Title: On reducing 3d supersymmetric theories to two dimensions

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Abstract: We study the dimensional reduction of 3d QFTs with N=2 supersymmetry. In particular, we are interested in deriving dualities between 2d N=(2,2) theories starting from 3d dualities. Our main tool is the supersymmetric index, ie, the partition function on S^2 x S^1, which formally reduces to the partition function on S^2 as the radius of the circle goes to zero. There are various technical subtleties in this limit of the index which reflect physical subtleties in the reduction of the theories. We reproduce several known 2d N=(2,2) dualities and find evidence for some new ones.

On reducing 3d supersymmetric theories to two dimensions

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based on work with S. Razamat

Motivation

- There have been many dualities discovered in various dimensions, with various amounts of supersymmetry.
- Some of these appear superficially very similar, eg:

$4d\ \mathcal{N}=1 : Sp(2N_c) + 2N_f$ flavors $\leftrightarrow$ [Seiberg,Intriligator-Pouliot]

$Sp(2(N_f - N_c - 2)) + 2N_f$ flavors + mesons

$3d\ \mathcal{N}=2 : Sp(2N_c) + 2N_f$ flavors $\leftrightarrow$ [Aharony]

$Sp(2(N_f - N_c - 1)) + 2N_f$ flavors + mesons+ monopole s.p.

$2d\ \mathcal{N}=(2,2) : Sp(2N_c) + 2N_f$ flavors $\leftrightarrow$ [Hori]

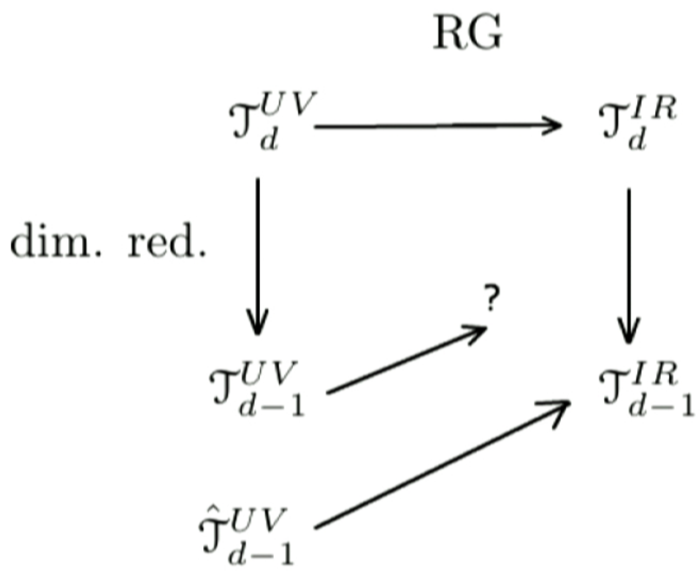
$Sp(2(N_f - N_c - \frac{1}{2})) + 2N_f$ flavors + mesons, $2N_f$ odd

- However, while clearly related, they are not connected by naive dimensional reduction.

Motivation (cont'd)

- This raises a general question - what does an IR duality in d dimensions imply upon compactification to $d-1$ dimensions?
- Problem: dimensional reduction and flowing to IR do not commute.
- More careful approach: Place a theory on $\mathbb{R}^{d-1} \times S^1_R$. For scales $\Lambda << \frac{1}{R}$, the theory is effectively $d-1$-dimensional.
- If $\frac{1}{R}$ is very small (compared to, eg, Λ_{QCD}), we flow to the IR before we feel the circle.
- Can we engineer a UV description in $d-1$ dimensions which flows to this effective theory?

Motivation (cont'd)



- IR duality for $\mathcal{T}_d^{UV,A}$, $\mathcal{T}_d^{UV,B}$ implies one for $\hat{\mathcal{T}}_{d-1}^{UV,A}$, $\hat{\mathcal{T}}_{d-1}^{UV,B}$.*
- Additionally, one must decide how to scale parameters of the theory.

