Title: Force-Free Electrodynamics around Extreme Kerr Black Holes

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Abstract: Plasma-filled magnetospheres can extract energy from a spinning black hole and provide the power source for a variety of observed astrophysical phenomena. These magnetospheres are described by the highly nonlinear equations of force-free electrodynamics, or FFE. Typically these equations can only be solved numerically. In this talk I will explain how to analytically obtain several infinite families of exact solutions of the full nonlinear FFE equations very near the horizon of a maximally spinning black hole, where the energy extraction takes place.

FORCE-FREE ELECTRODYNAMICS AROUND EXTREME KERR BLACK HOLES

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Perimeter Institute– 11th Sept 2014


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INTRODUCTION:
EXTRAVAGANT ENERGY SIGNALS IN THE SKY

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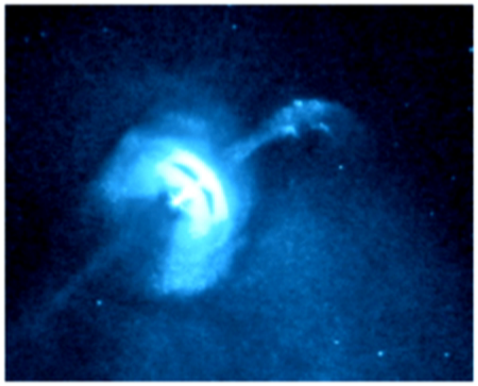
Pulsars



NASA's Chandra X-ray Observatory image shows a fast moving jet of particles produced by a rapidly rotating neutron star.

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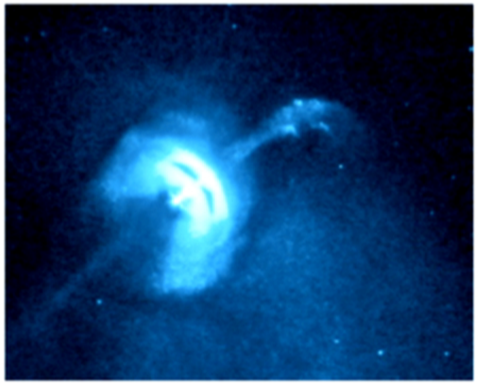

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Quasars

In this image, the lowest-energy X-rays Chandra detects are in red, while the medium-energy X-rays are green, and the highest-energy ones are blue.

EXTRAVAGANT ENERGY SIGNALS IN THE SKY

How are these jets generated?

Many of these powerful jets – quasars - are generated by the giant rotating black hole surrounded by a magnetosphere with a plasma at the galaxy's center.

What is our understanding of the physics involved?

Energy extraction from such a black hole is widely believed to be described by the highly nonlinear equations of force-free electromagnetism (FFE)

FORCE-FREE EQUATIONS

Maxwell's equations are

$$\nabla_\mu F^{\mu\nu} = \mathcal{J}^\nu .$$

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where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu .$$

is the matter charge current

The electromagnetic stress-energy tensor is

$$T^{\mu\nu}_{\mathrm{EM}} = F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} .$$

Which is not covariantly conserved by itself.

$$\nabla_\nu T^{\mu\nu}_{\mathrm{EM}} = -F_{\mu\nu} \mathcal{J}^\nu . \qquad (1)$$

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the relativistic form of the Lorentz force density

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Force-Free Electrodynamics from Extreme Kerr Black Holes

FORCE-FREE EQUATIONS

The full stress-energy tensor

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The full stress-energy tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{matter}}$$

is always conserved

$$\nabla_\nu T^{\mu\nu} = 0.$$

Force-free electrodynamics (FFE) describes systems in which most of the energy resides in the electrodynamical sector of the theory, so that

$$T^{\mu\nu} \approx T^{\mu\nu}_{\text{EM}}.$$

$$\nabla_\nu T^{\mu\nu}_{\text{EM}} = 0.$$

This approximation is known as the "force-free" condition, since by (1) it is equivalent to the requirement that the Lorentz force density vanishes

$$F_{\mu\nu}\mathcal{J}^\nu = 0. \qquad (2)$$

FORCE-FREE EQUATIONS

In the study of systems obeying this condition

$$F_{\mu\nu}\mathcal{J}^{\nu} = 0.$$

the current may be defined as the right hand side Maxwell's equation rather than independently specified.

