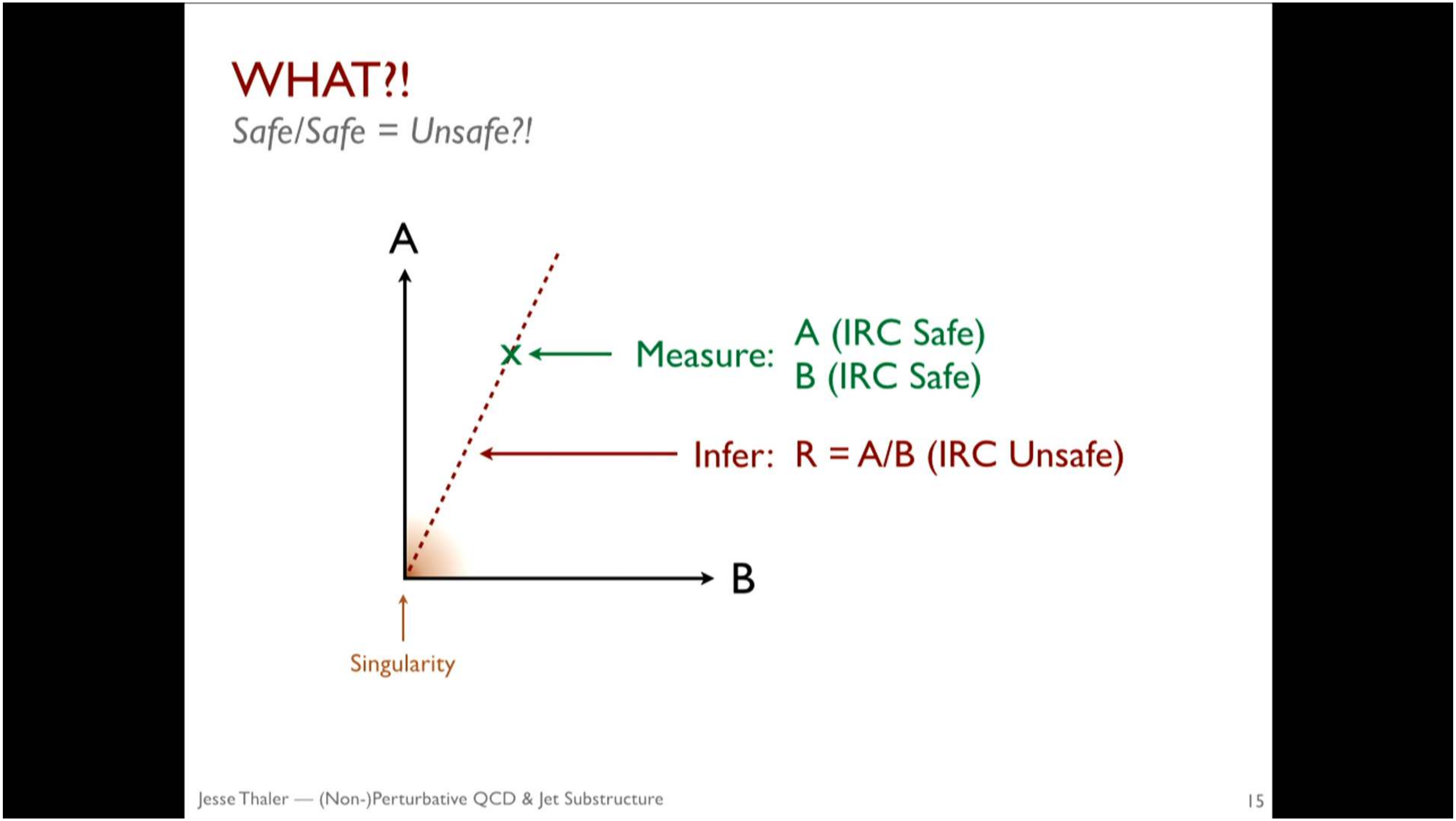
Safe/Safe = Unsafe?!



Measure: A (IRC Safe)
B (IRC Safe)

Infer: R = A/B (IRC Unsafe)

Singularity



WHAT?!

Safe/Safe = Unsafe?!

Measure: A (IRC Safe)
B (IRC Safe)

Infer: R = A/B (IRC Unsafe)

Simple Example

Ratios of Angularities

$$e_\beta \simeq \sum_{i \in \text{jet}} z_i \, (\theta_i)^\beta \qquad\qquad r = \frac{e_\alpha}{e_\beta}$$

momentum fraction   angle to axis

IR Limit
$z \rightarrow 0$

C Limit
$\theta \rightarrow 0$

[Berger, Kucs, Sterman, 2003; Ellis, Vermilion, Walsh, Hornig, Lee, 2010]
[Recoil-free Versions: Larkoski, Salam, JDT, 1305.0007; Larkoski, Neill, JDT, 1401.2158]

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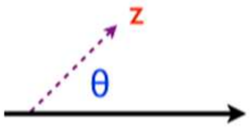
Single emission:

Order α_s (LO)



$$e_\beta = z\,\theta^\beta$$ IRC Safe

$$r = \theta^{\alpha-\beta}$$ IRC Unsafe

Simple Example

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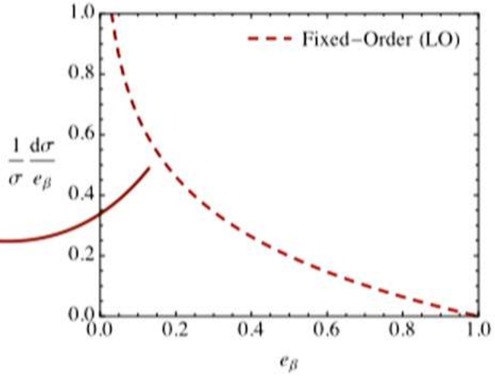

[Berger, Kucs, Sterman, 2003; Ellis, Vermilion, Walsh, Hornig, Lee, 2010]
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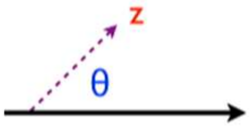

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Simple Example

Ratios of Angularities

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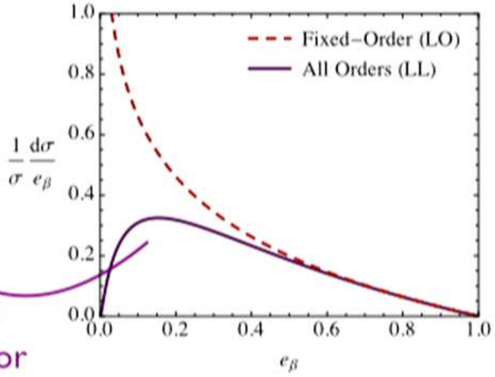
momentum fraction

angle to axis

IR Limit
$z \to 0$

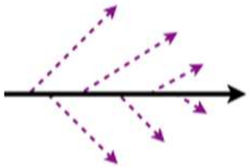
C Limit
$\theta \to 0$

Sudakov
Form Factor



[Berger, Kucs, Sterman, 2003; Ellis, Vermilion, Walsh, Hornig, Lee, 2010]
[Recoil-free Versions: Larkoski, Salam, JDT, 1305.0007; Larkoski, Neill, JDT, 1401.2158]

Many emissions:
All orders in α_s (LL)



$$e_\beta = z\,\theta^\beta$$ IRC Safe

$$r = \theta^{\cancel{\alpha-\beta}}$$ "Sudakov Safe"

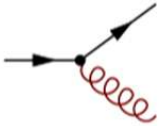
[Larkoski, JDT, 1307.1699]

The Key Realization

$$\frac{d\sigma}{dr} = \int de_\alpha \, de_\beta \, \frac{d^2\sigma}{de_\alpha de_\beta} \, \delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

↑ **IRC Unsafe**

Infinity at $O(\alpha_s)$



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"I can simultaneously measure e_α and e_β"

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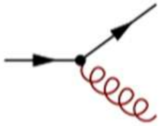


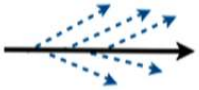
IRC Safe

"I can simultaneously measure e_α and e_β"

"Sudakov Safe"

Non-analytic in α_s

[Larkoski, JDT, 1307.1699]



Joint Resummable

"I can find a Sudakov form factor that resums large logarithms in e_α and e_β to *all orders* in α_s (e.g. a parton shower)"

After the Dust Settles

Ratio Cross Section (Details in Backup):

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha - \beta}\frac{1}{r}\left(1 - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\operatorname{erf}\left[\frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right] + 1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r}$$
$$-2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{\log r}{r}e^{-\frac{\alpha_s}{\pi}C_F\frac{\alpha}{(\alpha-\beta)^2}\log^2 r}$$

After the Dust Settles

Ratio Cross Section (Details in Backup):

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Expand in small α_s:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \sqrt{\alpha_s}\frac{\sqrt{C_F\beta}}{\alpha-\beta}\frac{1}{r} + \mathcal{O}(\alpha_s)$$

No Taylor expansion

Unsafe...

