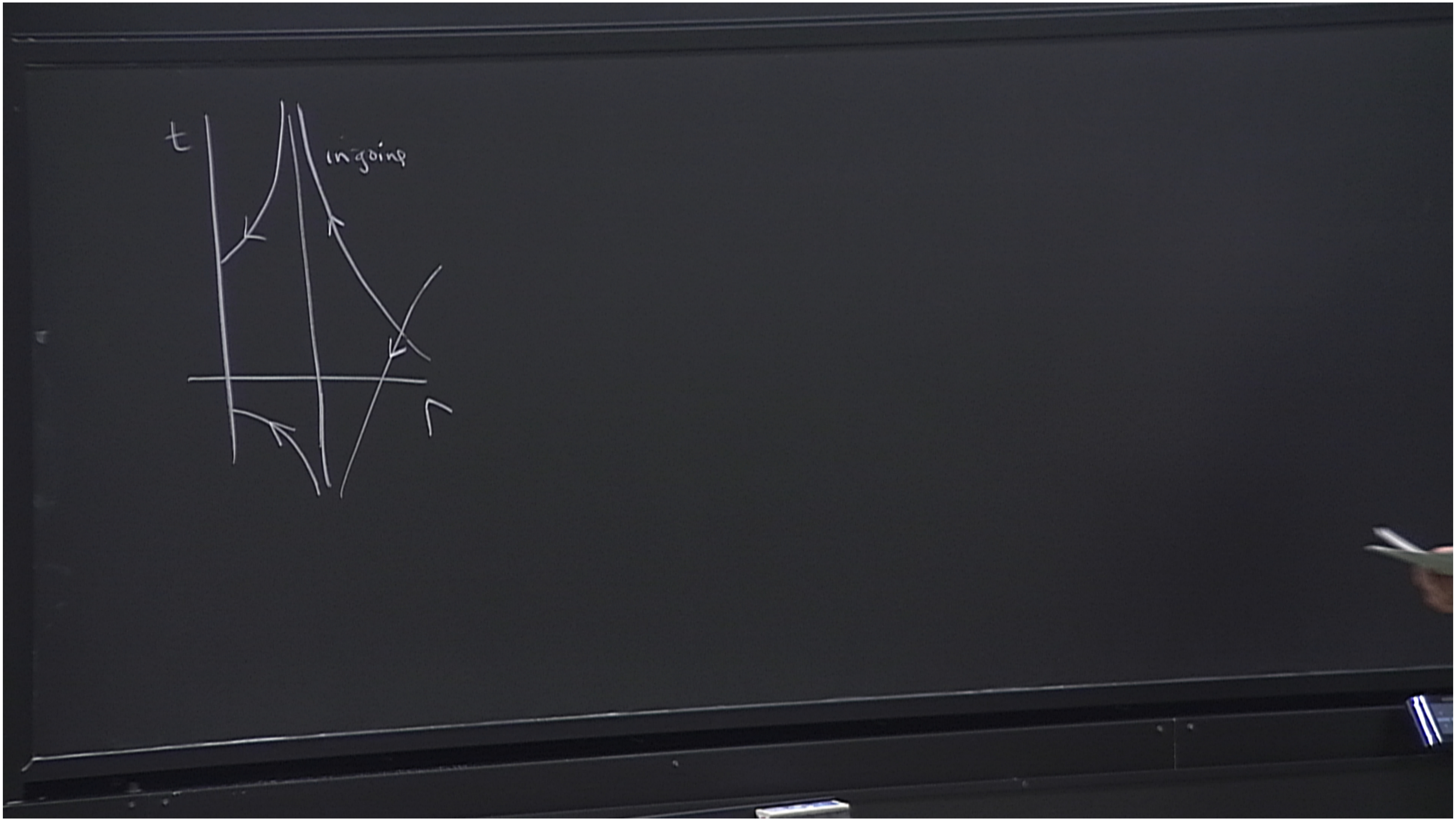
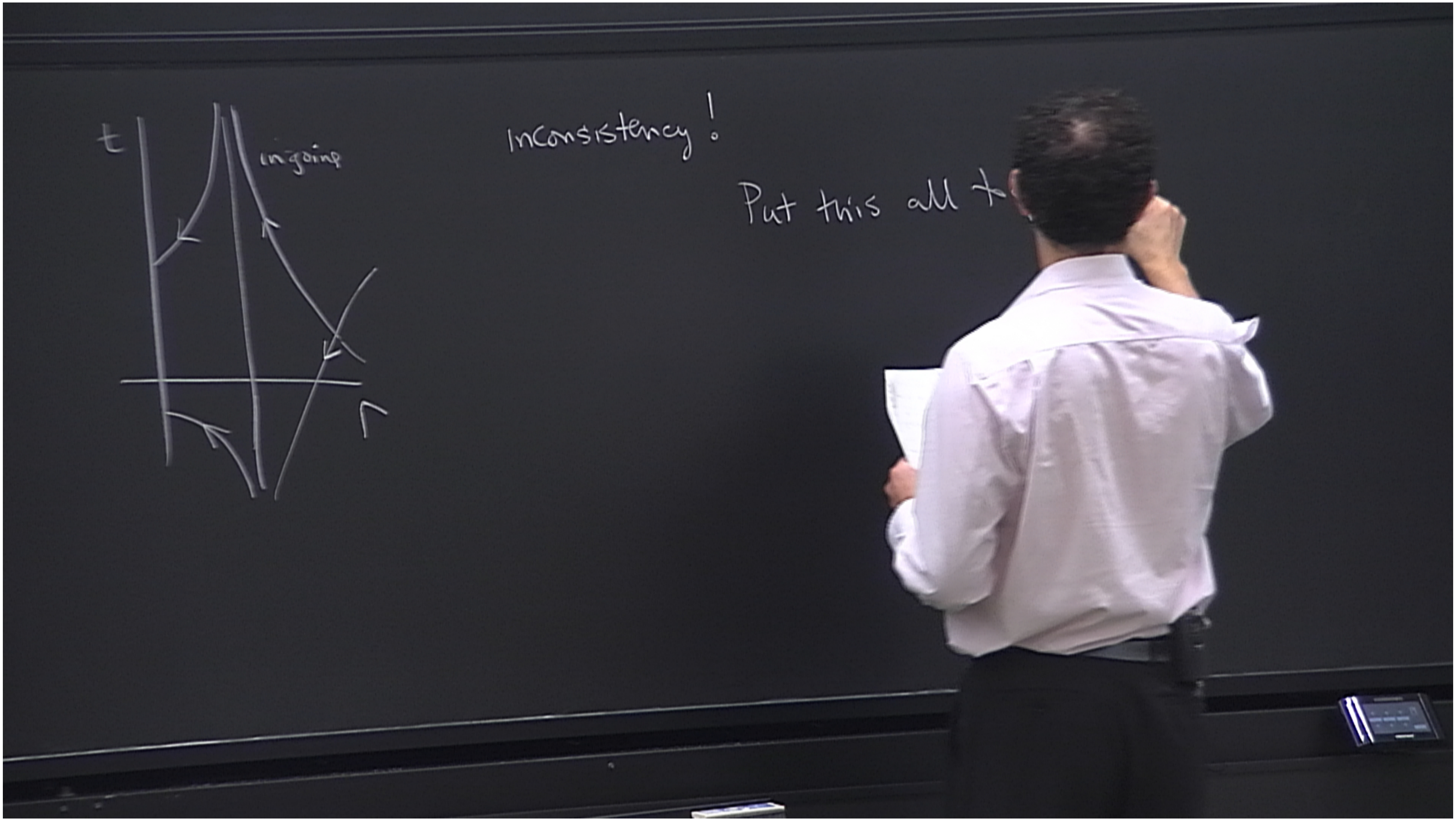
Title: 14/15 PSI - Relativity- Lecture 15