A complete set of equations of motion for the electromagnetic sector is obtained by appending to Maxwell's equations the force free condition

$$\begin{cases} \nabla_{[\alpha}F_{\mu\nu]} = 0 \\ \nabla_{\nu}F^{\mu\nu} = J^{\mu} \\ F_{\mu\nu}J^{\nu} = 0 \end{cases}$$

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in which

$F \equiv \mathrm{d}A$ denotes the electromagnetic field strength

$\star$ the Hodge dual $\quad \wedge$ the wedge product $\quad \mathrm{d}^{\dagger}$ the adjoint of the exterior derivative

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Force-Free Electrodynamics from Extreme Kerr Black Holes

FORCE-FREE EQUATIONS

Theoretical setup:

$$F_{\mu\nu}\mathcal{J}^{\nu} = 0.$$

It is widely believed that *astrophysical black holes* are typically surrounded by magneto-spheres composed of an electromagnetic plasma governed by these equations. Hence they are of both mathematical and physical interest.

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BACKGROUND

(1973) Michel found a monopole solution for Schwarzschild black hole

(1976) Blandford showed that for Kerr there are parabolic EM-configurations

(1977) Blandford-Znajek find energy extracting models for (slowly rotating) Kerr black hole

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(1985)-(2014) Numerical GRMHD simulations

⋮

A FULL ANALYTICAL SOLUTION IS NOT KNOWN and NUMERICAL RESULTS BREAK DOWN FOR EXTREME KERR

SYMMETRY IN THE UNIVERSE

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1. Energy extraction is possible only for rotating Kerr black holes, and the greater the rotation, the easier it becomes.

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1. Energy extraction is possible only for rotating Kerr black holes, and the greater the rotation, the easier it becomes.

2. Moreover it is a process that occurs near the black hole horizon, and is largely insensitive to the physics at spatial infinity

3. This suggests that much of the physics of force- free electrodynamic energy extraction can be captured by studying the near horizon region of maximally-rotating extreme Kerr black holes, such as the one in Cygnus X-1

SYMMETRY IN THE UNIVERSE

$$F_{\mu\nu}\mathcal{J}^{\nu} = 0.$$

Analytic solutions to FFE are only known for slowly rotating black holes.

Q1: What happens to the magenetospheres for extreme black holes?

e.g. can we solve the FFE equations in the NHEK region.

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Q2: Is there any symmetry realized in the Universe?

e.g. do solutions to FFE realize the symmetry enhancement of the NHEK geometry

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Q2: Is there any symmetry realized in the Universe?

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Main purpose: one hopes that this analytic approach will enable a better understanding of astrophysical black hole magnetospheres and energy extraction.

PLAN

- [x] Introduction: Force-Free Electrodynamis
- [x] NHEK
- [x] Technique and energy extraction
- [x] New solutions to FFE
  - [x] $F^2 \neq 0.$ solutions
  - [x] $F^2 = 0,$ solutions

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Force-Free Electrodynamics from Extreme Kerr Black Holes

FROM KERR TO NHEK

The Kerr metric describes a rotating black hole with angular momentum J and the mass M . In Boyer-Lindquist coordinates the line element is

$$ds^2 = -\frac{\Delta}{\Sigma}\left(\mathrm{d}\hat{t} - a\sin^2\hat{\theta}\,\mathrm{d}\hat{\phi}\right)^2 + \frac{\Sigma}{\Delta}\mathrm{d}\hat{r}^2 + \frac{\sin^2\hat{\theta}}{\Sigma}\left[\left(\hat{r}^2 + a^2\right)\,\mathrm{d}\hat{\phi} - a\,\mathrm{d}\hat{t}\right]^2 + \Sigma\,\mathrm{d}\hat{\theta}^2,$$

where

$$\Delta \equiv \hat{r}^2 - 2M\hat{r} + a^2, \qquad \Sigma \equiv \hat{r}^2 + a^2\cos^2\hat{\theta}, \qquad a \equiv \frac{J}{M}.$$

There is an event horizon at

$$\hat{r}_H = M + \sqrt{M^2 - a^2}, \qquad |a| \leq M.$$

This last bound is saturated by the so-called extreme Kerr solution, which carries the maximum allowed angular momentum

$$|J| = M^2.$$

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FROM KERR TO NHEK

We are interested in the region very near the horizon of extreme Kerr, described by the so-called Near-Horizon Extreme Kerr (NHEK) geometry