After the Dust Settles

Ratio Cross Section (Details in Backup):

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha-\beta}\frac{1}{r}\left(1-2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\mathrm{erf}\left[\frac{\sqrt{\alpha_s C_F\beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right]+1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r}$$
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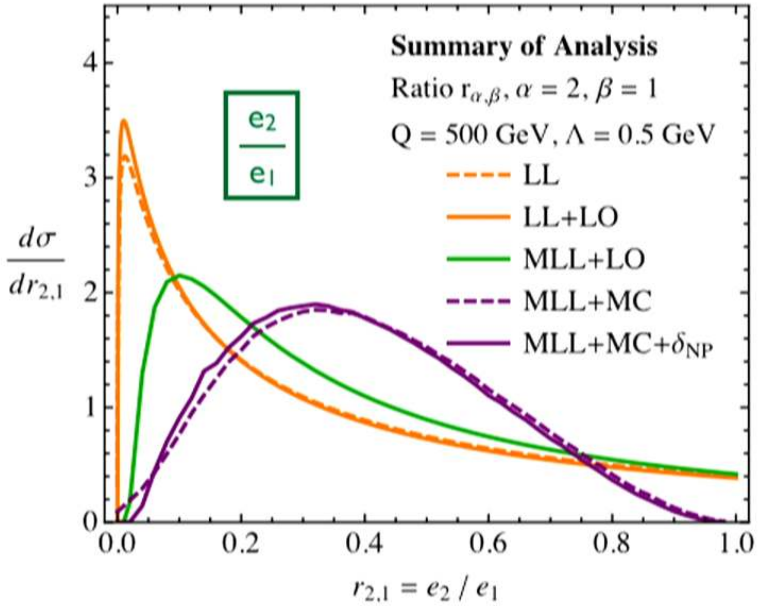
No Taylor expansion

Finite cross section

Unsafe... ...but Calculable

Systematically Improvable



Minimal set of effects for comparison to LHC data:

= Sudakov factor
+ tree-level matching
+ running α_s
+ multiple emissions
+ hadronization correction

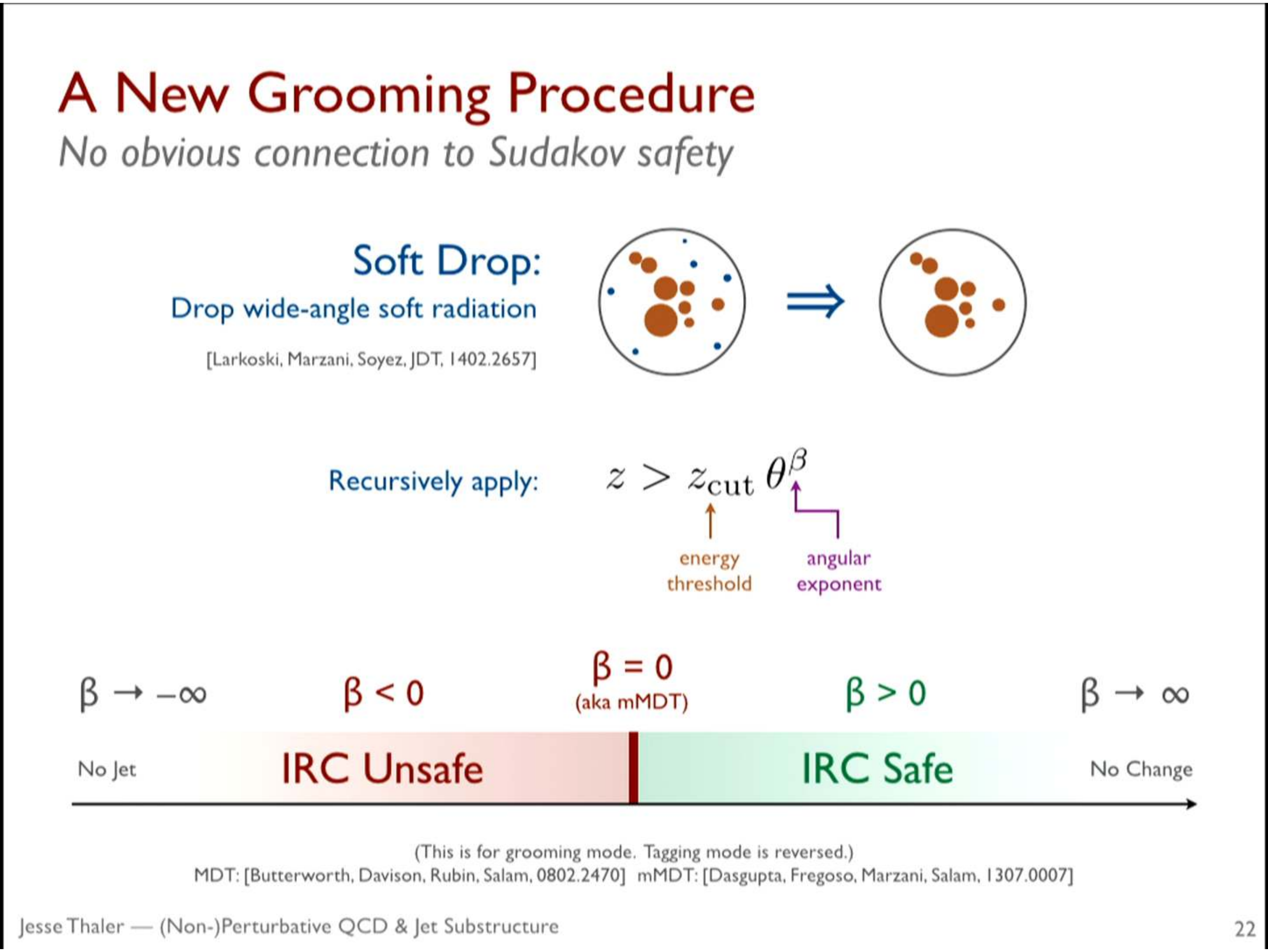
[Larkoski, JDT, 1307.1699]

NLL: [Larkoski, Moult, Neill, 1401.4458]

Predictions for jet substructure from first-principles QCD

“Interesting, but $\sqrt{\alpha_s}$
isn’t all that surprising…”

(In the Pink Book, $\sqrt{\alpha_s}$ appears in anomalous dimension of fragmentation function moments.)



A New Grooming Procedure

No obvious connection to Sudakov safety

Soft Drop:

Drop wide-angle soft radiation

[Larkoski, Marzani, Soyez, JDT, 1402.2657]

Recursively apply:

$$z > z_{\mathrm{cut}}\, \theta^{\beta}$$

energy threshold

angular exponent

(This is for grooming mode. Tagging mode is reversed.)

MDT: [Butterworth, Davison, Rubin, Salam, 0802.2470] mMDT: [Dasgupta, Fregoso, Marzani, Salam, 1307.0007]

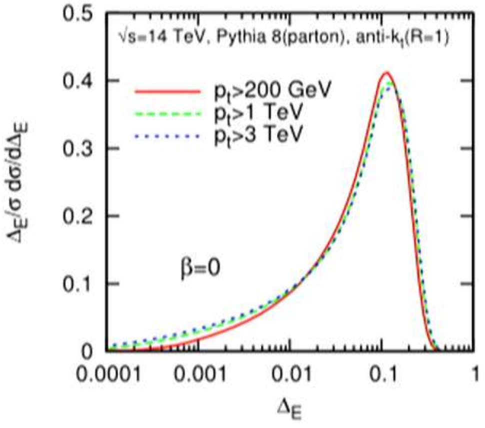
A Standard Candle for Jets?

