Title: Algebraic characterization of entanglement classes

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Abstract: Entanglement is a key feature of composite quantum system which is directly related to the potential power of quantum computers. In most computational models, it is assumed that local operations are relatively easy to implement. Therefore, quantum states that are related by local operations form a single entanglement class. In the case of local unitary operations, a finite set of polynomial invariants provides a complete characterization of the entanglement classes. Unfortunately, one faces the problem of combinatorial explosion so that computing such a complete set of invariants becomes difficult already for quite small system. The two main problems in this context are to compute invariants and to decide completeness, i.e., whether a given set of invariants generates the full invariant ring. Important tools are both univariate and multivariate Hilbert series which are already difficult to compute. We will also address computational aspects of these problems and techniques for showing completeness of a set of invariants.

Perimeter Institute Quantum Discussions
Perimeter Institute for Theoretical Physics
Waterloo, ON, Canada

Algebraic Characterization of Entanglement Classes

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Overview

- The main tool: polynomial invariants
- Computing polynomial invariants
  - Reynolds operator
  - commuting matrices
  - tensor contractions
- Hilbert-Poincaré series/Molien series
- Derksen's degree bound
- SAGBI bases
- more examples

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Prelude: Polynomial Invariants

A matrix group G acts on multivariate polynomials via linear transformation of the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$: $f(\boldsymbol{x}) \mapsto f(\boldsymbol{x})^g = f(\boldsymbol{x} \cdot g)$.

properties of the invariant ring
$\mathbb{K}[\boldsymbol{x}]^G := \{f(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}] | \forall g \in G \colon f(\boldsymbol{x})^g = f(\boldsymbol{x})\}$

- homogeneous polynomials remain homogeneous
  $\Longrightarrow$ homogeneous generators
- any linear combination of invariants is an invariant
- the product of invariants is an invariant
- for reductive groups $\mathbb{K}[\boldsymbol{x}]^G$ is finitely generated
- some invariants are algebraically independent (primary invariants)
- the other invariants obey some polynomial relations

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Main Problem

Characterize the non-local properties of quantum states & systems.

Various approaches

- entanglement measures/monotones:
  (real) functions on the state space, *e.g.* distance to product/separable states
- local equivalence:
  Given two quantum states

  $$|\psi\rangle \text{ and } |\phi\rangle \qquad (\rho \text{ and } \rho')$$

  on n particles (qudits), is there a local *unitary*[a] transformation $U = U_1 \otimes U_2 \otimes \ldots \otimes U_n$ with

  $$U|\psi\rangle = |\phi\rangle \qquad (U\rho U^{-1} = \rho')?$$

[a] We do not consider SLOCC here.

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Our Approach

Consider the polynomial invariants of the groups $SU(d)^n$ or $U(d)^n$ acting on pure or mixed quantum states.

This gives a *complete* description:

Theorem:
The orbits of a compact linear group acting in a *real* vector space are separated by the (polynomial) invariants.

(A. L. Onishchik, *Lie groups and algebraic groups*, Springer, 1990, Ch. 3, §4)

- pure states: identify $\mathbb{C}^m$ and $\mathbb{R}^{2m}$
- mixed states: Hermitian matrices form a real vector space

but: incomplete for SLOCC (related to $SL(d)^n$)

Entanglement "Coordinates"

Let $f_1, \dots, f_m$ be a generating set for all polynomial invariants. the first μ being an independent set of maximal size.

entanglement "coordinates":

$$|\psi\rangle \mapsto \big(\underbrace{f_1(|\psi\rangle), \dots, f_\mu(|\psi\rangle)}_{\text{algebraic independent}}, \underbrace{f_{\mu+1}(|\psi\rangle), \dots, f_m(|\psi\rangle)}_{\text{finitely many subclasses}}\big) \in \mathbb{C}^m$$

states in the same entanglement class have the same entanglement coordinates

Invariant Polynomials and Commuting Matrices

Every homogeneous polynomial $f(X) \in \mathbb{K}[x_{11}, \ldots, x_{dd}]$ of degree k can be expressed as

$$f_F(X) := \operatorname{tr}(F \cdot X^{\otimes k}) \quad \text{where } F \in \mathbb{K}^{kd \times kd}$$

(since $X^{\otimes k}$ contains all monomials of degree k).