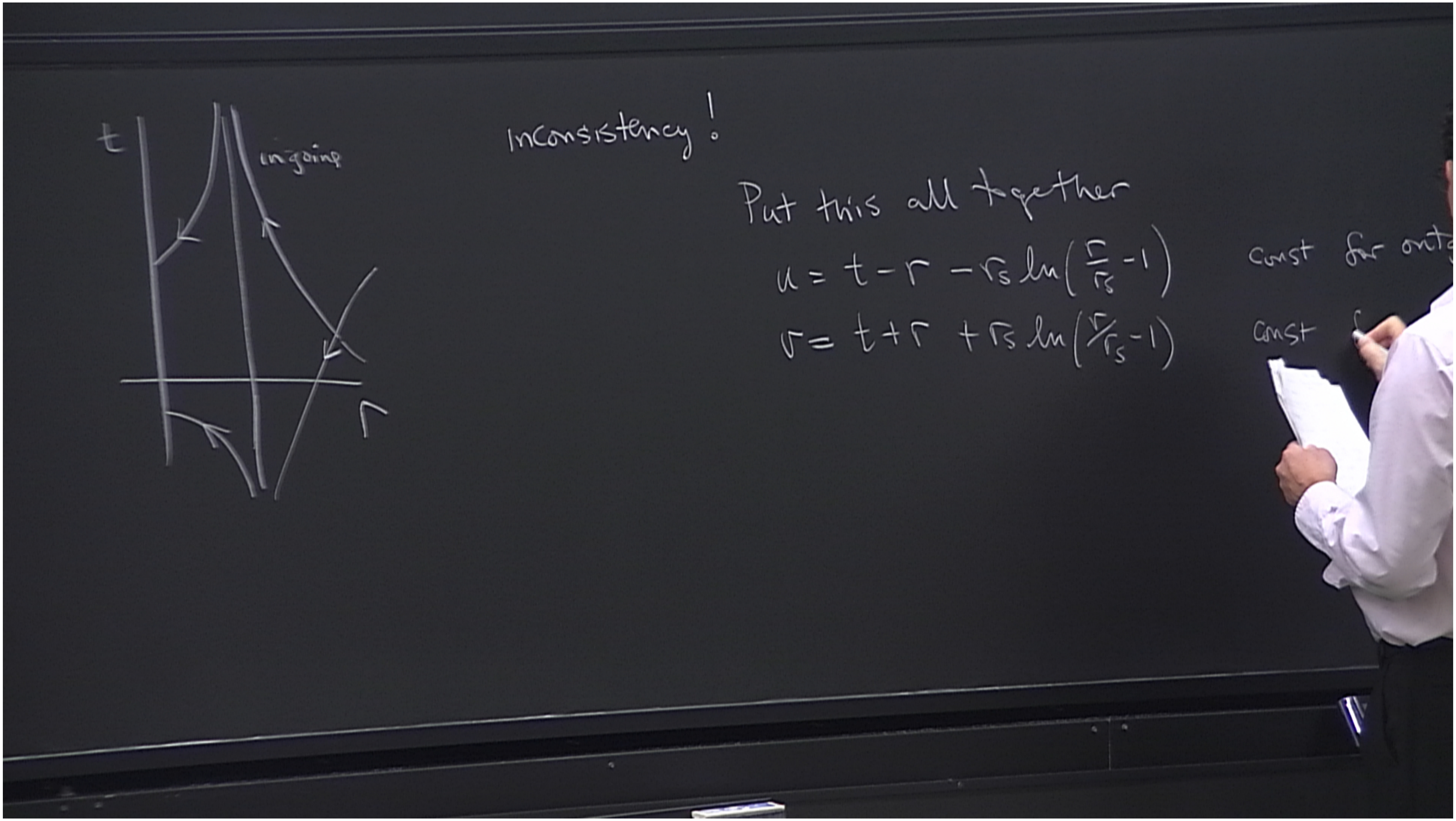
Date: Sep 26, 2014 09:00 AM

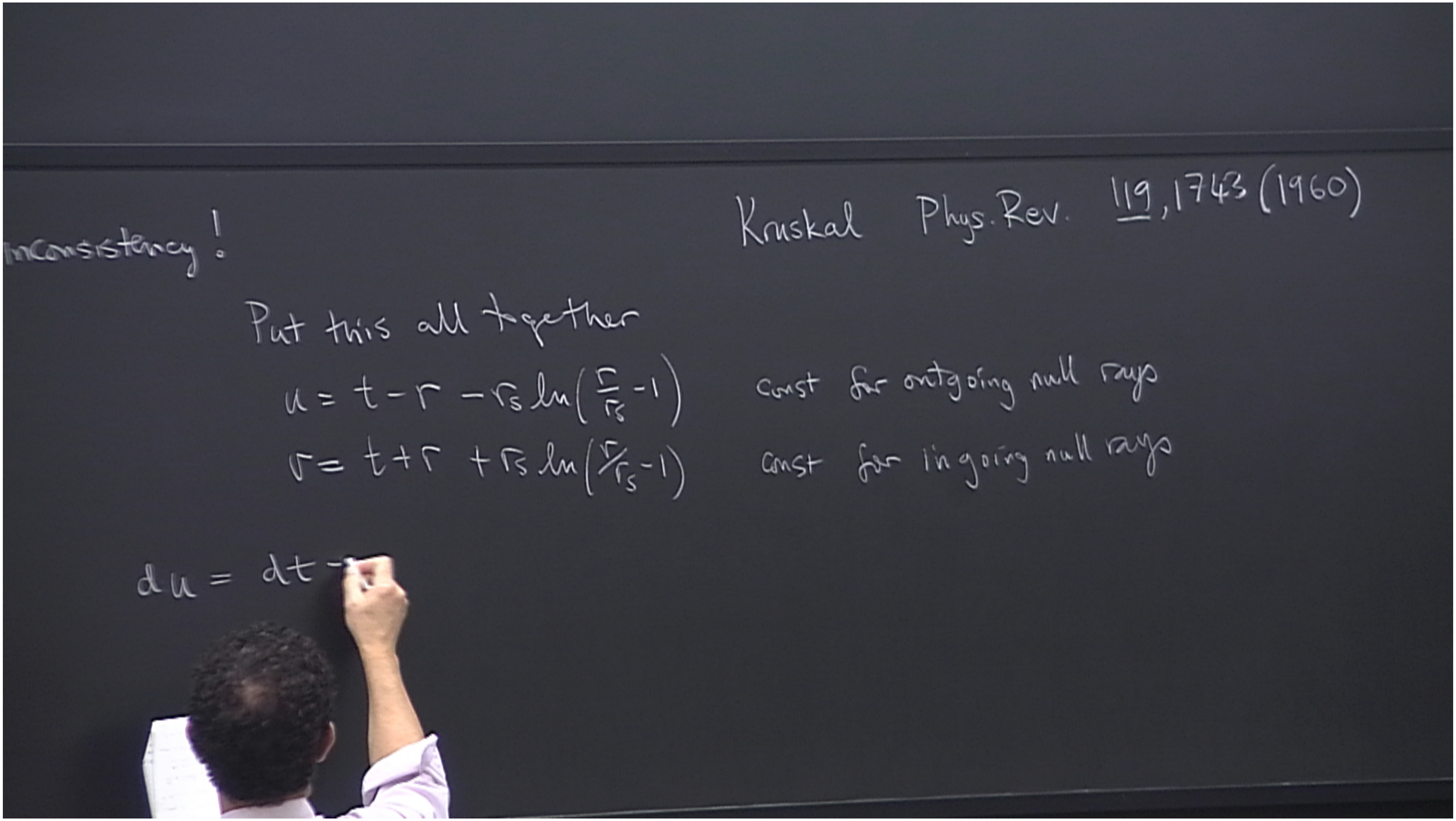
URL: http://pirsa.org/14090064

Abstract:


t
ingoing
r


t
ingoing
r
inconsistency!
Put this all to


t
ingoing
r
inconsistency!
Put this all together
$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ const for outg
$v = t + r + r_s \ln\left(r/r_s - 1\right)$ const f


inconsistency !
Kruskal Phys. Rev. 119, 1743 (1960)
Put this all together
$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ const for outgoing null rays
$v = t + r + r_s \ln\left(r/r_s - 1\right)$ const for ingoing null rays
$du = dt -$

inconsistency!

Kruskal Phys. Rev. **119**, 1743 (1960)

Put this all together

$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ const for outgoing null rays

$v = t + r + r_s \ln\left(r/r_s - 1\right)$ const for ingoing null rays

(°.°)

$$\left.\begin{aligned} du &= dt - \frac{dr}{1 - r_s/r} \\ dv &= dt + \frac{dr}{1 - \frac{r_s}{r}} \end{aligned}\right\} \quad ds^2 = -\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega^2$$

inconsistency !