Outline

- Motivation ✓
- Review of $4d \to 3d$
- General comments on $3d \to 2d$
- Supersymmetric partition functions and reduction
- Examples
- Conclusions

Brief review of $4d \to 3d$

- With O. Aharony, S. Razamat, N. Seiberg, we studied this problem for $d = 4$ and $\mathcal{N} = 1$ supersymmetric gauge theories.
- One finds that in a $4d$ gauge theory on $\mathbb{R}^3 \times S^1_R$, instanton effects generate a superpotential in the effective $3d$ description at low energies:

$$W = \eta Y, \qquad \eta = e^{-4\pi/(R{g_3}^2)}$$

- Here Y is a chiral operator parameterizing the $3d$ Coulomb branch.
- We can engineer a UV description in $3d$ by adding a superpotential $W = \eta Y_{mon}$, where Y_{mon} is a monopole operator which flows to Y at low energies.
- Claim: the $3d$ theories one gets by dimensionally reducing a $4d$ IR dual pair and adding the appropriate ηY superpotential are IR dual in $3d$.

Brief review of $4d \to 3d$ (cont'd)

- As an example, consider Seiberg duality between $4d$ $\mathcal{N}=1$ $SU(N_c)$ gauge theory with N_f fundamental flavors and $SU(N_f-N_c)$ with N_f flavors and N_f^2 mesons M, with $W = Mq\tilde{q}$.
- Claim 1: The $3d$ $\mathcal{N}=2$ dimensional reductions of these theories, with ηY superpotential, are IR dual.
- Claim 2: By starting with this $3d$ duality with N_f+1 flavors and giving one flavor a large real mass, one derives the following duality:

Theory A: $SU(N_c)$ with N_f flavors (and no superpotential)

Theory B: $U(N_f-N_c)$ with N_f flavors, mesons M, det fields $b, \tilde{b}$, and singlet Y, with $W = Mq\tilde{q} + Yb\tilde{b} + \tilde{V}_+ + \tilde{V}_-$

Review of 3d $\mathcal{N} = 2$ gauge theories

- Field content organized into chiral (ϕ, ψ, F) and vector $(A_\mu, \sigma, D, \lambda)$ multiplets.
- Action written in terms of D-terms (Kahler potential) and F-terms (superpotential). Also may include a Chern-Simons terms for gauge field:

$$\mathcal{L} = \frac{k}{4\pi} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A - 2D\sigma - \lambda\lambda)$$

- Real mass deformations associated to global symmetries - constant value m for scalar σ in background vector multiplet:

$$\mathcal{L} = ... + m^2 |\phi|^2 + im\psi^\dagger \psi$$

- We can also do this for the $U(1)_J$ topological symmetry, with current $J = \star Tr\ F$, leading to an FI term:

$$\mathcal{L} = ... + \zeta Tr\ D$$

Review of 2d $\mathcal{N} = (2,2)$ gauge theories

- Field content organized into chiral (ϕ, ψ, F) and vector multiplets $(A_\mu, \sigma, \eta, D, \lambda)$, as in 3$d$, plus twisted chiral multiplets, related to chirals by automorphism which exchanges $U(1)_V \times U(1)_A$ R-symmetry.
- Action written in terms of D-terms, F-terms, and twisted F-terms.
- Field strengths live in twisted chiral multiplets $\Sigma = \sigma + i\eta + ...$
- Twisted mass deformations associated to constant values for the complex scalar $\sigma + i\eta$ in background gauge multiplets.
- FI terms live in background twisted chiral multiplets $t = r + i\theta + ...$, contribute as:

$$\tilde{W} = tTr\,\Sigma \quad \Rightarrow \mathcal{L} = rTr\,D + \theta Tr\,F$$

3d theory on a circle - the "standard limit"

- Now we place a 3d $\mathcal{N}=2$ $U(1)$ gauge theory on $\mathbb{R}^2 \times S^1_R$ and write the dimensionally reduced 2d $\mathcal{N}=(2,2)$ theory, following [Aganagic et al].
- For vector multiplets, $A_3 \to \eta$, real mass parameters are complexified and become complex twisted masses.
- The Chern-Simons term and FI term contributions can be written in terms of a twisted superpotential, classically:

$$\tilde{W} = \frac{1}{2} k R \Sigma^2 + \zeta R \Sigma$$

- At 1-loop, the 2d FI term flows as:

$$r(\mu') = r(\mu) + \sum_a Q_a \log(\mu/\mu')$$

- Thus to obtain a well-defined limit, we must scale:

$$\zeta R \sim r(\mu) + \sum_a Q_a \log(\mu R)$$

- We call this the "standard" limit of the gauge theory.