Fractional Energy Loss:

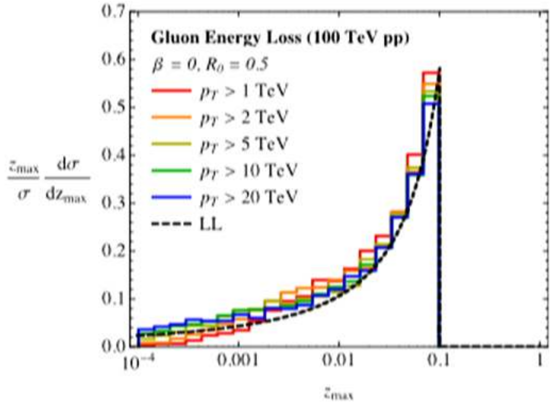
$$\left.\frac{d\log\sigma}{d\log\Delta_E}\right|_{\beta=0} \simeq \frac{\log\frac{1}{z_{\mathrm{cut}}}}{\log^2\Delta_E}$$

no α_s at fixed coupling (!)
≈ independent of quark vs. gluon
≈ independent of jet p_T, jet radius

14 TeV



100 TeV



[Larkoski, Marzani, Soyez, JDT, 1402.2657; Larkoski, JDT, 1406.7011]

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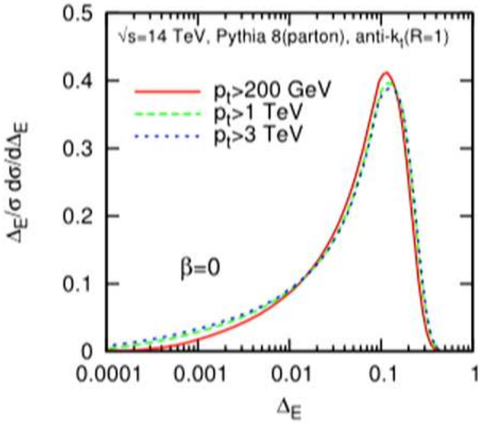
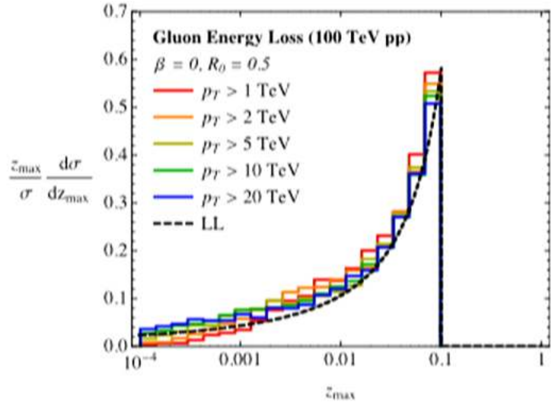
≈ independent of jet p_T, jet radius

14 TeV

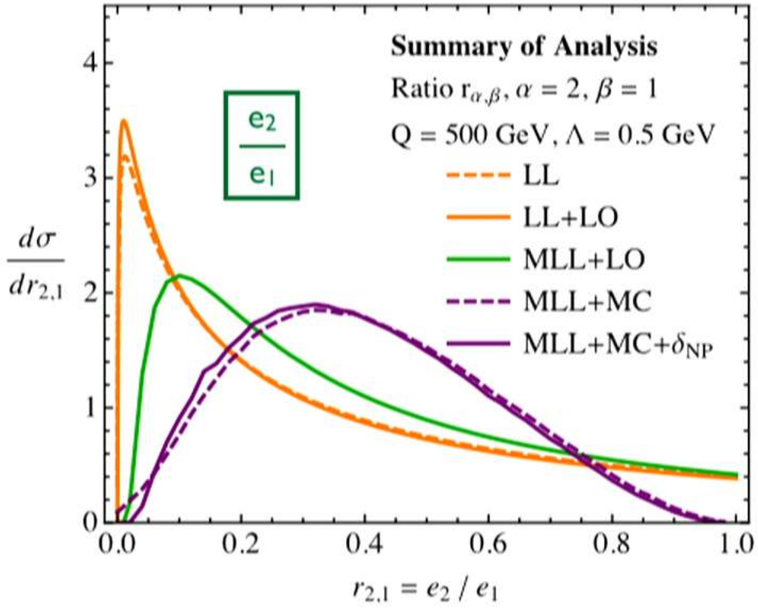


100 TeV



[Larkoski, Marzani, Soyez, JDT, 1402.2657; Larkoski, JDT, 1406.7011]

Systematically Improvable



Minimal set of effects for comparison to LHC data:

= Sudakov factor
+ tree-level matching
+ running α_s
+ multiple emissions
+ hadronization correction

[Larkoski, JDT, 1307.1699]

NLL: [Larkoski, Moult, Neill, 1401.4458]

Predictions for jet substructure from first-principles QCD

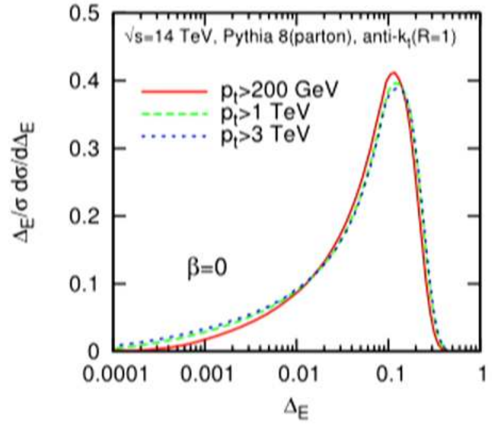
A Standard Candle for Jets?

Fractional Energy Loss:

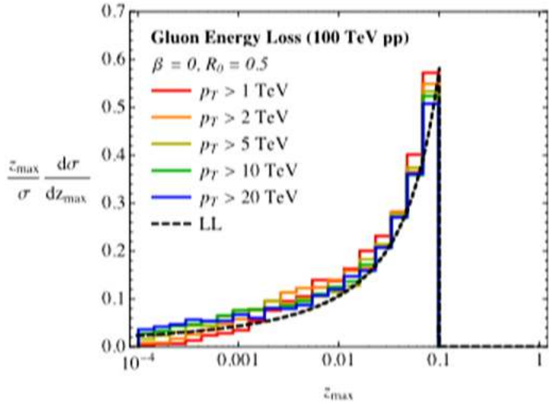
$$\frac{d\log\sigma}{d\log\Delta_E}\bigg|_{\beta=0} \simeq \frac{\log\frac{1}{z_{\rm cut}}}{\log^2\Delta_E}$$

no α_s at fixed coupling (!)
≈ independent of quark vs. gluon
≈ independent of jet p_T, jet radius

14 TeV



100 TeV



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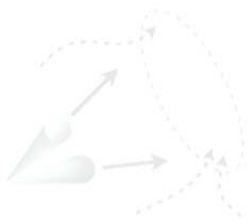
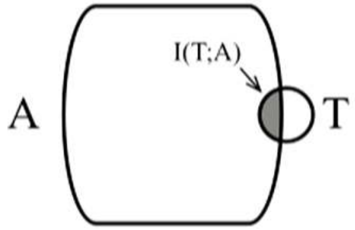


Why Jet Substructure?

$$\frac{\text{IRC Safe}}{\text{IRC Safe}} = \text{Sudakov Safe}$$

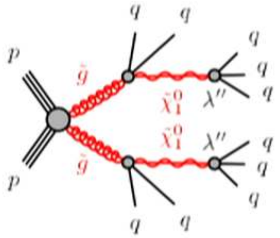
From IRC Safe to Sudakov Safe



Mutual Information with the Truth

Quark/Gluon Discrimination

The White Whale of Jet Substructure



[ATLAS CONF-2013-091]

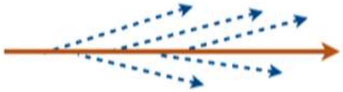
Often:

BSM Signals ⇒ Quarks

SM Backgrounds ⇒ Gluons

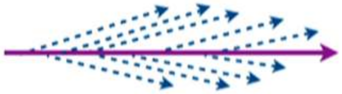
Jets, Simplified:

Quark Jet



$C_F = 4/3$

Gluon Jet



$C_A = 3$

Casimir Scaling:

Quark Efficiency = 50%

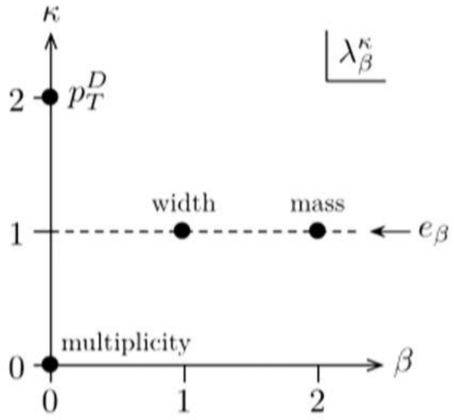
Gluon Mistag = $(50\%)^{9/4}$ = 21%

Generalized Angularities

$$\lambda_\beta^\kappa = \sum_{i\in\text{jet}} z_i^\kappa \theta_i^\beta$$

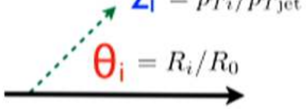
momentum fraction

angle to recoil-free axis



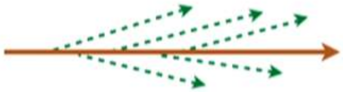
[Larkoski, JDT, Waalewijn, 1408.3122]