Kruskal Phys. Rev. 119, 1743 (1960)

Put this all together

$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ const for outgoing null rays

$v = t + r + r_s \ln\left(r/r_s - 1\right)$ const for ingoing null rays

$$du = dt - \frac{dr}{1 - r_s/r}$$

$$dv = dt + \frac{dr}{1 - \frac{r_s}{r}}$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du\, dv + r^2 d\Omega^2$$

r defined by $r + r_s \ln\left(r/r_s - 1\right) = \frac{v - u}{2}$ i.e.

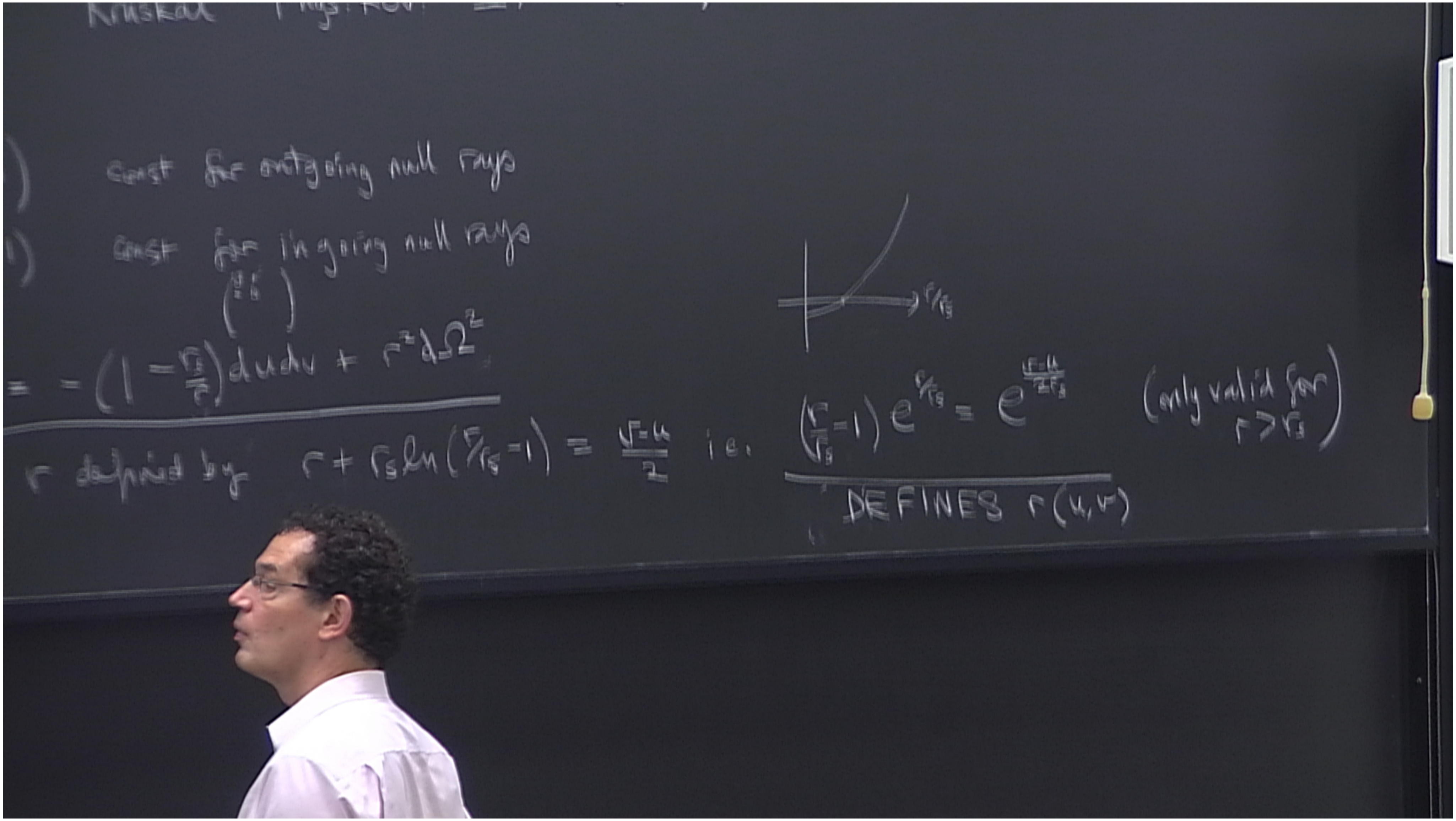
Kruskal Phys. Rev. 119, 1743 (1960)

) const for outgoing null rays

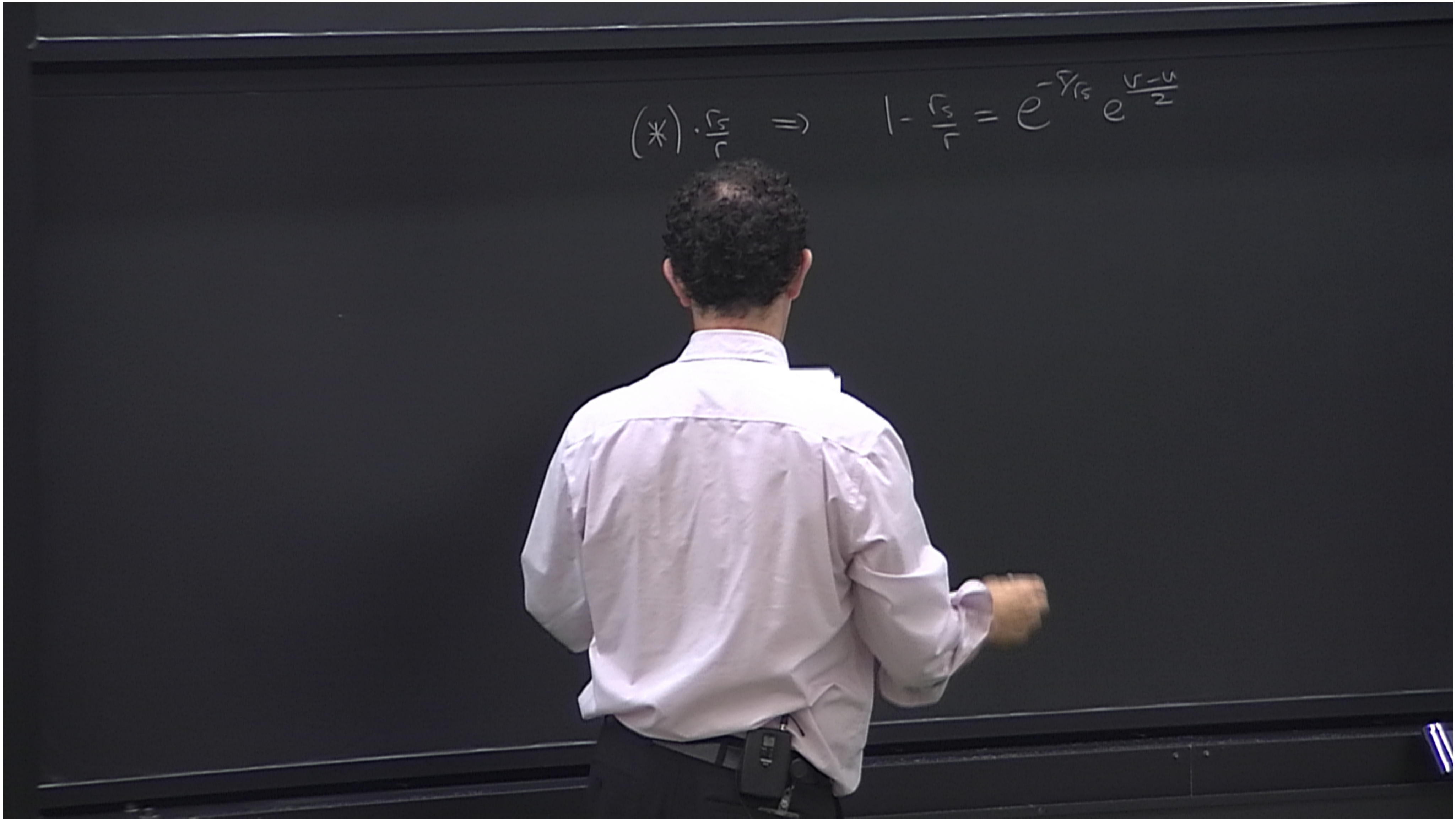
) const for ingoing null rays

$$= -\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega^2$$

r defined by $r + r_s \ln\left(r/r_s - 1\right) = \frac{v-u}{2}$ i.e. $\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = e^{\frac{v-u}{2r_s}}$ (only valid for $r > r_s$)



