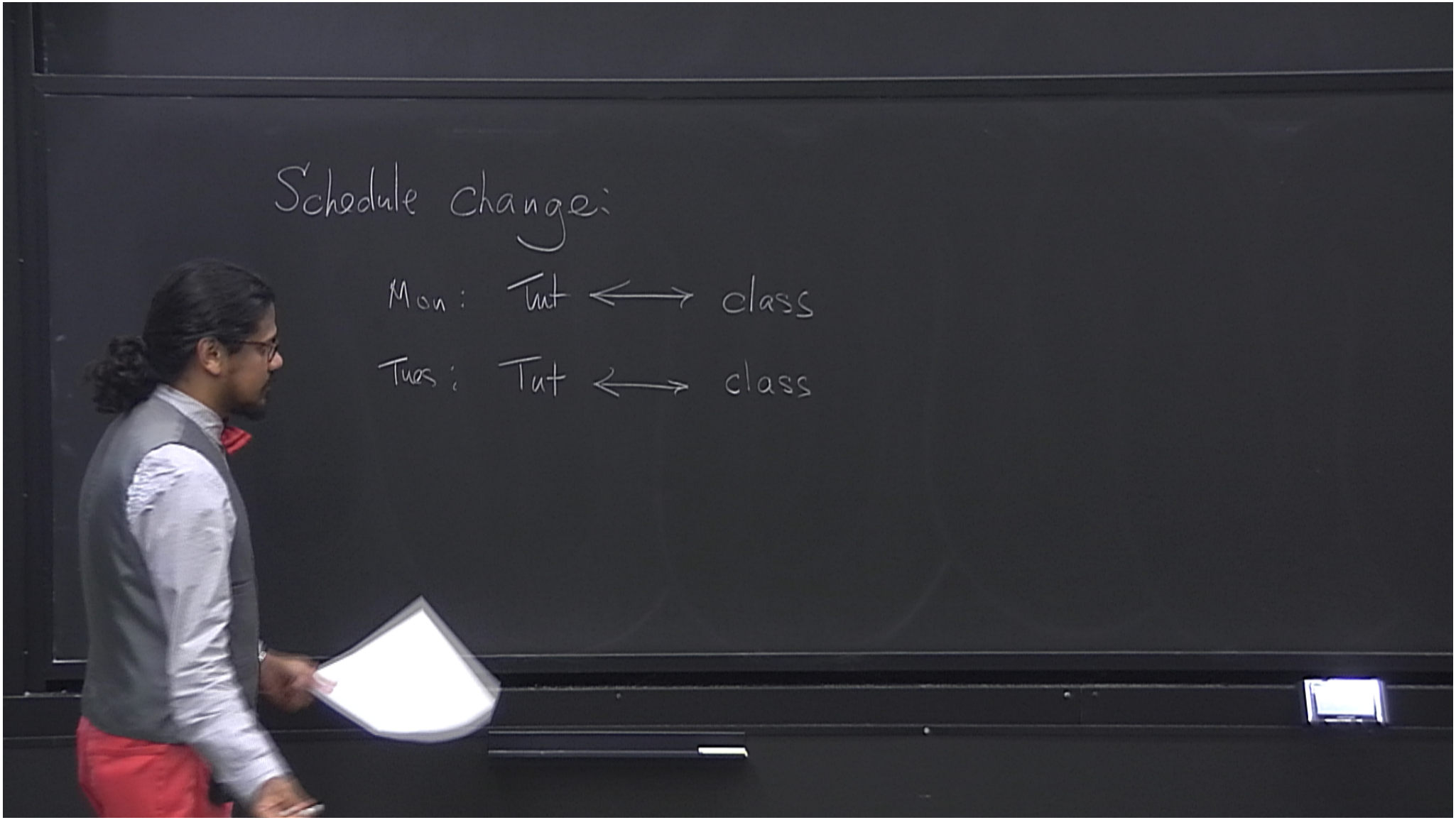
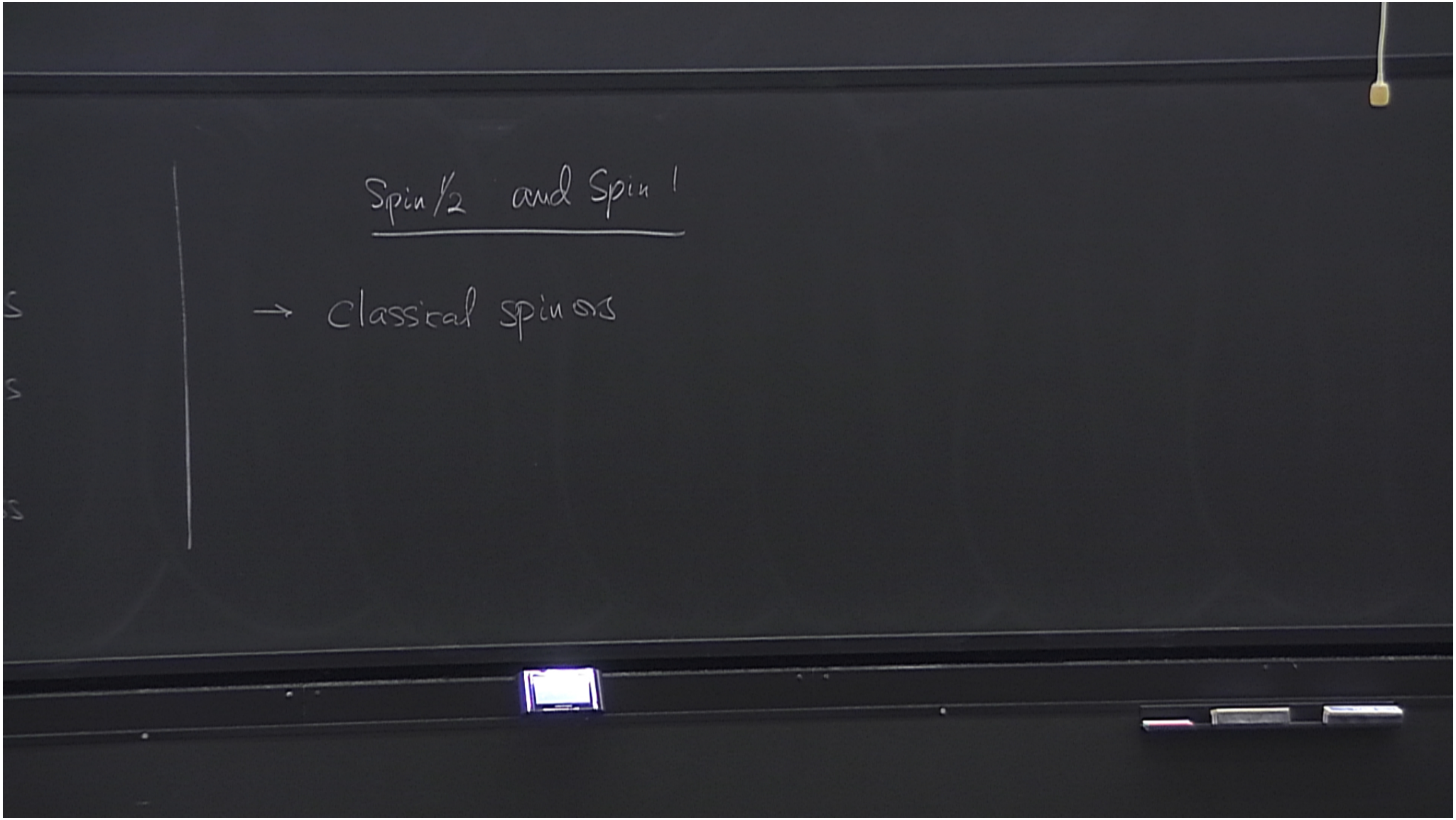
Title: 14/15 PSI - Quantum Field Theory I (A) - Lecture 8

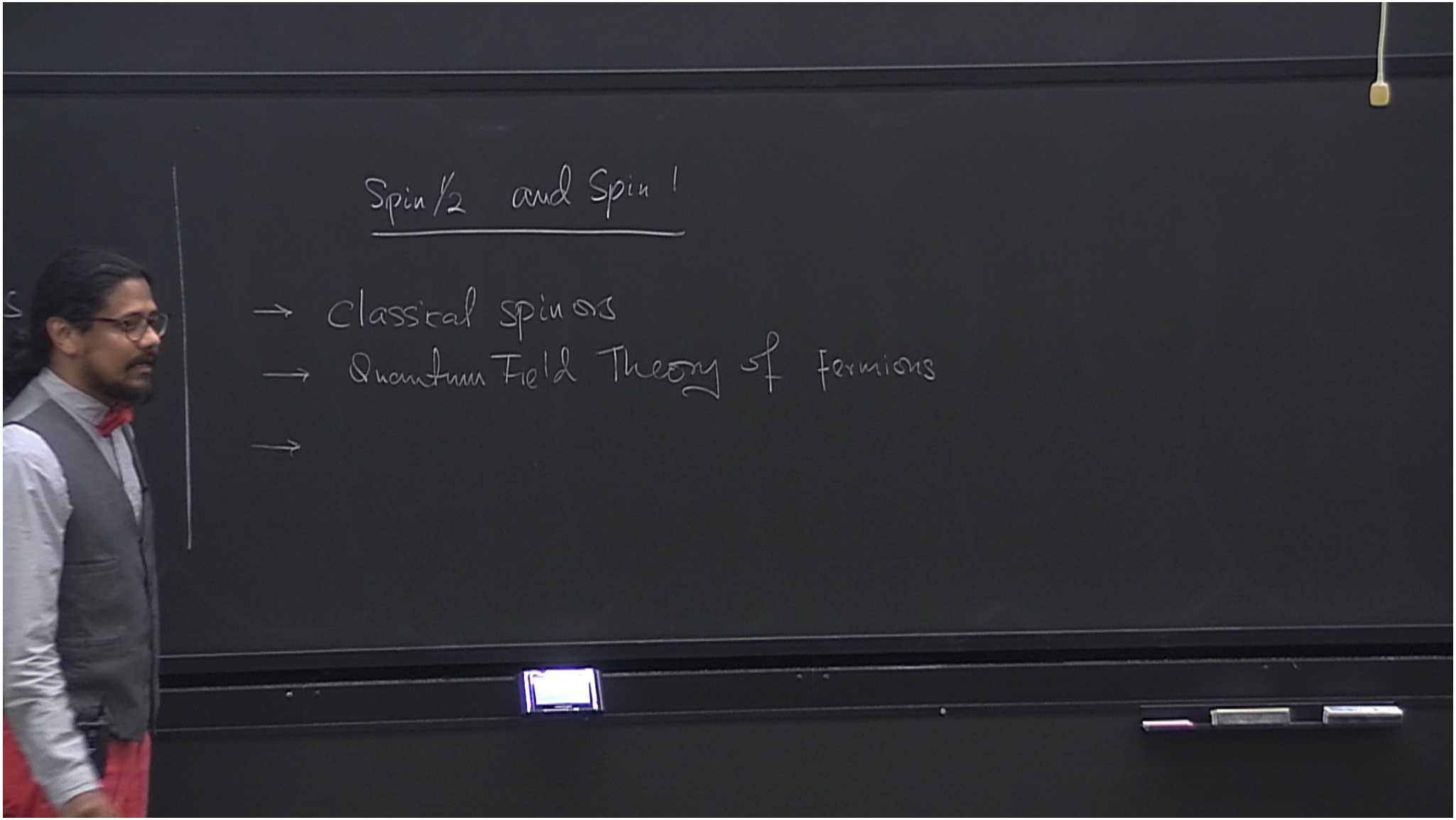
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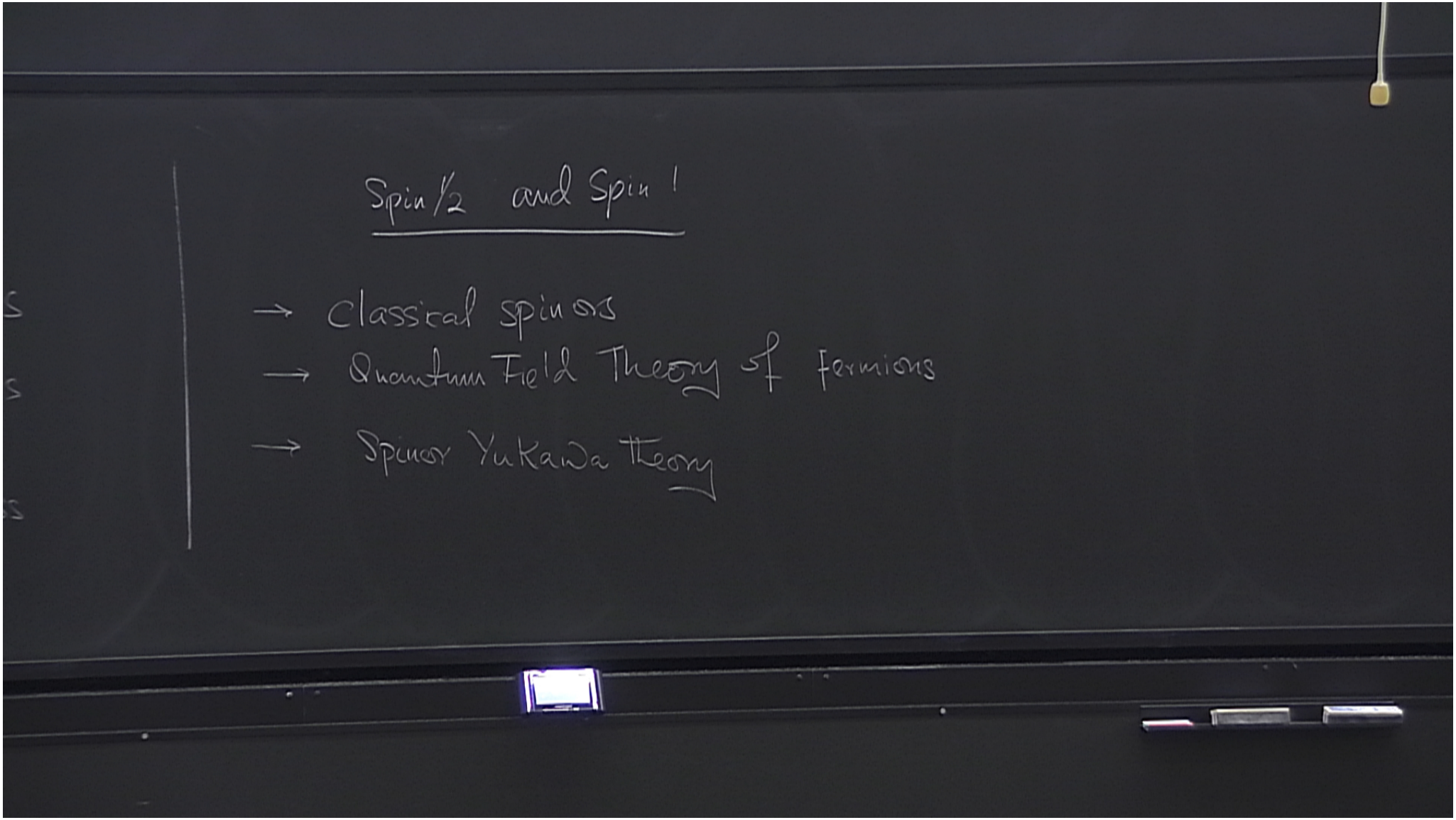
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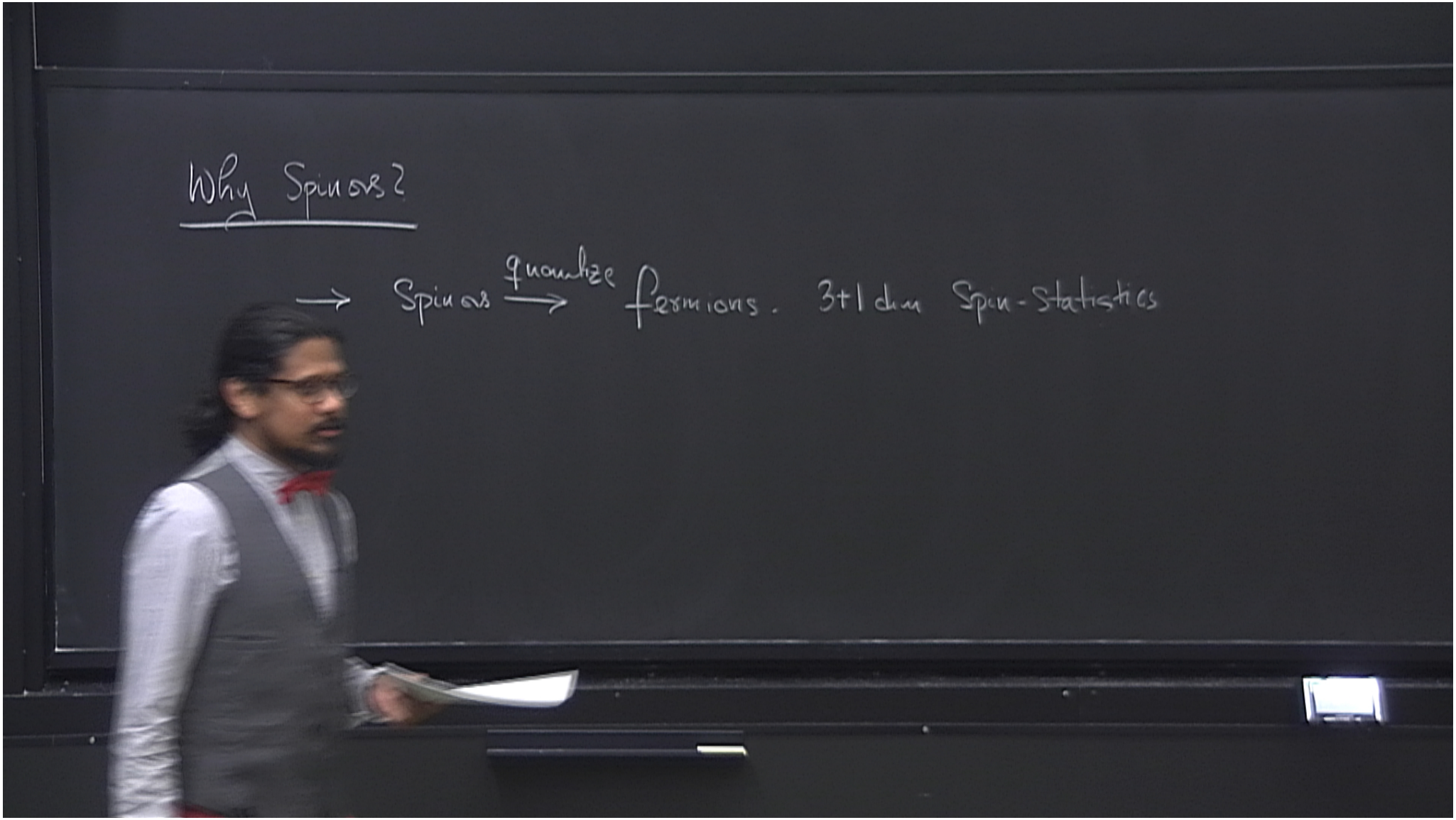
Abstract:

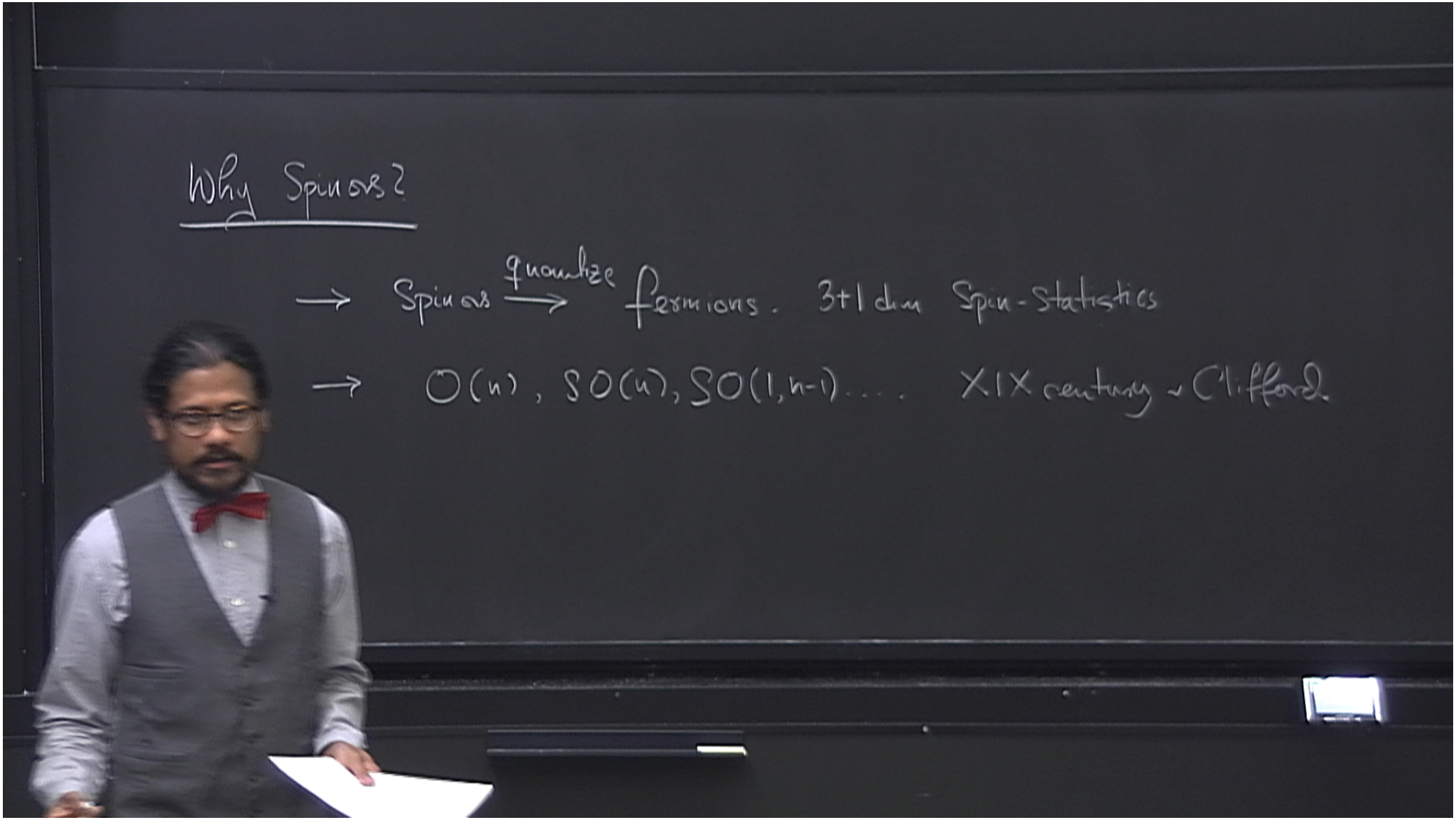

Schedule change:
Mon: Tut ⟷ class
Tues: Tut ⟷ class

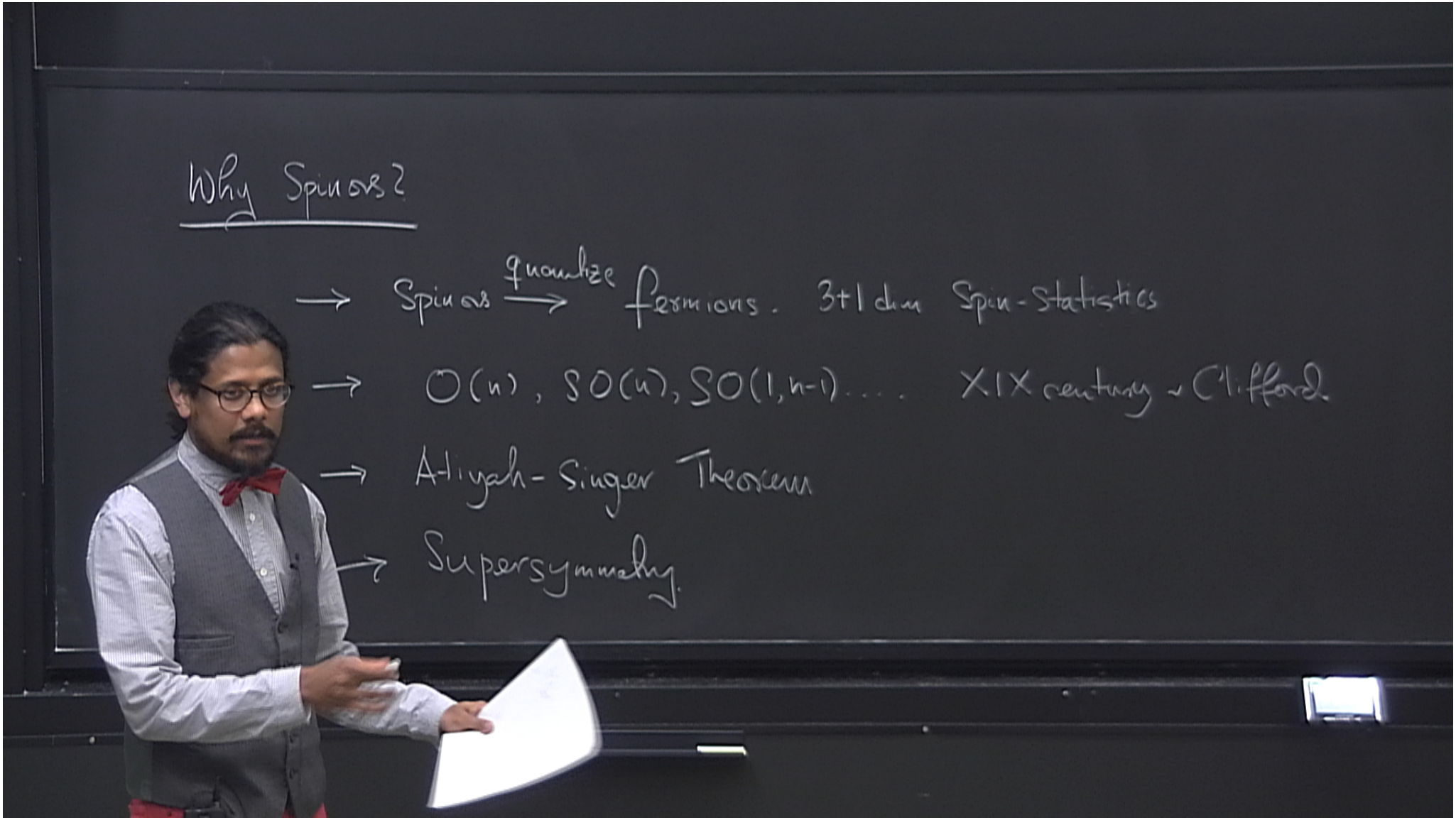

Spin 1/2 and Spin 1
→ classical spinors

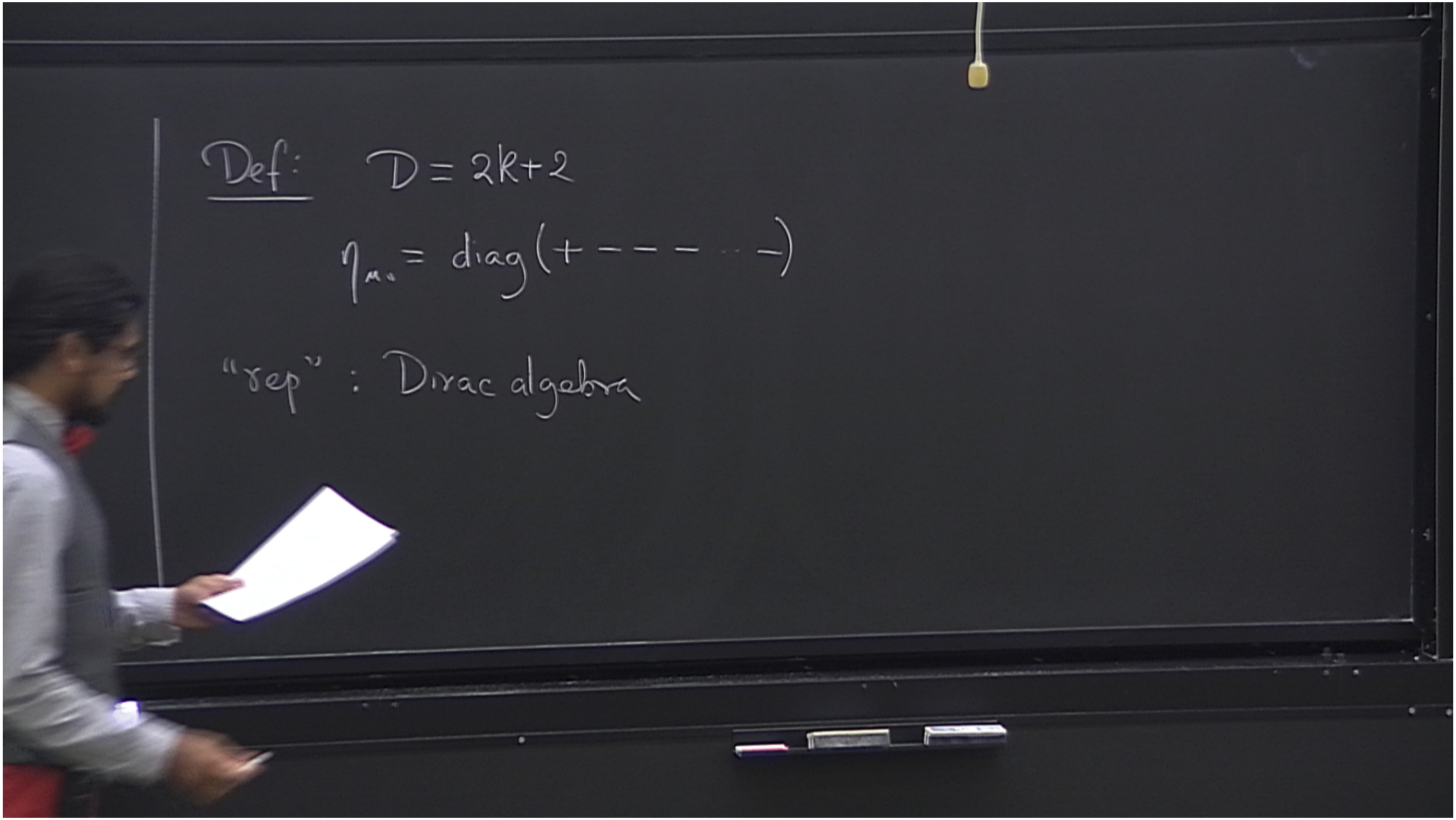

Spin 1/2 and Spin 1
→ Classical spinors
→ Quantum Field Theory of fermions
→

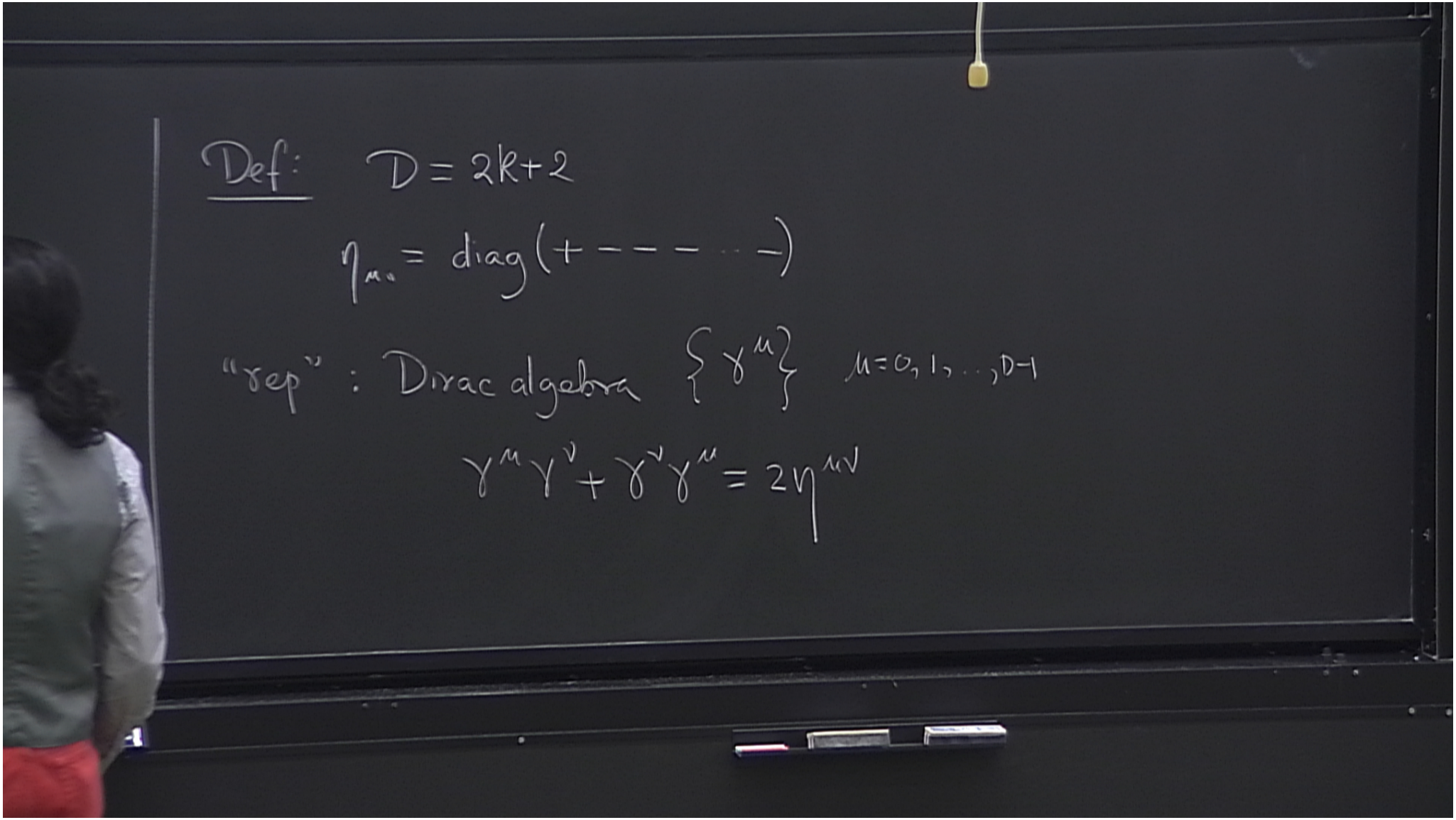

Spin 1/2 and Spin 1
→ Classical spinors
→ Quantum Field Theory of fermions
→ Spinor Yukawa Theory

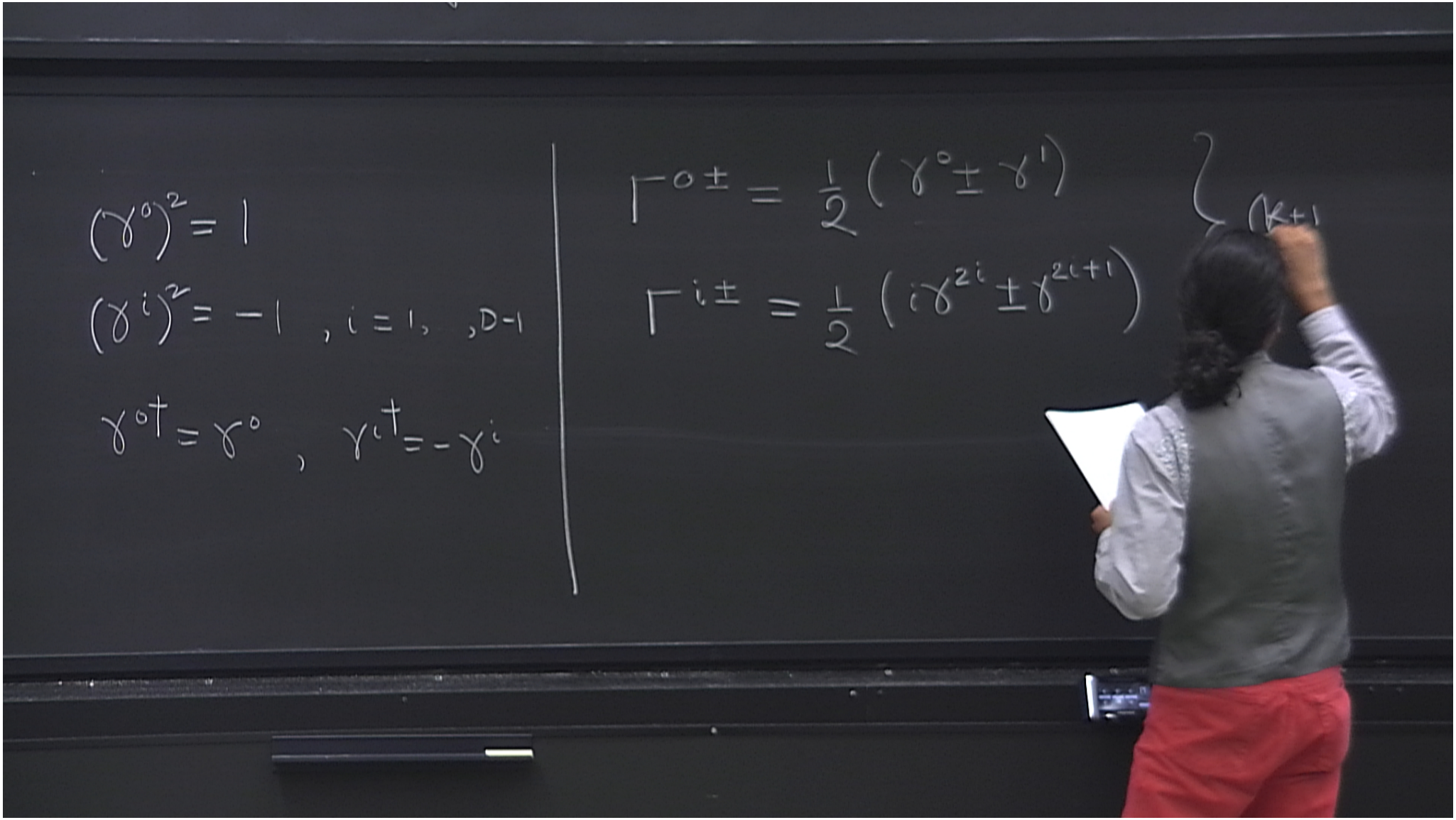

Why Spinors?
→ Spinors quantize fermions. 3+1 dim Spin-Statistics


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→ Spinors quantize fermions. 3+1 dim Spin-Statistics
→ O(n), SO(n), SO(1,n-1)... XIX century - Clifford.
→ Atiyah-Singer Theorem
→ Supersymmetry.


Def: $D = 2R+2$
$\eta_{\mu\nu} = \mathrm{diag}(+ - - - \cdots -)$
"rep" : Dirac algebra


Def: $D = 2k+2$
$\eta_{\mu\nu} = \mathrm{diag}(+---\cdots-)$
"rep": Dirac algebra $\{\gamma^\mu\}$ $\mu = 0, 1, \ldots, D-1$
$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$


$(\gamma^0)^2 = 1$
$(\gamma^i)^2 = -1 \quad , i = 1, \ldots, D-1$
$\gamma^{0\dagger} = \gamma^0 \quad , \quad \gamma^{i\dagger} = -\gamma^i$
$\Gamma^{0\pm} = \frac{1}{2}(\gamma^0 \pm \gamma^1)$
$\Gamma^{i\pm} = \frac{1}{2}(i\gamma^{2i} \pm \gamma^{2i+1})$
$(K+1$

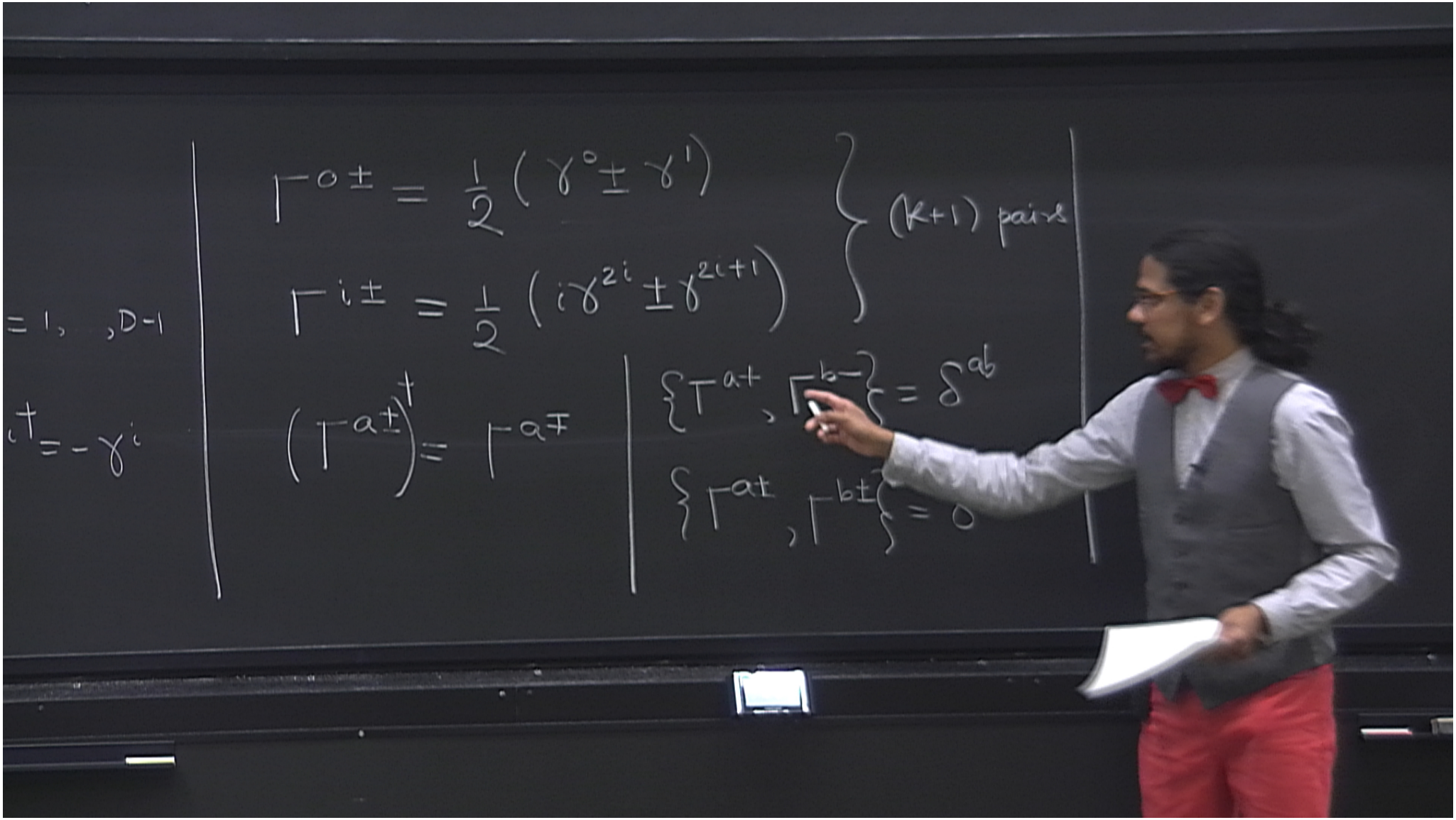
$$\Gamma^{0\pm} = \frac{1}{2}(\gamma^0 \pm \gamma^1)$$

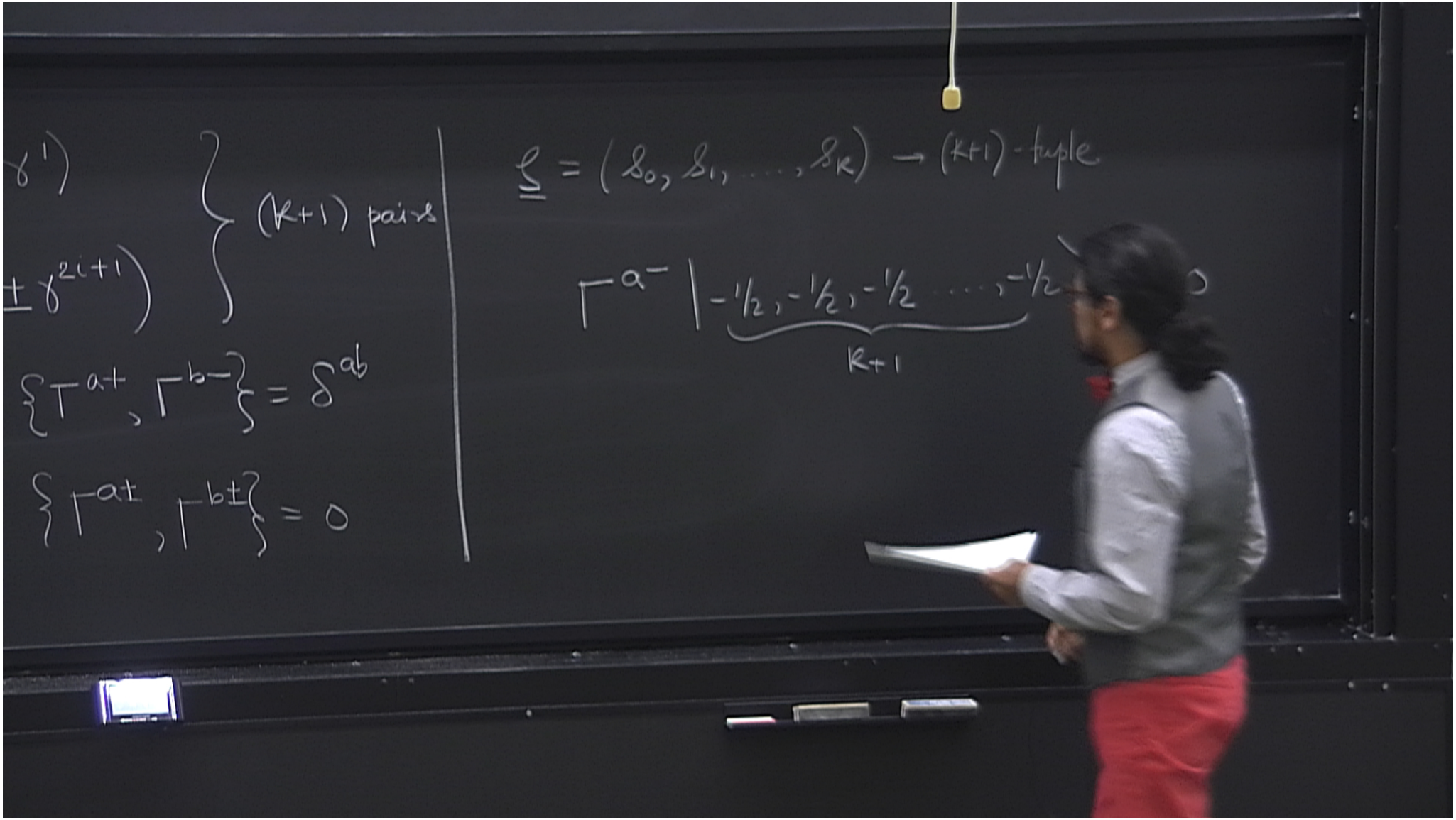
$$\Gamma^{i\pm} = \frac{1}{2}(i\gamma^{2i} \pm \gamma^{2i+1}) \quad \Big\} \ (k+1) \text{ pairs}$$

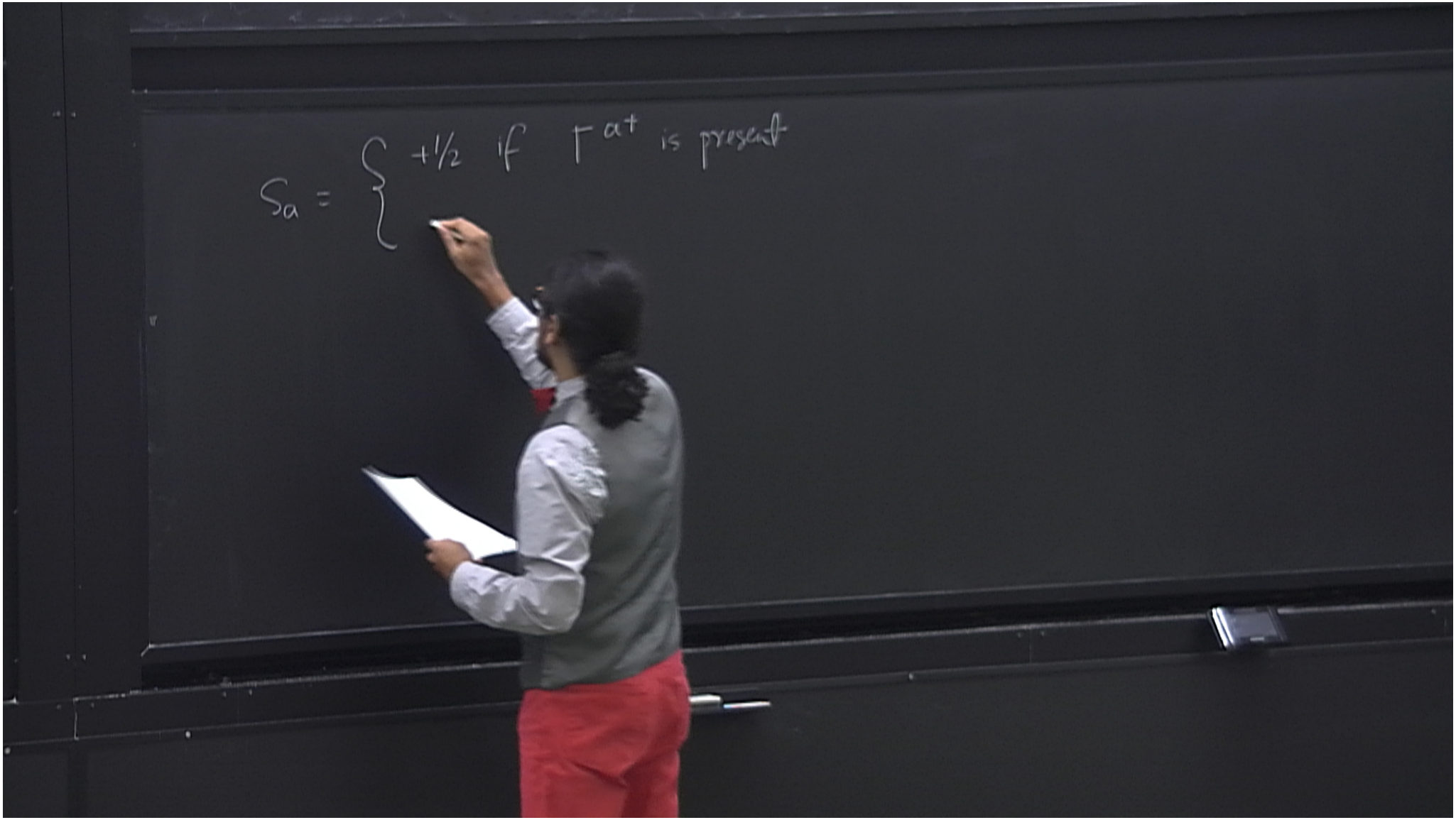
$$(\Gamma^{a\pm})^\dagger = \Gamma^{a\mp}$$

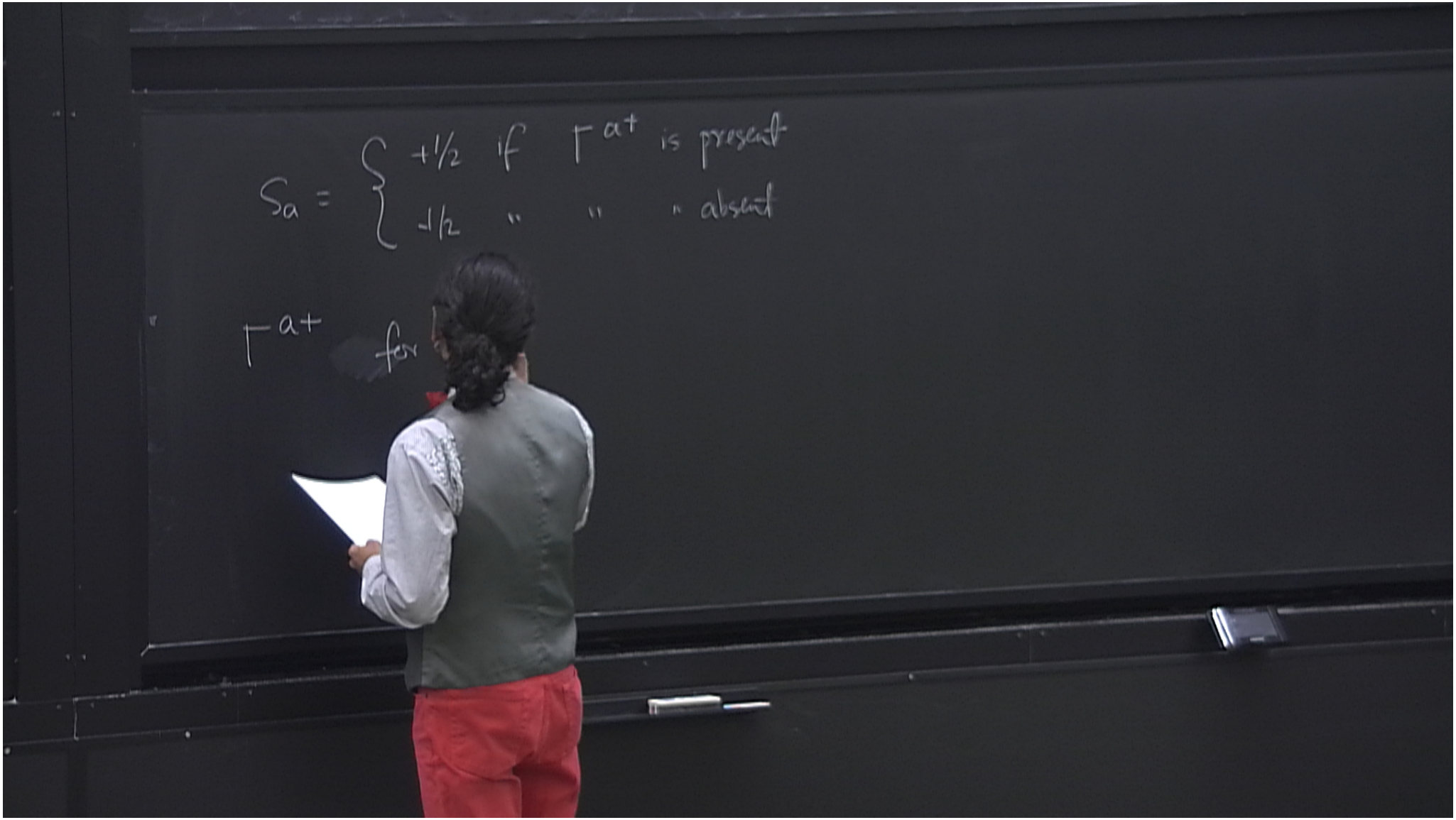
$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$$