Measure of gluon radiation about hard jet core

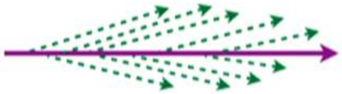

$z_i = p_{Ti}/p_{T\text{jet}}$

$\theta_i = R_i/R_0$

Quark Jet



Gluon Jet



[For a more complete catalog, see Gallicchio, Schwartz, 1106.3076, 1211.7038]

(NLL calculations in backup)

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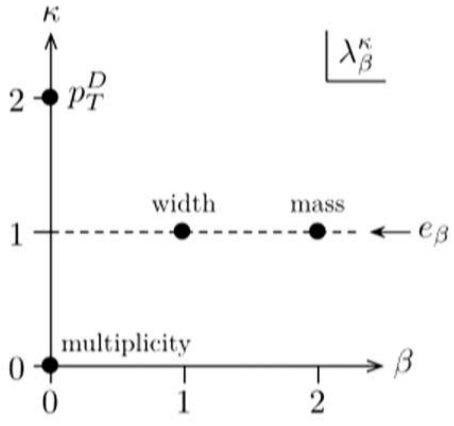

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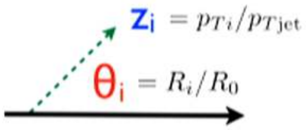


[For a more complete catalog, see Gallicchio, Schwartz, 1106.3076, 1211.7038]

Recoil-Free:

Measurement Axis ≈ Hard Parton

Jet Axis (Mean)

e.g. WTA Axis (Median)

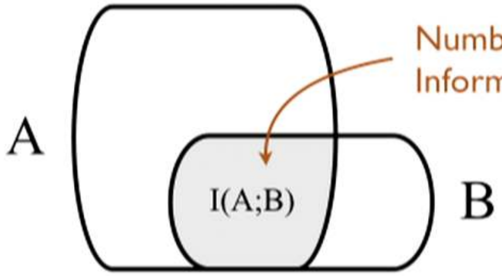
[Bertolini, Chan, JDT, 1310.7584; Larkoski, Neill, JDT, 1401.2158; Salam, unpublished]

Can we understand quark/gluon separation from first principles?

Mutual Information

Robust Measure of Correlations

$$I(A;B) = \int \mathrm{d}a\,\mathrm{d}b\,p(a,b)\log_2\frac{p(a,b)}{p(a)p(b)}$$



Number of Shared Information "Bits"

I is for "Intersection"

Can obtain directly from double differential cross section

$$p(a,b) = \frac{1}{\sigma}\frac{\mathrm{d}^2\sigma}{\mathrm{d}a\,\mathrm{d}b}$$

[see, e.g. Larkoski, Moult, Neill, 1401.4458]

Related to (binned) Shannon entropy

$$H(A) = -\sum_{a\in A} p(a)\log_2 p(a)$$

$$I(A;B) = H(A) + H(B) - H(A,B)$$

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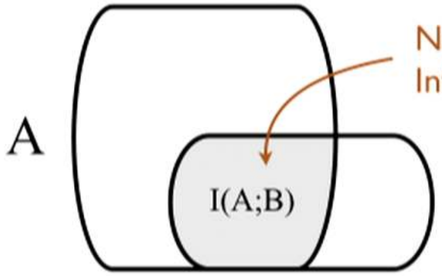

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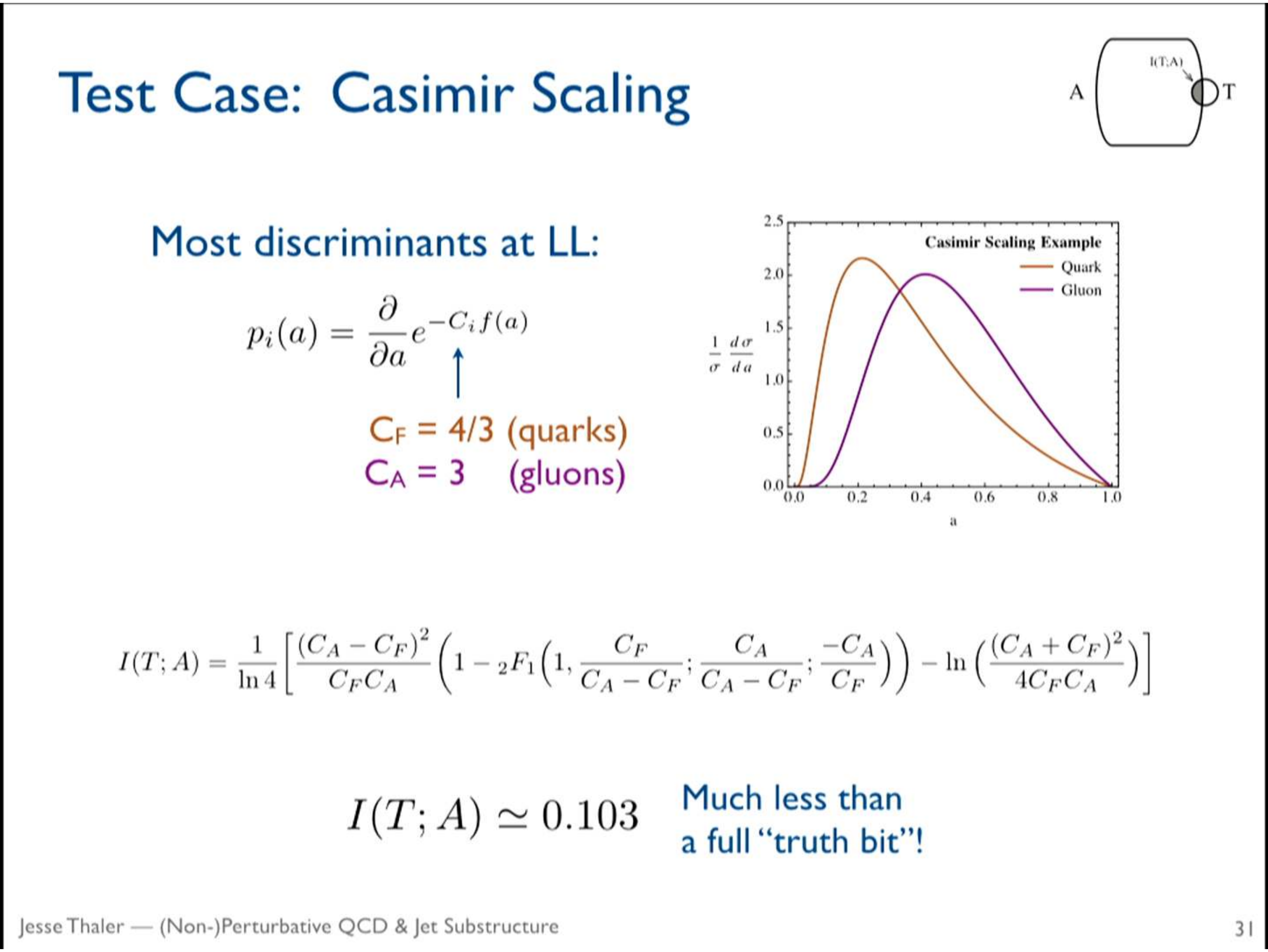


Test Case: Casimir Scaling

Most discriminants at LL:

$$p_i(a) = \frac{\partial}{\partial a} e^{-C_i f(a)}$$

↑

C_F = 4/3 (quarks)
C_A = 3 (gluons)

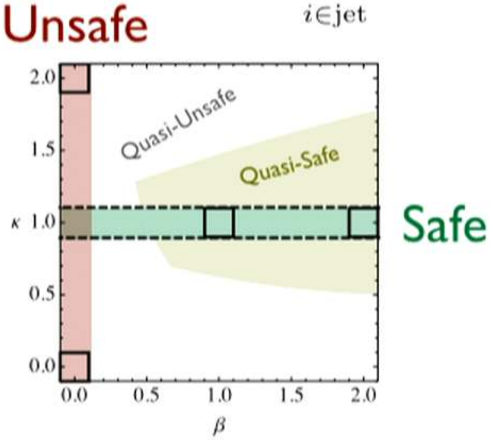
$$I(T;A) = \frac{1}{\ln 4}\left[\frac{(C_A - C_F)^2}{C_F C_A}\left(1 - {}_2F_1\left(1, \frac{C_F}{C_A - C_F}; \frac{C_A}{C_A - C_F}; \frac{-C_A}{C_F}\right)\right) - \ln\left(\frac{(C_A + C_F)^2}{4 C_F C_A}\right)\right]$$

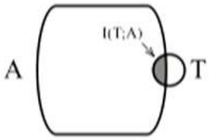
$$I(T;A) \simeq 0.103$$

Much less than a full "truth bit"!