It can be obtained by a limiting procedure from the Kerr metric in usual Boyer-Lindquist coordinates

$$t = \frac{\lambda \hat{t}}{2M}, \qquad r = \frac{\hat{r} - M}{\lambda M}, \qquad \theta = \hat{\theta}, \qquad \phi = \hat{\phi} - \frac{\hat{t}}{2M}.$$

$$\lambda \to 0$$

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$$\lambda \to 0$$

This procedure yields the NHEK line element in Poincare coordinates

$$ds^2 = 2J\Gamma\left[-r^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \Lambda^2(\,\mathrm{d}\phi + r\,\mathrm{d}t)^2\right] \qquad (4)$$

where $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \sim \phi + 2\pi$

$$\Lambda(\theta) \equiv \frac{2\sin\theta}{1+\cos^2\theta}. \quad \text{and} \quad \Gamma(\theta) \equiv \frac{1+\cos^2\theta}{2},$$

FROM KERR TO NHEK: ISOMETRIES

A crucial feature of the NHEK region is that the

$$\underbrace{U(1) \times U(1)}_{\text{Kerr isometry group}} \xrightarrow{\text{enhanced}} SL(2,\mathbb{R}) \times U(1)$$

This enhanced symmetry governs the dynamics of the near- horizon region of extreme Kerr

The U(1) rotational symmetry is generated by the Killing vector field

$$W_0 = \partial_\phi .$$

The time translation symmetry becomes part of an SL(2,R) isometry group generated by the Killing vector fields

$$H_0 = t\,\partial_t - r\,\partial_r ,$$

$$H_+ = \sqrt{2}\,\partial_t ,$$

$$H_- = \sqrt{2}\left[\frac{1}{2}\left(t^2 + \frac{1}{r^2}\right)\partial_t - tr\,\partial_r - \frac{1}{r}\,\partial_\phi\right]$$

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these satisfy the $SL(2,\mathbb{R}) \times U(1)$ commutation relations,

$$[H_0, H_\pm] = \mp H_\pm , \qquad [H_+, H_-] = 2H_0 ,$$

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FROM KERR TO NHEK: GLOBAL COORDINATES

The Poincare coordinates (4) cover only the part of the NHEK geometry outside the horizon of the original extreme Kerr. Global coordinates in NHEK are found by

$$r = \frac{\cos\tau - \cos\psi}{\sin\psi}, \qquad t = \frac{\sin\tau}{\cos\tau - \cos\psi}, \qquad \phi = \varphi + \ln\left|\frac{\cos\tau - \sin\tau\cot\psi}{1 + \sin\tau\csc\psi}\right|$$

In these new coordinates the line element becomes

$$ds^2 = 2J\Gamma\left[\left(-\,d\tau^2 + d\psi^2\right)\csc^2\psi + d\theta^2 + \Lambda^2\left(d\varphi - \cot\psi\, d\tau\right)^2\right] \qquad (5)$$

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where $\tau \in (-\infty, \infty)$, $\psi, \theta \in [0, \pi]$ and $\varphi \sim \varphi + 2\pi$

In global coordinates a useful complex basis for the SL(2,R)xU(1) Killing vectors is

$$L_\pm = ie^{\pm i\tau}\sin\psi\left(-\cot\psi\,\partial_\tau \mp i\,\partial_\psi + \partial_\varphi\right)$$
$$L_0 = i\,\partial_\tau,$$
$$Q_0 = -i\,\partial_\varphi.$$

obeying

$$[L_0, L_\pm] = \mp L_\pm, \qquad [L_+, L_-] = 2L_0,$$
$$[Q_0, L_\pm] = 0, \qquad [Q_0, L_0] = 0,$$

TECHNIQUE: EXPLOITING THE SYMMETRIES

In general FFE equations are highly nonlinear and can only be solved numerically.

However in NHEK the symmetries can be exploited to simplify the analysis.

Given one solution of the force-free equations, another can always be generated by the action of an isometry. Therefore the solutions must lie in representations of SL(2,R)xU(1)

We look for axisymmetric solutions which lie in the so-called highest-weight representations of SL(2,R) obeying

$$\begin{aligned} \mathcal{L}_{L_+} F &= 0, \\ \mathcal{L}_{L_0} F &= hF, \\ \mathcal{L}_{Q_0} F &= 0, \end{aligned} \qquad (6)$$

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where L_V is the Lie derivative w/respect to the vector field V and h is a constant characterizing the representation

The last condition requires that F be U(1)-invariant, while the first two conditions state that F is in a highest-weight representation of SL(2, R) with weight h.

Since L_+ is complex, all of these solutions are complex. However we will show that the real and imaginary parts of these solutions surprisingly also solve the force-free equations and hence provide physical field configurations.

In the ensuing analysis we will find force-free solutions obeying (6)

ENERGY AND ANGULAR MOMENTUM FLUX

The NHEK geometry possesses an axial U(1) symmetry generated by

$$W_0 = \partial_\phi$$

as well as a time-translation symmetry generated by

$$H_+ = \sqrt{2}\,\partial_t$$

It is therefore natural to define energy and angular momentum in NHEK as the conserved quantities associated with these vectors respectively.