Kruskal Phys. Rev.
const for outgoing null rays
const for ingoing null rays
$= -(1-\frac{r_s}{r})\,du\,dv + r^2 d\Omega^2$
r defined by $r + r_s \ln(r/r_s - 1) = \frac{v-u}{2}$ i.e. $(\frac{r}{r_s}-1)e^{r/r_s} = e^{\frac{v-u}{2r_s}}$
(only valid for $r > r_s$)
DEFINES $r(u,v)$
r/r_s



$(*) \cdot \frac{r_s}{r} \implies 1 - \frac{r_s}{r} = e^{-\bar{r}/r_s} e^{\frac{v-u}{2}}$

$$(*) \cdot \frac{r_s}{r} \Rightarrow \quad 1 - \frac{r_s}{r} = \frac{r_s}{r} e^{-r/r_s} e^{\frac{v-u}{2}}$$

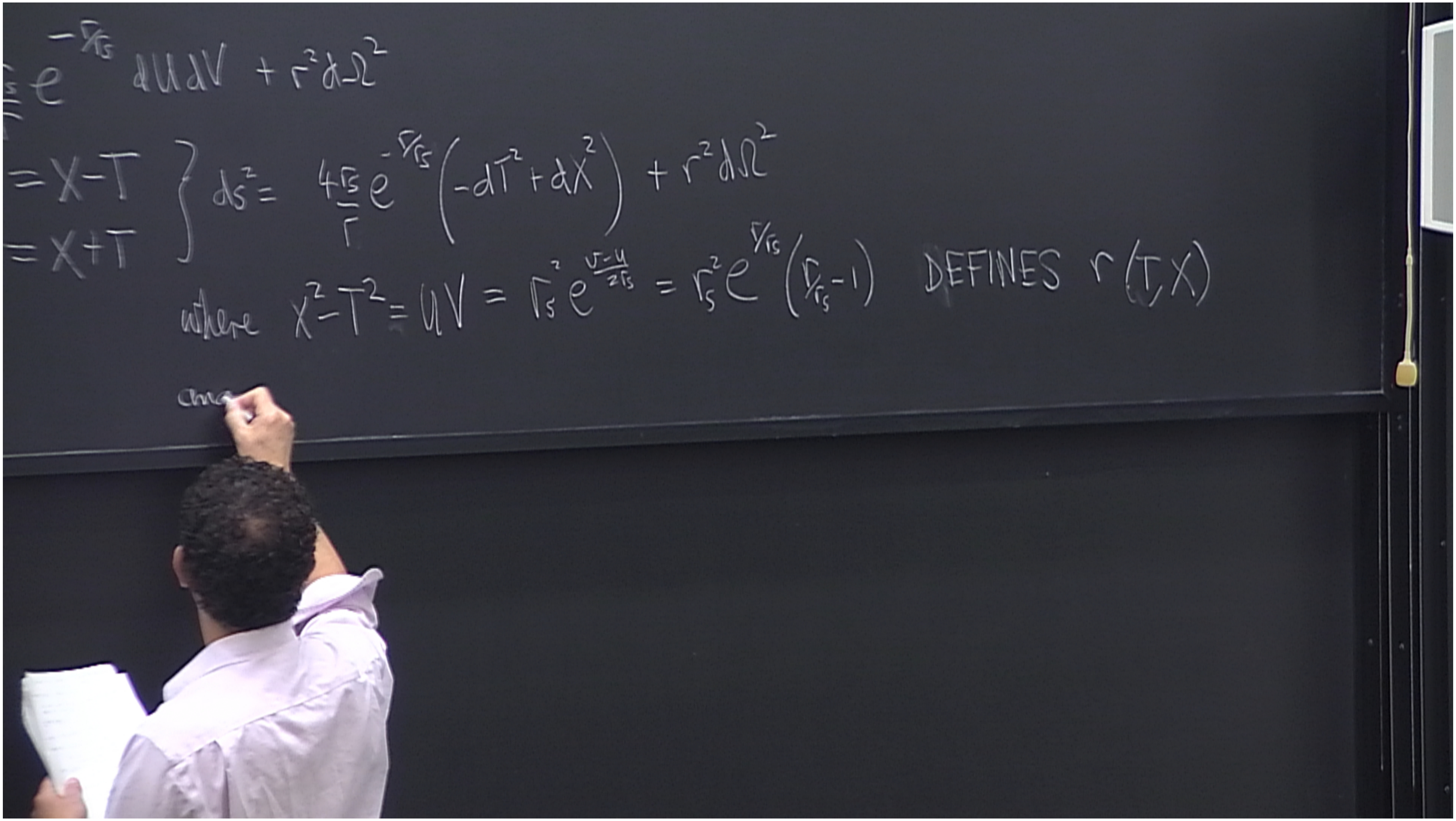
$$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{v-u}{2r_s}} du\, dv + r^2 d\Omega^2$$

$$(*)\cdot\frac{r_s}{r} \Rightarrow 1-\frac{r_s}{r} = \frac{r_s}{r}e^{-r/r_s}e^{\frac{v-u}{2}}$$

$$\Rightarrow ds^2 = -\frac{r_s}{r}e^{-r/r_s}e^{\frac{v-u}{2r_s}}du\,dv + r^2 d\Omega^2$$

Now set $\underbrace{U}_{\text{extrapolate to negative values.}} = r_s e^{-u/2r_s}$ $\Rightarrow ds^2$

$$V = \underbrace{r_s e^{v/2r_s}}_{>0}$$



$e^{-r/r_s} \, dU dV + r^2 d\Omega^2$
$= X - T$
$= X + T$
$ds^2 = \frac{4r_s}{r} e^{-r/r_s} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2$
where $X^2 - T^2 = UV = r_s^2 e^{\frac{V-U}{2r_s}} = r_s^2 e^{r/r_s} \left(r/r_s - 1\right)$ DEFINES $r(T,X)$