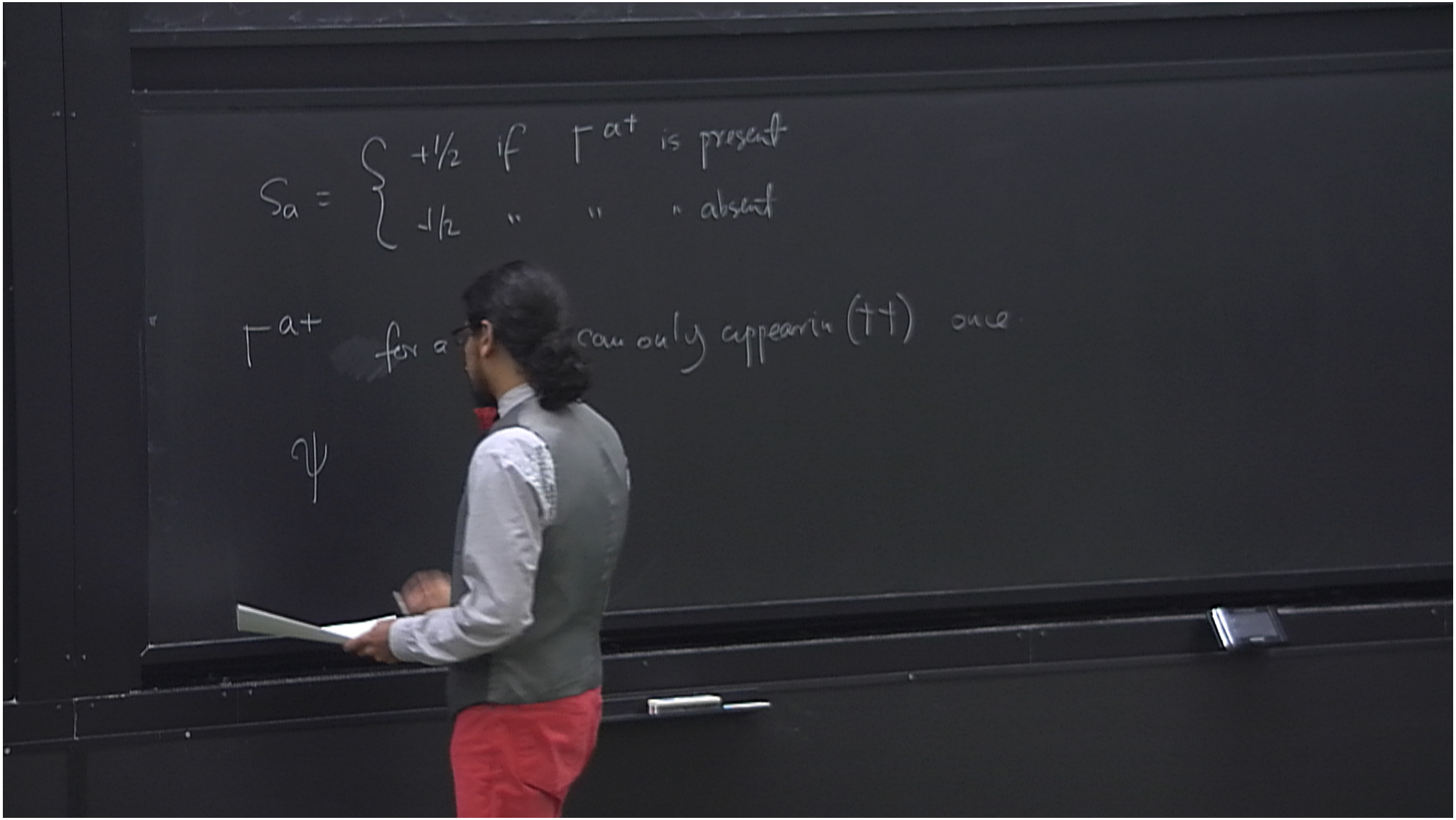
$$\{\Gamma^{a+}, \Gamma^{b+}\} = 0$$

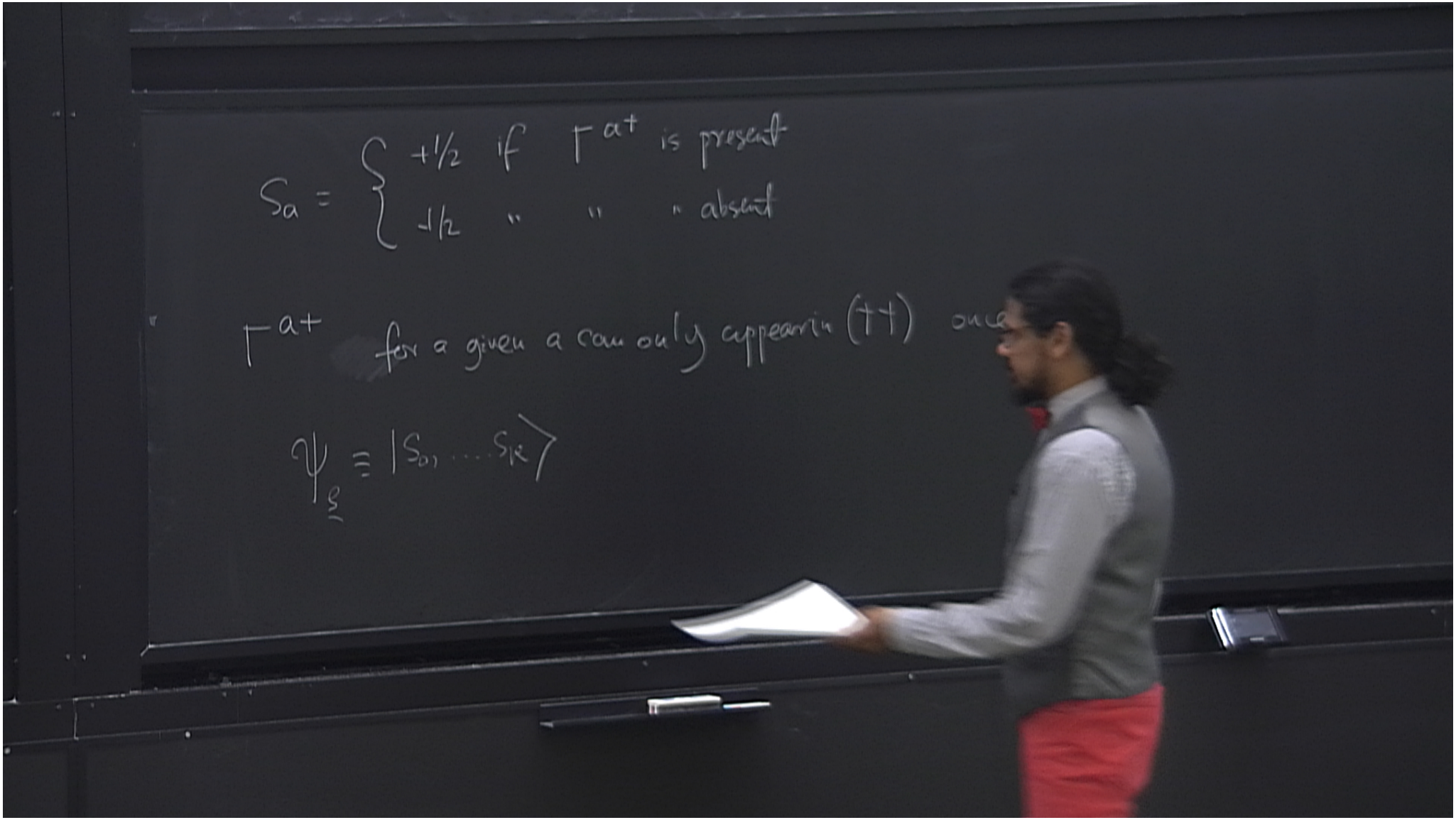

$\Gamma^{0\pm} = \frac{1}{2}(\gamma^0 \pm \gamma^1)$
$\Gamma^{i\pm} = \frac{1}{2}(i\gamma^{2i} \pm \gamma^{2i+1})$
(K+1) pairs
$(\Gamma^{a\pm})^\dagger = \Gamma^{a\mp}$
$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$
$\{\Gamma^{a+}, \Gamma^{b+}\} = 0$
$= 1, \ldots, D-1$
$= -\gamma^i$


$\gamma^1)$
$\pm\gamma^{2(k+1)})$
$(k+1)$ pairs
$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$
$\{\Gamma^{at}, \Gamma^{-bt}\} = 0$
$\underline{s} = (s_0, s_1, \ldots, s_k) \to (k+1)$-tuple
$\Gamma^{a-}|-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \ldots, -\frac{1}{2}$
$k+1$


$S_a = \begin{cases} +1/2 \text{ if } \Gamma^{a\dagger} \text{ is present} \end{cases}$


$S_a = \begin{cases} +1/2 & \text{if } \Gamma^{a+} \text{ is present} \\ -1/2 & \text{" " " absent} \end{cases}$
Γ^{a+} for


$S_a = \begin{cases} +1/2 & \text{if } \Gamma^{a+} \text{ is present} \\ -1/2 & \text{" " " absent} \end{cases}$
Γ^{a+} for a can only appear in (++) once
ψ

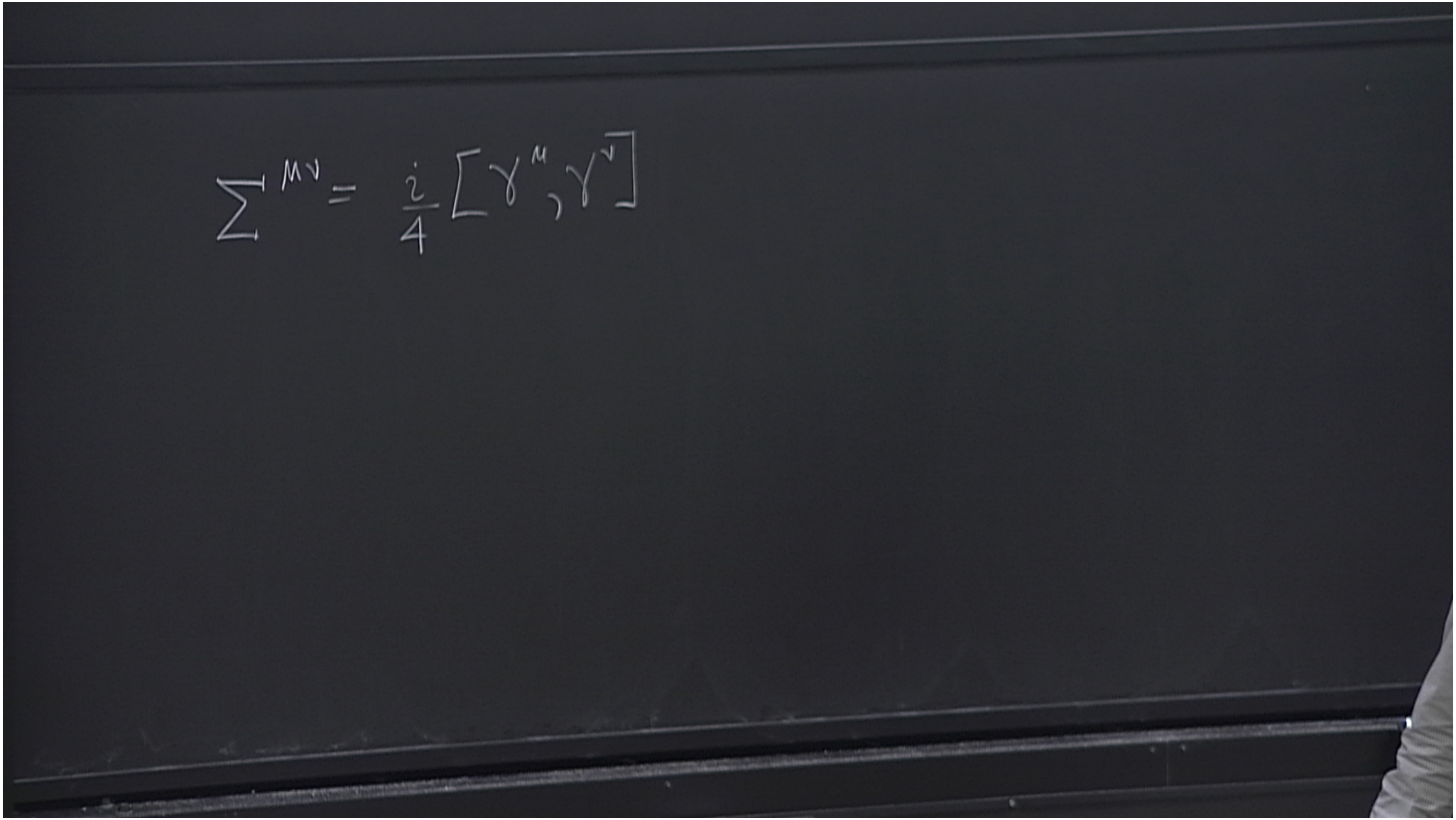

$S_a = \begin{cases} +1/2 & \text{if } \Gamma^{a+} \text{ is present} \\ -1/2 & \text{" " " absent} \end{cases}$
Γ^{a+} for a given a can only appear in (++) once
$\psi_{\underline{s}} = |s_{a_1} \dots s_{a_k}\rangle$

$$s_a = \begin{cases} +1/2 & \text{if } \Gamma^{a+} \text{ is present} \\ -1/2 & \text{" " " absent} \end{cases}$$

Γ^{a+} for a given a can only appear in $(++)$ once.

$$\psi_{\underline{s}} \equiv |s_0 \dots s_K\rangle \qquad 2^{K+1} = 2^{D/2}$$

The vect. space spanned by $\psi_{\underline{s}}$ over $\mathbb{C}$ is called a Dirac spinor.


$\sum^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

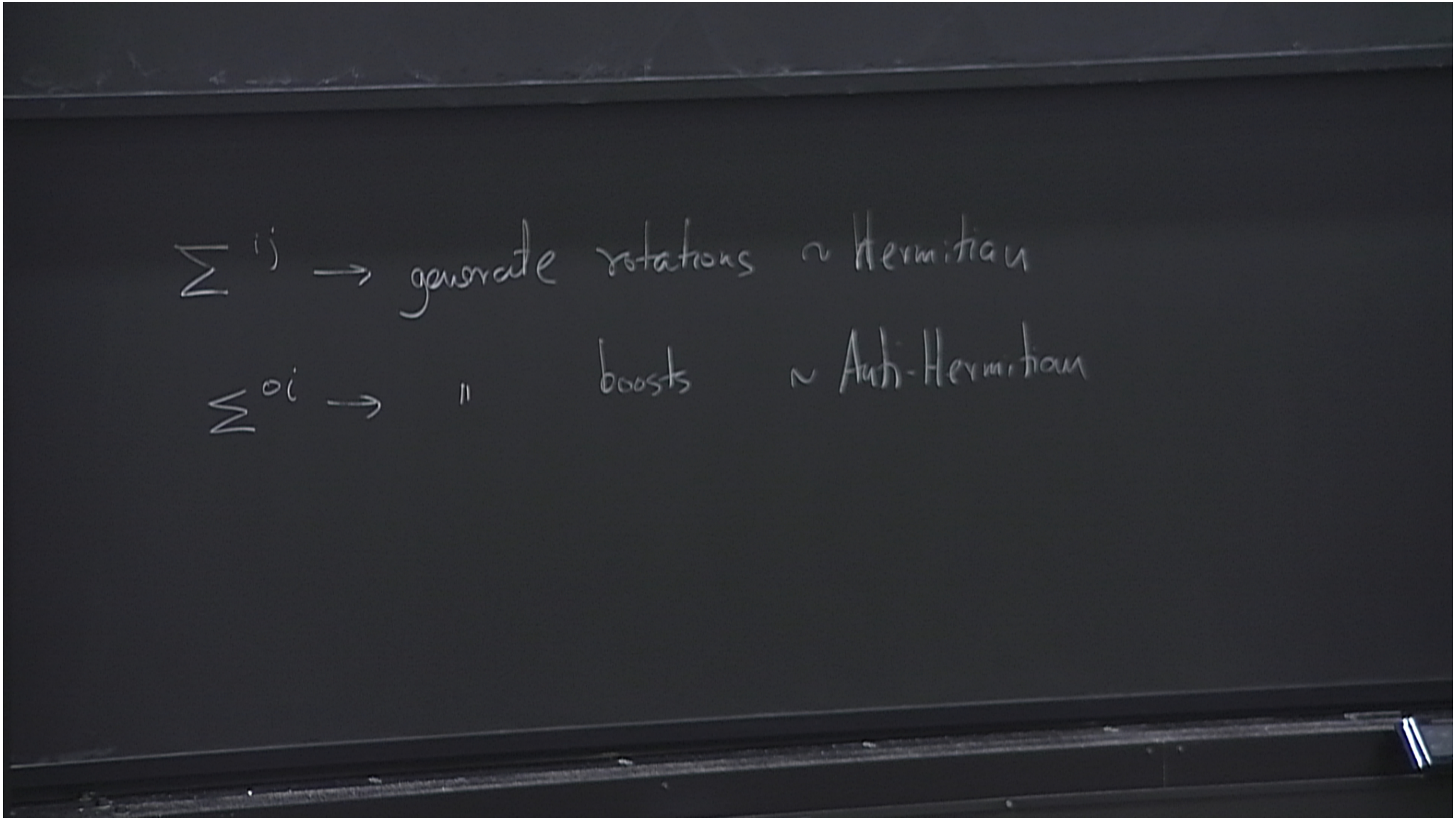
$$[\Sigma^{\mu\nu}, \gamma^\rho] = i(\eta^{\rho\nu}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu)$$

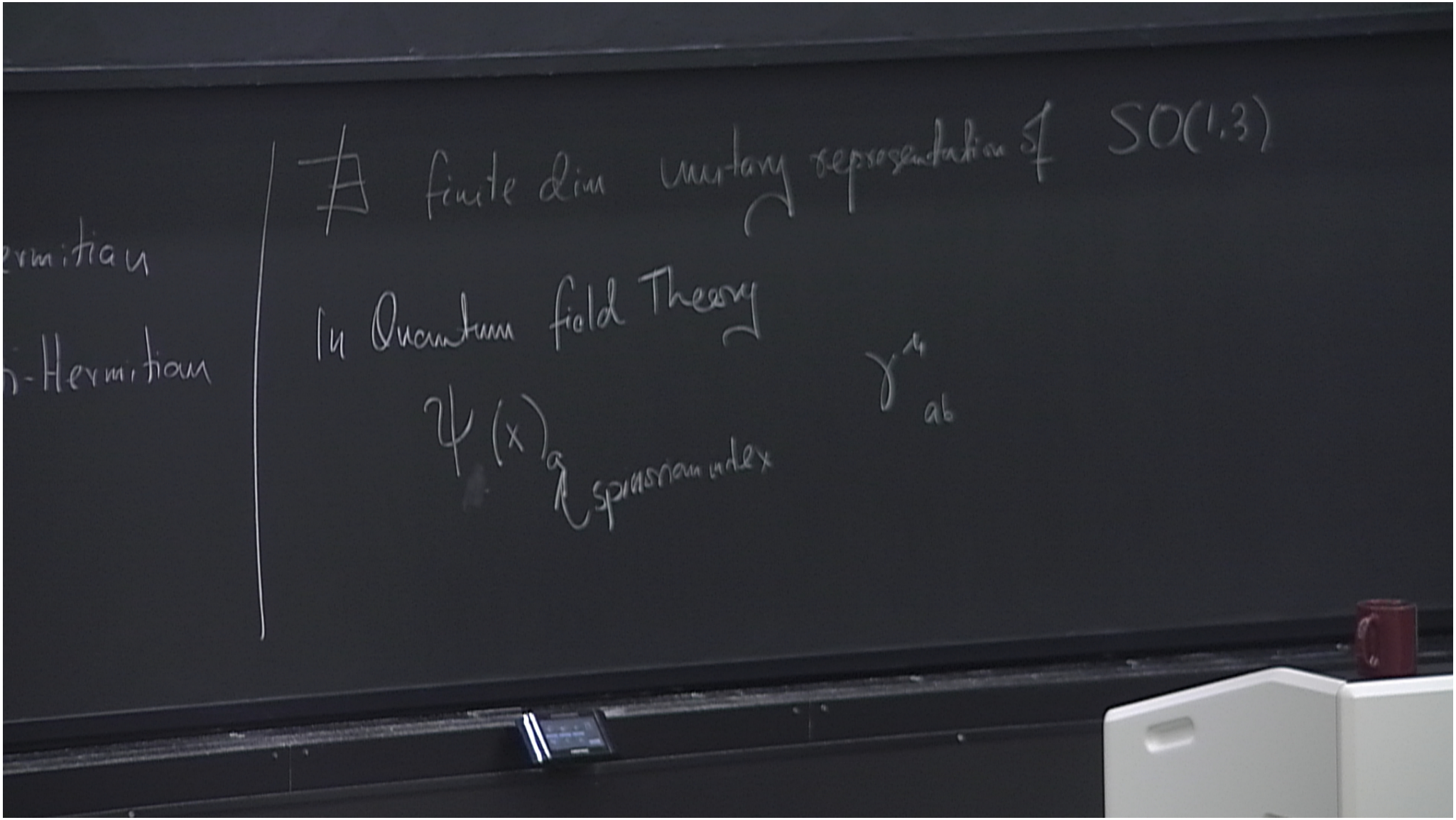
$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(\eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\sigma})$$

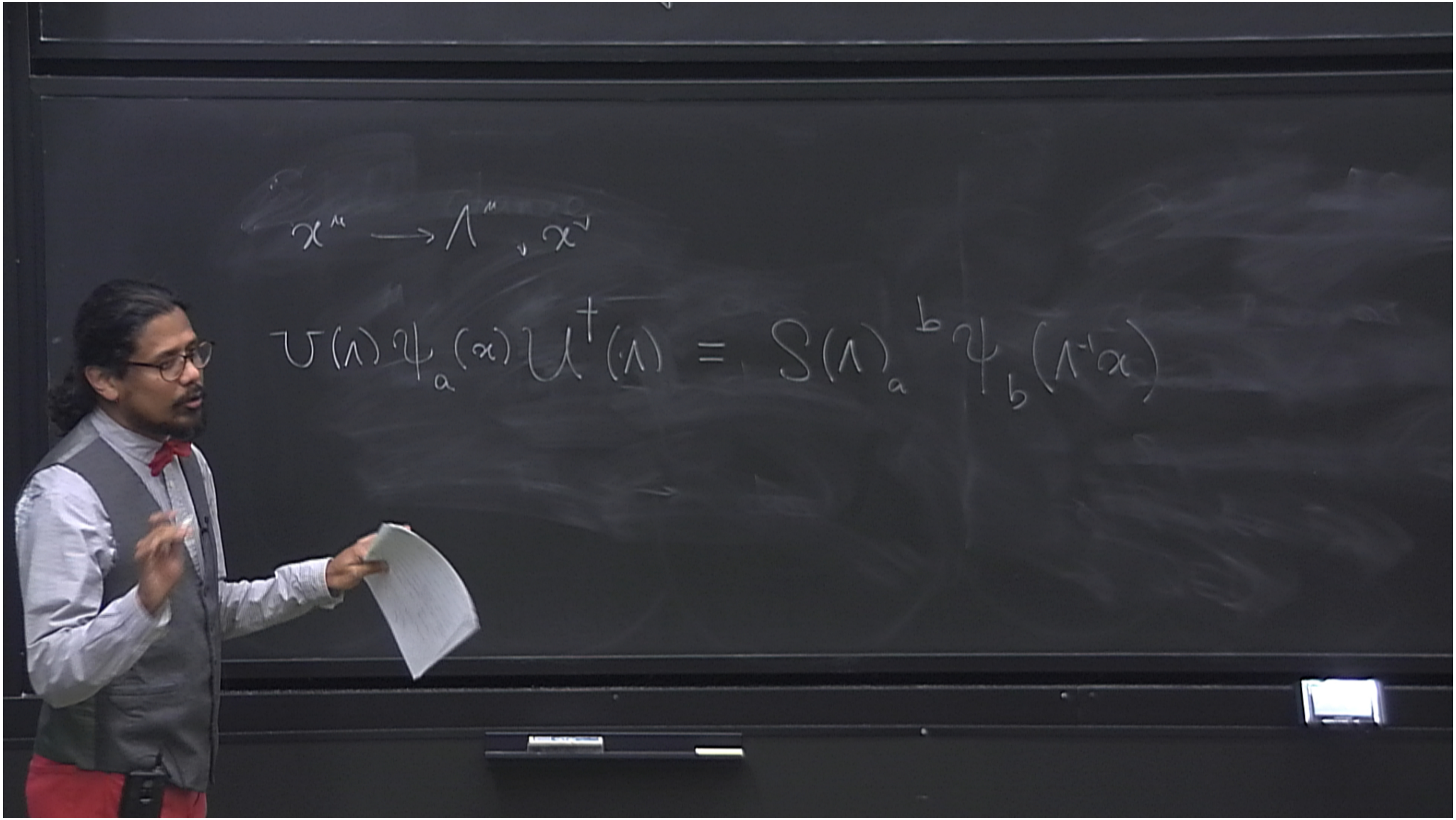
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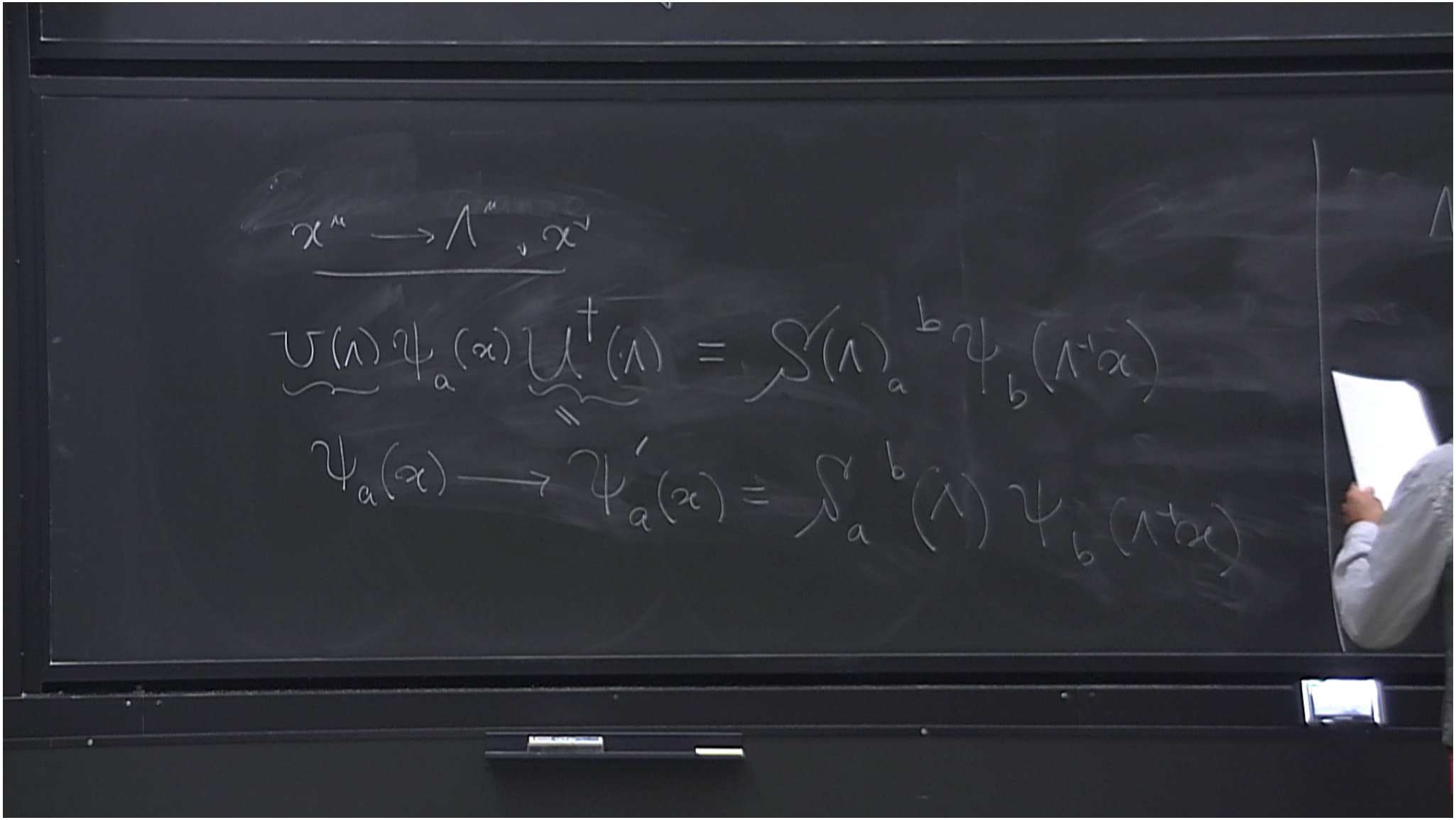
$$[\Sigma^{\mu\nu}, \gamma^\rho] = i(\eta^{\rho\nu}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu)$$