Legend

$$\lambda_\beta^\kappa = \sum_{i\in\text{jet}} z_i^\kappa \theta_i^\beta$$





gluon mistag @
50% quark efficiency

0.3 ← ≈ 5%

← ≈ 10%

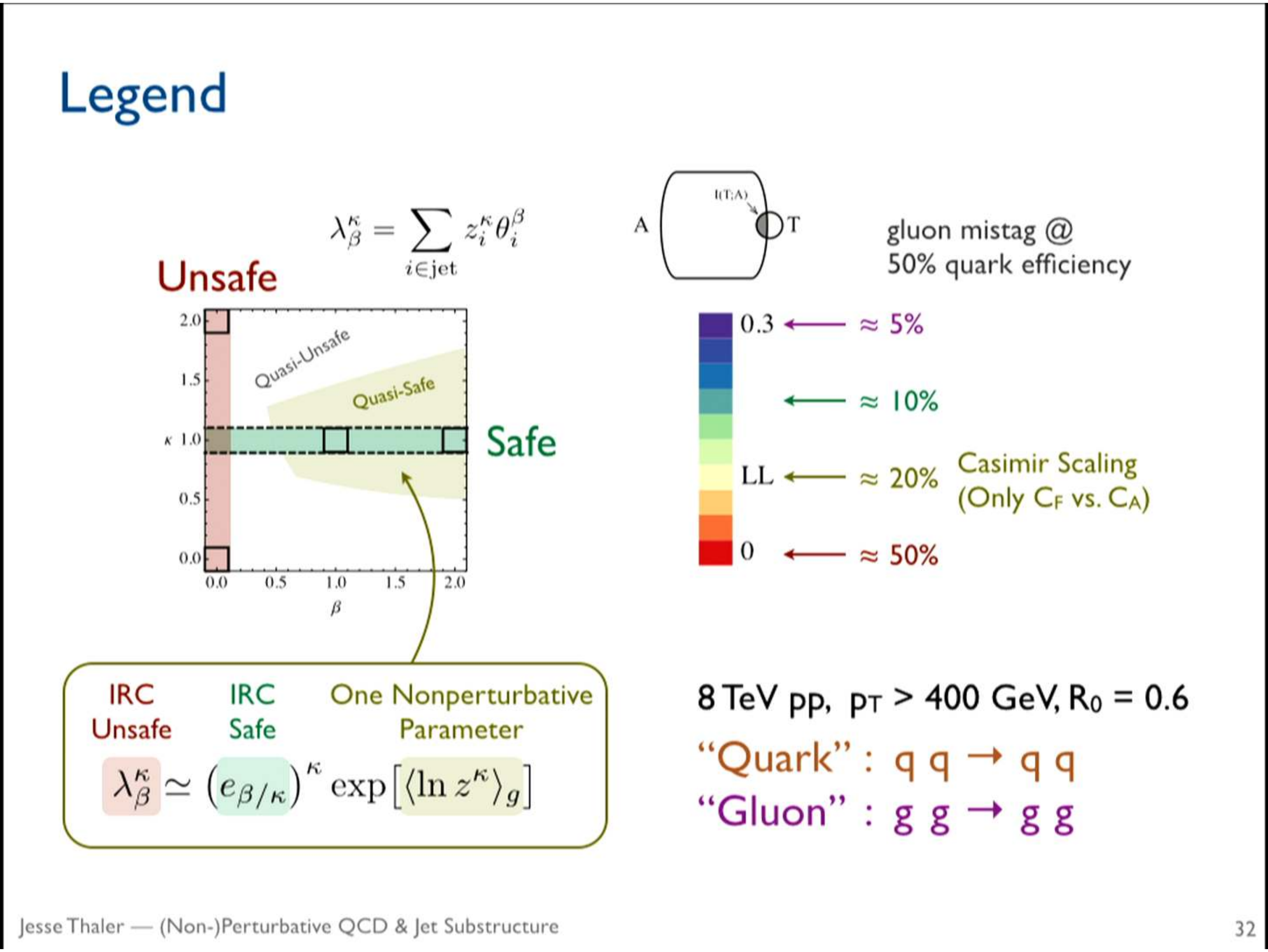
LL ← ≈ 20% Casimir Scaling (Only C_F vs. C_A)

0 ← ≈ 50%

8 TeV pp, p_T > 400 GeV, R_0 = 0.6

"Quark" : q q → q q

"Gluon" : g g → g g



Legend

$$\lambda_\beta^\kappa = \sum_{i\in\text{jet}} z_i^\kappa \theta_i^\beta$$

Unsafe

Safe

IRC Unsafe

IRC Safe

One Nonperturbative Parameter

$$\lambda_\beta^\kappa \simeq \left(e_{\beta/\kappa}\right)^\kappa \exp\left[\langle \ln z^\kappa \rangle_g\right]$$

gluon mistag @ 50% quark efficiency

← ≈ 5%

← ≈ 10%

← ≈ 20%

Casimir Scaling (Only C_F vs. C_A)

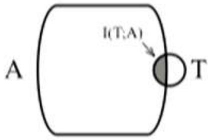
← ≈ 50%

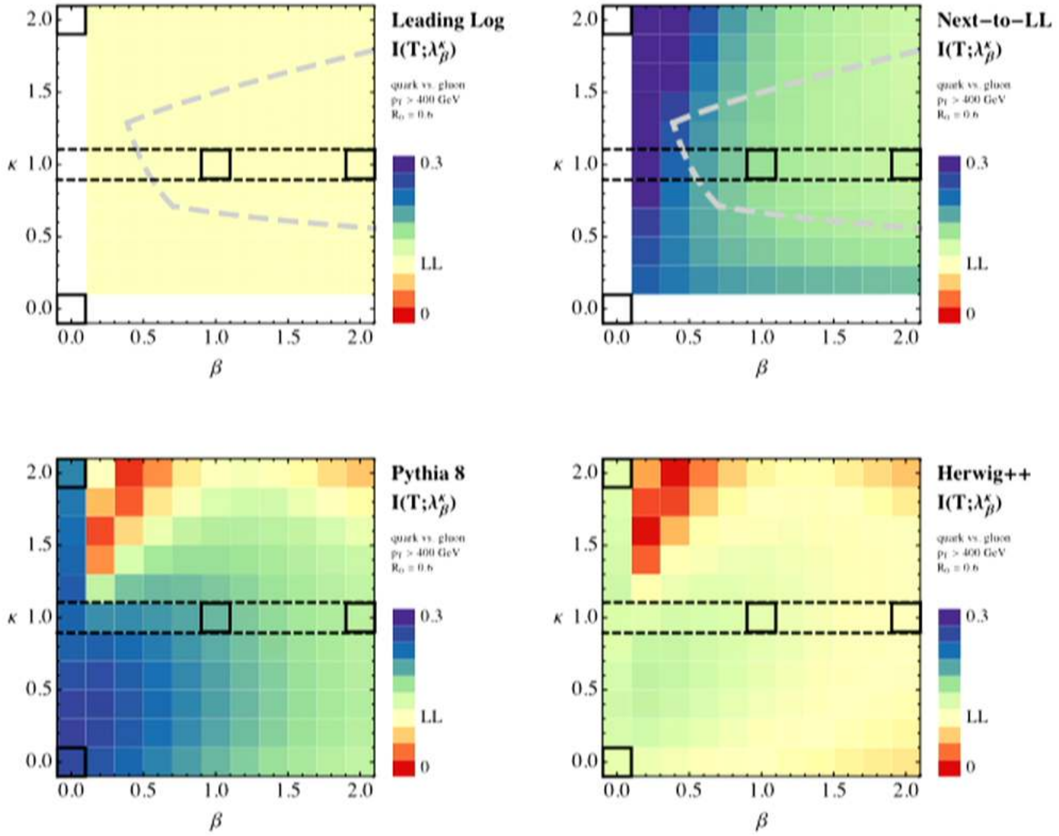
8 TeV pp, $p_T > 400$ GeV, $R_0 = 0.6$

"Quark" : q q → q q

"Gluon" : g g → g g

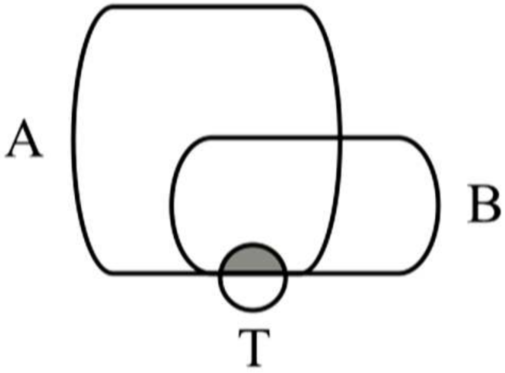
Truth Overlap for One Angularity



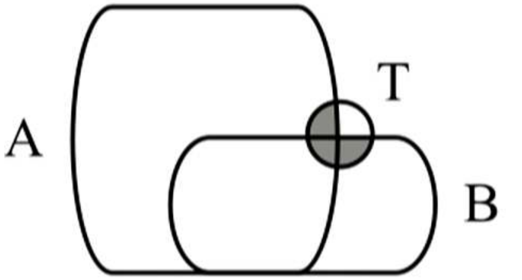


Joint Discrimination Power

"Redundant Variables"