Given a solution to the force-free equations (3) one can compute the stress-tensor

$$T_{\mu\nu} = T_{\mu\nu}^{\text{EM}}$$

and thence obtain the associated NHEK energy current

$$\mathcal{I}_\nu^E \equiv H_+^\mu T_{\mu\nu},$$

$$\mathcal{I}_\nu^L \equiv W_0^\mu T_{\mu\nu}.$$

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ENERGY FLUX

R is the entirety of the NHEK Poincare patch, then by Stokes' Theorem, the previous equation implies the energy conservation relation

$$\Delta E_H^+ + \Delta E_H^- + \Delta E_B = 0,$$

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total energy crossing into the future horizon ($\psi = +\tau$)

is minus the energy coming out of the past horizon ($\psi = -\tau$),

Total energy extracted from the boundary of the throat ($\psi = \pi$)

These quantities, smooth across the horizon, are most conveniently computed in global coordinates, as

$$\Delta E_H^+ = \int_0^{\pi} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \; \mathcal{E}_H^+,$$

$$\Delta E_H^- = \int_{-\pi}^{0} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \; \mathcal{E}_H^-,$$

$$\Delta E_B = \int_{-\pi}^{\pi} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \; \mathcal{E}_\infty,$$

ENERGY FLUX

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$$\Delta E_H^+ + \Delta E_H^- + \Delta E_B = 0,$$

- ΔE_H^+: total energy crossing into the future horizon ($\psi = +\tau$)
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$$\Delta E_B = \int_{-\pi}^\pi d\tau \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, \mathcal{E}_\infty,$$

where the integrands correspond to the energy flux density per solid angle on the horizon and the boundary of the throat

$$\mathcal{E}_H^\pm \equiv \sqrt{\gamma}\left(\pm\sqrt{2}H_+^\nu\right)\mathcal{I}_\nu^E\Big|_{\psi=\pm\tau},$$
$$\mathcal{E}_\infty \equiv \sqrt{-\sigma}\, n^\nu \mathcal{I}_\nu^E\Big|_{\psi=\pi}.$$

Where σ is the induced 3-metric on the boundary of the throat and n is the outward unit vector normal to this boundary, while γ denotes the 2-metric on the event horizon, which has null generator H_+

A completely analogous story holds for the angular momentum flux, with $\mathcal{L}$ and L replacing $\mathcal{E}$ and E, respectively.

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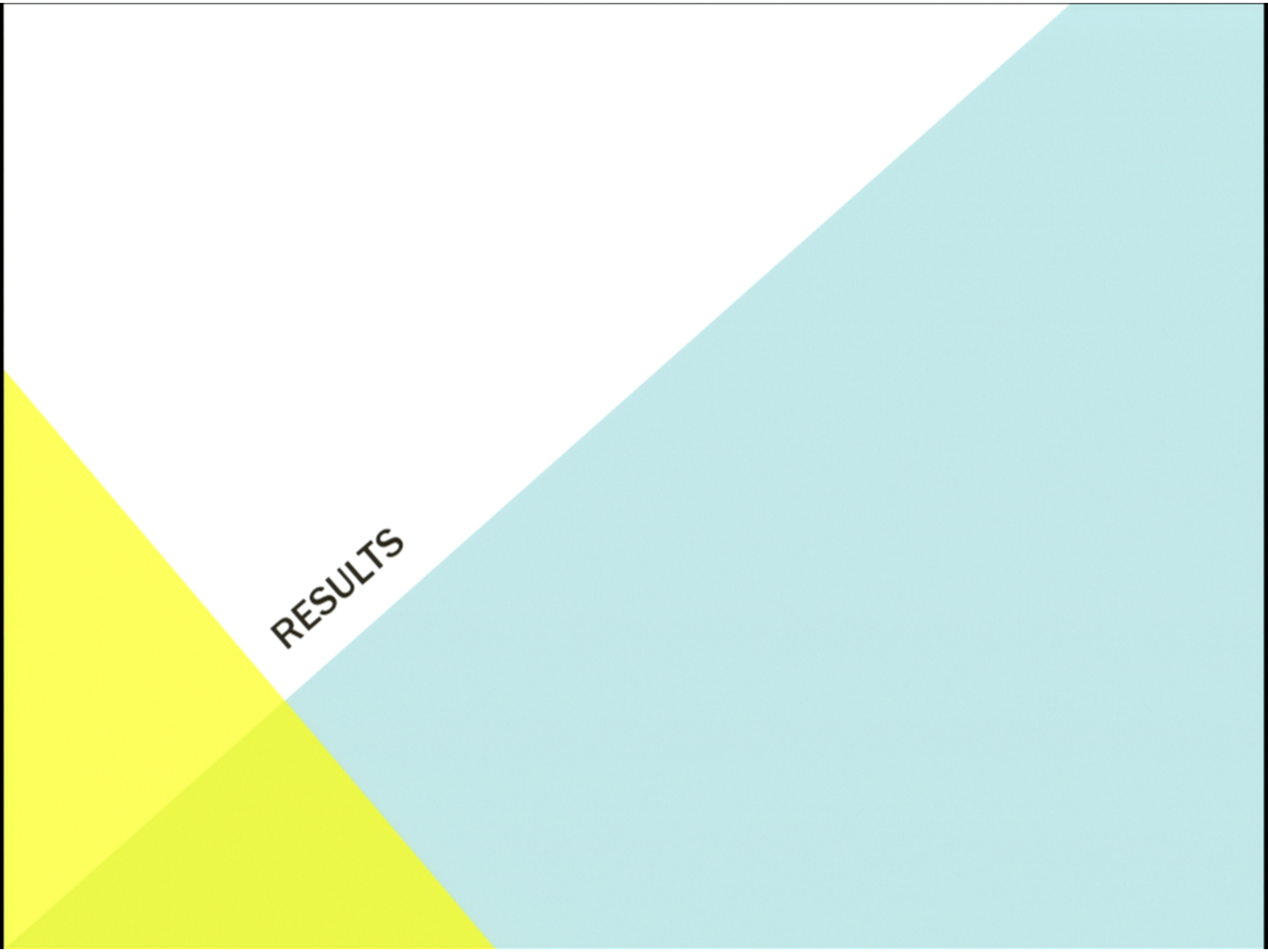
RESULTS