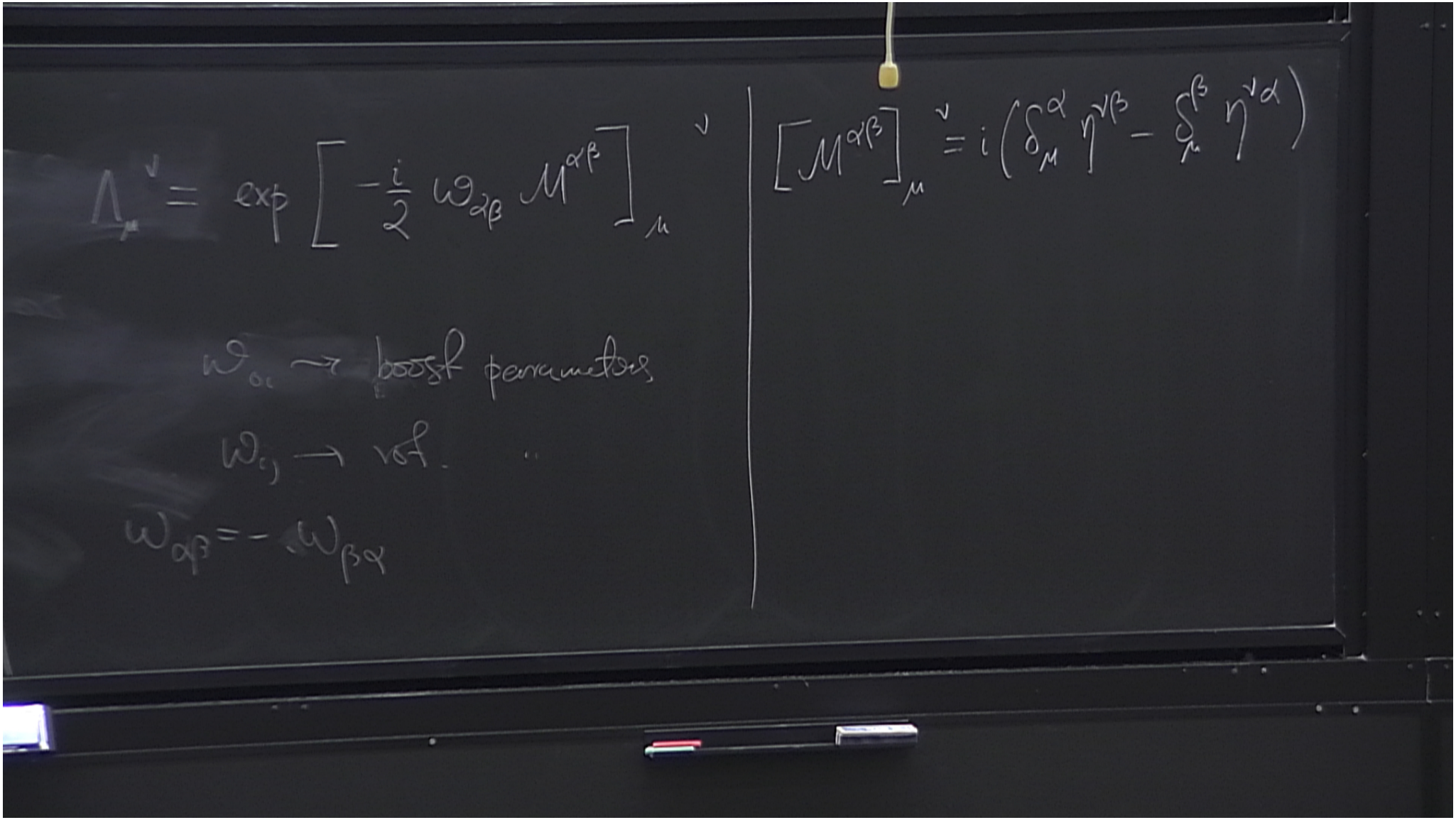
$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(\eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho}) \sim \text{Lor}$$

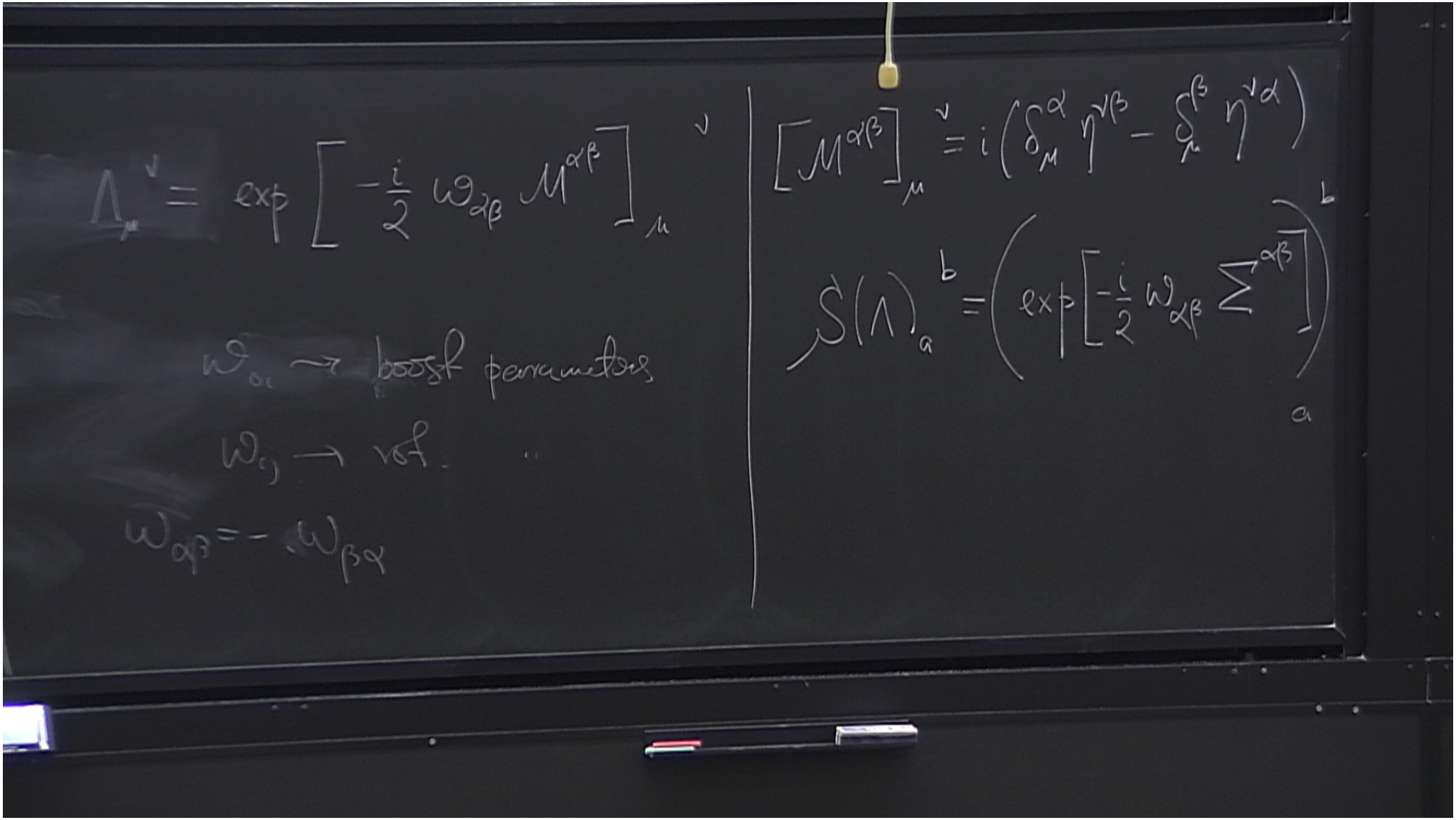

Σ^{ij} → generate rotations ~ Hermitian
Σ^{oi} → " boosts ~ Anti-Hermitian

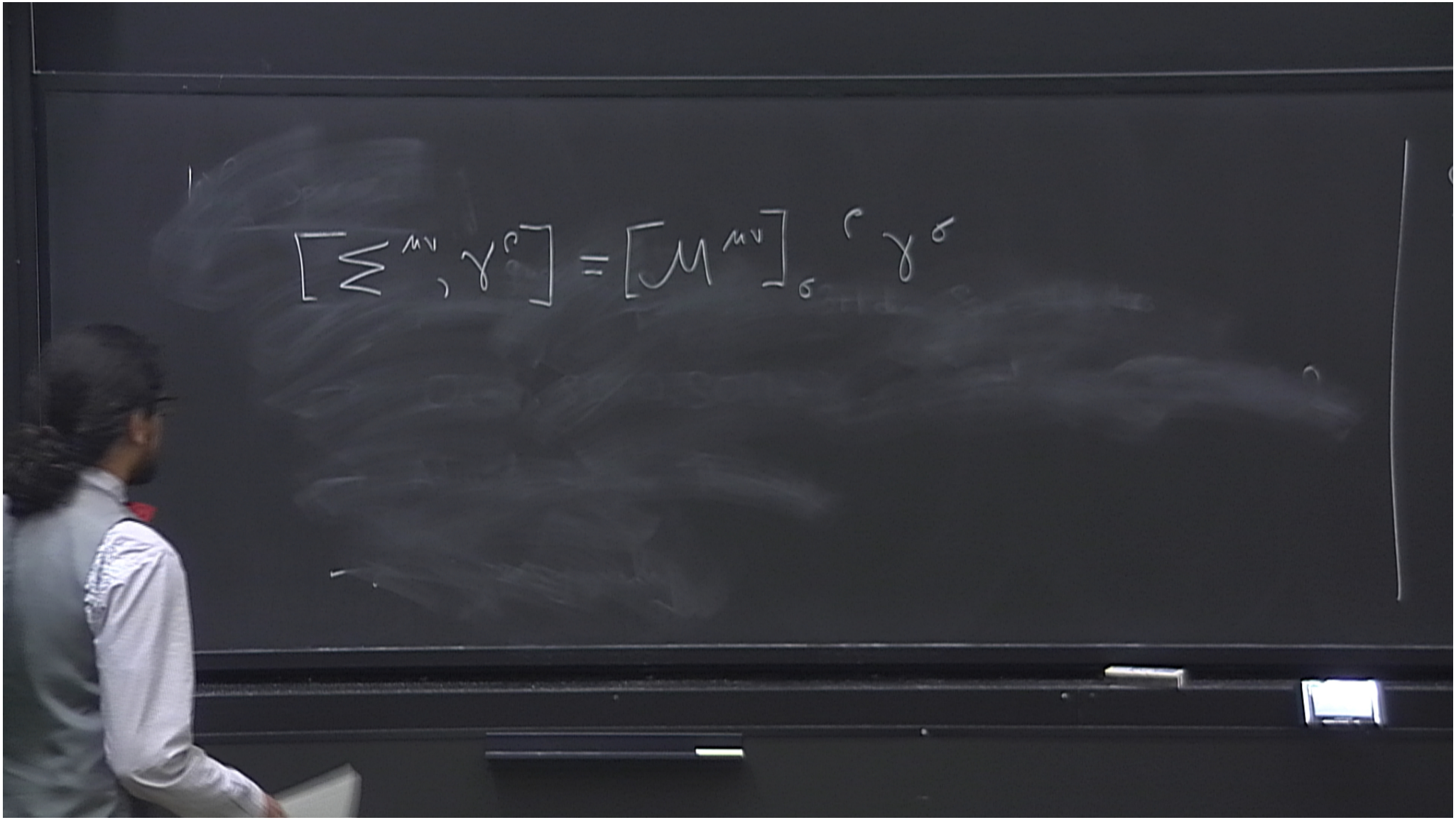

ermitian
n-Hermitian
∄ finite dim unitary representation of SO(1,3)
in Quantum field Theory
ψ(x)_a
spinorian index
γ^μ_ab

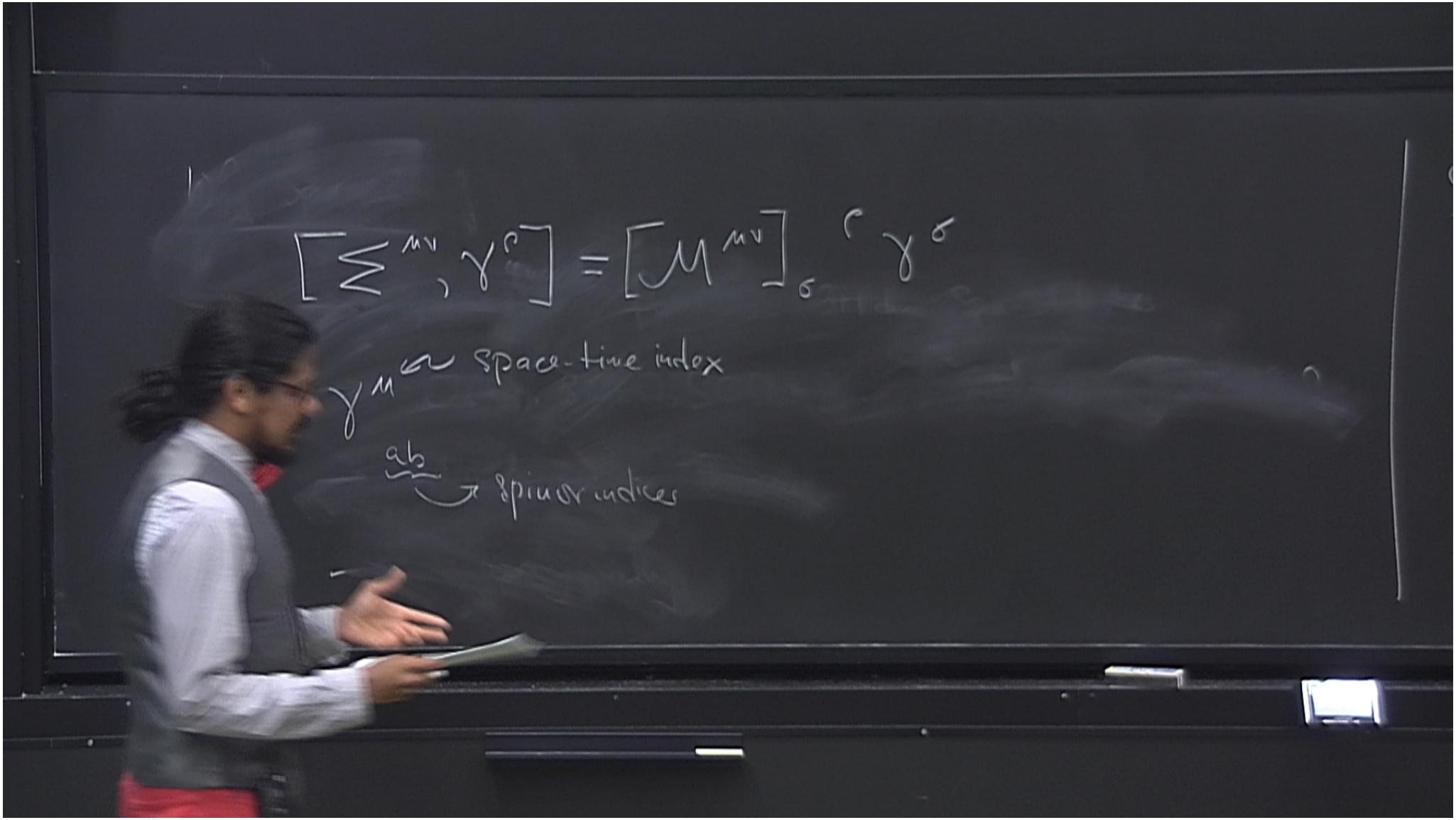

$x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$
$U(\Lambda)\psi_a(x)U^\dagger(\Lambda) = S(\Lambda)_a{}^b \psi_b(\Lambda^{-1}x)$

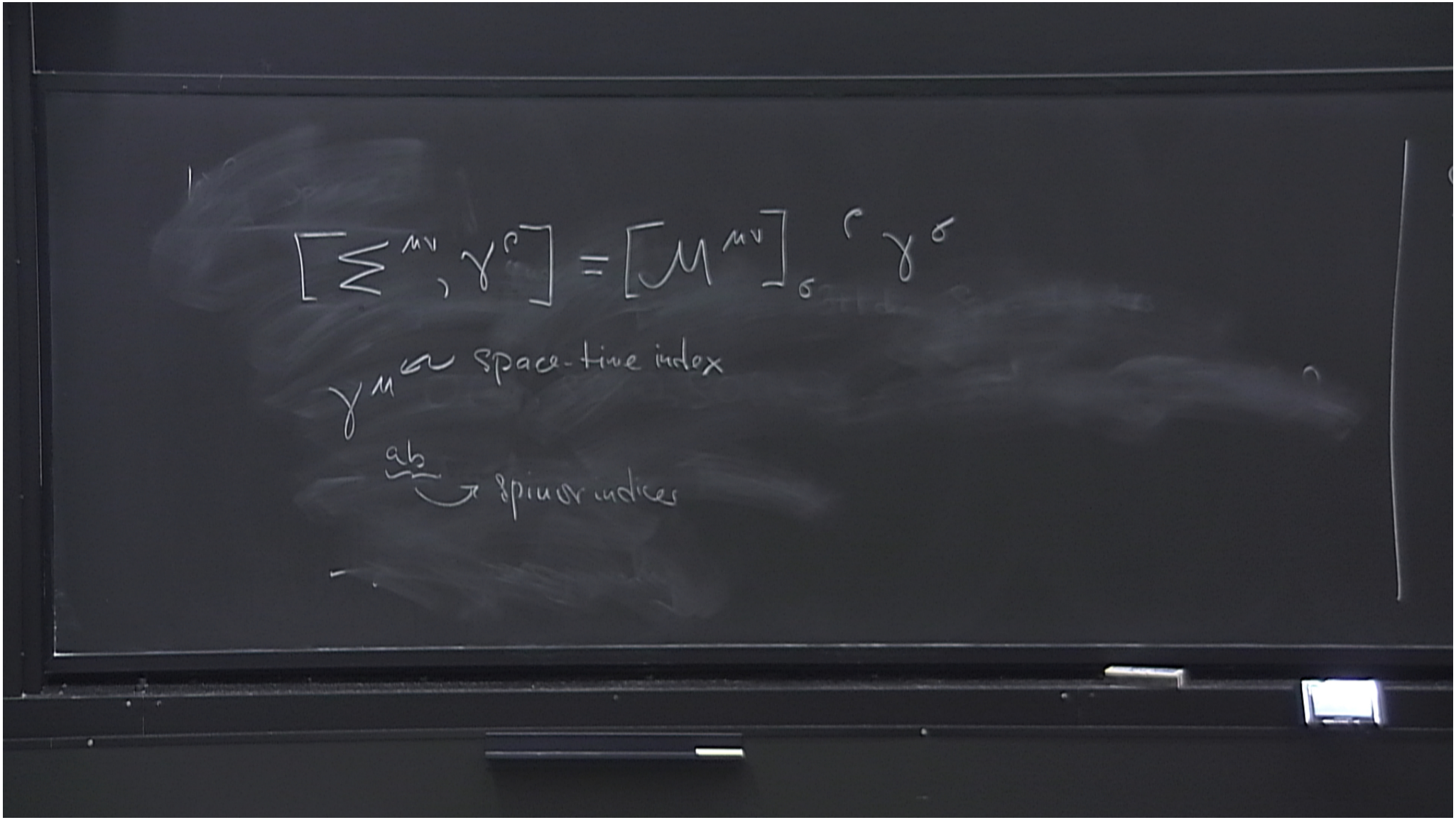

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$U(\Lambda)\psi_a(x)U^\dagger(\Lambda) = S(\Lambda)_a{}^b \psi_b(\Lambda^{-1}x)$
$\psi_a(x) \longrightarrow \psi'_a(x) = S_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x)$

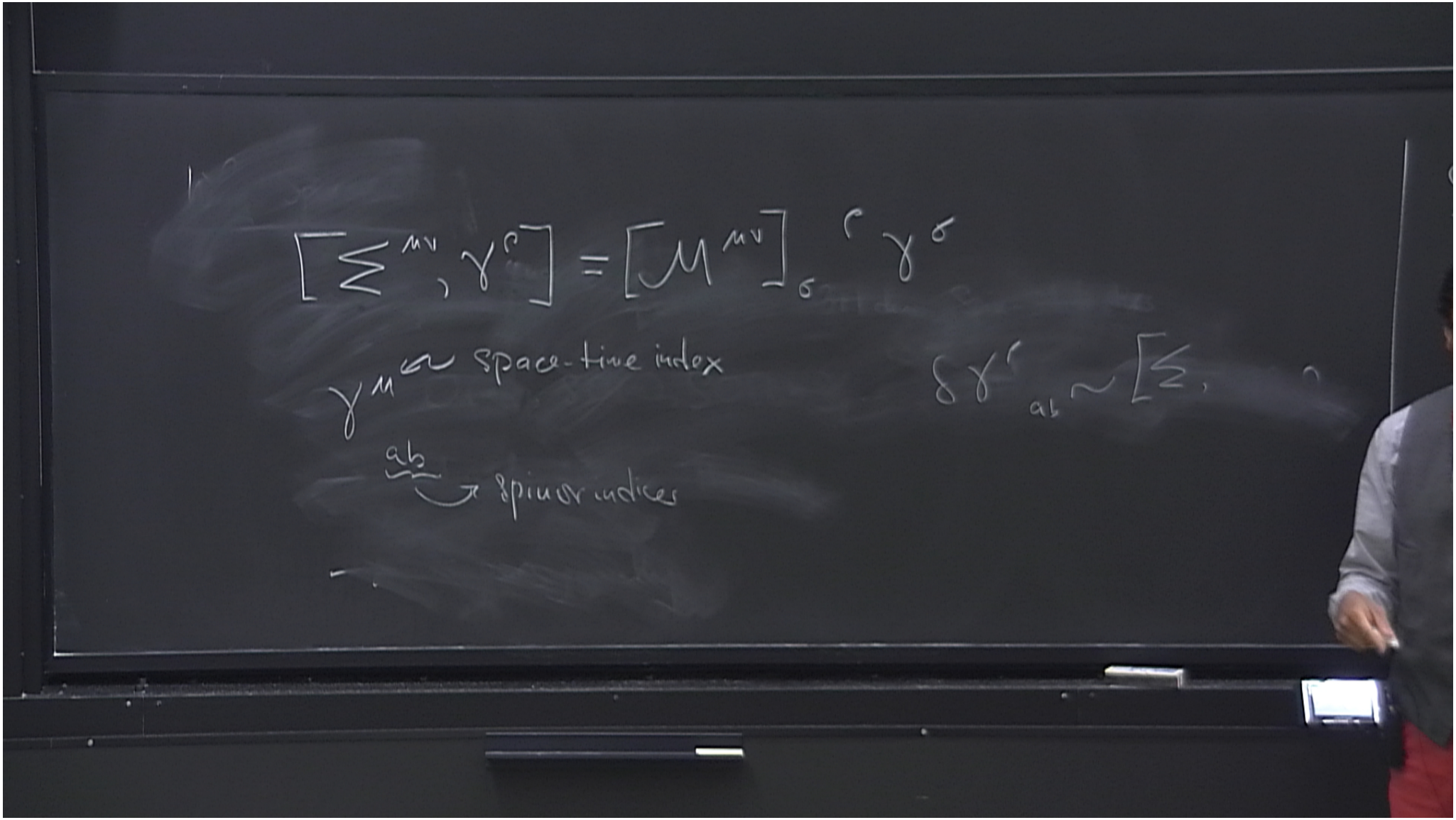

$\Lambda_\mu{}^\nu = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right]_\mu{}^\nu$
$\omega_{0i} \rightarrow$ boost parameters
$\omega_{ij} \rightarrow$ rot.
$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$
$\left[M^{\alpha\beta}\right]_\mu{}^\nu = i\left(\delta_\mu^\alpha \eta^{\nu\beta} - \delta_\mu^\beta \eta^{\nu\alpha}\right)$