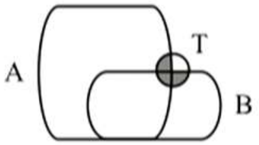


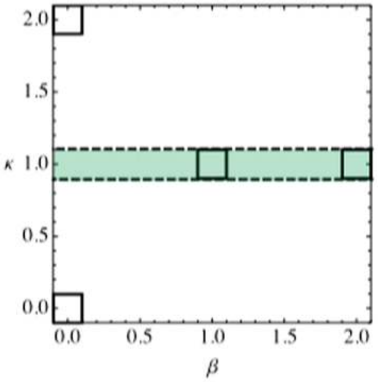
"Complementary Variables"

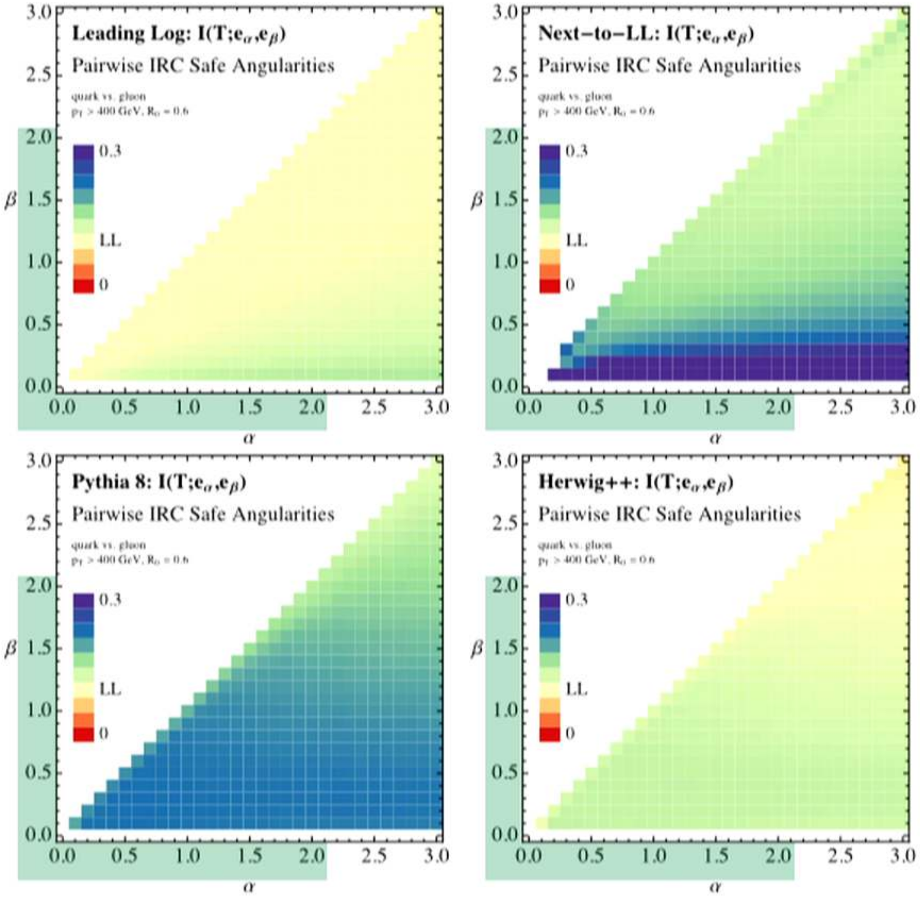


I(A;B): Same correlations
I(T;A) and I(T;B): Same individual discrimination power
I(T;A,B): Different joint discrimination power