Scaling of real masses

- Note we hold real masses for flavor symmetries finite, but scale FI terms to infinity.
- However these are on the same footing. Eg, they are exchanged under mirror symmetry. Thus in the mirror theory we must take masses for chirals very large.
- [Aganagic et al] showed that the twisted superpotential one obtains after integrating them out reproduces precisely the Hori-Vafa mirror dual of the gauge theory in $2d$ (more below).
- We would like to study this phenomenon in more complicated theories, and to gain more of a handle on the problem, we will use supersymmetric partition functions.

Supersymmetric partition functions

- An important tool for studying supersymmetric theories and dualities between them is the partition function on a compact manifold $\mathcal{M}$.
- In many cases, these can be computed by localization of the path integral to a finite subspace of BPS field configurations .
- In addition to integrating over configurations for dynamical fields, we can also turn on fixed BPS configurations for background fields schematically:

$$\mathcal{Z}^{\mathcal{M}}(\mu_a) = \int [dz_j] \mathcal{Z}^{\mathcal{M}}_{1-loop}(z_j; \mu_a)$$

- It also depends on (a subset of) geometric parameters of $\mathcal{M}$ [Closset, Dumitrescu, Festuccia, Komargodski].
- These are RG invariant, and so give a powerful test of IR dualities.

Supersymmetric partition functions and dimensional reduction

- We will be interested in the cases $\mathcal{M} = S^d$ and $S^d \times S^1_\tau$.
- In the latter case, $\mathcal{Z}^{S^d \times S^1_\tau}$ depends on τ, and one expects:

$$\mathcal{Z}^{S^{d-1} \times S^1_\tau}[\mathcal{T}_d^{UV}] \underset{\tau \to 0}{\longrightarrow} f_{div}(\tau) \mathcal{Z}^{S^{d-1}}[\hat{\mathcal{T}}_{d-1}^{UV}] + \ldots$$

  where $f_{div}(\tau)$ is some divergent prefactor [Di Pietro, Komargodski]
- In particular, if we have an IR dual pair in d dimensions, the identity of $\mathcal{Z}^{S^{d-1} \times S^1_\tau}$ implies the identity of $\mathcal{Z}^{S^{d-1}}$ for their IR reductions.
- One must also choose how to scale the parameters μ_a as $\tau \to 0$, and this will affect what theory we obtain.

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Supersymmetric partition functions and dimensional reduction (cont'd)

- We can attempt to use this fact to reconstruct $\hat{\mathcal{T}}_{d-1}^{UV}$.
- Namely, we would like to write the RHS of the limit in a form where we can read off the UV ingredients which give rise to it $\rightarrow$ but limitations.
- Since we are taking a limit of an integral, in order to write the result as an integral, we must find the correct way to scale the integration variables in order to commute the operations of taking a limit and integral.
- Toy examples:

$$\int dx\; e^{-\tau x^2} \rightarrow \frac{1}{\sqrt{\tau}} \int dy\; e^{-y^2}, \qquad \int dx\; e^{-(x-1/\tau)^2} \rightarrow \int dy\; e^{-y^2}$$

- One can then conjecture this gives the correct scaling for the fields in the full theory.

3d supersymmetric index

- The 3d index, or partition function on $S^2 \times S^1_\tau$, localizes to the following BPS vector multiplet configurations:

$$A = \mathbf{m} A_{Dirac} + Z dx_3, \quad \sigma = \frac{\mathbf{m}}{2}$$

These are labeled the integer $\mathbf{m}$ and the holonomy $\mathbf{z} = e^{i\tau Z}$.