Example ($d = 2$, $k = 2$):

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$$X^{\otimes 2} = \begin{pmatrix} x_{11}^2 & x_{11}x_{12} & x_{12}x_{11} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22} & x_{12}x_{21} & x_{12}x_{22} \\ x_{21}x_{11} & x_{21}x_{12} & x_{22}x_{11} & x_{22}x_{12} \\ x_{21}^2 & x_{21}x_{22} & x_{22}x_{21} & x_{22}^2 \end{pmatrix}$$

Invariant Polynomials and Commuting Matrices

$$\begin{aligned} f_F(X)^g &= \operatorname{tr}(F \cdot (g^{-1} \cdot X \cdot g)^{\otimes k}) \\ &= \operatorname{tr}(F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k} \cdot g^{\otimes k}) \\ &= \operatorname{tr}(g^{\otimes k} \cdot F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k}) \\ &= \operatorname{tr}(F^{(g^{-1})^{\otimes k}} \cdot X^{\otimes k}) \end{aligned}$$

$$f_F(X)^g = f_F(X) \iff f_F(X) = f_{F'}(X) \quad \text{and} \quad F' \cdot g^{\otimes k} = g^{\otimes k} \cdot F'$$

transformed question

Which matrices commute with each $g^{\otimes k}$ for $g \in G$?

R. Brauer (1937):

The algebra $\mathcal{A}_{d,k}$ of matrices that commute with each $U^{\otimes k}$ for $U \in U(d)$ is generated by a certain representation of the symmetric group S_k.

Computing Invariants

(see E. Rains, quant-ph/9704042[a]; Grassl et al., quant-ph/9712040[b])

Computing the homogeneous polynomial invariants of degree k for an n particle system with density operator ρ:

for each n tuple $\pi = (\pi_1, \ldots, \pi_n)$ of permutations $\pi_\nu \in S_k$ compute

$$f_{\pi_1,\ldots,\pi_n}(\rho_{ij}) := \operatorname{tr}\left(T_{d,k}^{(n)}(\pi) \cdot \rho^{\otimes k}\right)$$

- all homogeneous polynomial invariants of degree k
- in general, $(k!)^n$ invariants to compute
- not necessarily linearly independent, not even distinct
- it is sufficient to consider certain tuples of permutations

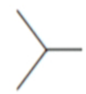
[a]E. Rains, IEEE Transactions on Information Theory, vol. 46, no. 1, pp. 54–59 (2000)
[b]M. Grassl, M. Roetteler & Th. Beth, Physical Review A 58, 1833–1839 (1998)

Invariant Tensors

- use local basis for the 4×4 two-qubit density matrix:

$$\rho = \frac{1}{4} I + \sum_{\ell=x,y,z} s_\ell (\sigma_\ell \otimes I) + \sum_{r=x,y,z} p_r (I \otimes \sigma_r) + \sum_{\ell,r=x,y,z} \beta_{\ell r} (\sigma_\ell \otimes \sigma_r)$$

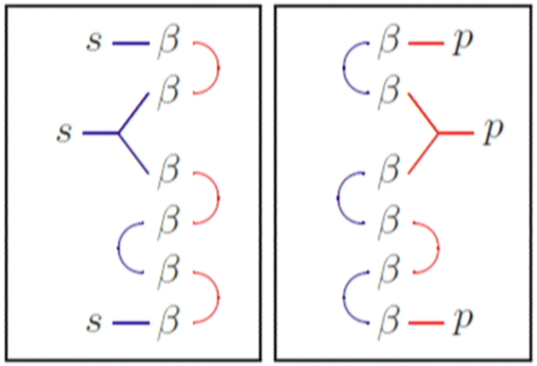
- $SU(2) \otimes SU(2)$ acts as $SO(3) \times SO(3)$ on the coefficient vectors s, p and the coefficient matrix β
- contract copies of the coefficient tensors with tensors that are invariant under $SO(3)$ resp. $SO(3) \times SO(3)$

δ_{ij}	inner product	
ϵ_{ijk}	determinant	

- create all possible contractions modulo the relations of the tensors

for two qubits, there is only a finite number of such contractions

$\Longrightarrow$ complete set of invariants, resp. a set of generators for all invariants