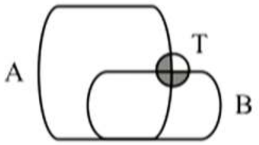
Truth Overlap for Two Angularities

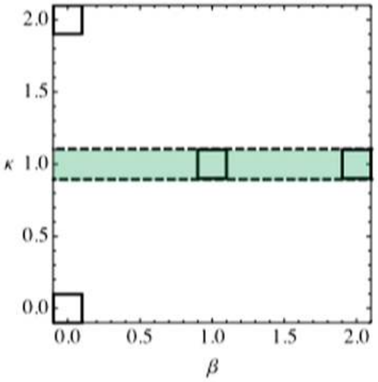


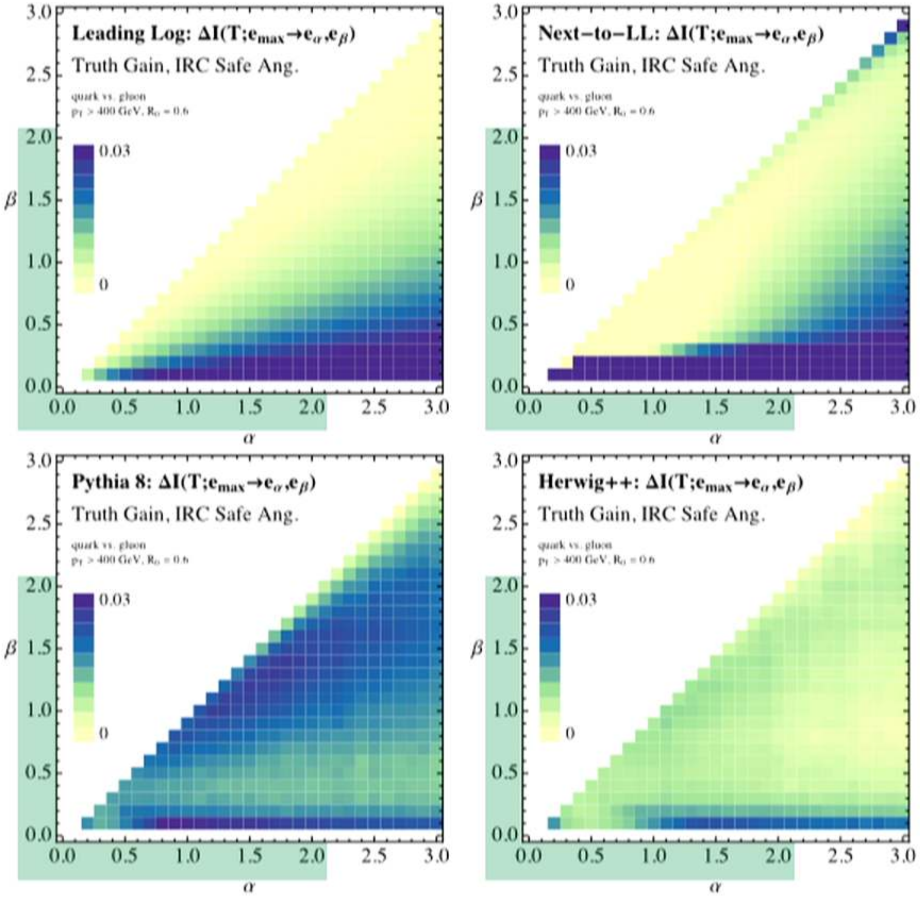




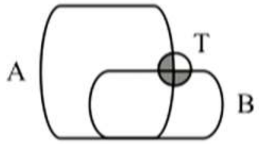
Truth Gain from One ⇒ Two

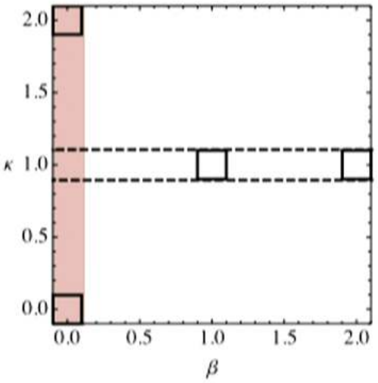


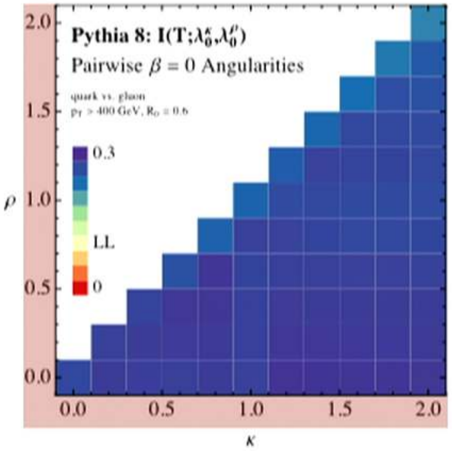


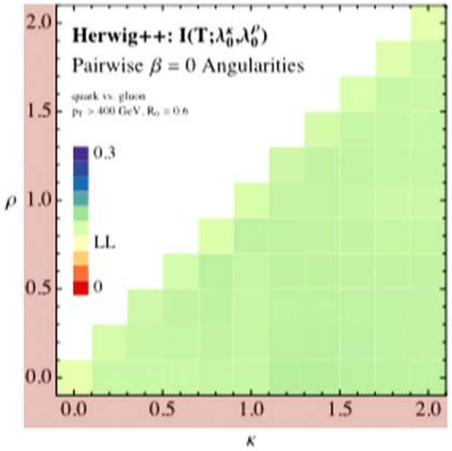


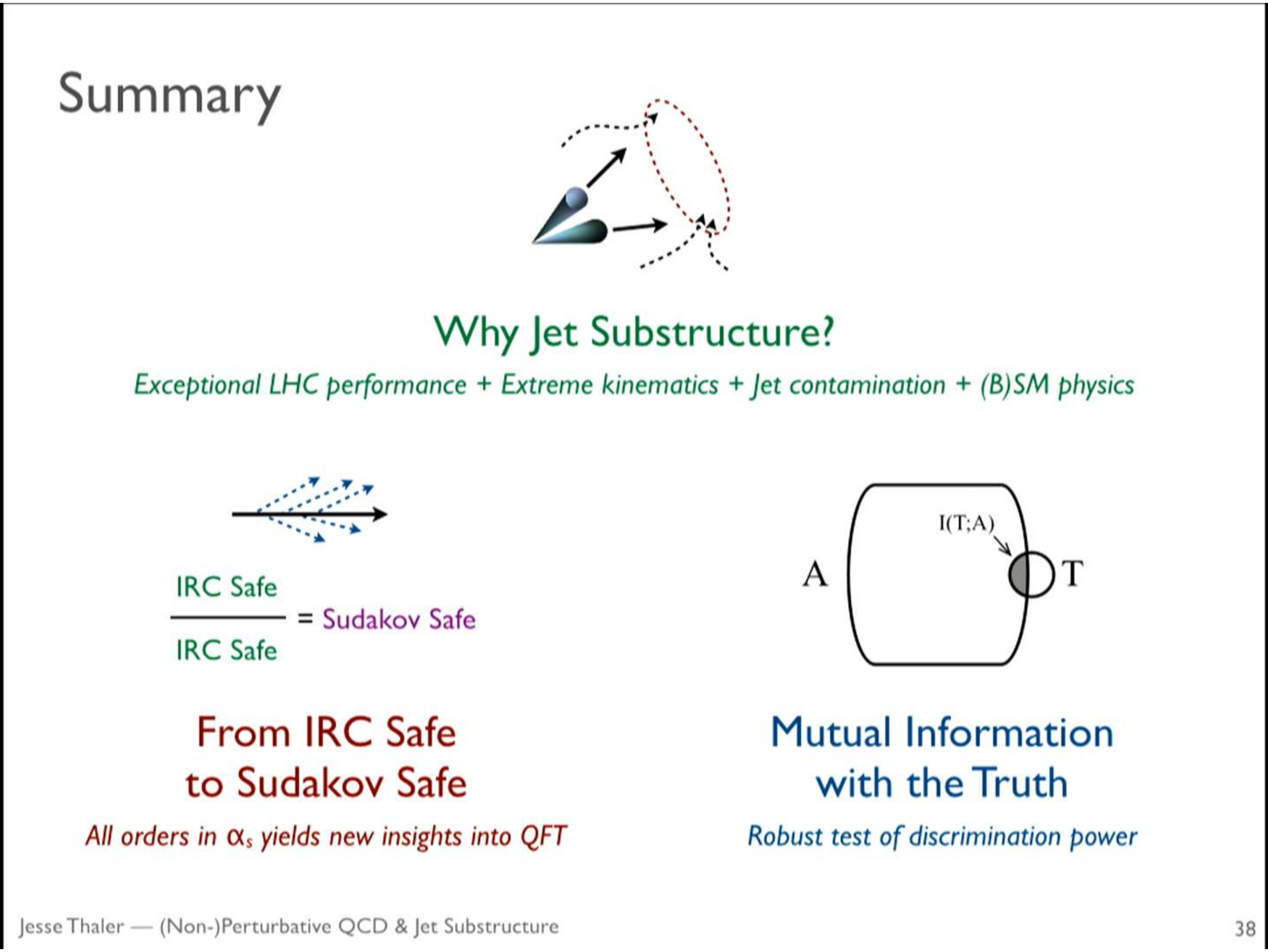
Truth Overlap for Two Angularities











Summary

Why Jet Substructure?

Exceptional LHC performance + Extreme kinematics + Jet contamination + (B)SM physics

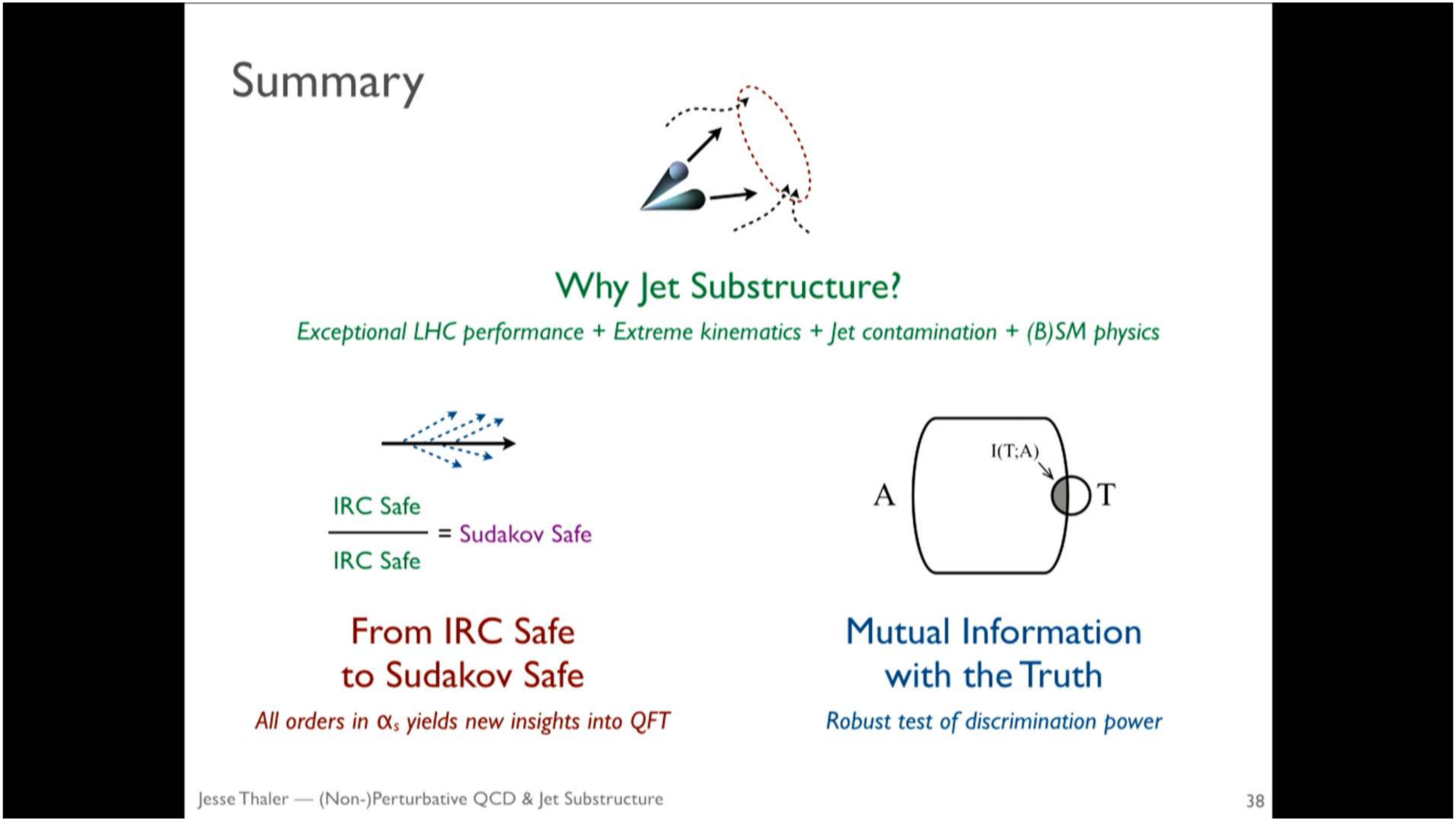
$$\frac{\text{IRC Safe}}{\text{IRC Safe}} = \text{Sudakov Safe}$$