$\ldots du\, dv + r^2 d\Omega^2$

$\ldots e^{-r/r_s}\, dU\, dV + r^2 d\Omega^2$

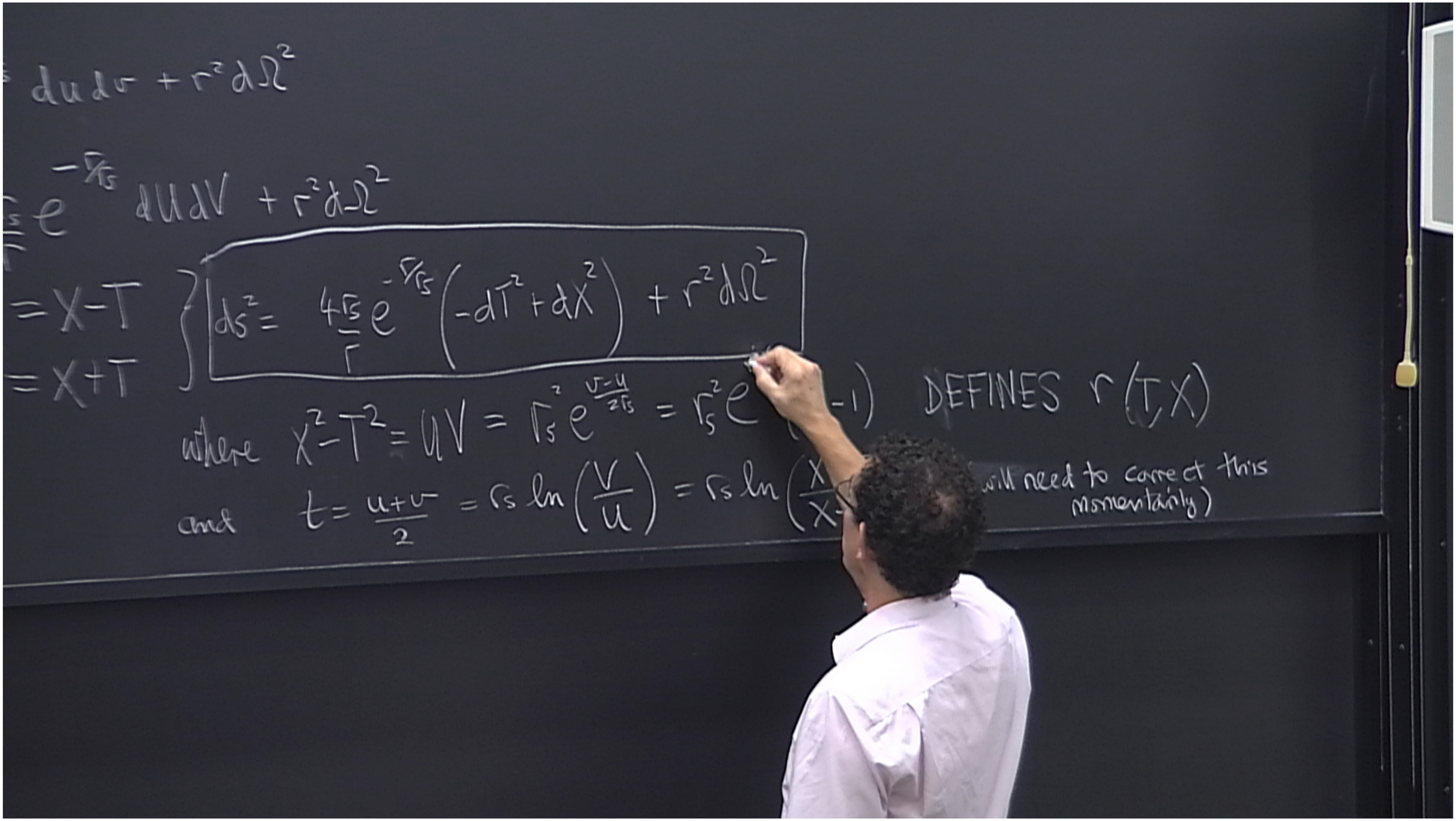
$= X - T$

$= X + T$

$$\boxed{ds^2 = \frac{4 r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2}$$

where $X^2 - T^2 = UV = r_s^2\, e^{\frac{v-u}{2r_s}} = r_s^2\, e^{r/r_s}\left(r/r_s - 1\right)$ DEFINES $r(T,X)$

and $t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$ (will need to correct this momentarily)


du dv + r² dΩ²
e^{-r/rs} dU dV + r² dΩ²
= X − T
= X + T
ds² = (4rs/r) e^{-r/rs} (−dT² + dX²) + r² dΩ²
where X² − T² = UV = rs² e^{(v−u)/2rs} = rs² e
-1) DEFINES r(T, X)
and t = (u+v)/2 = rs ln(V/U) = rs ln(X
will need to correct this
momentarily)

$$du\,dv + r^2 d\Omega^2$$

$$e^{-r/r_s}\,dU\,dV + r^2 d\Omega^2$$

$$= X - T$$
$$= X + T$$

$$\boxed{ds^2 = \frac{4r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2}$$

where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(r/r_s - 1\right)$ DEFINES $r(T, X)$

and $t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$ (will need to correct this momentarily)

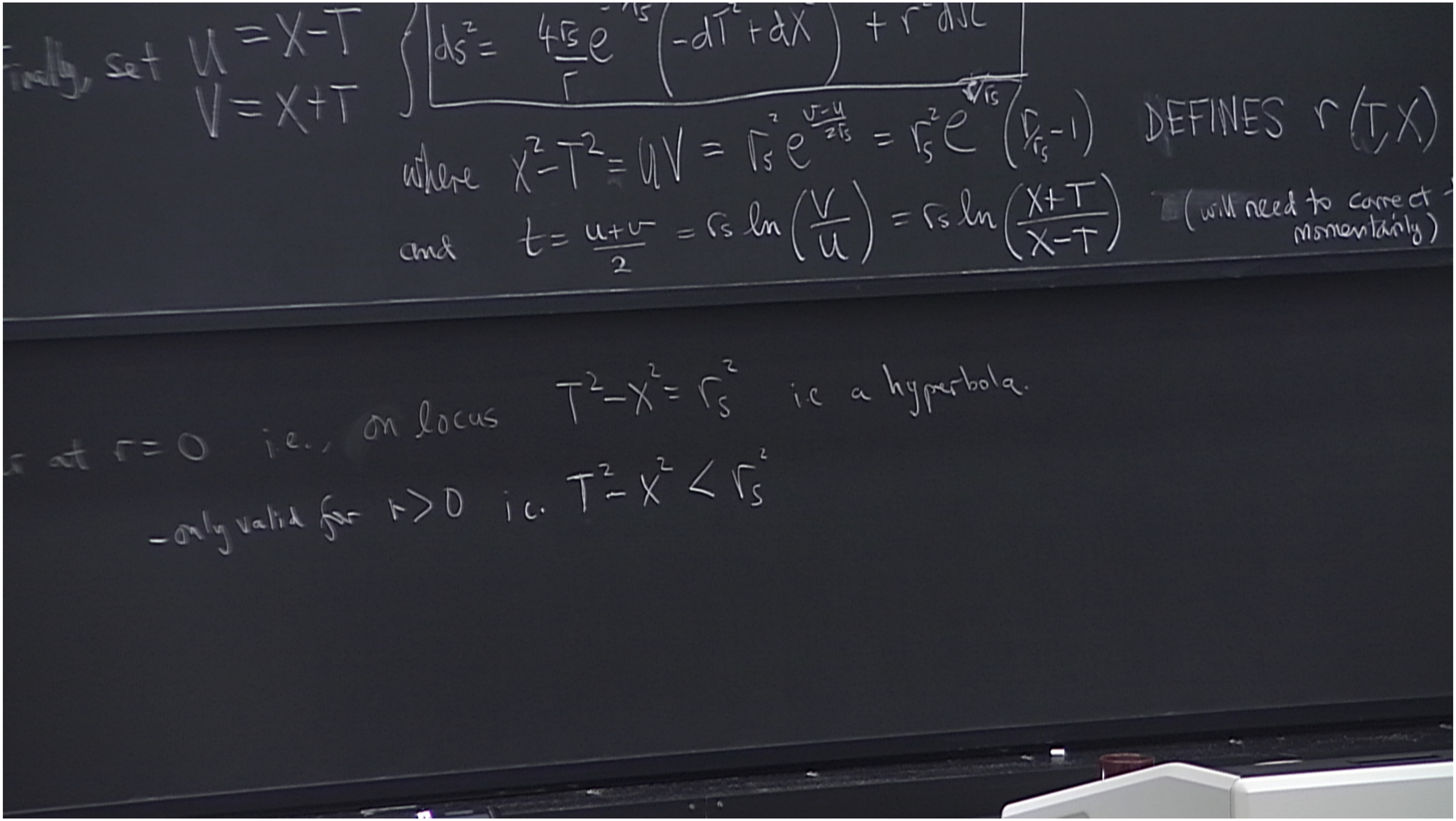
$$= e^{-r/r_s} \, dU dV + r^2 d\Omega^2$$

$$\left.\begin{array}{l} = X - T \\ = X + T \end{array}\right\} \boxed{ds^2 = \frac{4r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2}$$

where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(\frac{r}{r_s} - 1\right)$ DEFINES $r(T, X)$

and $t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$ (will need to correct this momentarily)

i.e., on locus $T^2 - X^2 = r_s^2$


Finally, set $U = X - T$
$V = X + T$
$ds^2 = \frac{4 r_s}{r} e^{-r/r_s} (-dT^2 + dX^2) + r^2 d\Omega^2$
where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s} (r/r_s - 1)$ DEFINES $r(T,X)$
and $t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$ (will need to correct momentarily)
at $r = 0$ i.e., on locus $T^2 - X^2 = r_s^2$ i.e a hyperbola.
– only valid for $r > 0$ i.e. $T^2 - X^2 < r_s^2$