- The index of a chiral multiplet in such a background is (here $\mathbf{q} = e^{-\tau}$):

$$\mathcal{I}_\chi^\Delta(\mathbf{z}, \mathbf{m}; \mathbf{q}) = (\mathbf{q}^{\frac{1-\Delta}{2}} \mathbf{z}^{-1})^{|\mathbf{m}|/2} \prod_{\ell=0}^{\infty} \frac{1 - \mathbf{q}^{1-\frac{\Delta}{2}+|\mathbf{m}|/2+\ell} \mathbf{z}^{-1} (-1)^{\mathbf{m}}}{1 - \mathbf{q}^{\frac{\Delta}{2}+|\mathbf{m}|/2+\ell} \mathbf{z} (-1)^{\mathbf{m}}}$$

- The index of a general gauge theory is given by:

$$\mathcal{I}_{gauge}(\mu_a, \mathbf{s}_a) = \sum_{\mathbf{m}_j \in \Lambda_G} \int_{\mathbb{T}_G} \frac{d\mathbf{z}_j}{2\pi i \mathbf{z}_j} \mathcal{I}_{CS} \mathcal{I}_{matt}(\mathbf{z}_j, \mathbf{m}_j; \mu_a, \mathbf{s}_a)$$

2d sphere partition function

- The partition function on S^2 localizes to the following configurations for gauge multiplets:

$$A = mA_{Dirac}, \quad \sigma = \frac{m}{2}, \quad \eta = Z$$

- The p.f. of a chiral multiplet in this background is:

$$\mathcal{Z}_\chi^\Delta(Z, m) = \frac{\Gamma(\frac{\Delta}{2} - iZ - \frac{m}{2})}{\Gamma(1 - \frac{\Delta}{2} + iZ - \frac{m}{2})}$$

- Twisted chiral fields also localize to constant configurations, and contribute as:

$$\mathcal{Z}_{tc} = \int \frac{dYd\bar{Y}}{\pi} e^{\tilde{W}(Y) - \tilde{W}(\bar{Y})}$$

- The p.f. of a general gauge theory is given by:

$$\mathcal{Z}_{gauge}(M_a, s_a) = \sum_{\mathbf{m}_j \in \Lambda_G} \int \frac{dZ_j}{2\pi} \mathcal{Z}_{tc} e^{2irZ + i\theta m} \mathcal{Z}_{matt}(Z_j, m_j; M_a, s_a)$$

Standard limit in the index

- A basic relation between these objects is:

$$\mathcal{I}_\chi^\Delta(\mathbf{z}=e^{i\tau Z},\mathbf{m}=m) \underset{\tau\to 0}{\longrightarrow} \tau^{\Delta-1-2iZ}\mathcal{Z}_\chi^\Delta(Z,m)$$

- Consider the $3d$ index of a general $U(1)$ gauge theory:

$$\mathcal{I}(\mathbf{w},\mathbf{n};\mu_a,\mathbf{n}_a)=\sum_{\mathbf{m}\in\mathbb{Z}}\oint\frac{d\mathbf{z}}{2\pi i\mathbf{z}}\mathbf{z}^{k\mathbf{m}}\mathbf{w}^{\mathbf{m}}\mathbf{z}^{\mathbf{n}}\prod_a\mathcal{I}_\chi^{\Delta_a}(\mathbf{z}^{Q_a}\mu_a,Q_a\mathbf{m}+\mathbf{s}_a)$$

- And the following S^2 partition function of its dimensional reduction:

$$\mathcal{Z}(r,\theta;M_a,s_a)=\sum_{m\in\mathbb{Z}}\int\frac{dZ}{2\pi}e^{2irZ+i\theta m}\prod_a\mathcal{Z}_\chi^{\Delta_a}(Q_aZ+M_a,Q_am+s_a)$$

- As we take $\tau\to 0$, we scale:

$$\mu_a=e^{i\tau M_a},\ \mathbf{s}_a=s_a,\ \mathbf{w}=e^{i\theta},\ \mathbf{n}=2\tau^{-1}(r+\sum_a Q_a\log\tau)$$

- Gauge variables scale* as $\mathbf{z}=e^{i\tau Z}, \mathbf{m}=m$, and we find:

$$\mathcal{I}(\mathbf{w},\mathbf{n},\mu_a,\mathbf{s}_a)\underset{\tau\to 0}{\longrightarrow}\tau^{1+\sum_a\Delta_a-1-2iM_a}\mathcal{Z}(r,\theta,M_a,s_a)$$