SUMMARY OF RESULTS

- [x] $F^2 \neq 0.$ solutions

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Force-Free Electrodynamics
from Extreme Kerr Black Holes

SUMMARY OF RESULTS

$F^2 \neq 0.$ solutions

$$A_{(h,k)} = -\frac{iP_h}{2J\Gamma}\sum_{n=0}^{k}\binom{k}{n}\Phi_{(h,k-n)}\mathcal{L}^n_{L_-}\left(\Phi\, L_+ + Q_0\right),$$

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- Max. Symmetric: $A_{(0,0)}$ with h=0 and k=0
- Highest Weight: $A_{(h,0)}$ with h non 0 and k=0
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$F^2 = 0,$ solutions

$$A_{(1,0)} = \Psi\Lambda\tilde{P}\,\mathrm{d}\theta,$$

- Highest Weight: $A_{(1,0)}$ with h =1 and k=0
- Descendants $A_{(1,k)}$ with h=1 and k non 0

Linear superposition are solutions

SUMMARY OF RESULTS

$F^2 \neq 0.$ solutions

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- Descendants $A_{(h,k)}$ with h and k non 0

$F^2 = 0,$ solutions

$$A_{(1,0)} = \Psi\Lambda\tilde{P}\,\mathrm{d}\theta,$$

- Highest Weight: $A_{(1,0)}$ with h =1 and k=0
- Descendants $A_{(1,k)}$ with h=1 and k non 0

Linear superposition are solutions

$$A(c,d) = \int_{-\infty}^{\infty}\mathrm{d}h\sum_{k=0}^{\infty}\left[c_k(h)\Re A_{(h,k)} + d_k(h)\Im A_{(h,k)}\right]$$

All these solutions will have non trivial energy and angular momentum currents but vanishing total flux @ the boundary.

Max. Symmetric

RESULTS

Consider the vector potential

$$A_{(0,0)} \equiv P_0 \left(\cot\psi \, \mathrm{d}\tau - i\, \mathrm{d}\psi\right) = -\frac{iP_0}{2J\Gamma}\left(\Phi\, L_+ + Q_0\right),$$

For the maximally symmetric case we could actually eliminate the ΦL_+ term here by a gauge transformation we keep it to facilitate the generalizations of the next section.

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where P_0 is a function of θ only and

$$\Phi(\tau, \psi) \equiv e^{-i\tau} \sin\psi.$$

$A_{(0,0)}$ is $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{U}(1)$ invariant

$$\mathcal{L}_{L_\pm} A_{(0,0)} = \mathcal{L}_{L_0} A_{(0,0)} = \mathcal{L}_{Q_0} A_{(0,0)} = 0.$$

Highest Weight

RESULTS

We construct large families of U(1) axisymmetric solutions to the force-free equations in highest-weight representations labeled by a real parameter h.