Finally, set $U = X - T$, $V = X + T$

$$\boxed{ds^2 = \frac{4r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2}$$

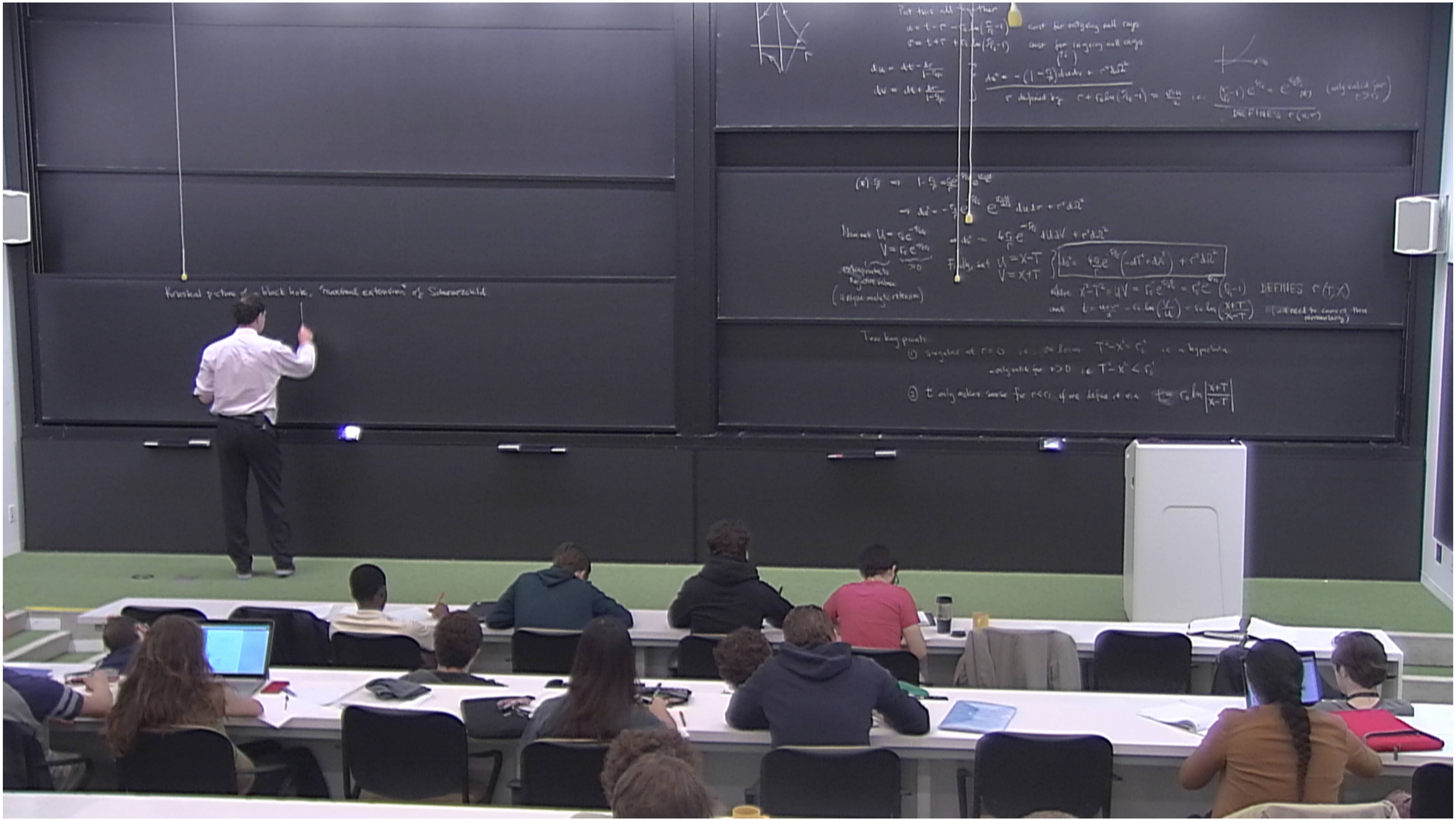
where $X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(\frac{r}{r_s} - 1\right)$ DEFINES $r(T, X)$

and $t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right)$ (will need to correct momentarily)

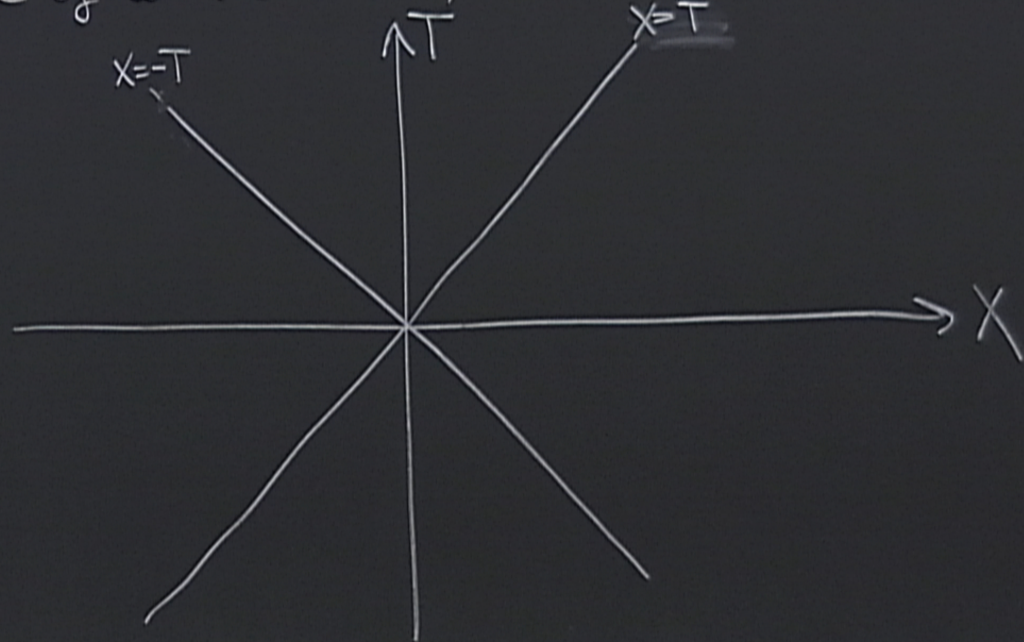
at $r = 0$ i.e., on locus $T^2 - X^2 = r_s^2$ i.e. a hyperbola.