Standard limit in the index

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$$\mathcal{I}_\chi^\Delta(\mathbf{z} = e^{i\tau Z}, \mathbf{m} = m) \underset{\tau\to 0}{\longrightarrow} \tau^{\Delta-1-2iZ} \mathcal{Z}_\chi^\Delta(Z, m)$$

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- And the following S^2 partition function of its dimensional reduction:

$$\mathcal{Z}(r, \theta; M_a, s_a) = \sum_{m\in\mathbb{Z}} \int \frac{dZ}{2\pi} e^{2irZ + i\theta m} \prod_a \mathcal{Z}_\chi^{\Delta_a}(Q_a Z + M_a, Q_a m + s_a)$$

- As we take $\tau \to 0$, we scale:

$$\mu_a = e^{i\tau M_a}, \ \mathbf{s}_a = s_a, \ \mathbf{w} = e^{i\theta}, \ \mathbf{n} = 2\tau^{-1}(r + \sum_a Q_a \log\tau)$$

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$$\mathcal{I}(\mathbf{w}, \mathbf{n}, \mu_a, \mathbf{s}_a) \underset{\tau\to 0}{\longrightarrow} \tau^{1+\sum_a \Delta_a - 1 - 2iM_a} \mathcal{Z}(r, \theta, M_a, s_a)$$

$\mathcal{N} = 2$ mirror symmetry

- Under mirror symmetry, this is mapped to a limit where flavor fugacities are taken large and $U(1)_J$ kept small.
- As a simple example, consider the duality between a free chiral and $U(1)$, $k = \frac{1}{2}$ with a charge one chiral. The indices are:

$$\mathcal{I}_A(\mathbf{w}, \mathbf{n}) = \mathcal{I}_\chi^\Delta(\mathbf{w}, \mathbf{n})$$

$$\mathcal{I}_B(\mathbf{w}, \mathbf{n}) = \sum_{\mathbf{m} \in \mathbb{Z}} \oint \frac{d\mathbf{z}}{2\pi i \mathbf{z}} \mathbf{z}^{\mathbf{n}} \mathbf{w}^{\mathbf{m}} \mathbf{z}^{\mathbf{m}/2} \mathcal{I}_\chi^\Delta(\mathbf{z}, \mathbf{m})$$

- Take the standard limit on theory B, which means $\mathbf{w} = e^{i\theta}, \mathbf{n} = 2\tau^{-1}(r + \log\tau)$, and we find $\mathcal{I}_B \underset{\tau \to 0}{\longrightarrow} \tau^\Delta \mathcal{Z}_B$.

$\mathcal{N} = 2$ mirror symmetry (cont'd)

- On the dual side, we find:

$$\mathcal{I}_{\chi}^{\Delta}(\mathbf{w} = e^{i\theta}, \mathbf{n} = \frac{2}{\tau}(r + \log \tau)) \underset{\tau \to 0}{\longrightarrow} \tau^{\Delta} \exp\left(\Delta r - e^{r+i\theta} + e^{r-i\theta}\right)$$

- Compare to the predictions of Hori-Vafa mirror symmetry. The charged chiral is T-dualized to a twisted chiral Y, with twisted superpotential:

$$\tilde{W} = \Sigma(Y - t) - e^{Y}$$

- After integrating out Σ, this gives just a classical contribution $\tilde{W} = e^{-t}$, agreeing with above.

$\mathcal{N} = 2$ mirror symmetry (cont'd)

- There is another limit we can take, $\mathbf{w} = e^{i\tau W}$, $\mathbf{n} = n$.
- Chiral reduces in standard way. On the gauge side, we find that to commute the limit and integration, one must define:

$$\mathbf{z} = e^{i\beta}, \quad \mathbf{m} = \frac{2}{\tau}(\alpha - \log\tau)$$

- Then the index reduces to:

$$\tau^{\Delta-1-2iW} \int \frac{d\alpha d\beta}{\pi} \exp\left((2iW-\Delta)\alpha + is\beta - e^{-\alpha-i\beta} + e^{-\alpha+i\beta}\right)$$

- This is an LG model: $Y = \alpha + i\beta = \tau\Sigma + \log\tau$ and:

$$\tilde{W} = \Sigma_W Y - e^{-Y}$$

- Similarly, one can reduce general $\mathcal{N} = 2$ abelian $3d$ mirror symmetry to general Hori-Vafa mirror dual pairs, reproducing the argument of [Aganagic et al] in the index. Also can be checked directly on S^2 [Gomis, Lee].
- A useful check is the matching of $f_{div}(\tau)$.