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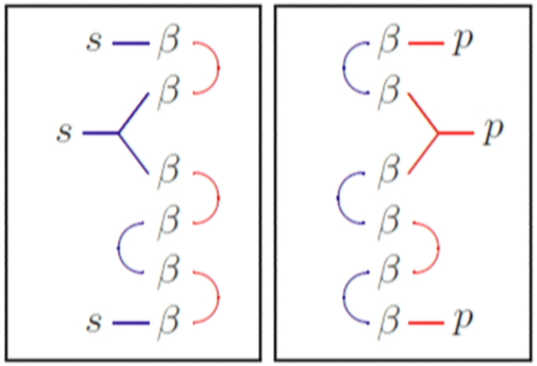

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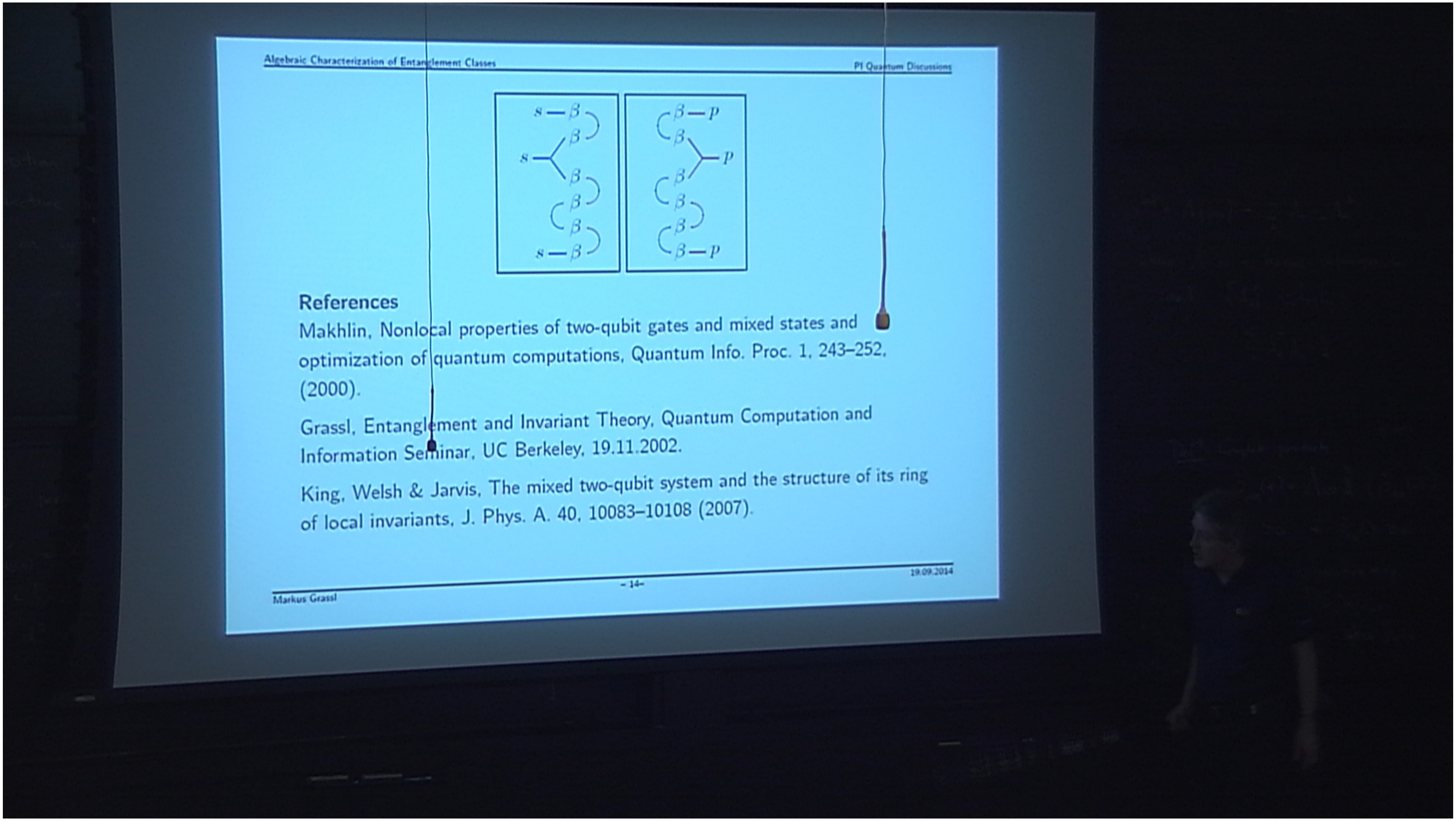
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19.09.2014