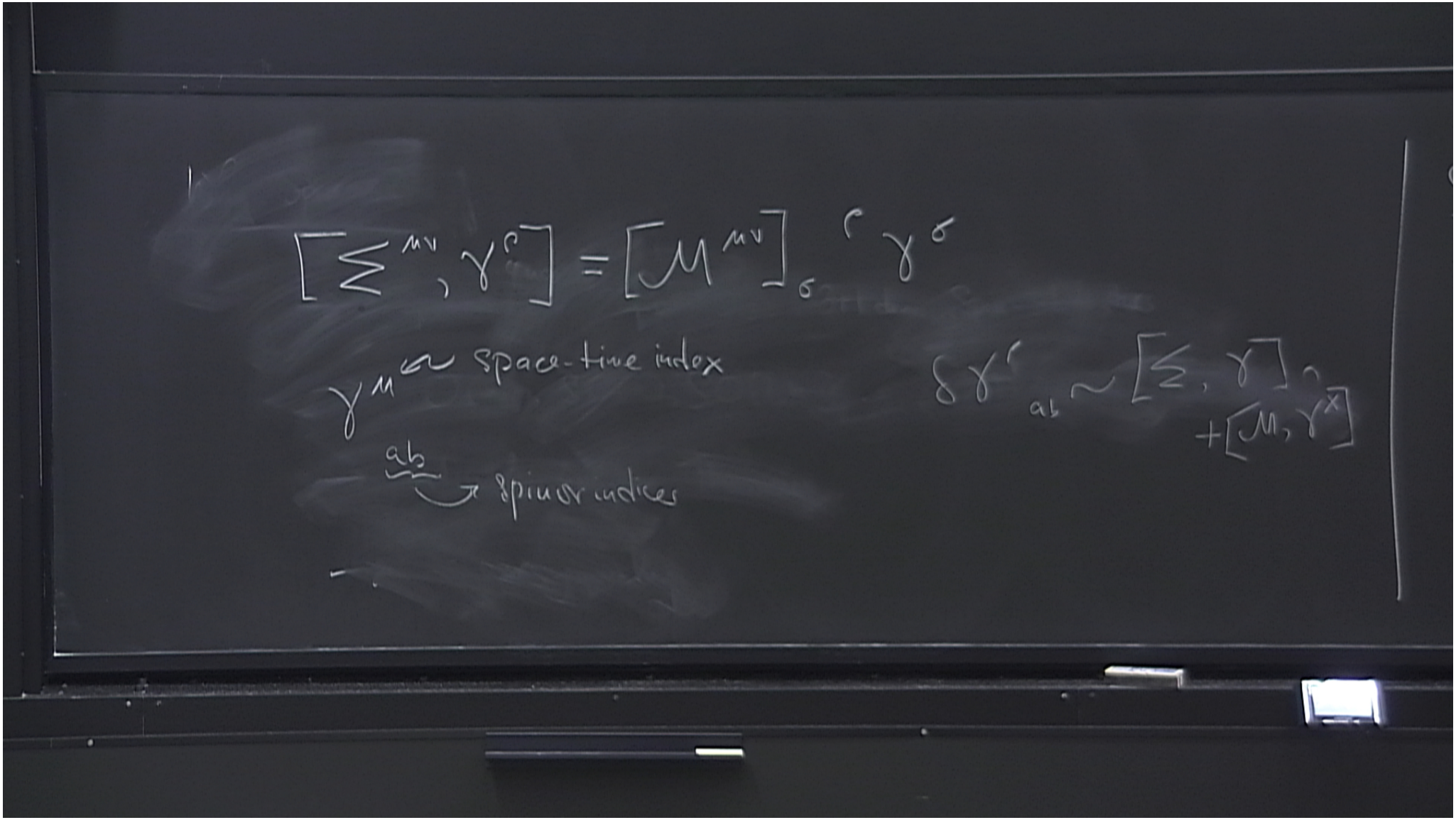

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$S(\Lambda)_a{}^b = \left(\exp\left[-\frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\right]\right)_a{}^b$

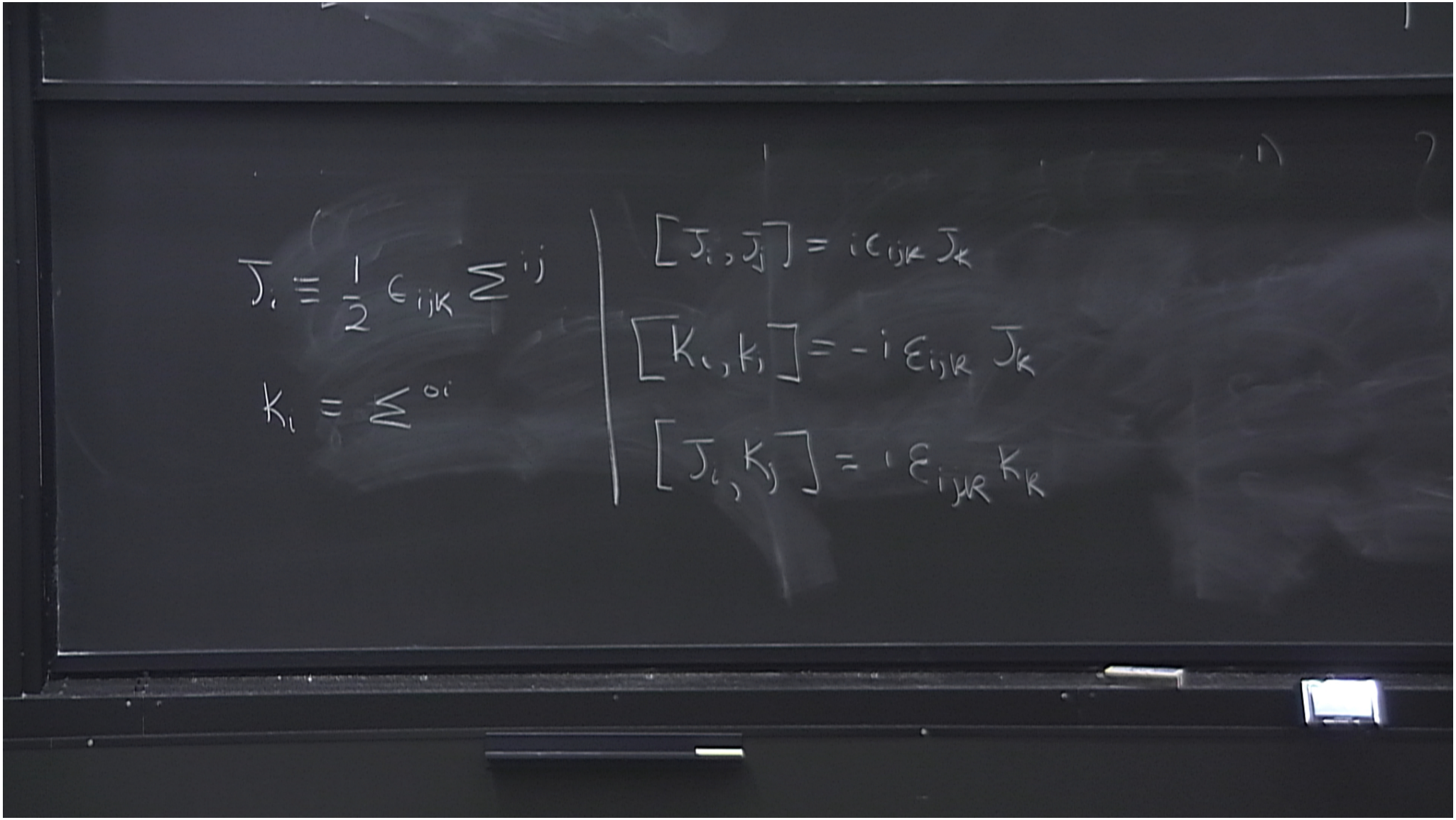

$[\Sigma^{\mu\nu}, \gamma^\rho] = [M^{\mu\nu}]^\rho{}_\sigma \gamma^\sigma$

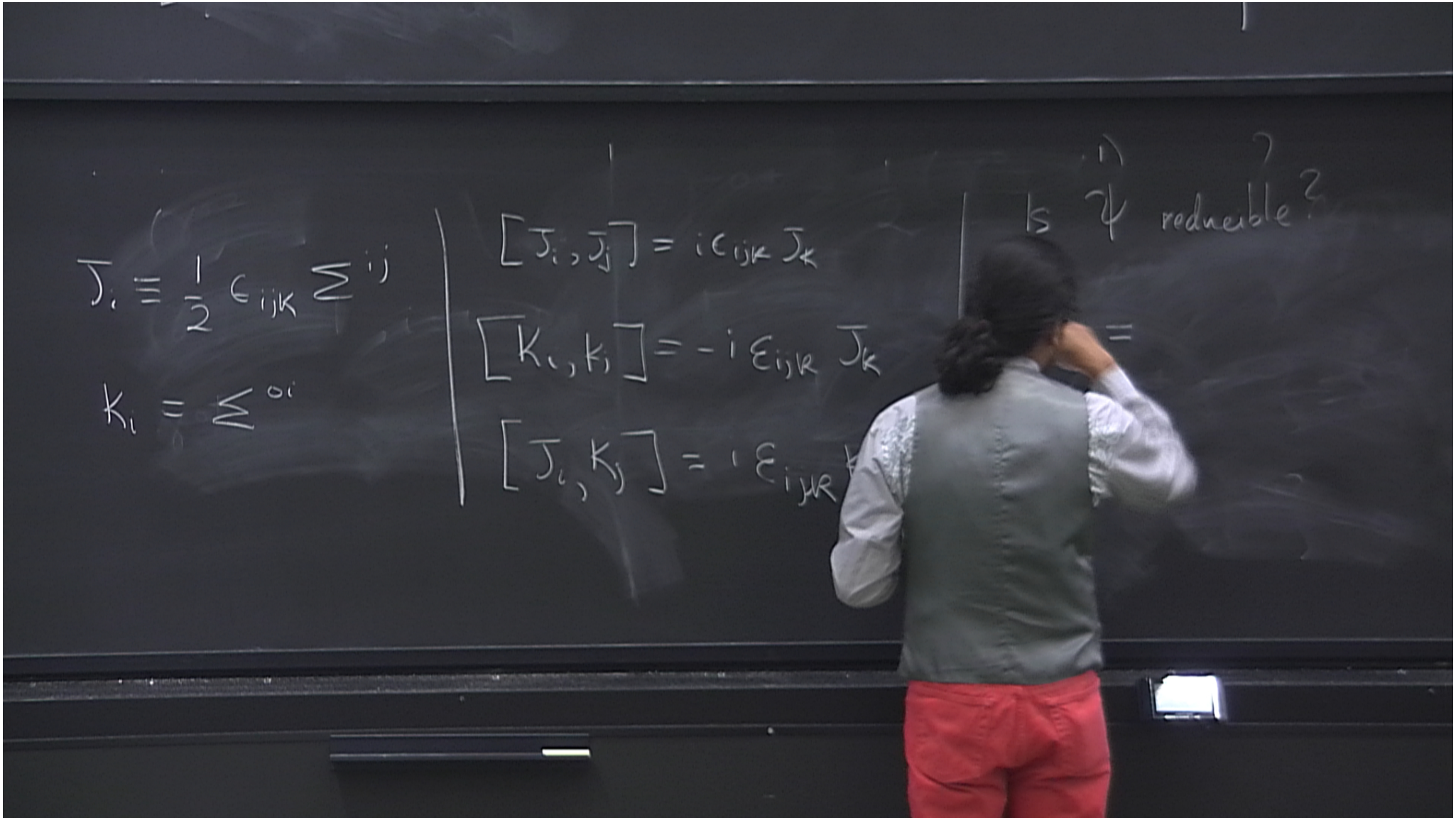

$[\Sigma^{\mu\nu}, \gamma^{\rho}] = [M^{\mu\nu}]_{\sigma}{}^{\rho}\gamma^{\sigma}$
γ^{μ} space-time index
$\alpha\beta$
spinor indices

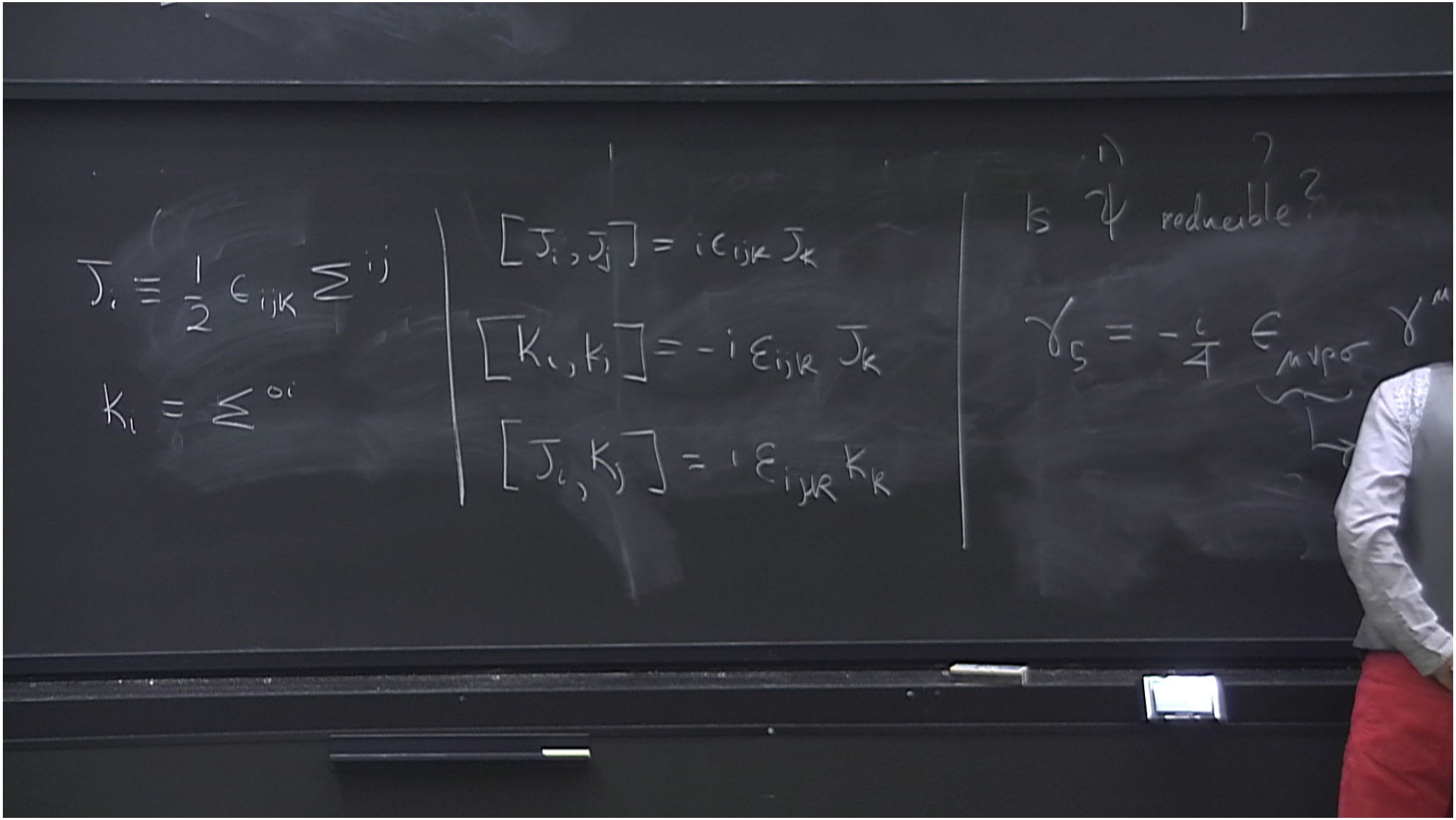

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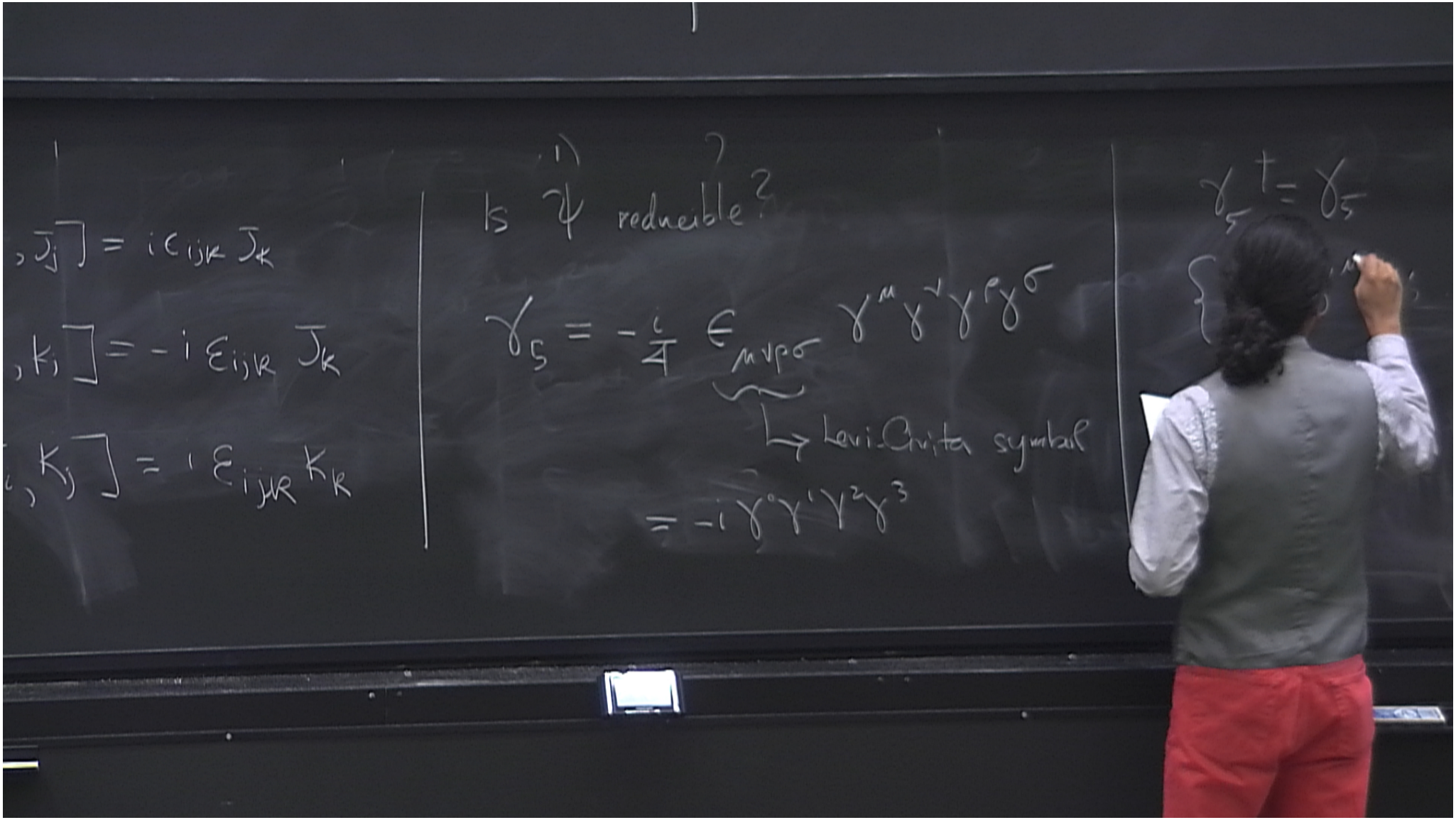

$[\Sigma^{\mu\nu}, \gamma^\rho] = [\mathcal{M}^{\mu\nu}]^\rho{}_\sigma \gamma^\sigma$
γ^μ space-time index
$\alpha\beta$ spinor indices
$\delta\gamma^\rho{}_{\alpha\beta} \sim [\Sigma,$

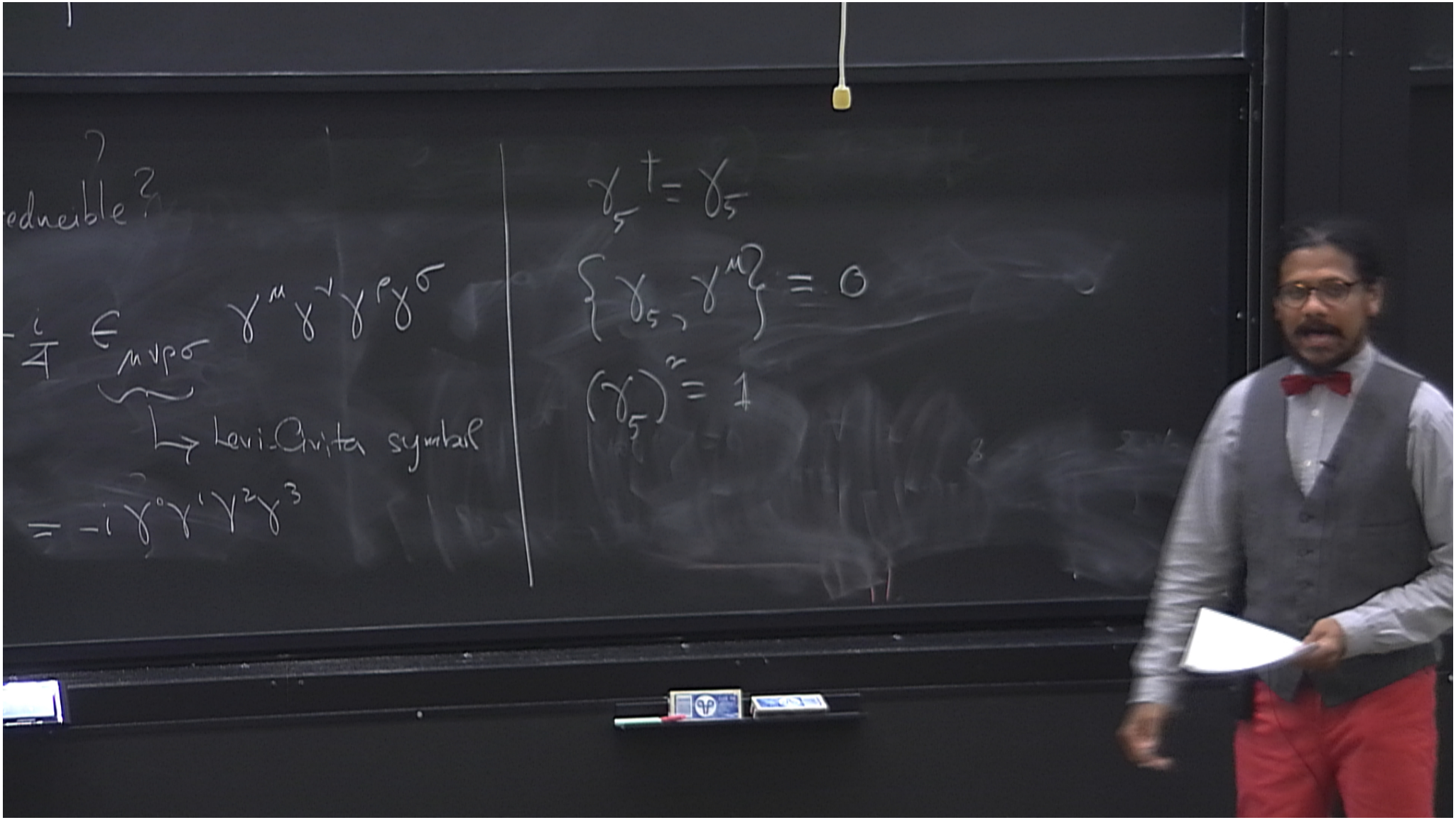

$[\Sigma^{\mu\nu}, \gamma^\rho] = [\mathcal{M}^{\mu\nu}]^\rho{}_\sigma \gamma^\sigma$
γ^μ space-time index
$\alpha\beta$ spinor indices
$\delta\gamma^\rho{}_{\alpha\dot{\beta}} \sim [\Sigma, \gamma] + [\mathcal{M}, \gamma^x]$

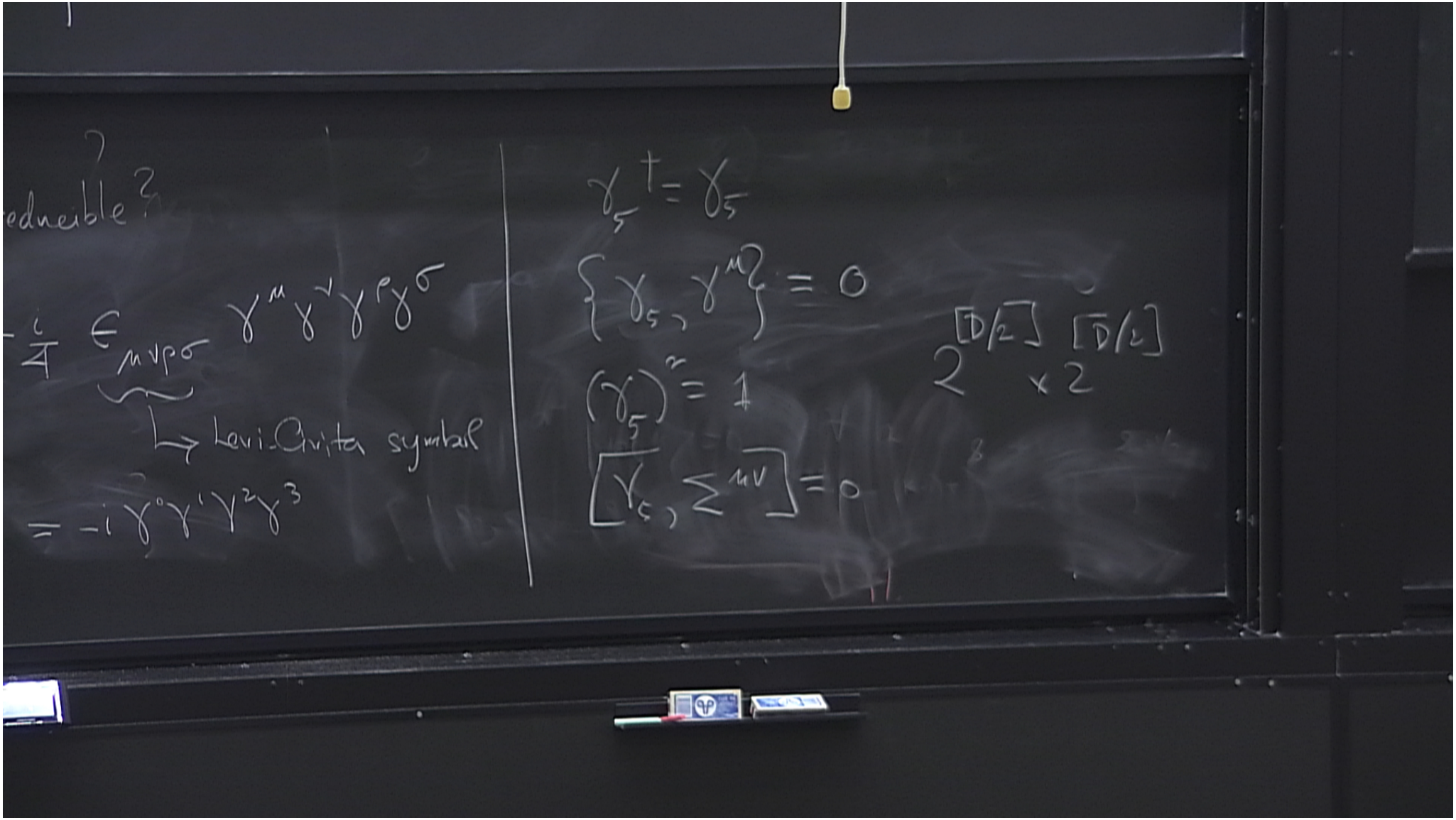

$J_i \equiv \frac{1}{2} \epsilon_{ijk} \Sigma^{jj}$
$K_i = \Sigma^{oi}$
$[J_i, J_j] = i \epsilon_{ijk} J_k$
$[K_i, K_j] = -i \epsilon_{ijk} J_k$
$[J_i, K_j] = i \epsilon_{ijk} K_k$