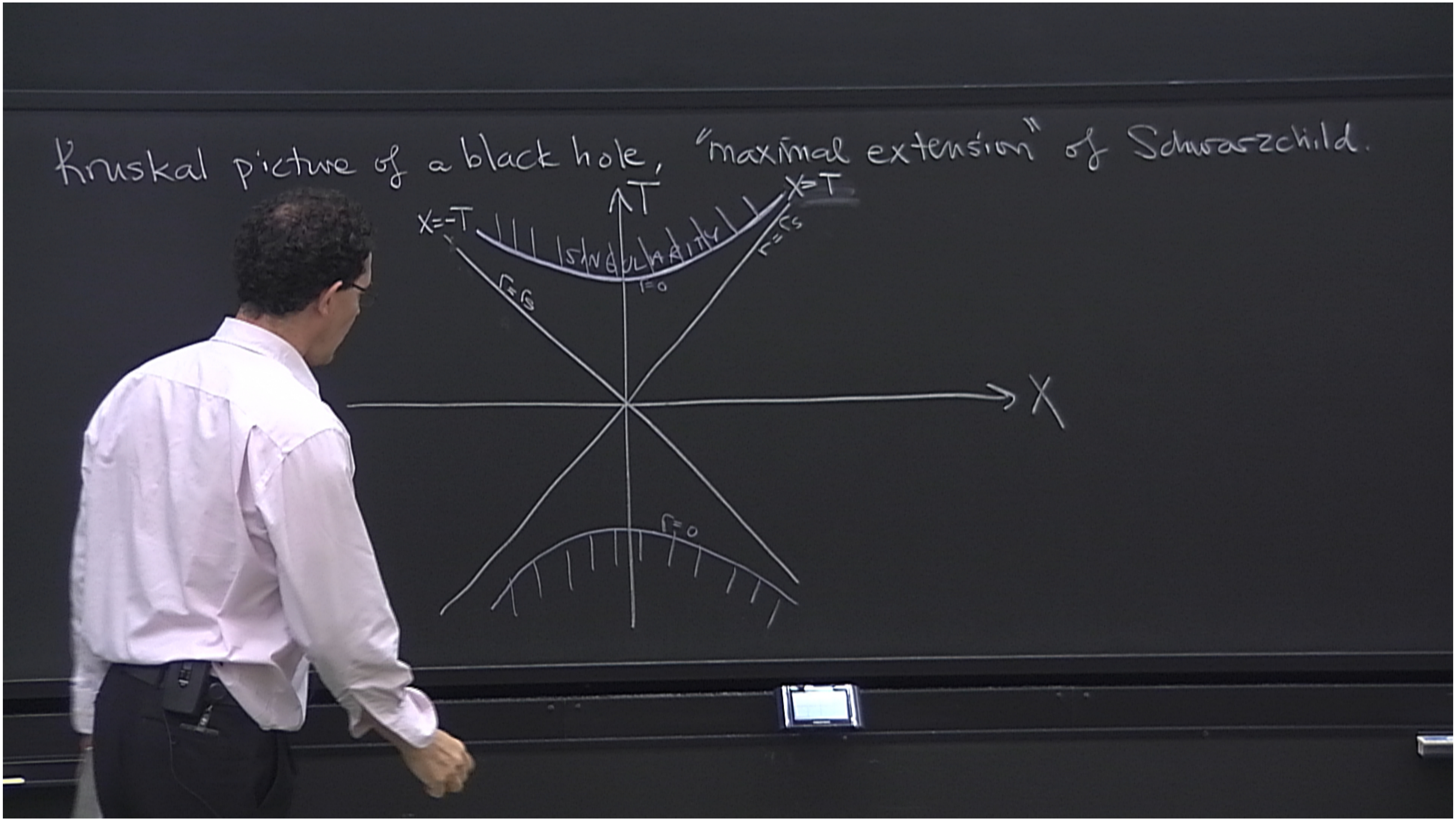
- only valid for $r > 0$ i.e. $T^2 - X^2 < r_s^2$

makes sense for $r < r_s$ if we define it via $t = r_s \ln\left|\frac{X+T}{X-T}\right|$

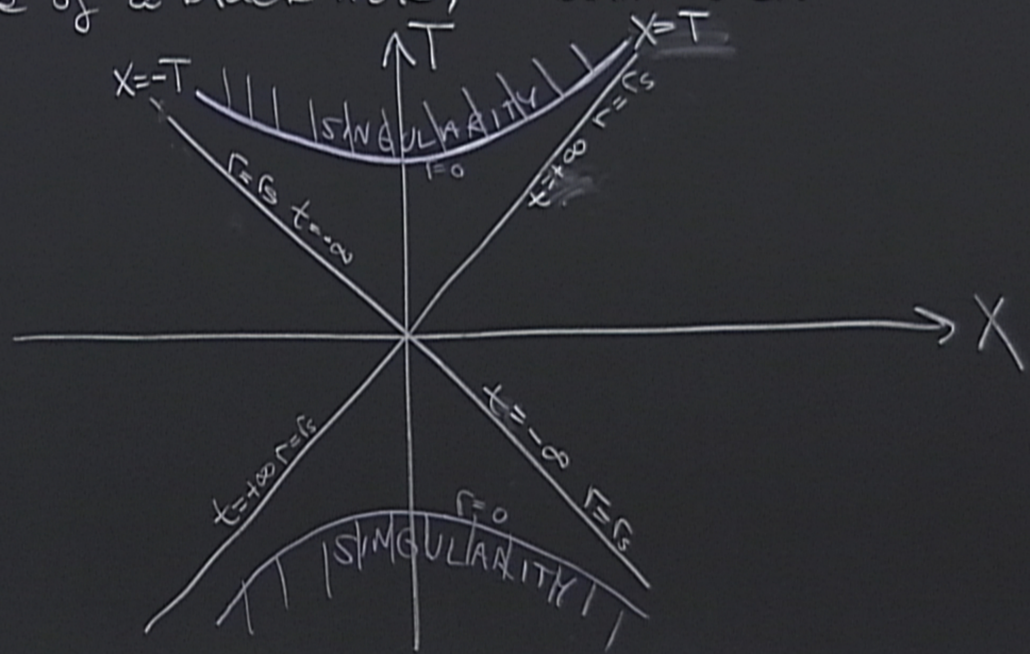


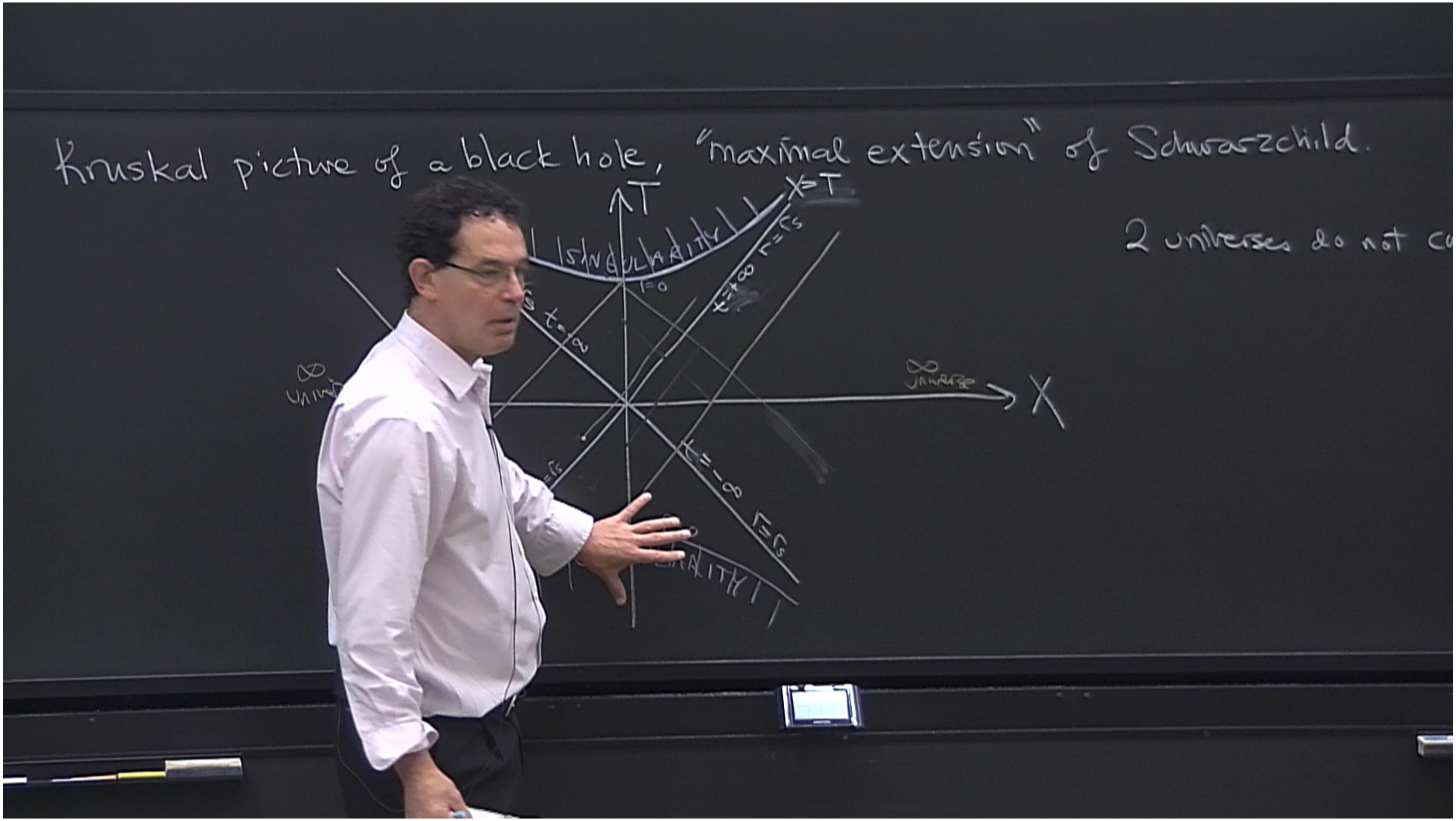
Kruskal picture of a black hole, "maximal extension" of Schwarzschild.