Pure State of Two Qubits

pure state

$$|\psi\rangle = x_{00}|00\rangle + x_{01}|01\rangle + x_{10}|10\rangle + x_{11}|11\rangle$$

Invariants

$$\mathrm{tr}(|\psi\rangle\langle\psi|) = x_{00}\overline{x}_{00} + x_{01}\overline{x}_{01} + x_{10}\overline{x}_{10} + x_{11}\overline{x}_{11}$$

$$\begin{aligned}\mathrm{tr}\left((\mathrm{tr}_i\,|\psi\rangle\langle\psi|)^2\right) &= x_{00}^2\overline{x}_{00}^2 + x_{01}^2\overline{x}_{01}^2 + x_{10}^2\overline{x}_{10}^2 + x_{11}^2\overline{x}_{11}^2\\ &\quad +2x_{00}x_{01}\overline{x}_{00}\overline{x}_{01} + 2x_{00}x_{10}\overline{x}_{00}\overline{x}_{10} + 2x_{00}x_{11}\overline{x}_{01}\overline{x}_{10}\\ &\quad +2x_{01}x_{10}\overline{x}_{00}\overline{x}_{11} + 2x_{01}x_{11}\overline{x}_{01}\overline{x}_{11} + 2x_{10}x_{11}\overline{x}_{10}\overline{x}_{11}\end{aligned}$$

Remark

We have to introduce new variables which are the “complex conjugated variables.”

Three Pure Qubits: Series for $SU(2)^{\otimes 3}$ and $U(2)^{\otimes 3}$

$$M_{SU}(z,\overline{z}) = \frac{z^5\overline{z}^5 + z^3\overline{z}^3 + z^2\overline{z}^2 + 1}{(1-z\overline{z})(1-z^4)(1-\overline{z}^4)(1-z^2\overline{z}^2)^2(1-z\overline{z}^3)(1-z^3\overline{z})}$$

$$= 1 + z\overline{z} + z^4 + z^3\overline{z} + 4z^2\overline{z}^2 + z\overline{z}^3 + \overline{z}^4 + z^5\overline{z} + z^4\overline{z}^2 + 5z^3\overline{z}^3 + z^2\overline{z}^4 + z\overline{z}^5$$
$$+ z^8 + z^7\overline{z} + 5z^6\overline{z}^2 + 5z^5\overline{z}^3 + 12z^4\overline{z}^4 + 5z^3\overline{z}^5 + 5z^2\overline{z}^6 + z\overline{z}^7 + \overline{z}^8$$
$$+ z^9\overline{z} + z^8\overline{z}^2 + 6z^7\overline{z}^3 + 6z^6\overline{z}^4 + 15z^5\overline{z}^5 + z\overline{z}^9 + z^2\overline{z}^8 + 6z^3\overline{z}^7 + 6z^4\overline{z}^6$$
$$+ z^{12} + z^{11}\overline{z} + 5z^{10}\overline{z}^2 + 6z^9\overline{z}^3 + 16z^8\overline{z}^4 + 16z^7\overline{z}^5 + 30z^6\overline{z}^6$$
$$+ \overline{z}^{12} + z\overline{z}^{11} + 5z^2\overline{z}^{10} + 6z^3\overline{z}^9 + 16z^4\overline{z}^8 + 16z^5\overline{z}^7$$
$$+ \ldots$$

$$M_U(z) = \frac{z^{12}+1}{(1-z^2)(1-z^4)^3(1-z^6)(1-z^8)}$$
$$= 1 + z^2 + 4z^4 + 5z^6 + 12z^8 + 15z^{10} + 30z^{12} + 37z^{14} + 65z^{16} + 80z^{18}$$
$$+ 128z^{20} + 156z^{22} + 234z^{24} + 282z^{26} + 402z^{28} + 480z^{30} + \ldots$$

Four Pure Qubits: Ansatz for Series of $SU(2)^{\otimes 4}$

$$M_{SU}(\overline{z}, z) = \int_{U \in G} d\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot \overline{U})}$$

$$= \alpha \oint_{\Gamma_u} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-u^2)(1-v^2)(1-w^2)(1-x^2)}{\prod_{a,b,c,d \in \{1,-1\}} (1 - z \cdot u^a v^b w^c x^d)(1 - \overline{z} \cdot u^a v^b w^c x^d)} \frac{du}{u} \frac{dv}{v} \frac{dw}{w} \frac{dx}{x}$$