Duality Appetizer

- Next we consider the duality appetizer. This is relates an $SU(2)_{k=\frac{1}{2}}$ theory with an adjoint chiral ϕ to a free chiral.
- Here we take a standard limit for the free chiral. For the gauge theory, we find the appropriate scaling:

$$Y = \tau^{1/2}\Sigma$$

- This can be traced to the effective twisted superpotential:

$$\tilde{W} = \frac{1}{2}kRTr\Sigma^2$$

- Then we find the S^2 matrix model of an LG model coupled to a chiral multiplet. After applying HV duality, we obtain a theory with chirals X, Y and $W = XY^2$.
- This follows from $N = N_f = 1$ case of a duality of [Hori]:

$$SO(N)\ N_f \text{ flavors} \quad \leftrightarrow \quad O(N_f - N + 1)\ N_f \text{ flavors} + \text{mesons}$$

- One can also check elliptic genus of this duality; unlike S^2 this is sensitive to the $\mathbb{Z}_2$ orbifold.

$U(N)$ Aharony duality

- Next we start in $3d$ with the following theories:
  - Theory A: $U(N_c)$ with N_f fundamental hypermultipelts $Q, \tilde{Q}$
  - Theory B: $U(N_f - N_c)$ with N_f fundamental hypermultiplets $q, \tilde{q}$, ${N_f}^2$ mesons M, singlets $V_\pm$, and superpotential:

$$W = Mq\tilde{q} + V_+ \tilde{V}_- + \tilde{V}_+ V_-$$

- Take the standard limit on theory A. Then since $V_\pm$ are charged under the $U(1)_J$, they become massive and can be integrated out, generating a superpotential:

$$\tilde{W} = -2N_f \Phi \log(e^{t/2} + (-1)^{N_f - N_c} e^{-t/2})$$

- This reproduces a duality discovered recently by [Benini, Park, Zhao].

$Sp(2N)$ duality

- To recover $Sp(2N)$ Hori duality, we start in 3d from:
  - Theory A: $Sp(2N_c)$ level k with $2N_f$ fundamentals
  - Theory B: $Sp(2(k+N_f-N_c-1))$ level $-k$ with $2N_f$ fundamentals and ${N_f}^2$ mesons, $W = Mq\tilde{q}$.
- Then one observes that, for $k=\frac{1}{2}$, the functions $f_{div}(\tau)$ match for the standard limit. One obtains in 2d precisely the duality of Hori.
- The 2d Sp theories with odd number of chirals are "regular," in the sense that their Coulomb branch is lifted by quantum effects.

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$Sp(2)$ duality

- A simple example of a proposal for an irregular duality can be derived starting from the following $3d$ pair:
  - Theory A: $Sp(2)$ with four fundamentals
  - Theory B: Chiral fields M_{ab}, $1 \leq a < b \leq 4$, and Y, with $W = Y\, Pf\, M$
- One finds $f_{div}(\tau)$ matches with the standard reduction, and so the S^2 matrix models of the dimensionally reduced theories agree.
- However, the elliptic genera do not agree.
- Either the duality is wrong, or needs some modifications that cannot be seen on S^2.
- Can generate similar S^2 identities for $SU(N)$ theories.
- May need a better understanding of $2d$ theories with non-compact moduli spaces.

Summary

- We have seen various physical subtleties arise when studying the compactification of IR dualities.
- The theory we obtain depends crucially on how we scale background parameters.
- Moreover, it is a difficult problem to determine how to scale dynamical fields to obtain a non-singular description - the index can help answer this.
- We have derived evidence, the matching of S^2 partition functions, for several known $2d$ dualities, and some new ones, but also puzzles.

Future directions

- Understand better the physics underlying the non-trivial scalings and non-compact moduli spaces.
- Reduce more dualities from 3 to 2, eg, nonabelian mirror symmetry, dualities derived from class S in $4d$.
- Study compactification from $d+2 \to d$. Here one has many more possibilities (non-trivial topologies and fluxes).