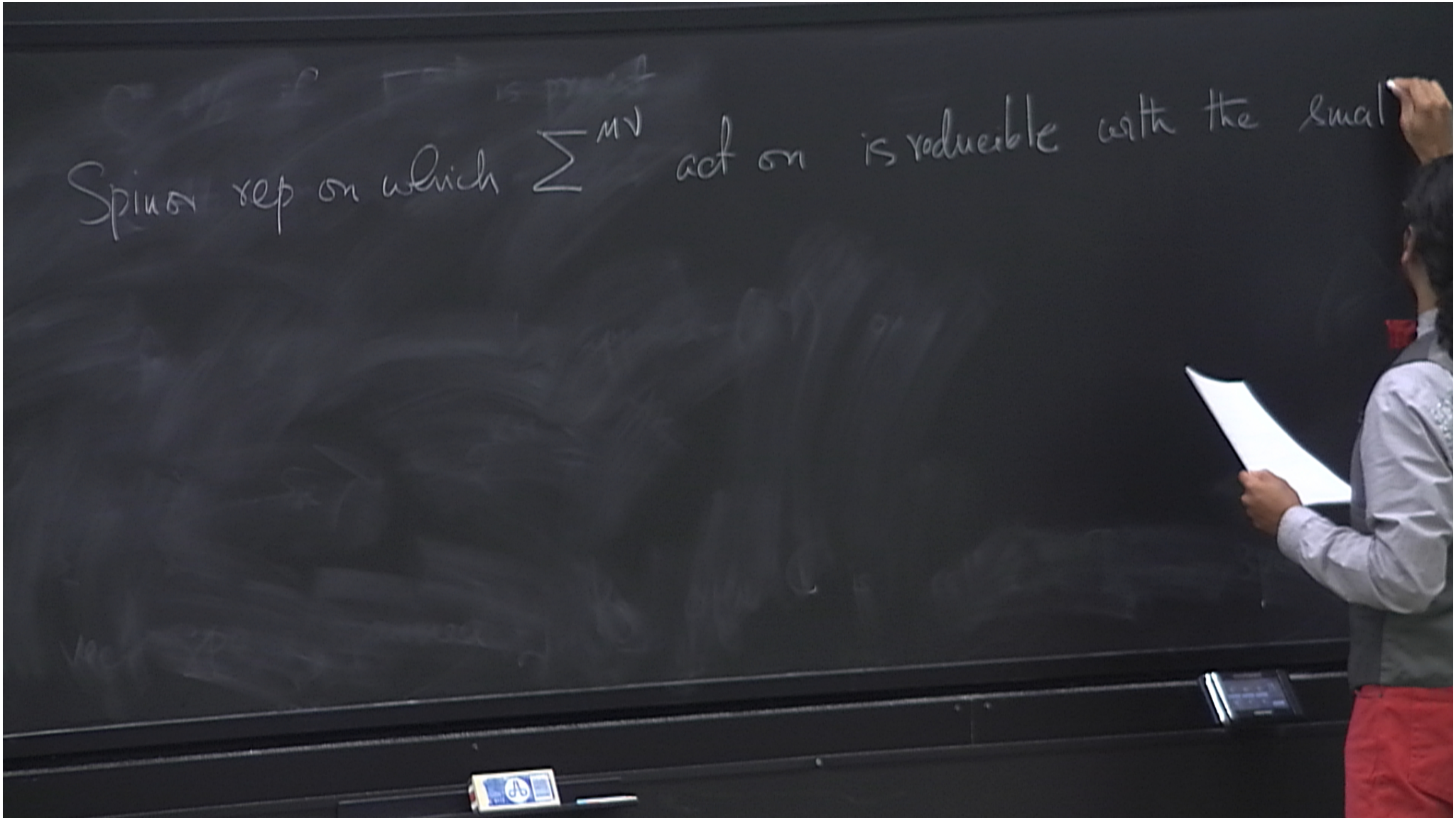

$J_i \equiv \frac{1}{2} \epsilon_{ijk} \Sigma^{ij}$
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$[J_i, K_j] = i \epsilon_{ijk} K_k$
Is ψ reducible?


$J_i \equiv \frac{1}{2} \epsilon_{ijk} \Sigma^{ij}$
$K_i = \Sigma^{0i}$
$[J_i, J_j] = i \epsilon_{ijk} J_k$
$[K_i, K_j] = -i \epsilon_{ijk} J_k$
$[J_i, K_j] = i \epsilon_{ijk} K_k$
Is ψ reducible?
$\gamma_5 = -\frac{i}{4} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu$


$J_j] = i C_{ijk} J_k$
$K_j] = -i \varepsilon_{ijk} J_k$
$K_j] = i \varepsilon_{ijk} K_k$
Is ψ reducible?
$\gamma_5 = -\frac{i}{4} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$
Levi-Civita symbol
$= -i \gamma^0 \gamma^1 \gamma^2 \gamma^3$
$\gamma_5^\dagger = \gamma_5$


educible?
$\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$
Levi-Civita symbol
$= -i\gamma^0\gamma^1\gamma^2\gamma^3$
$\gamma_5{}^\dagger = \gamma_5$
$\{\gamma_5, \gamma^\mu\} = 0$
$(\gamma_5)^2 = 1$


reducible?
$\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$
$\epsilon_{\mu\nu\rho\sigma}$
↳ Levi-Civita symbol
$= -i\gamma^0\gamma^1\gamma^2\gamma^3$
$\gamma_5^\dagger = \gamma_5$
$\{\gamma_5, \gamma^\mu\} = 0$
$(\gamma_5)^2 = 1$
$[\gamma_5, \Sigma^{\mu\nu}] = 0$
$2^{[D/2]}$ $[D/2]$
$\times 2$


Spinor rep on which $\sum^{MV}$ act on is reducible with the

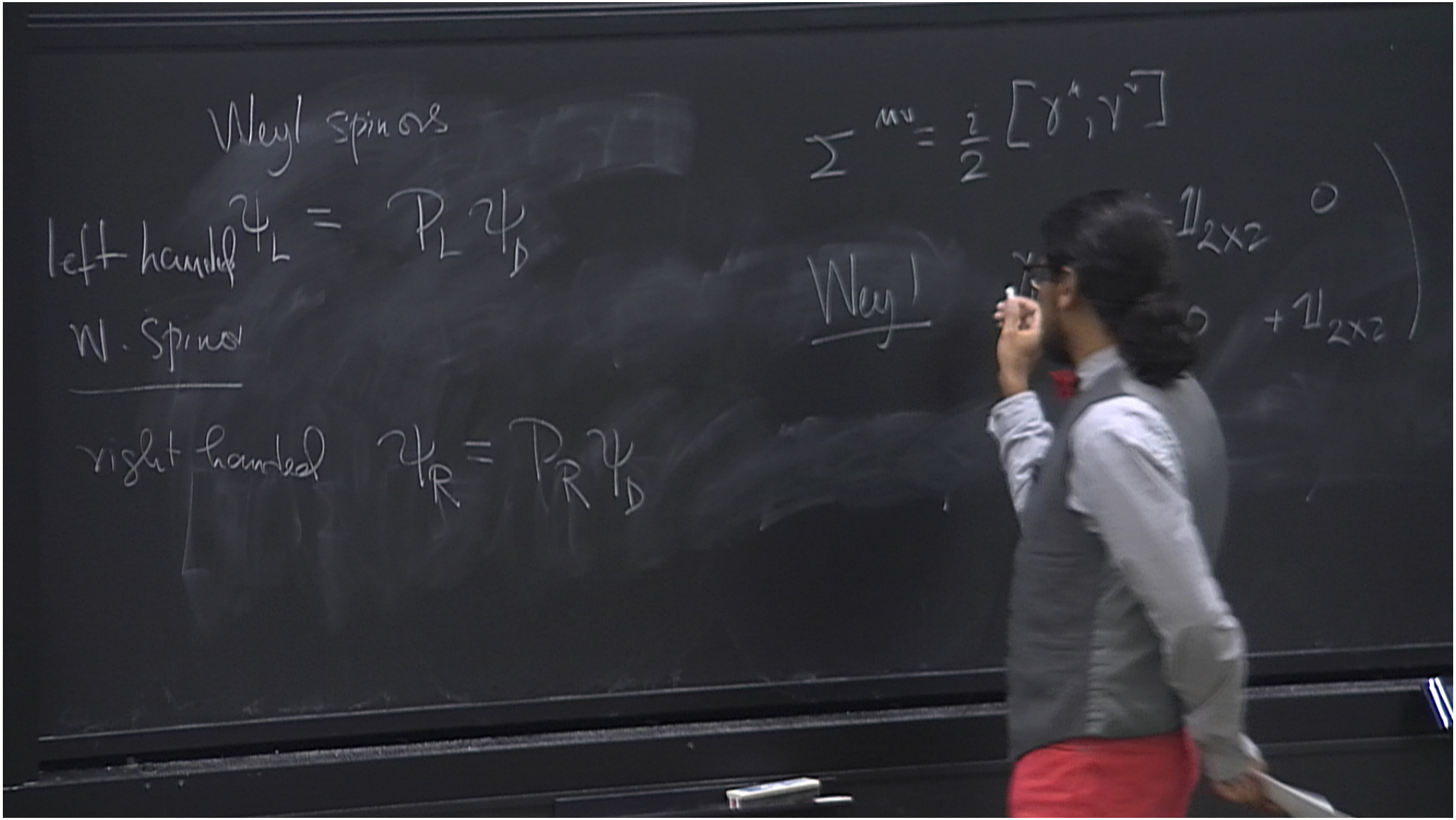
Weyl spinors

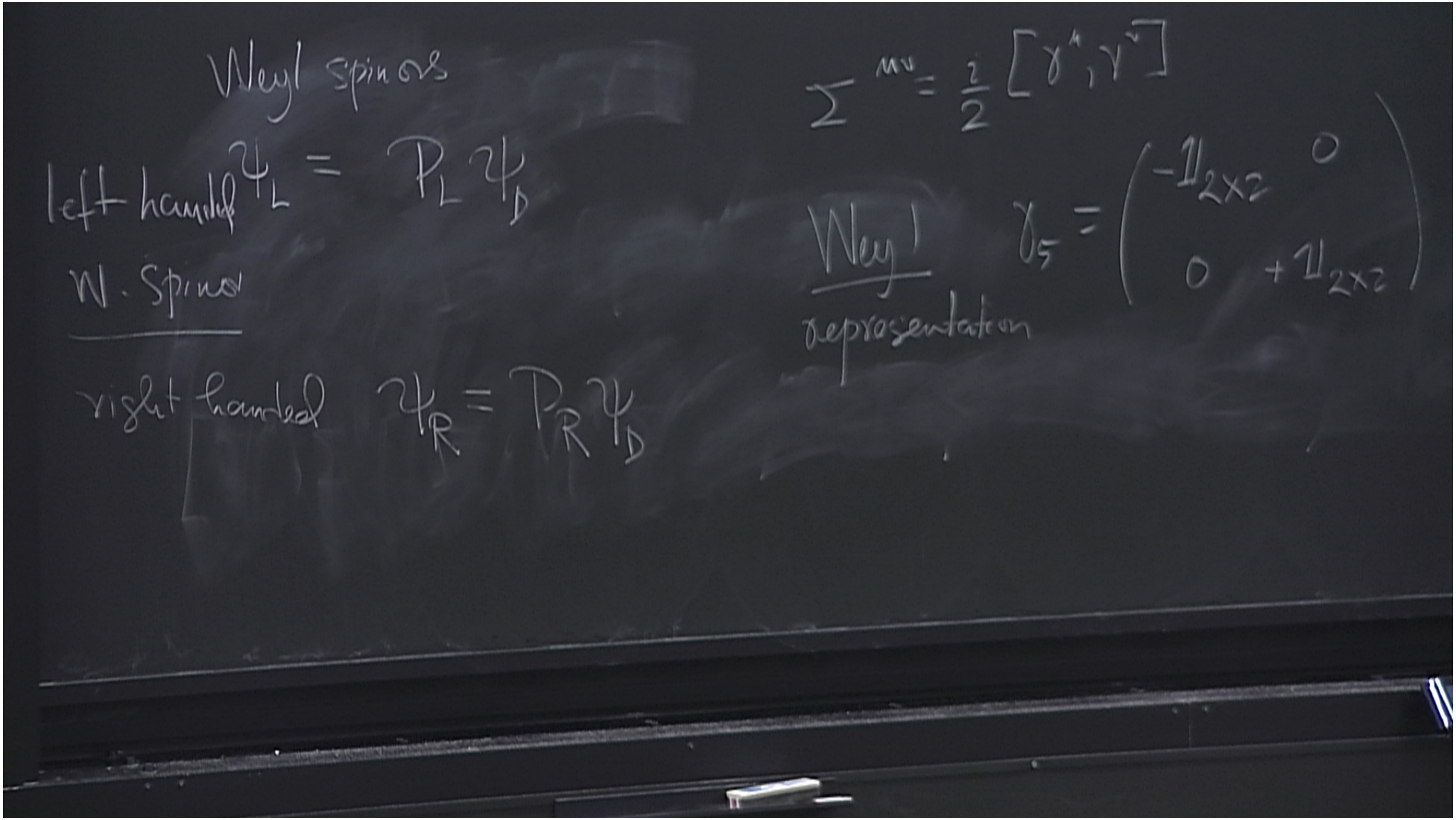
$$\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

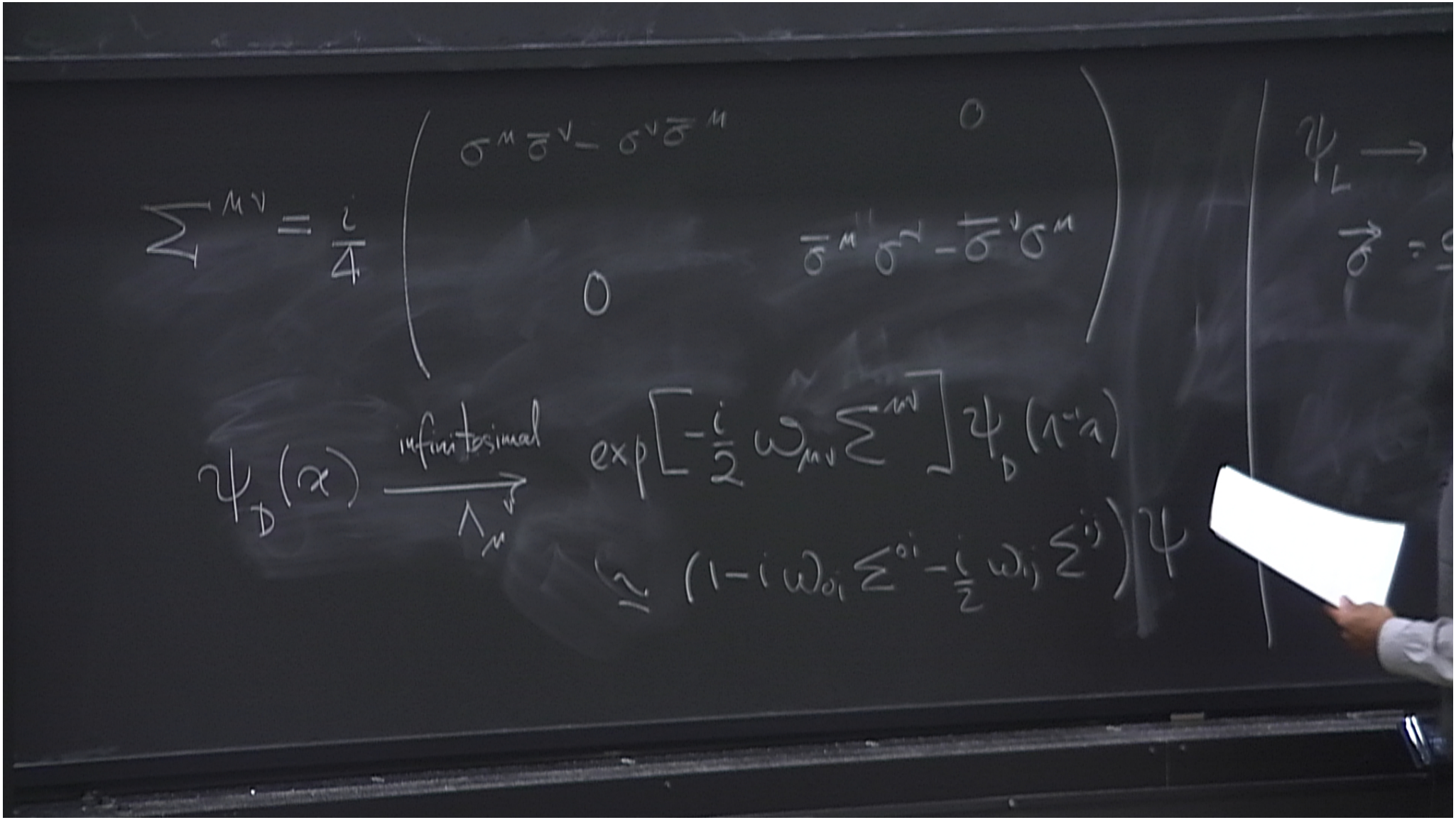
left handed $\psi_L = P_L \psi_D$

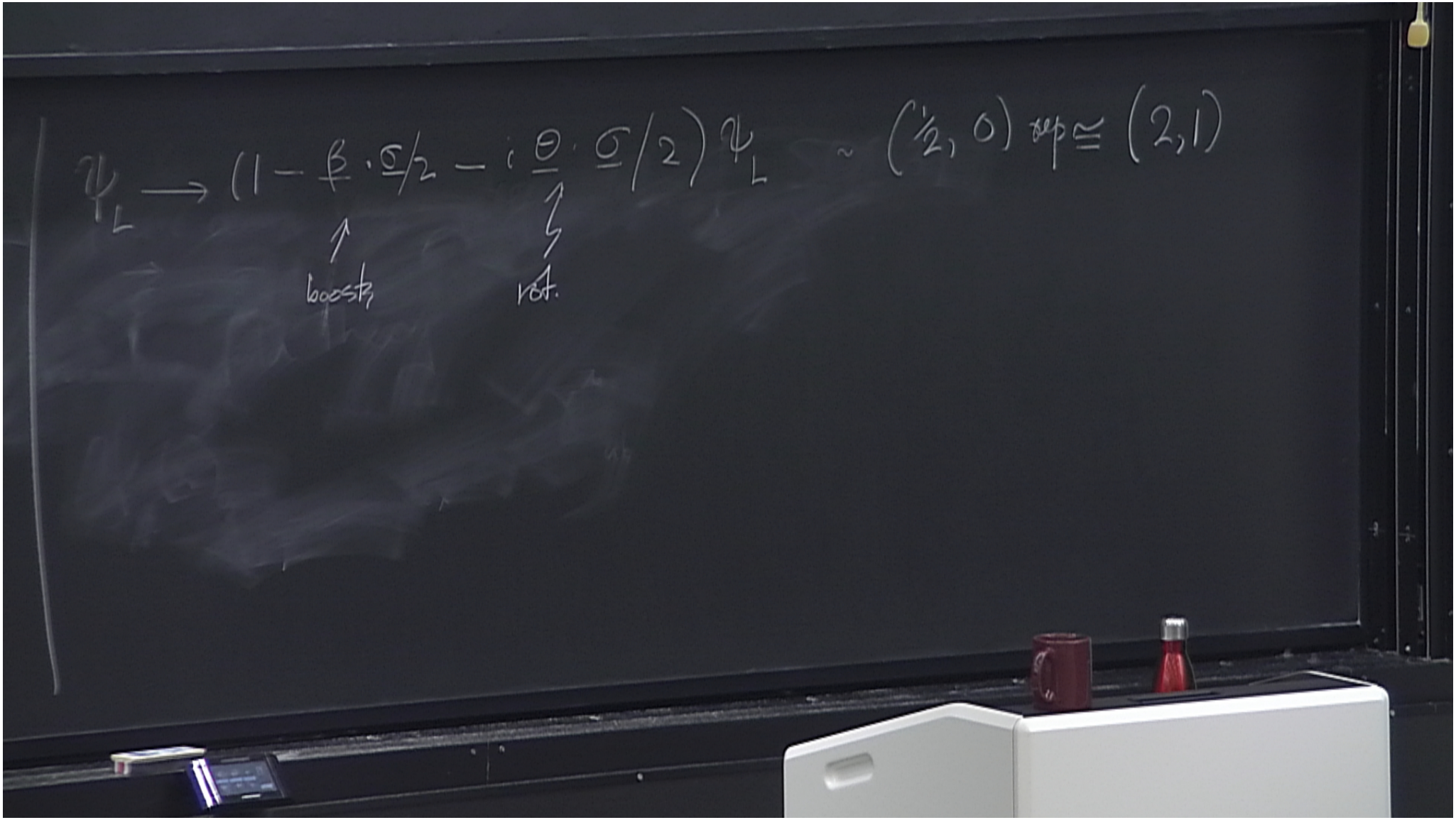
W. Spinor

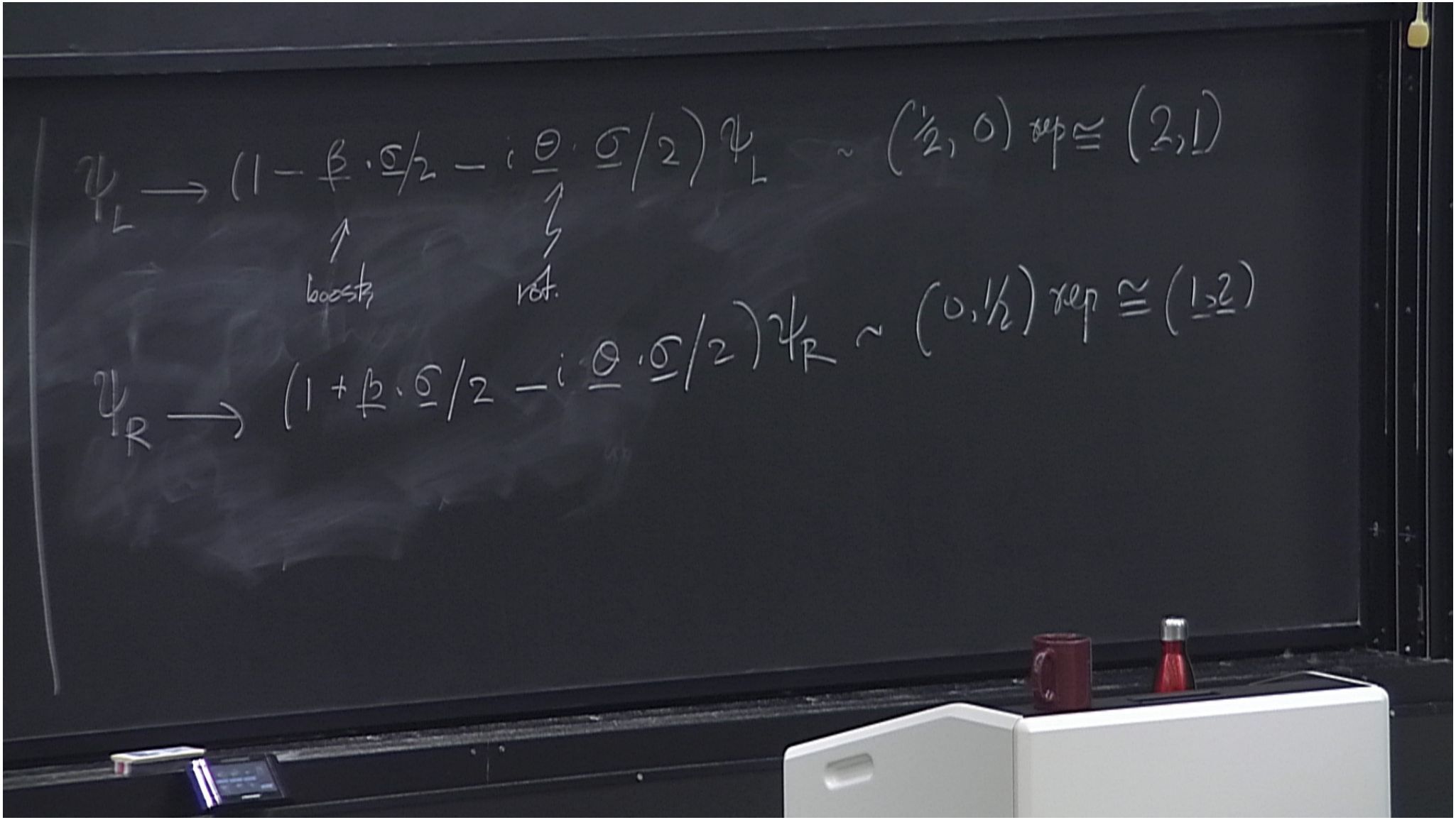
right handed $\psi_R = P_R \psi_D$


Weyl spinors
left handed $\psi_L = P_L \psi_D$
W. Spinor
right handed $\psi_R = P_R \psi_D$
$\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
Weyl
$\mathbb{1}_{2\times 2}$ 0
0 $+\mathbb{1}_{2\times 2}$


Weyl spinors
$\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
left handed $\psi_L = P_L \psi_D$
W. Spinor
Weyl representation
$\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & +\mathbb{1}_{2\times2} \end{pmatrix}$
right handed $\psi_R = P_R \psi_D$