Four Pure Qubits: Hilbert Series of $U(2)^{\otimes 4}$

$$
\begin{aligned}
M_U(z) = \big(&z^{76}+6z^{70}+46z^{68}+110z^{66}+344z^{64}+844z^{62}+2154z^{60}+4606z^{58}+9397z^{56}\\
&+16848z^{54}+28747z^{52}+44580z^{50}+65366z^{48}+88036z^{46}+111909z^{44}\\
&+131368z^{42}+145676z^{40}+149860z^{38}+145676z^{36}+131368z^{34}\\
&+111909z^{32}+88036z^{30}+65366z^{28}+44580z^{26}+28747z^{24}+16848z^{221}\\
&+9397z^{20}+4606z^{18}+2154z^{16}+844z^{14}+344z^{12}+110z^{10}+46z^{8}+6z^{6}\\
&+1\big) \Big/ \Big((1-z^{10})\,(1-z^{8})^4\,(1-z^{6})^6\,(1-z^{4})^7\,(1-z^{2})\Big)\\
= {}& 1+z^2+8z^4+20z^6+98z^8+293z^{10}+1\,128z^{12}+3\,409z^{14}\\
&+10\,846z^{16}+30\,480z^{18}+84\,652z^{20}+217\,677z^{22}+544\,312z^{24}\\
&+1\,289\,225z^{26}+2\,961\,626z^{28}+6\,528\,284z^{30}+13\,980\,717z^{32}\\
&+28\,963\,980z^{34}+58\,464\,510z^{36}+114\,806\,429z^{38}+\dots
\end{aligned}
$$

later independently computed by:

Nolan R. Wallach, The Hilbert Series of Measures of Entanglement for 4 Qubits, Acta Applicandae Mathematicae 86:203–220 (2005)

Derksen's Degree Bound

[H. Derksen, Proc. Am. Math. Soc., 129(4):955–963 (2000)]

Let G be a linearly reductive algebraic group over $\mathbb{K}$, $\mathbb{K}$ algebraically closed, $\mathrm{char}(\mathbb{K}) = 0$.

- G is defined via polynomials $h_i \in \mathbb{K}[Z_1, \ldots, Z_t]$
- representation $\rho\colon G \to GL(V)$ defined via polynomials $a_{i,j} \in \mathbb{K}[Z_1, \ldots, Z_t]$
- $H := \max_i \deg(h_i)$, $A := \max_{i,j} \deg(a_{i,j})$, and $m := \dim(G)$

If ρ has finite kernel, then the degree of primary invariants is bounded by

$$\sigma(V) \leq H^{t-m} A^m.$$

The degree of generators for the invariant ring is bounded by

$$\beta(V) \leq \max\{2, \frac{3}{8} \dim \mathcal{O}(V)^G \sigma^2(V)\}.$$

Derksen's Degree Bound: $SU(2)$

Consider the group $G = SU(2)^{\otimes n}$ acting via conjugation on $2^n \times 2^n$ matrices.

- $\dim SU(2) = 3$, defined via polynomials of degree 2 in 4 variables
  $\Longrightarrow m = 3n,\ H = 2,\ t = 4n$
- action is given by $M \mapsto (g_1 \otimes \ldots \otimes g_n)M(g_1 \otimes \ldots \otimes g_n)^{-1}$
  $\Longrightarrow A = 2n$

degree bound $\sigma(V)$ for the primary invariants:

$$\sigma(V) \leq H^{t-m}A^m = 2^{4n-3n}(2n)^{3n} = 2^{4n}n^{3n}$$

already for $n = 2$, we get $\sigma(V) \leq 2^{14} = 16384$

Relation Ideal

Problem:

Given some invariants $f_1, \ldots, f_m$, do they generate the full invariant ring?

evaluation homorphism: $\mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[x_1, \ldots, x_n]$

$$g(y_1, \ldots, y_m) \mapsto g(f_1, \ldots, f_m)$$

relation ideal:

$$\mathrm{Rel}(f_1, \ldots, f_m) = \{g(y_1, \ldots, y_m) \colon g(f_1, \ldots, f_m) = 0\} \trianglelefteq \mathbb{K}[y_1, \ldots, y_m]$$

$$\mathcal{A} = \langle f_1, \ldots, f_m \rangle \cong \mathbb{K}[y_1, \ldots, y_m] / \mathrm{Rel}(f_1, \ldots, f_m)$$

Hilbert series: $\mathrm{Hilb}(\mathcal{A}) = \mathrm{Hilb}(\mathrm{Rel})$

computed (in principle) as

$$\mathrm{Rel}(f_1, \ldots, f_m) = \langle f_1 - y_1, \ldots, f_m - y_m \rangle \cap \mathbb{K}[y_1, \ldots, y_m]$$

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Sturmfels' Conjecture

Conjecture: The invariant ring of a connected reductive affine algebraic group has a finite SAGBI basis with respect to some term order.