An axisymmetric highest weight vector potential with weight h obeys

$$\mathcal{L}_{L_+} A_{(h,0)} = 0,$$
$$\mathcal{L}_{L_0} A_{(h,0)} = h\, A_{(h,0)},$$
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The solutions degenerate for the case h = 1.

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The solutions degenerate for the case h = 1.

These conditions are solved by

$$A_{(h,0)} \equiv \Phi^h P_h \left(\cot\psi\, \mathrm{d}\tau - i\, \mathrm{d}\psi\right) = -\frac{i\Phi^h P_h}{2J\Gamma}\left(\Phi\, L_+ + Q_0\right),$$

where P_h is a function of the θ and Φ(τ,ψ) obeys

$$\mathcal{L}_{L_+}\Phi^h = 0, \qquad \mathcal{L}_{L_0}\Phi^h = h\Phi^h, \qquad \mathcal{L}_{Q_0}\Phi^h = 0.$$

Highest Weight

RESULTS

The field strength $F_{(h,0)} \equiv \mathrm{d}A_{(h,0)}$ is given by

$$F_{(h,0)} = -\Phi^h \left[(h-1)P_h \csc^2\psi\, \mathrm{d}\tau \wedge \mathrm{d}\psi + P_h' \left(\cot\psi\, \mathrm{d}\tau \wedge \mathrm{d}\theta - i\, \mathrm{d}\psi \wedge \mathrm{d}\theta\right)\right]$$
$$= \frac{i\Phi^h}{(2J\Gamma)^2} \left\{(h-1)P_h \left[(\Phi\, L_+ + Q_0) \wedge L_0 - \Phi \cot\psi\, L_+ \wedge Q_0\right] + P_h' \left(\Phi\, L_+ + Q_0\right) \wedge \Theta\right\}.$$

The Hodge dual of this expression is

$$\star F_{(h,0)} = -\frac{i\Phi^h}{(2J\Gamma)^2\Lambda} \left[(h-1)P_h\, Q_0 \wedge \Theta + P_h' \left(\Phi \cot\psi\, L_+ \wedge Q_0 + L_0 \wedge Q_0\right)\right].$$

$$\mathcal{J}_{(h,0)} = \frac{i\Phi^h P_h}{(2J\Gamma)^2}(h-1)Q_0,$$

Observe that when h = 1 the current vanishes – it is a solution to free Maxwell eqs. hence a trivial solution to FFE

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Highest Weight

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FFE $\mathcal{J}_{(h,0)} \wedge \star F_{(h,0)} = 0.$

If the function P_h satisfies

$$\partial_\theta \left(\Lambda\, \partial_\theta P_h\right) + h(h-1)\Lambda P_h = 0,$$

One has to still solve for P_h

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Highest Weight

RESULTS

Descendants

RESULTS

SL(2, R) invariance of NHEK guarantees that any finite SL(2, R) transformation of the above highest weight solutions are also solutions.

If the equations were linear, this would immediately imply that the SL(2, R) descendants (i.e. the fields obtained by acting with the raising operator L_{L_-}) of these solutions, which are infinitesimal transformations, are also solutions.

Despite the nonlinearity of the equations, the descendants also turn out to solve the force-free equations!!!

The reason for this is simple. If we start with the vector potential given by the k[th] descendant,

$$A_{(h,k)} \equiv \mathcal{L}_{L_-}^k A_{(h,0)} = -\frac{iP_h}{2J\Gamma} \sum_{n=0}^{k} \binom{k}{n} \Phi_{(h,k-n)} \mathcal{L}_{L_-}^n \left(\Phi\, L_+ + Q_0\right),$$

$$\mathcal{L}_{L_-} Q_0 = 0,$$

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Superposition

REALITY CONDITION AND SUPERPOSITION

Reality condition

So far the solutions have been complex. Physically we are interested in real solutions.
In general the *real or imaginary part of a solution to a nonlinear equation will not itself solve the equation*.
However the real part of the vector potential leads to dual field strengths and currents which are the real parts of the original ones. Since Q_0 has constant phase, the real or imaginary parts of anything proportional to Q_0 is itself proportional to Q_0. It follows that the real or imaginary parts of all the solutions, $\mathrm{Re}[A_{(h,k)}]$ and $\mathrm{Im}[A_{(h,k)}]$, are themselves solutions, although no longer simple descendants of a highest-weight solution.