From IRC Safe to Sudakov Safe

All orders in α_s yields new insights into QFT

Mutual Information with the Truth

Robust test of discrimination power



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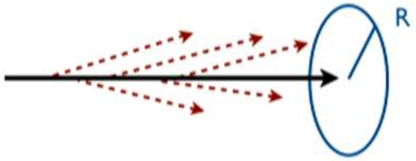
Jets, Simplified

Eikonal Hard Quark/Gluon...



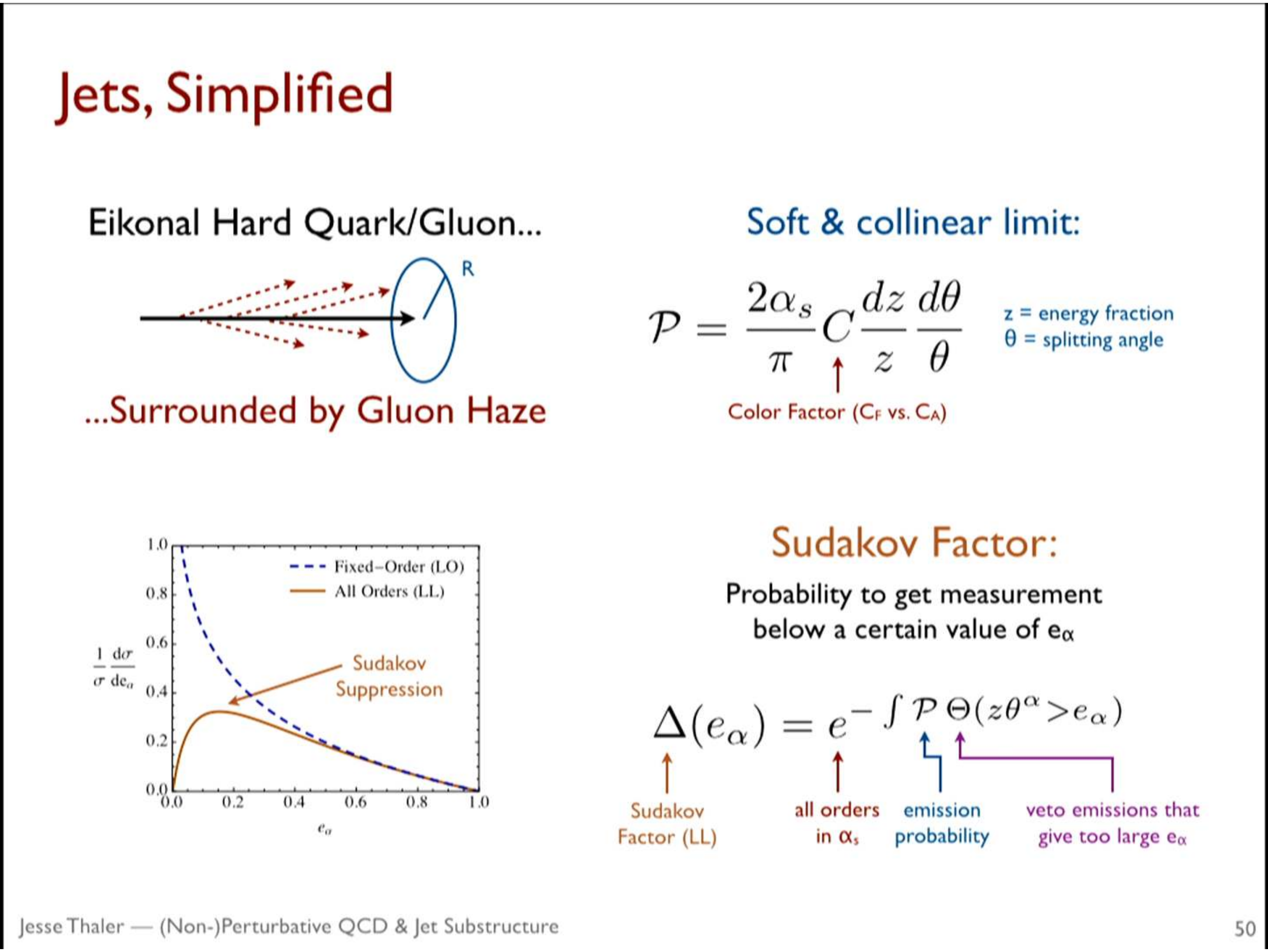
...Surrounded by Gluon Haze

Soft & collinear limit:

$$\mathcal{P} = \frac{2\alpha_s}{\pi} C \frac{dz}{z} \frac{d\theta}{\theta}$$

↑ Color Factor (C_F vs. C_A)

z = energy fraction
θ = splitting angle



Jets, Simplified

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Color Factor (C_F vs. C_A)

z = energy fraction
θ = splitting angle

Sudakov Factor:

Probability to get measurement below a certain value of e_α

$$\Delta(e_\alpha) = e^{-\int \mathcal{P}\,\Theta(z\theta^\alpha > e_\alpha)}$$

Sudakov Factor (LL)

all orders in α_s

emission probability

veto emissions that give too large e_α

Turning the Crank

Sudakov Factor:

$$\Delta(e_\alpha, e_\beta) =$$ Probability to get measurement below a certain value of both e_α & e_β

Double Differential Cross Section:

$$\frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha\, de_\beta} = \frac{\partial}{\partial e_\alpha}\frac{\partial}{\partial e_\beta}\Delta(e_\alpha, e_\beta)$$

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \int de_\alpha\, de_\beta\, \frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha de_\beta}\,\delta\left(r - \frac{e_\alpha}{e_\beta}\right)$$

Turning the Crank

Sudakov Factor:

$$\Delta(e_\alpha, e_\beta) = \exp\left[-\frac{\alpha_s}{\pi} C_F \left(\frac{1}{\beta}\log^2 e_\beta + \frac{1}{\alpha-\beta}\log^2\frac{e_\alpha}{e_\beta}\right)\right]$$

Double Differential Cross Section:

$$\frac{d^2\sigma^{\mathrm{LL}}}{de_\alpha\, de_\beta} = \left(\frac{2\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{1}{e_\alpha e_\beta} + \frac{4\alpha_s^2}{\pi^2}\frac{C_F^2}{\beta(\alpha-\beta)^2}\frac{1}{e_\alpha e_\beta}\log\frac{e_\beta}{e_\alpha}\log\frac{e_\alpha^\beta}{e_\beta^\alpha}\right)\Delta(e_\alpha, e_\beta)$$

(Cross check: Reduces to known single differential)

Ratio Cross Section:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \frac{\sqrt{\alpha_s C_F \beta}}{\alpha-\beta}\frac{1}{r}\left(1 - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r\right)\left(\mathrm{erf}\left[\frac{\sqrt{\alpha_s C_F \beta}}{\sqrt{\pi}(\alpha-\beta)}\log r\right]+1\right)e^{-\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\log^2 r} - 2\frac{\alpha_s}{\pi}\frac{C_F}{\alpha-\beta}\frac{\log r}{r}e^{-\frac{\alpha_s}{\pi}C_F\frac{\alpha}{(\alpha-\beta)^2}\log^2 r}$$

Turning the Crank

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Expand in small α_s:

$$\frac{d\sigma^{\mathrm{LL}}}{dr} = \sqrt{\alpha_s}\frac{\sqrt{C_F\beta}}{\alpha-\beta}\frac{1}{r} + \mathcal{O}(\alpha_s)$$

Not a valid
Taylor expansion

Finite cross section
for all r (even r = 0)

Unsafe... ...but Calculable