(see M. Stillman & H. Tsai, Using SAGBI bases to compute invariants, J. Pure and Appl. Algebra 139:285–302 (1999))

Bernd Sturmfels, email on 5 September 2006:

> *I did indeed conjecture, some time ago, that for a connected reductive group over C, the ring of invariants has a finite SAGBI basis. However, I don't [think] this conjecture ever made it into writing. Also, it was based mainly on "wishful thinking." To the best of my knowledge, it's still open.*

Using SAGBI Bases

assume $B = \{g_1, \ldots, g_\ell\}$ is a SAGBI basis of the polynomial algebra $\mathcal{A}$

all relevant information is given by the leading monomials

- $\mathrm{Hilb}(\mathcal{A}) = \mathrm{Hilb}(\langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_\ell) \rangle)$
- the Hilbert series can be computed from the ideal

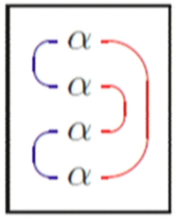
$$\mathrm{Rel}(\mathrm{LM}(B)) = \langle \mathrm{LM}(g_1) - t_1, \ldots, \mathrm{LM}(g_\ell) - t_\ell \rangle \cap \mathbb{K}[t_1, \ldots, t_\ell]$$

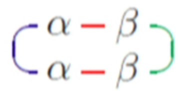
- if B has been computed only up to degree d, we can still compare the Hilbert series

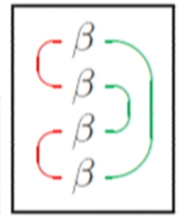
$\Longrightarrow$ direct proof of completeness for two-qubit mixed state < 1 min

$\Longrightarrow$ proof of completeness for $SU(2)^{\otimes 3}$
("private communication" in Luque, Thibon & Toumazet (2007))

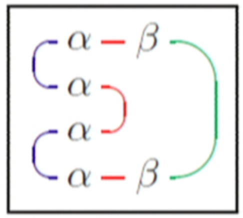
degree 4:

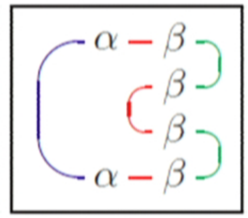


$\mathrm{Tr}(\alpha\beta\beta^t\alpha^t) =$ 

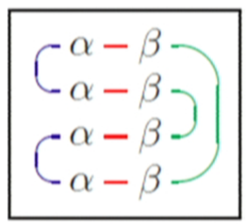


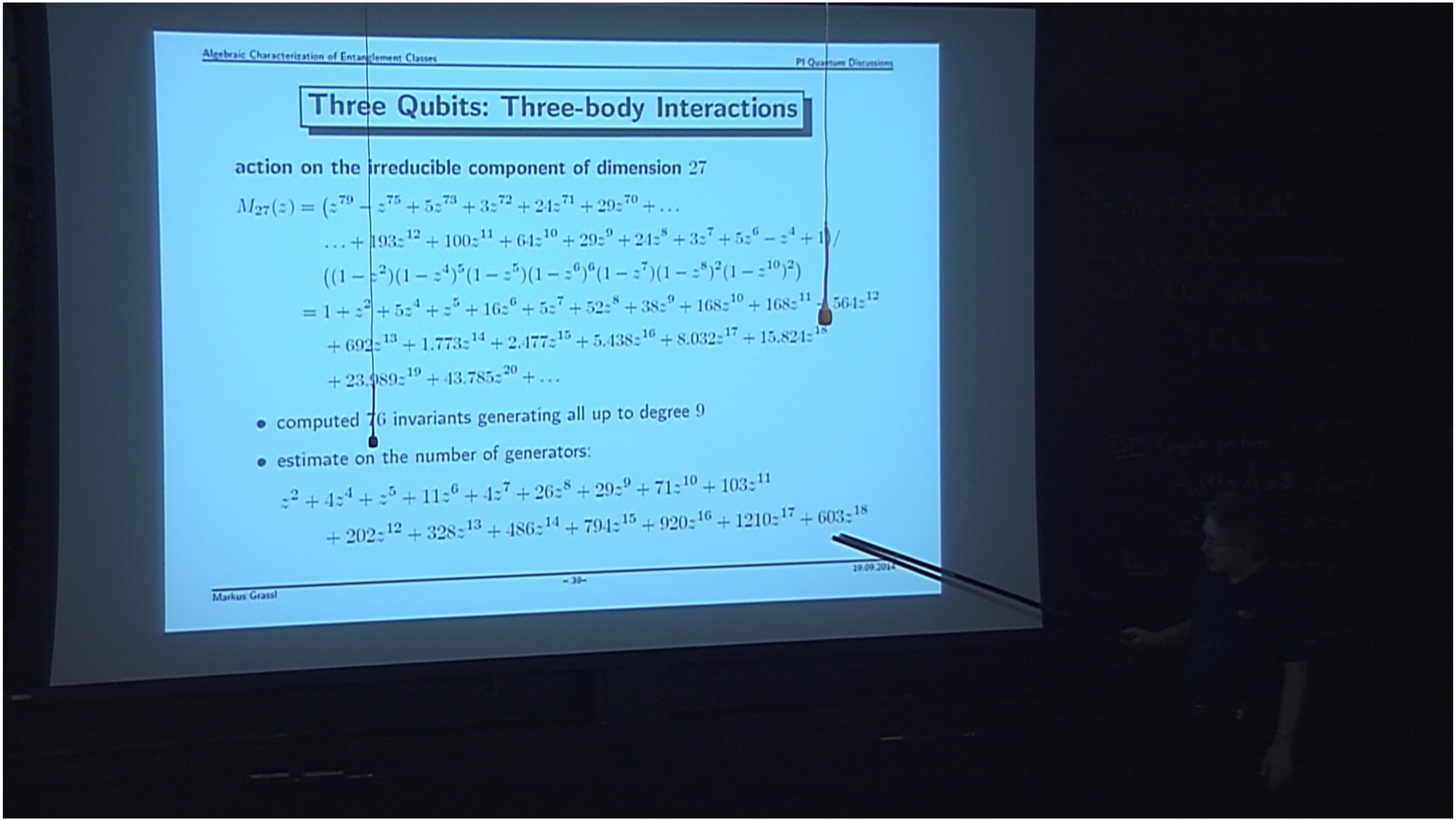
degree 6:





degree 8:




Algebraic Characterization of Entanglement Classes
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Three Qubits: Three-body Interactions
action on the irreducible component of dimension 27
$M_{27}(z) = (z^{79} - z^{75} + 5z^{73} + 3z^{72} + 24z^{71} + 29z^{70} + \dots$
$\dots + 193z^{12} + 100z^{11} + 64z^{10} + 29z^{9} + 24z^{8} + 3z^{7} + 5z^{6} - z^{4} + 1)/$
$((1-z^2)(1-z^4)^5(1-z^5)(1-z^6)^6(1-z^7)(1-z^8)^2(1-z^{10})^2)$
$= 1 + z^2 + 5z^4 + z^5 + 16z^6 + 5z^7 + 52z^8 + 38z^9 + 168z^{10} + 168z^{11} + 564z^{12}$
$+ 692z^{13} + 1.773z^{14} + 2.477z^{15} + 5.438z^{16} + 8.032z^{17} + 15.824z^{18}$
$+ 23.989z^{19} + 43.785z^{20} + \dots$
• computed 76 invariants generating all up to degree 9
• estimate on the number of generators:
$z^2 + 4z^4 + z^5 + 11z^6 + 4z^7 + 26z^8 + 29z^9 + 71z^{10} + 103z^{11}$
$+ 202z^{12} + 328z^{13} + 486z^{14} + 794z^{15} + 920z^{16} + 1210z^{17} + 603z^{18}$
Markus Grassl
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19.09.2014

Summary

- invariant theory provides a means to describe all entanglement classes
- information about the invariants via the Hilbert series
- already for small systems, we are facing combinatorial explosion
- proof of completeness via Hilbert series and SAGBI bases

Open Problems/Outlook

- Does Sturmfels' conjecture hold, at least for the representations considered here?
- Is it possible to find fewer invariants separating the orbits?
- Extend the approach to the group $SL(d)^n$, including covariants.
- Coarse-graining of the entanglement classes.

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