It is important to note that these physical solutions no longer have a complex F^2. Rather, we find that F^2 may be positive or negative at different points in the spacetime.

Linear superposition are solutions

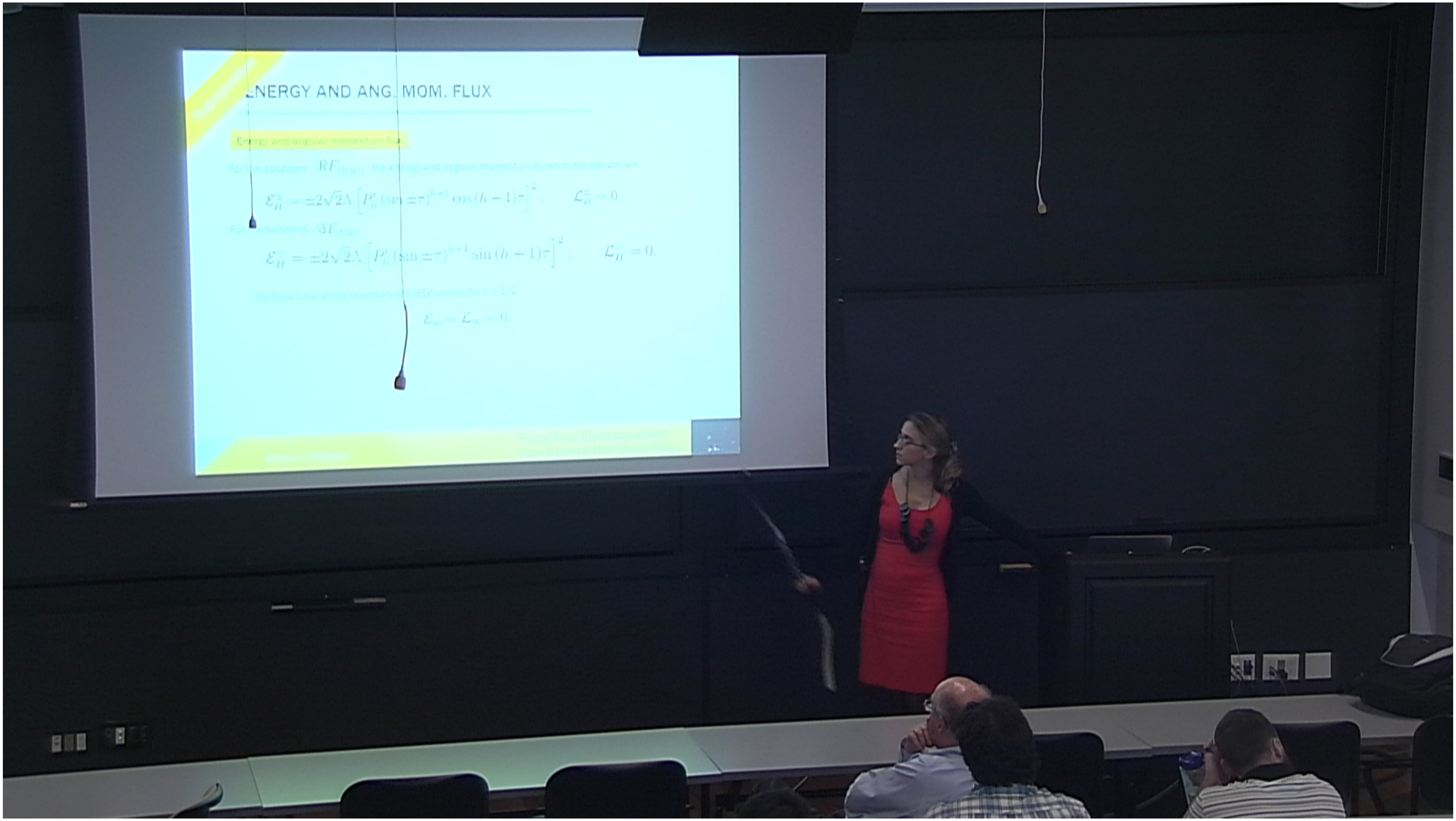
The arguments of the preceding two subsections are readily generalized to imply that the general linear combination

$$A(c,d) = \int_{-\infty}^{\infty} \mathrm{d}h \sum_{k=0}^{\infty} \left[c_k(h)\Re A_{(h,k)} + d_k(h)\Im A_{(h,k)}\right] \qquad (7)$$

or arbitrary real functions $c_k(h)$ and $d_k(h)$ is a real solution to the force free equations.