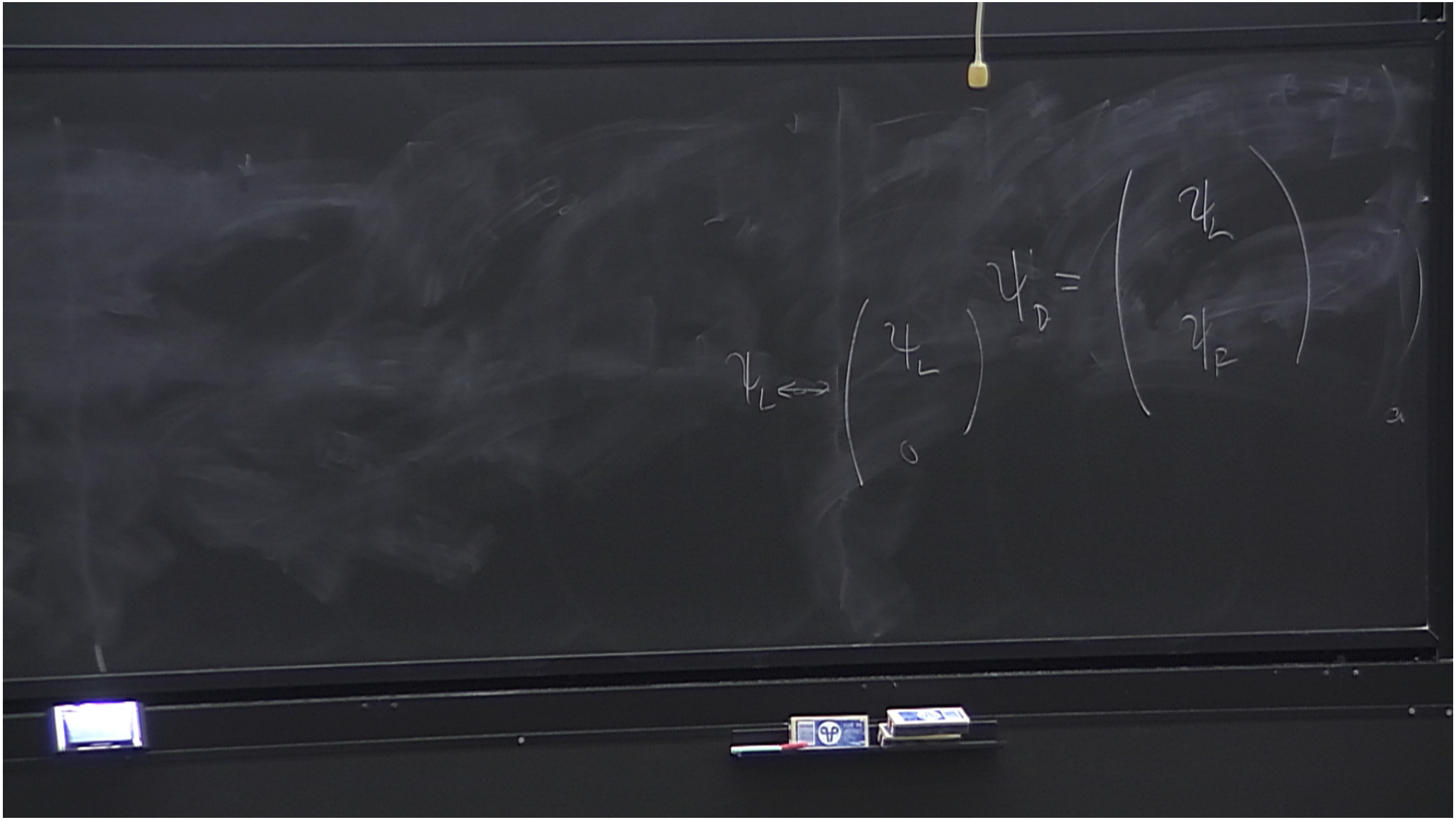

$\Sigma^{\mu\nu} = \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}$
$\psi_D(x) \xrightarrow[\Lambda^\mu{}_\nu]{\text{infinitesimal}} \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]\psi_D(\Lambda^{-1}x)$
$\simeq \left(1 - i\omega_{0i}\Sigma^{0i} - \frac{i}{2}\omega_{ij}\Sigma^{ij}\right)\psi$
$\psi_L \rightarrow$

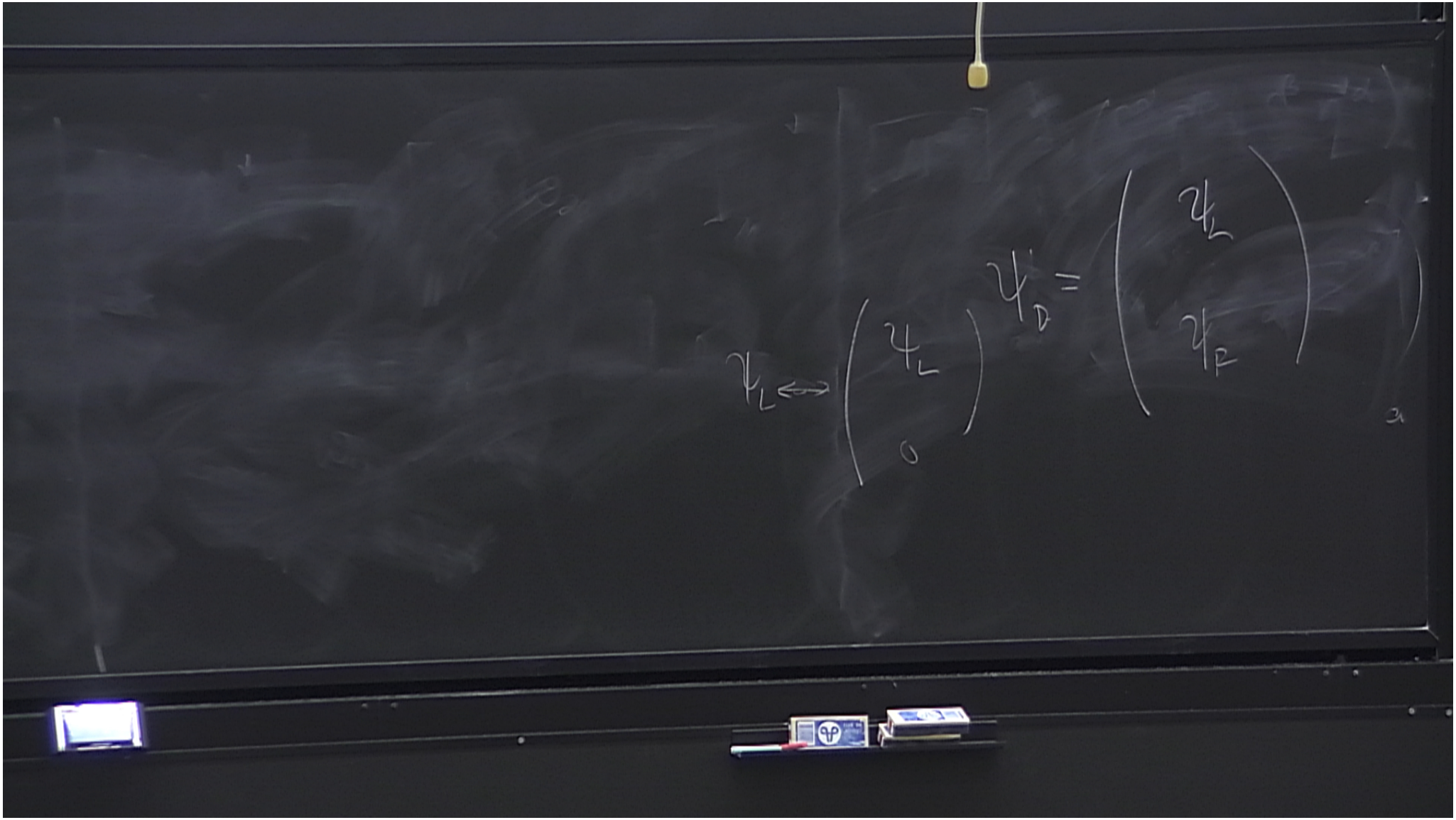

$\psi_L \longrightarrow (1 - \underline{\beta}\cdot\underline{\sigma}/2 - i\,\underline{\theta}\cdot\underline{\sigma}/2)\,\psi_L$
$\sim (\frac{1}{2}, 0)$ ⊕ $\cong (2,1)$
boosts
rot.

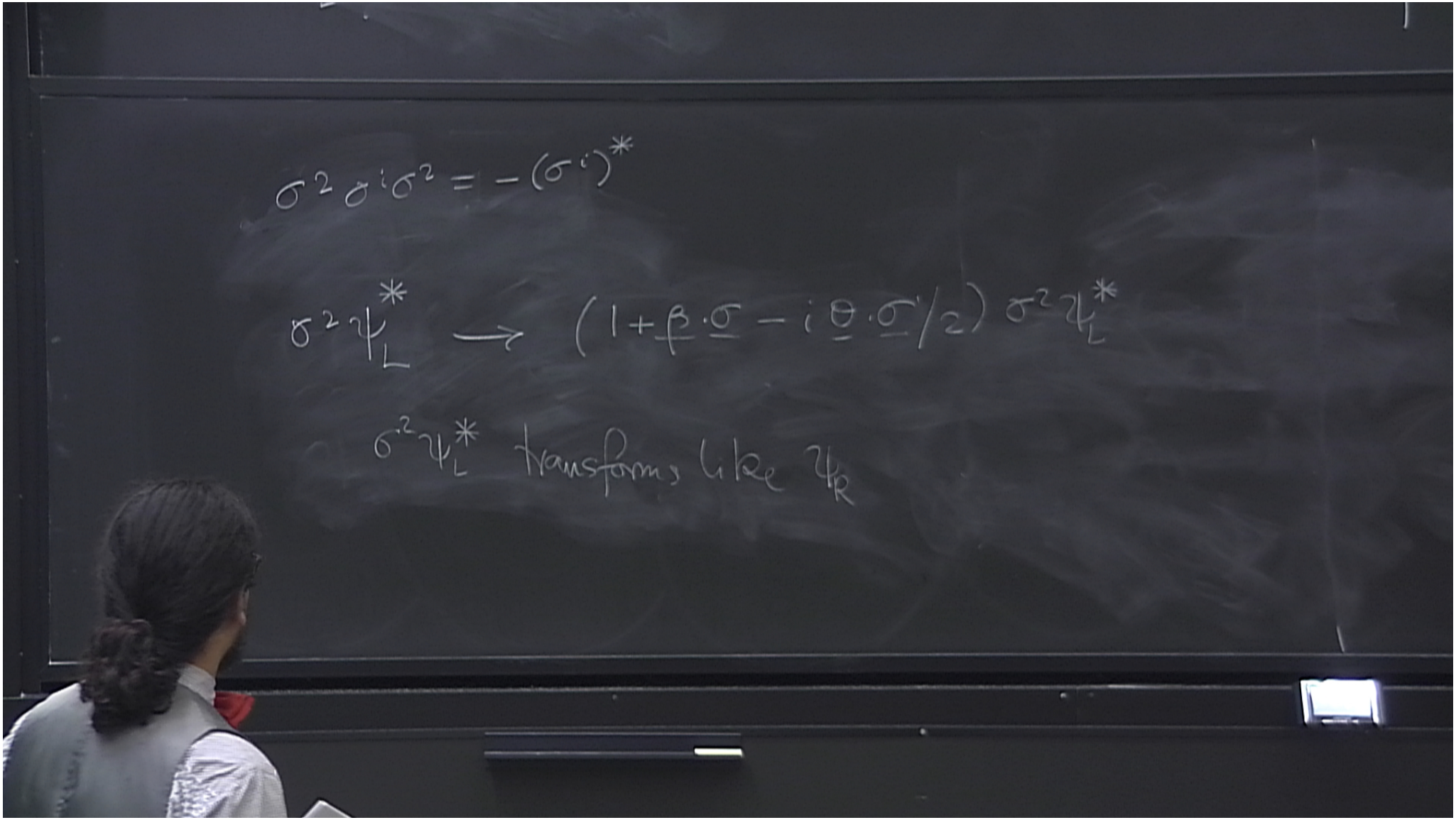

$\psi_L \longrightarrow (1 - \underline{\beta}\cdot\underline{\sigma}/2 - i\,\underline{\theta}\cdot\underline{\sigma}/2)\,\psi_L \sim (\tfrac{1}{2}, 0)$ rep $\cong (\underline{2}, \underline{1})$
boosts
rot.
$\psi_R \longrightarrow (1 + \underline{\beta}\cdot\underline{\sigma}/2 - i\,\underline{\theta}\cdot\underline{\sigma}/2)\,\psi_R \sim (0, \tfrac{1}{2})$ rep $\cong (\underline{1}, \underline{2})$

$$\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$$

$$\sigma^2 \psi_L^* \rightarrow (1 + \underline{\beta} \cdot \underline{\sigma} - i \underline{\theta} \cdot \underline{\sigma}/2) \sigma^2 \psi_L^*$$


$\psi_L \leftrightarrow \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$
$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

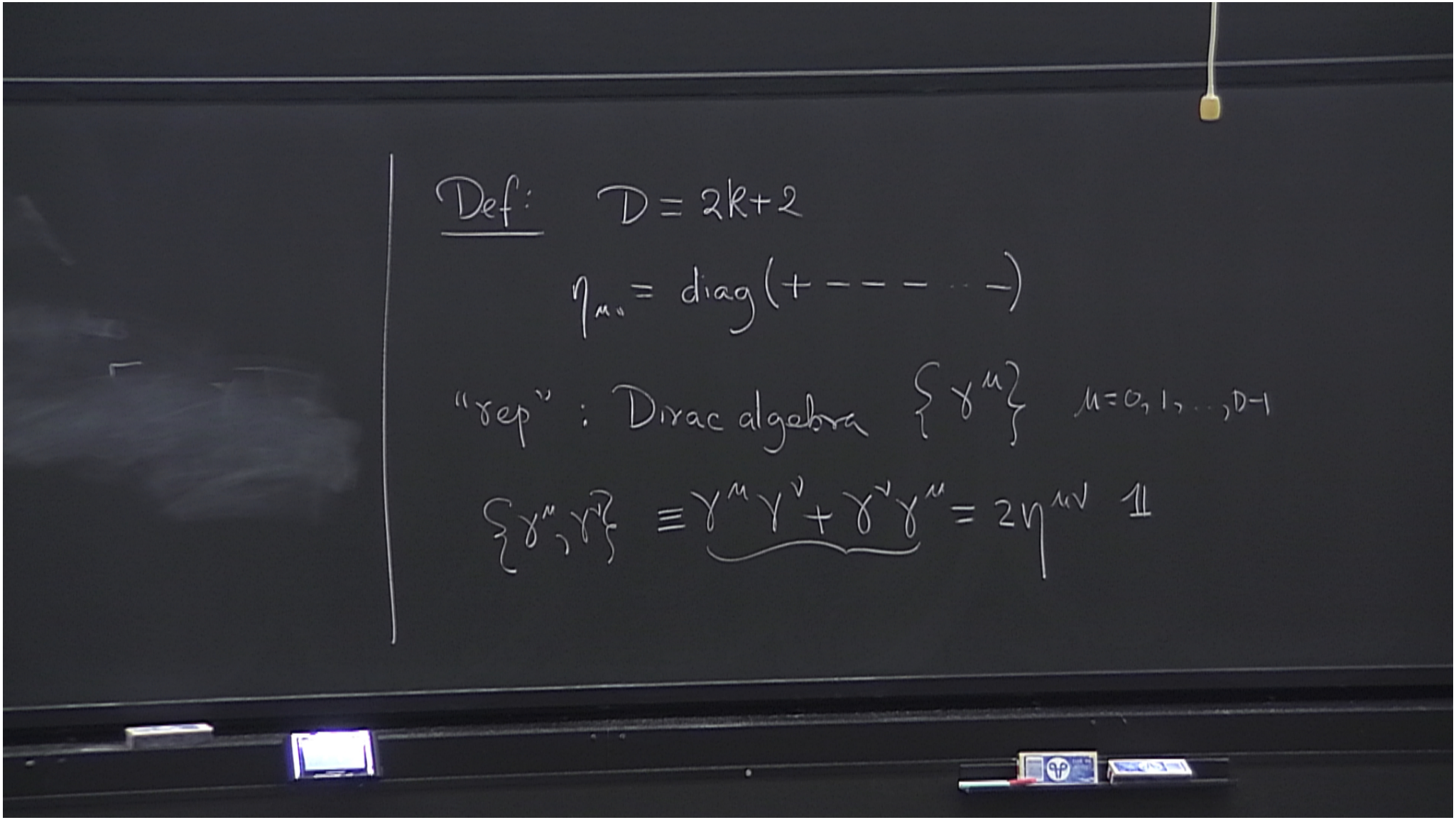

$\psi_L \leftrightarrow \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$
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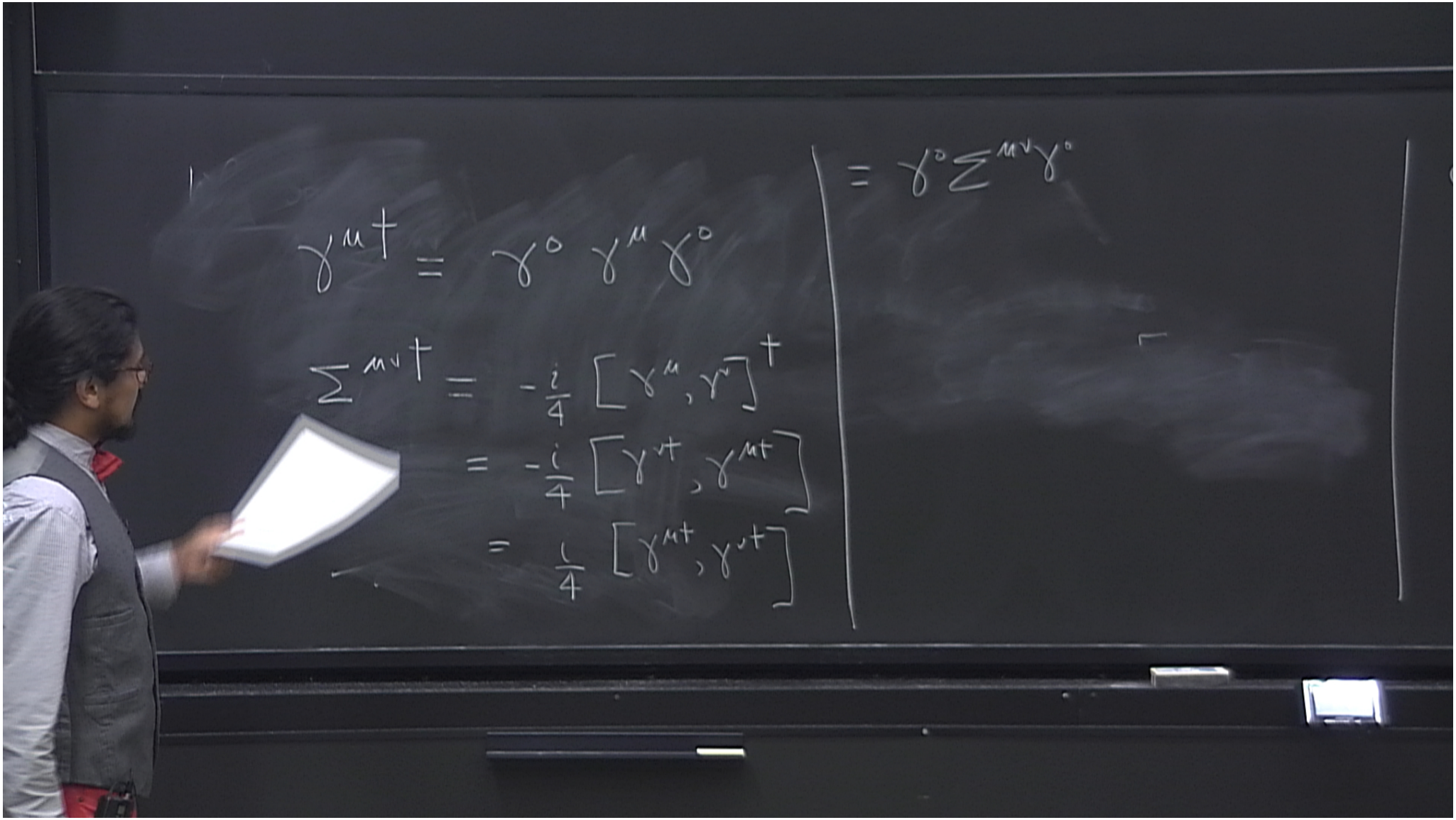

$\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$
$\sigma^2 \psi_L^* \rightarrow (1 + \underline{\beta} \cdot \underline{\sigma} - i \underline{\theta} \cdot \underline{\sigma}/2) \sigma^2 \psi_L^*$
$\sigma^2 \psi_L^*$ transforms like ψ_R

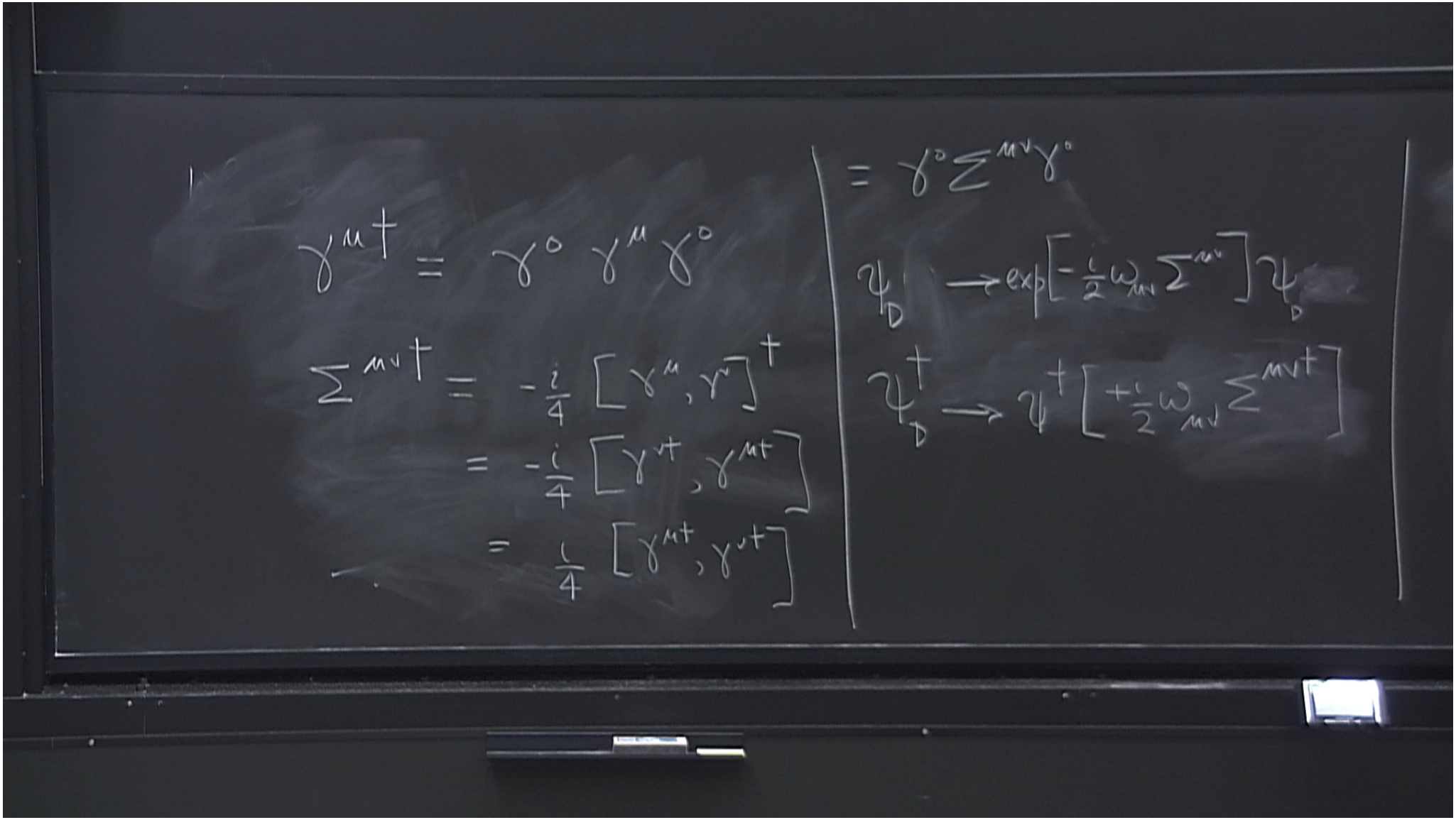
$$\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$$

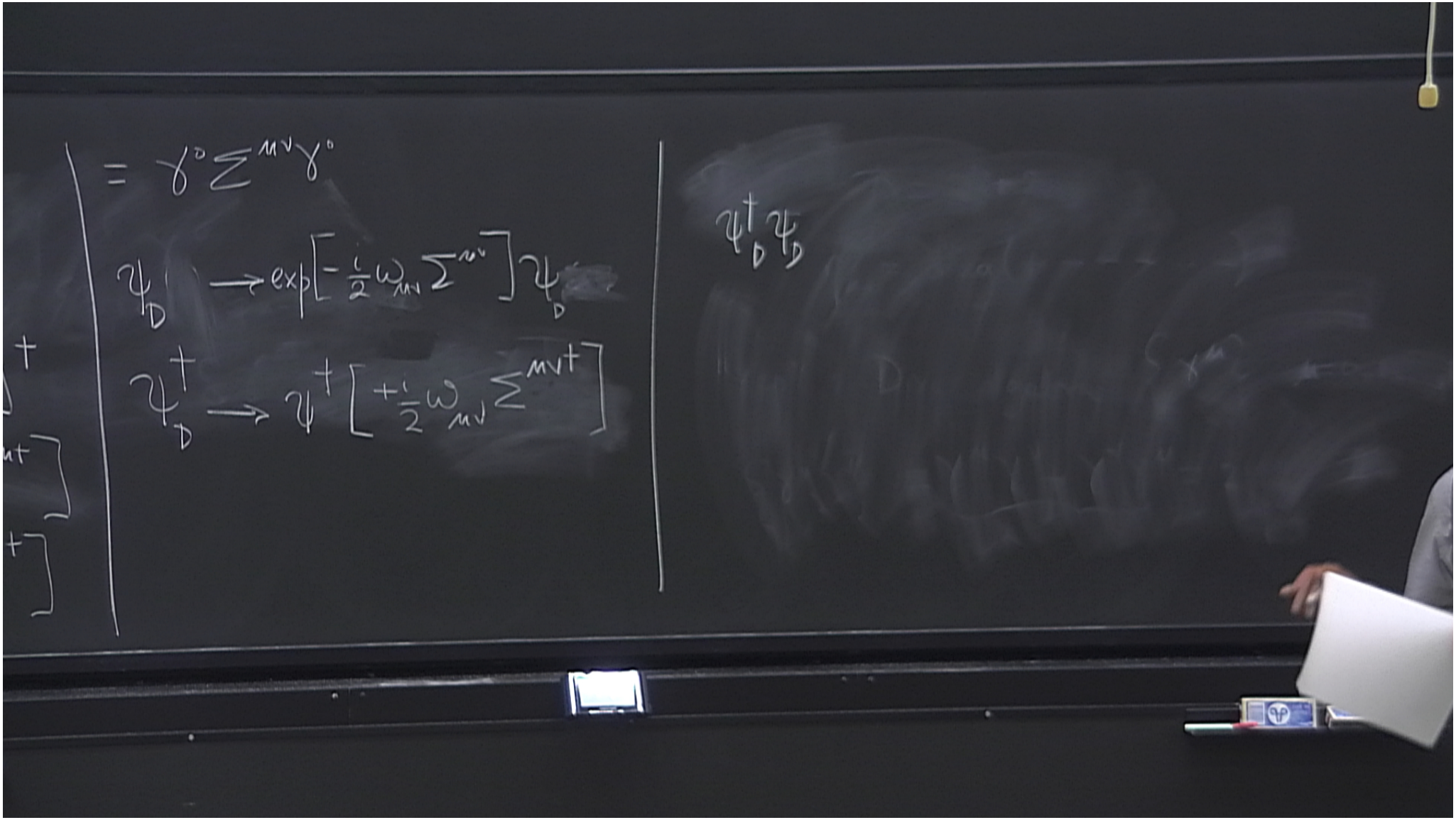
$$\sigma^2 \psi_L^* \longrightarrow (1 + \underline{\beta} \cdot \underline{\sigma} - i \underline{\theta} \cdot \underline{\sigma}/2)\, \sigma^2 \psi_L^*$$