This follows because every term on the r.h.s of (7) gives both a $\star F$ and a J proportional to Q_0, hence FFE are satisfied.

What has happened here is that we have effectively linearized the equations: the conditions that $\star F$ and J be proportional to Q_0 are linear conditions which imply the full nonlinear equation.


ENERGY AND ANG. MOM. FLUX
Energy and angular momentum flux

Null solution

RESULTS: NULL SOLUTIONS

A different highest-weight solution with h = 1, which is nontrivial but has "null" $F^2 = 0$.

We suspect that it is some kind of limit of the null solutions for full Kerr found in Jacobson et. al., but have not verified the details.

Consider the gauge field

$$A_{(1,0)} = \Psi \Lambda \tilde{P}\, \mathrm{d}\theta,$$

$\tilde{P}(\theta)$ can be an arbitrary regular function and

$$F_{(1,0)} = -i\Psi\Lambda\tilde{P}\, \mathrm{d}\theta \wedge (\,\mathrm{d}\tau + \mathrm{d}\psi) = \frac{i\Phi\Lambda\tilde{P}}{2J\Gamma}\, \mathrm{d}\theta \wedge (\Psi L_+ - L_0 - Q_0)\,,$$

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$$\Psi(\tau, \psi) \equiv -e^{-i(\tau+\psi)},$$

$$F_{(1,0)} = -i\Psi\Lambda\tilde{P}\, \mathrm{d}\theta \wedge (\,\mathrm{d}\tau + \,\mathrm{d}\psi) = \frac{i\Phi\Lambda\tilde{P}}{2J\Gamma}\, \mathrm{d}\theta \wedge (\Psi L_+ - L_0 - Q_0)\,,$$

a scalar function U(1)xU(1) eigenfunction

$$\mathcal{L}_{L_0}\Psi = \Psi,$$
$$\mathcal{L}_{Q_0}\Psi = 0.$$

It does not lie in a scalar highest-weight representation of SL(2,R) because it is not annihilated by L+.

$$\mathcal{J}_{(1,0)} = \frac{i\Psi\left(\Lambda\tilde{P}' + 2\Lambda'\tilde{P}\right)}{2J\Gamma}(\,\mathrm{d}\tau + \,\mathrm{d}\psi) = \frac{i\Phi\left(\Lambda\tilde{P}' + 2\Lambda'\tilde{P}\right)}{(2J\Gamma)^2}(\Psi L_+ - L_0 + Q_0)\,.$$

$$\star F_{(1,0)} = -i\Psi\Lambda^2\tilde{P}\,[\cot\psi\, \mathrm{d}\tau \wedge \mathrm{d}\psi + (\,\mathrm{d}\tau + \,\mathrm{d}\psi) \wedge \mathrm{d}\varphi] = \frac{i\Phi\tilde{P}}{(2J\Gamma)^2} Q_0 \wedge (\Psi L_+ - L_0 + Q_0)\,.$$

Since both are proportional to $d\tau + d\psi$ (or $\Psi L_+ - L_0 + Q_0$)

$$\mathcal{J}_{(1,0)} \wedge \star F_{(1,0)} = 0$$

FFE

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Null solution

DESCENDANTS, REALITY, SUPERPOSITION AND FLUXES

The situation here is similar to the non-null case.

Using the relation

$$\mathcal{L}_{L_-}(\,\mathrm{d}\tau + \mathrm{d}\psi) = \Psi\,(\,\mathrm{d}\tau + \mathrm{d}\psi)\,,$$

it is easily seen that all descendants of both

$\mathcal{J}_{(1,0)}$ propto $\mathrm{d}\tau + \mathrm{d}\psi$ $\star F_{(1,0)}$ propto $\mathrm{d}\tau + \mathrm{d}\psi$

ELECTRIC AND MAGNETIC FIELDS

To visualize the physical properties of these solutions, we animate the electric and magnetic field strengths

Figure 1: Electric field strength E^2 (left) and magnetic field strength B^2 (right) evaluated at Poincare time for a non-null solution for the solution $\Re F_{(2,0)}$. The black hole is the point at the center of the box

$$E^2 = E_\nu E^\nu \qquad E_\nu \equiv -U^\mu F_{\mu\nu},$$

where

$$B^2 = B_\nu B^\nu \qquad B_\nu \equiv U^\mu (\star F)_{\mu\nu},$$

$$U^\mu = (1,0,0,0)$$

is the 4-vector of a static observer in Poincare coordinates

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FORCE-FREE ELECTRODYNAMICS
AROUND EXTREME KERR BLACK HOLES
MARIA J. RODRIGUEZ
Thanks!


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