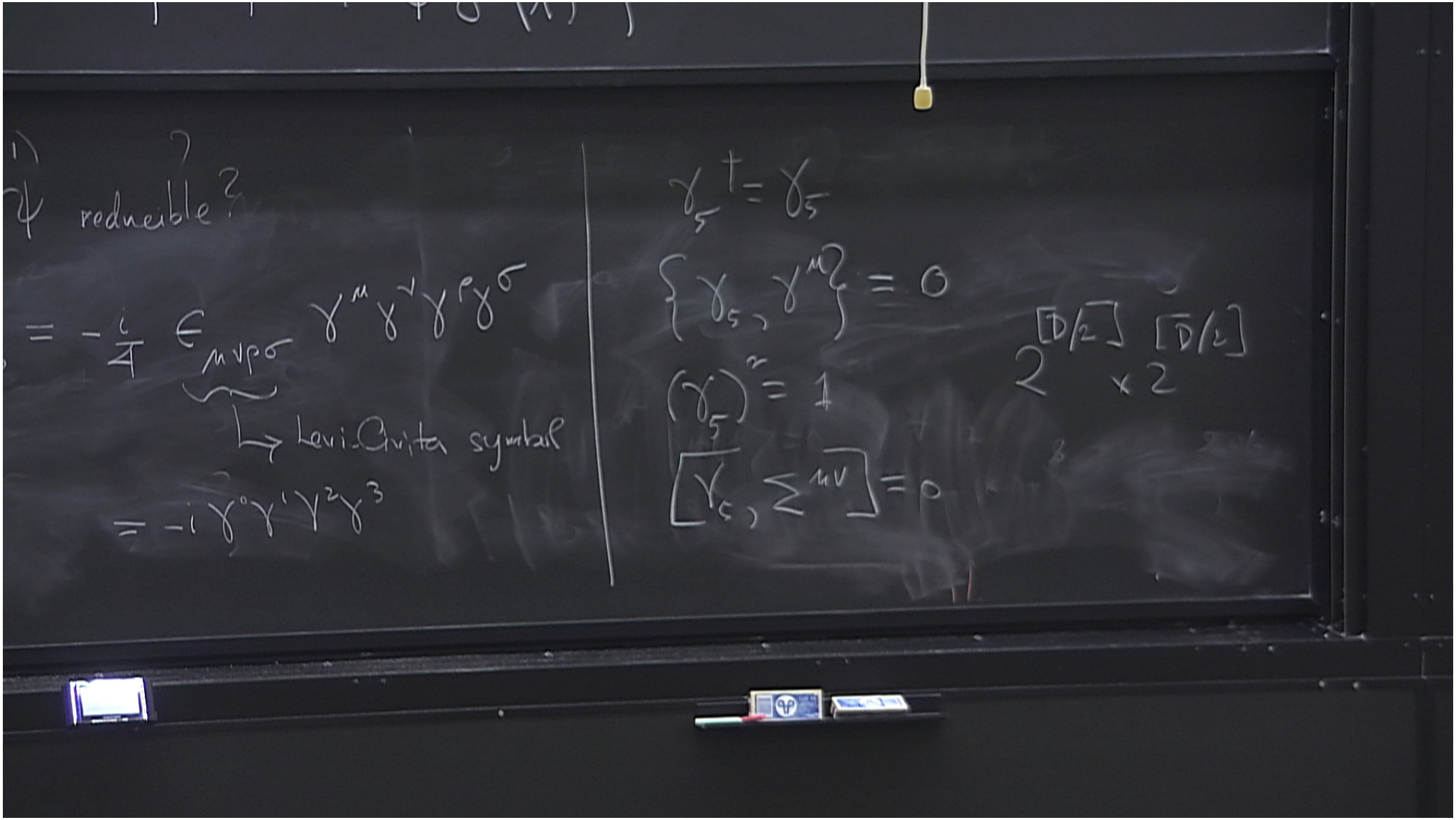
$\sigma^2 \psi_L^*$ transforms like ψ_R

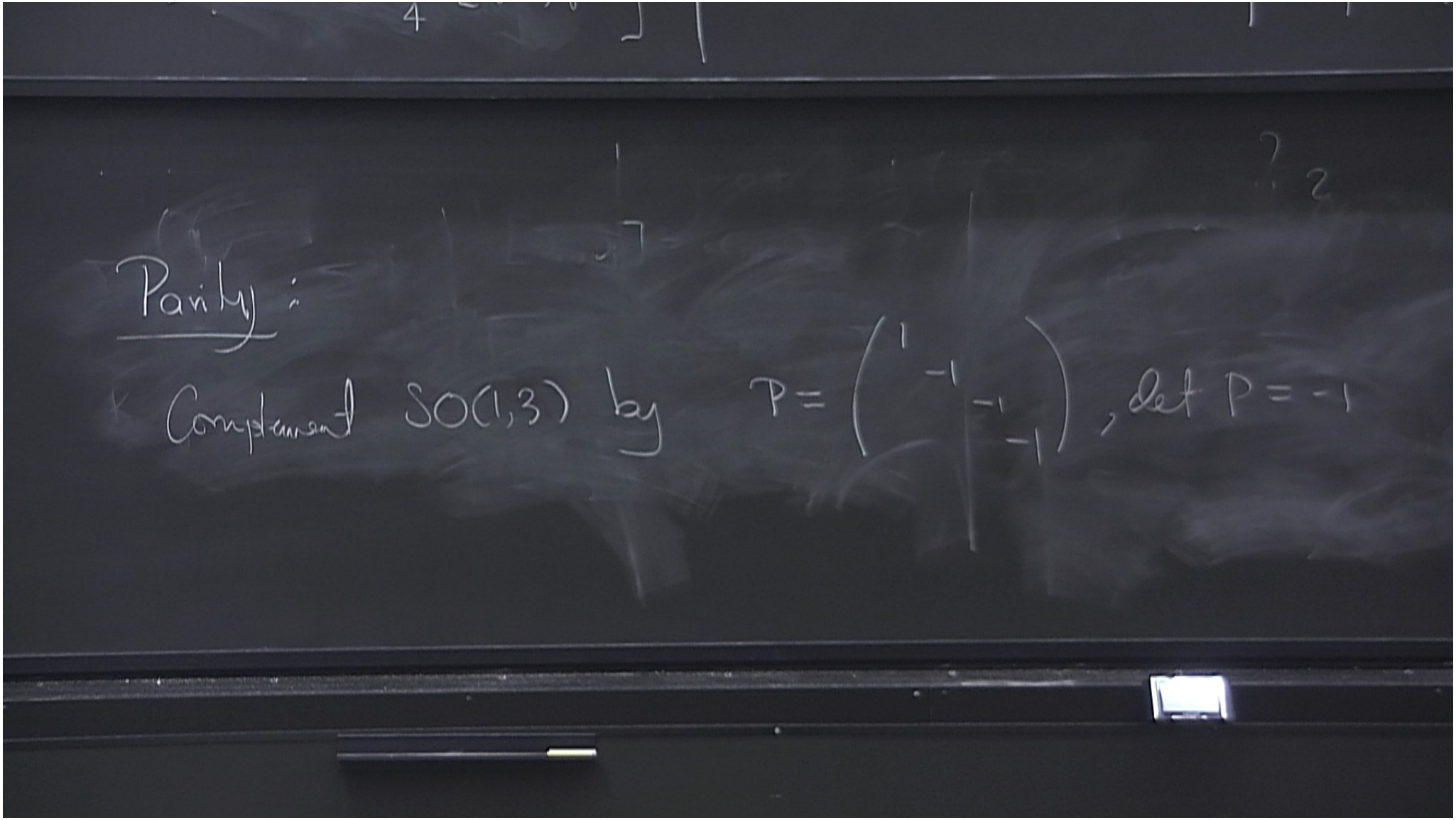

Def: $D = 2k+2$
$\eta_{\mu\nu} = \mathrm{diag}(+ - - - \cdots -)$
"rep": Dirac algebra $\{\gamma^\mu\}$ $\mu = 0, 1, \ldots, D-1$
$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,\mathbb{1}$

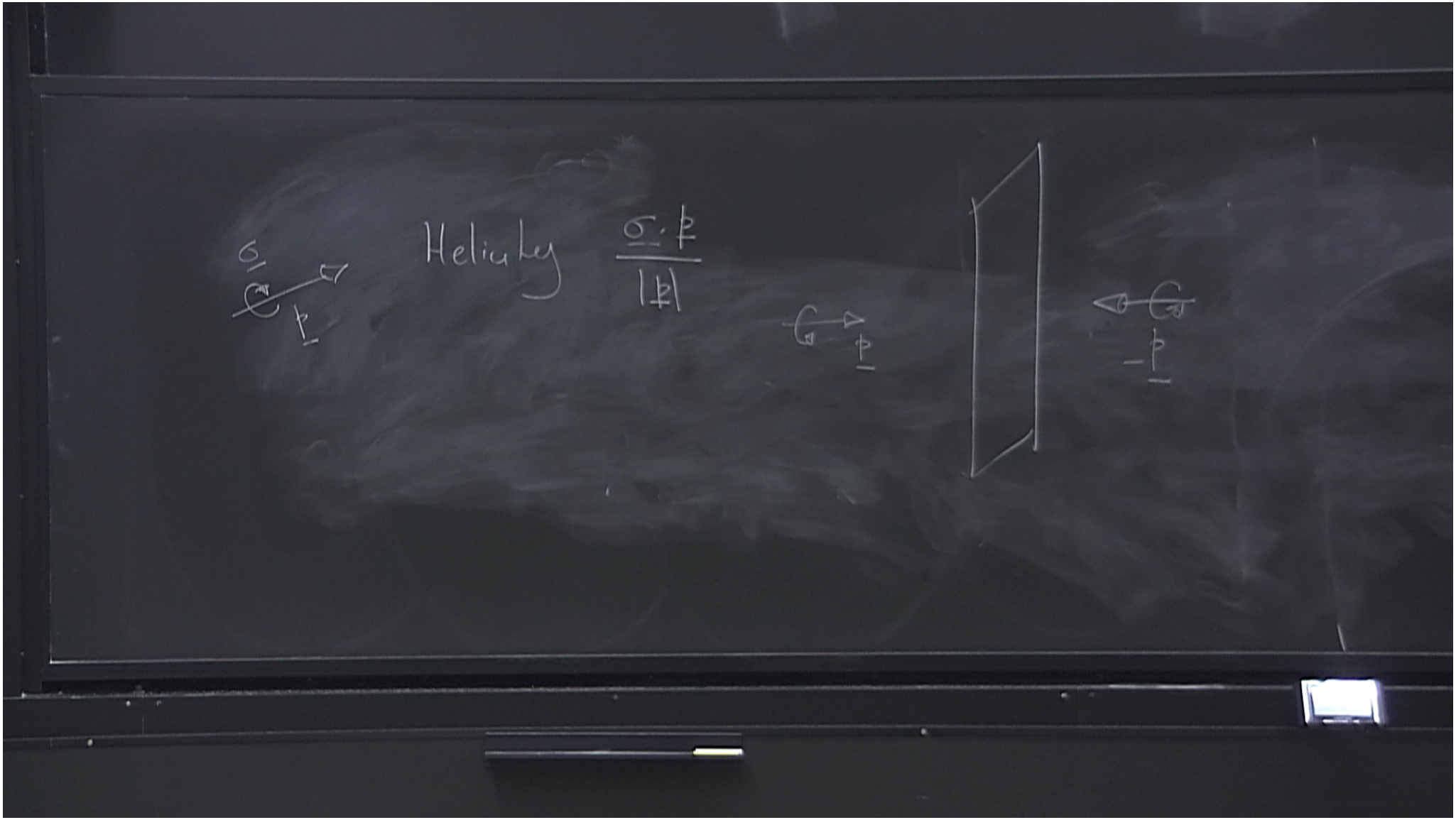

$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$
$\Sigma^{\mu\nu\dagger} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger$
$= -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$
$= \frac{i}{4}[\gamma^\mu, \gamma^\nu]$
$= \gamma^0 \Sigma^{\mu\nu} \gamma^0$

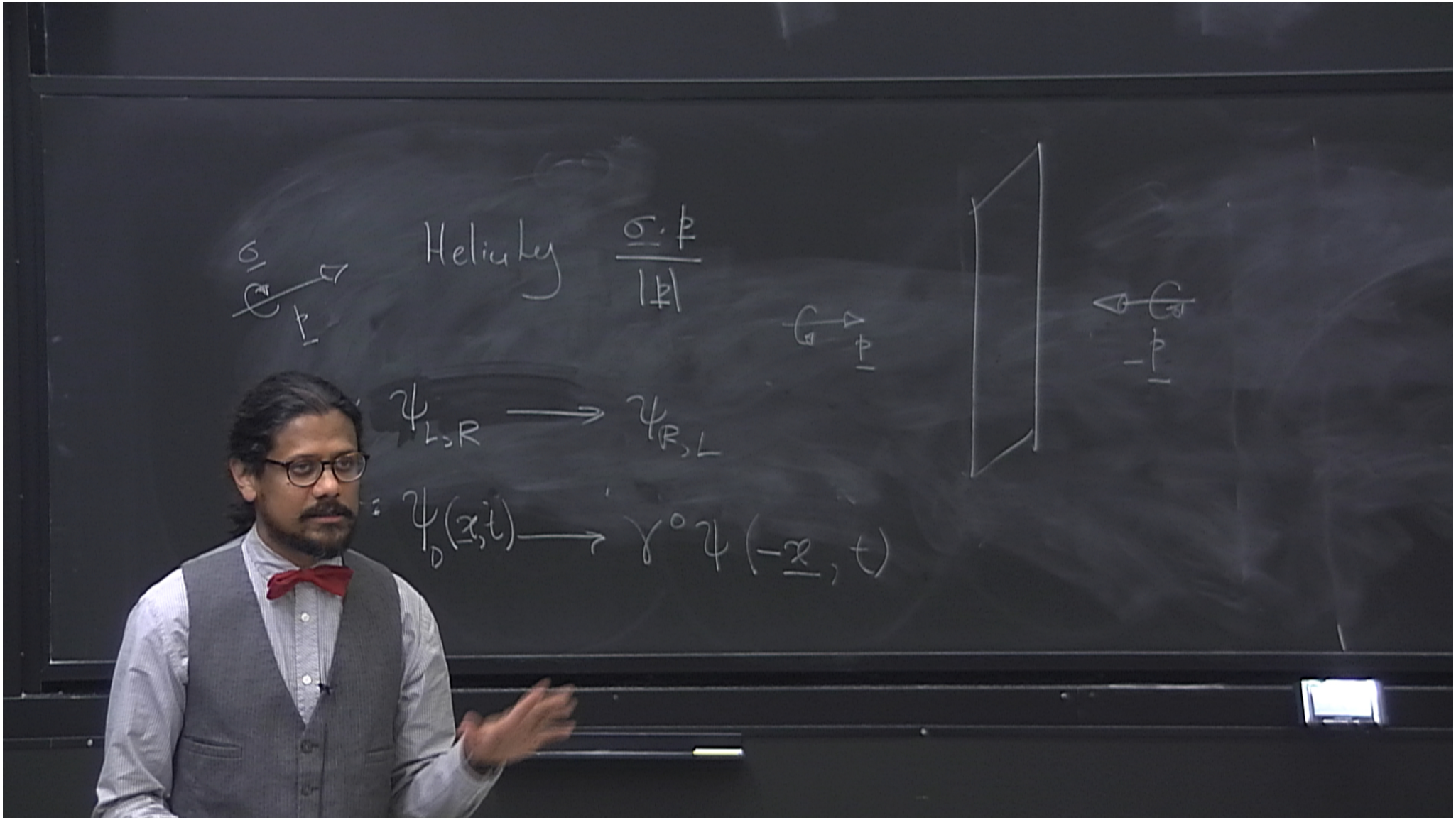

$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$
$\Sigma^{\mu\nu\dagger} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger$
$= -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$
$= \frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]$
$= \gamma^0 \Sigma^{\mu\nu} \gamma^0$
$\psi_D \to \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]\psi_D$
$\psi_D^\dagger \to \psi^\dagger\left[+\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\right]$

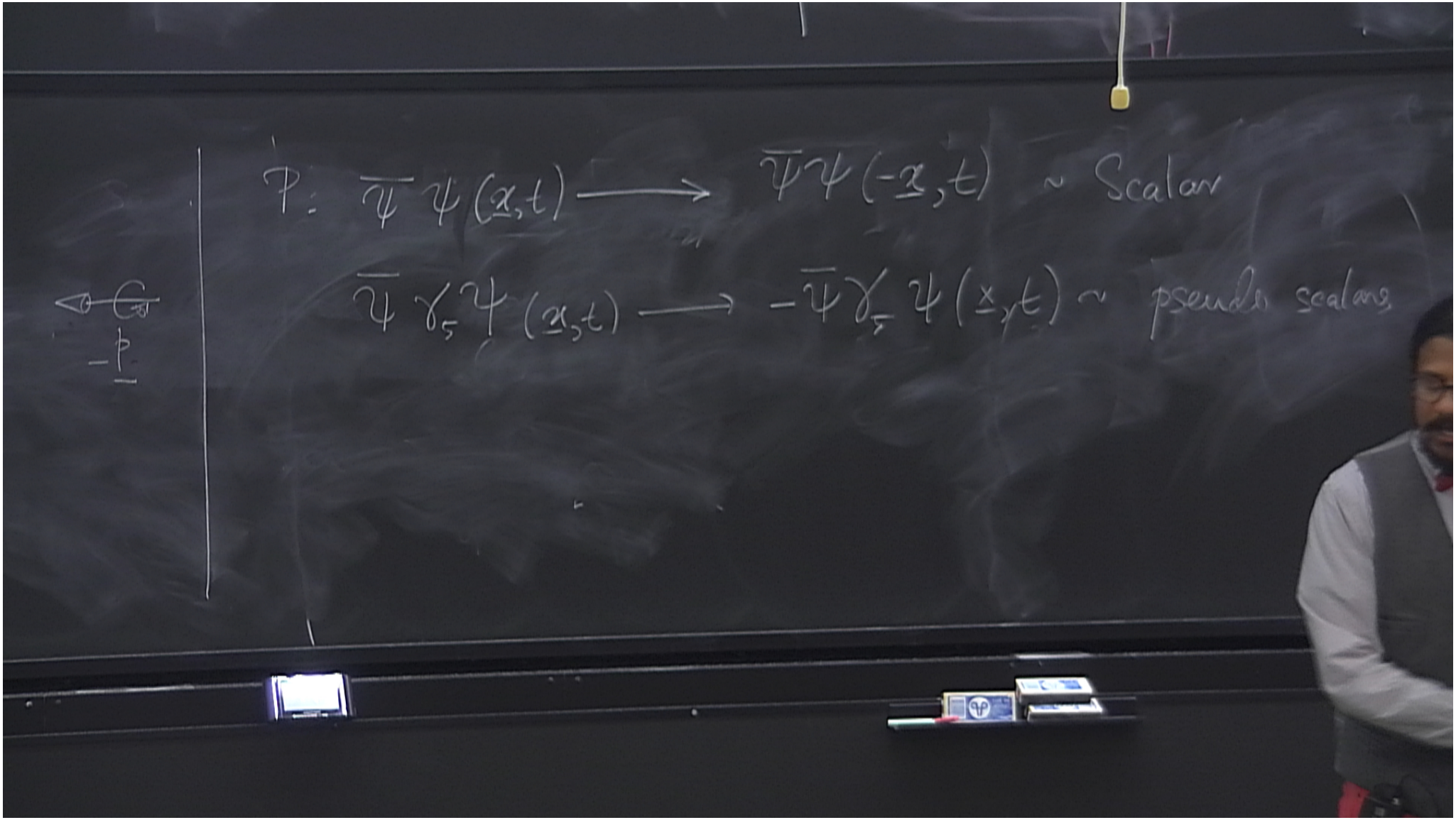

$= \gamma^0 \Sigma^{\mu\nu} \gamma^0$
$\psi_D \rightarrow \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]\psi_D$
$\psi_D^\dagger \rightarrow \psi^\dagger\left[+\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\right]$
$\psi_D^\dagger \psi_D$

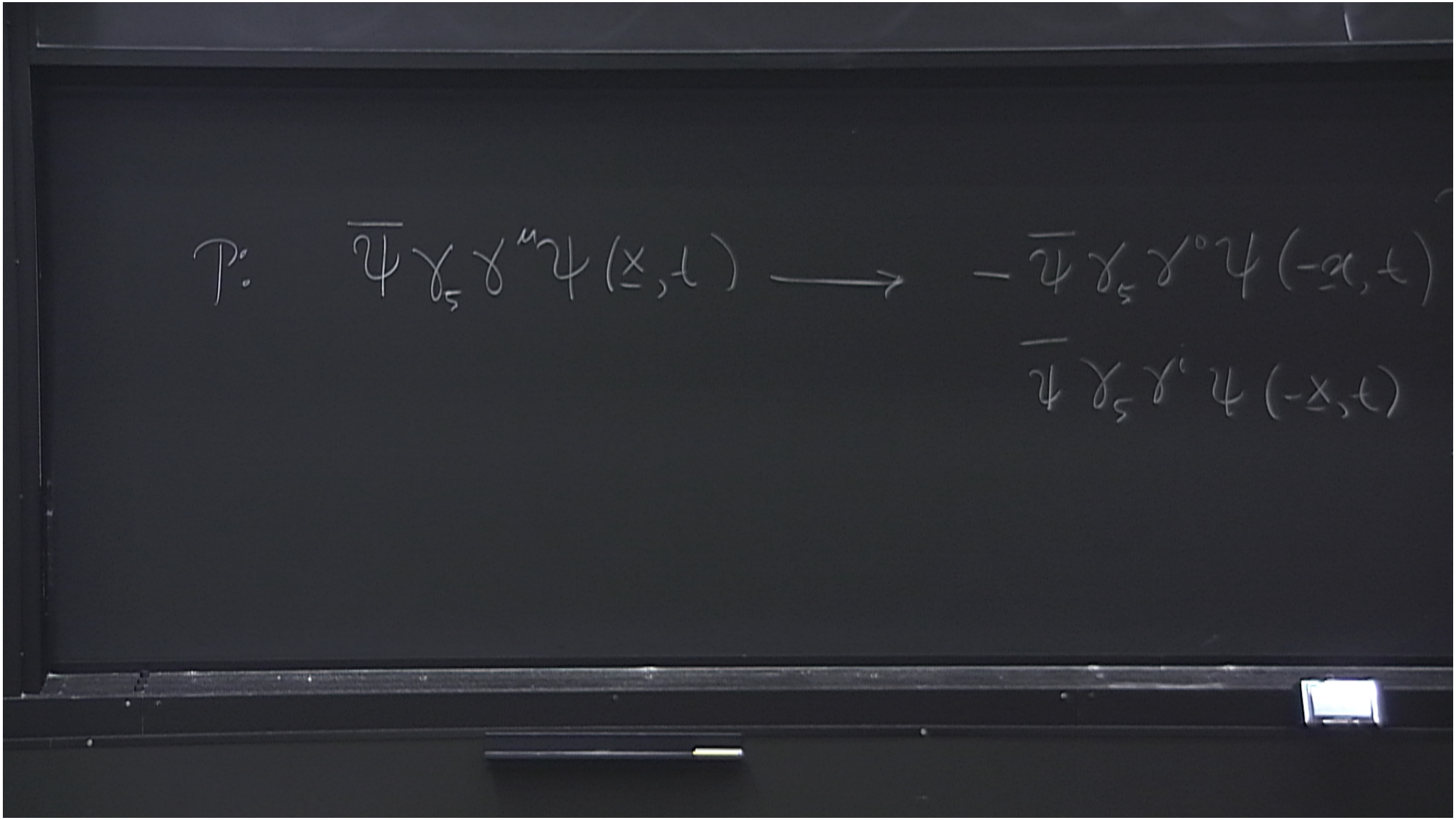

reducible?
$= -\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$
↳ Levi-Civita symbol
$= -i\gamma^0\gamma^1\gamma^2\gamma^3$
$\gamma_5^\dagger = \gamma_5$
$\{\gamma_5, \gamma^\mu\} = 0$
$(\gamma_5)^2 = 1$
$[\gamma_5, \Sigma^{\mu\nu}] = 0$
$2^{[D/2]} \times 2^{[D/2]}$


Parity :
Complement SO(1,3) by P = (1 -1 -1 -1), det P = -1


Helicity
σ·p̂/|p̂|
σ
p
p
p


Helicity
$\frac{\sigma \cdot p}{|p|}$
$\psi_{L,R} \longrightarrow \psi_{R,L}$
$\psi_D(x,t) \longrightarrow \gamma^0 \psi(-x,t)$


P: $\bar{\psi}\psi(\underline{x},t) \longrightarrow \bar{\psi}\psi(-\underline{x},t)$ ~ Scalar
$\bar{\psi}\gamma_5\psi(\underline{x},t) \longrightarrow -\bar{\psi}\gamma_5\psi(-\underline{x},t)$ ~ pseudo scalar
P


P: $\bar{\psi}\gamma_5\gamma^\mu\psi(\underline{x},t) \longrightarrow -\bar{\psi}\gamma_5\gamma^0\psi(-\underline{x},t)$
$\bar{\psi}\gamma_5\gamma^i\psi(-\underline{x},t)$