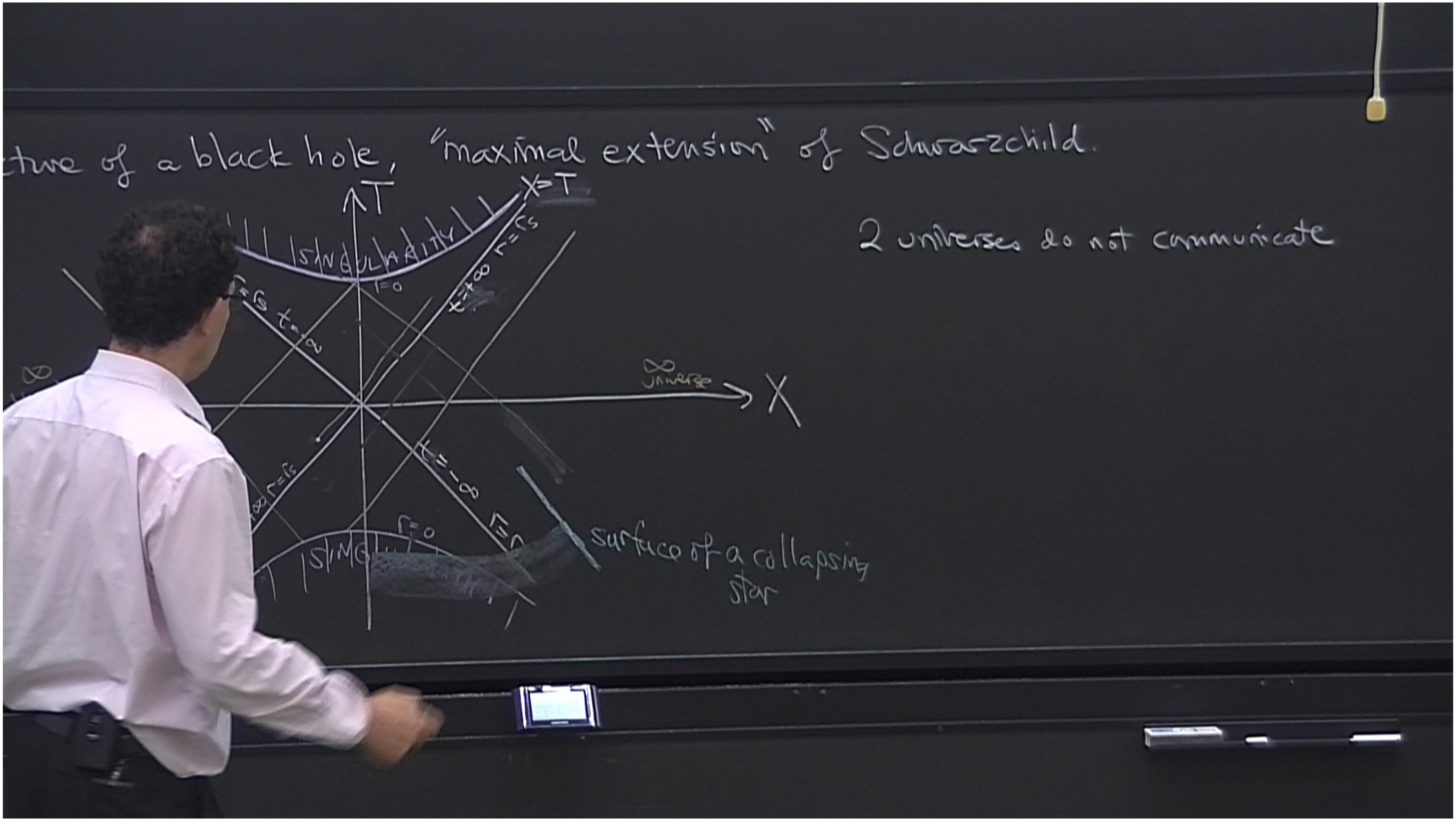


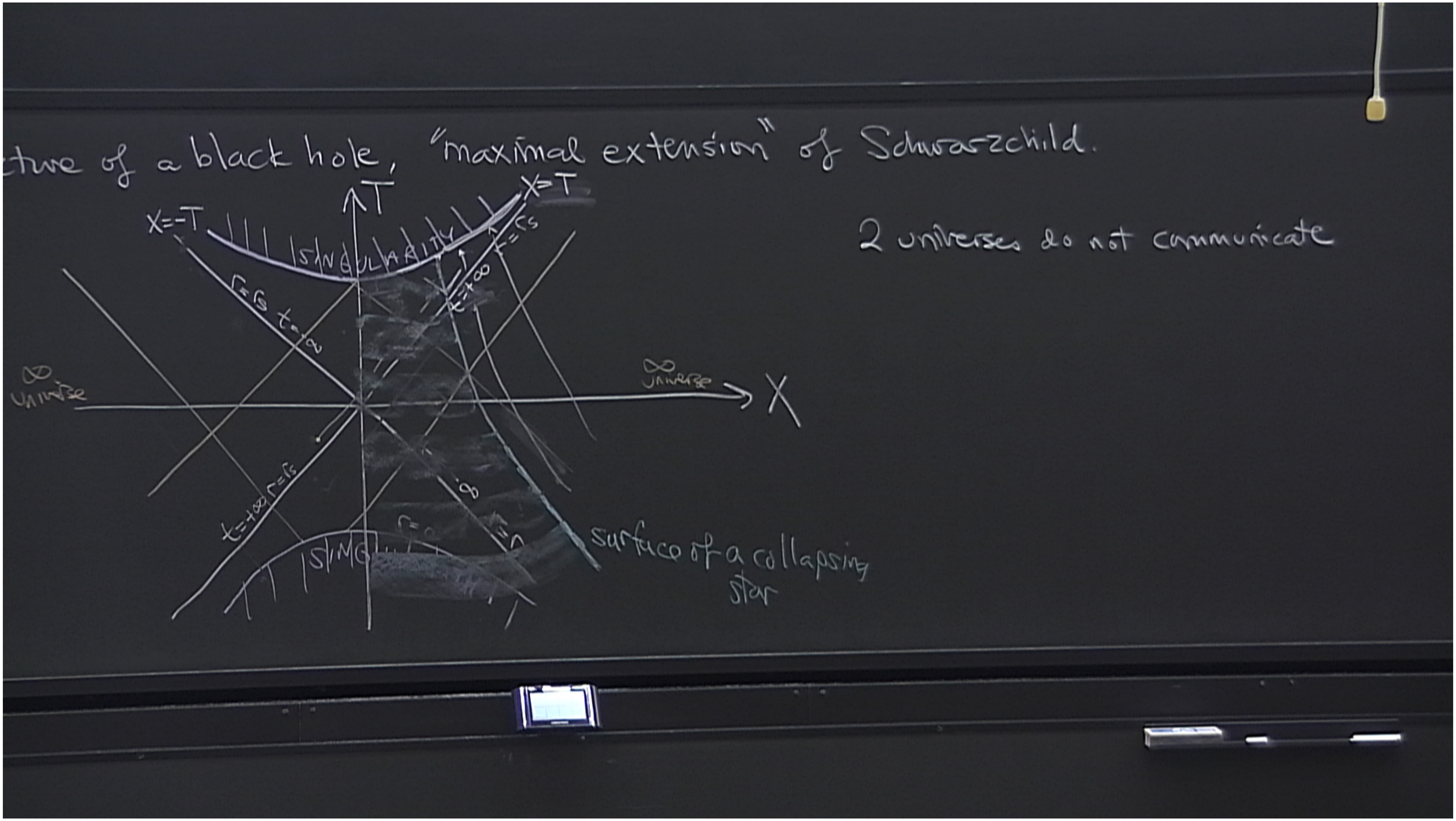

Kruskal picture of a black hole, "maximal extension" of Schwarzschild.
T
X
X=-T
X=T
r=rs
r=rs
SINGULARITY
r=0
r=0

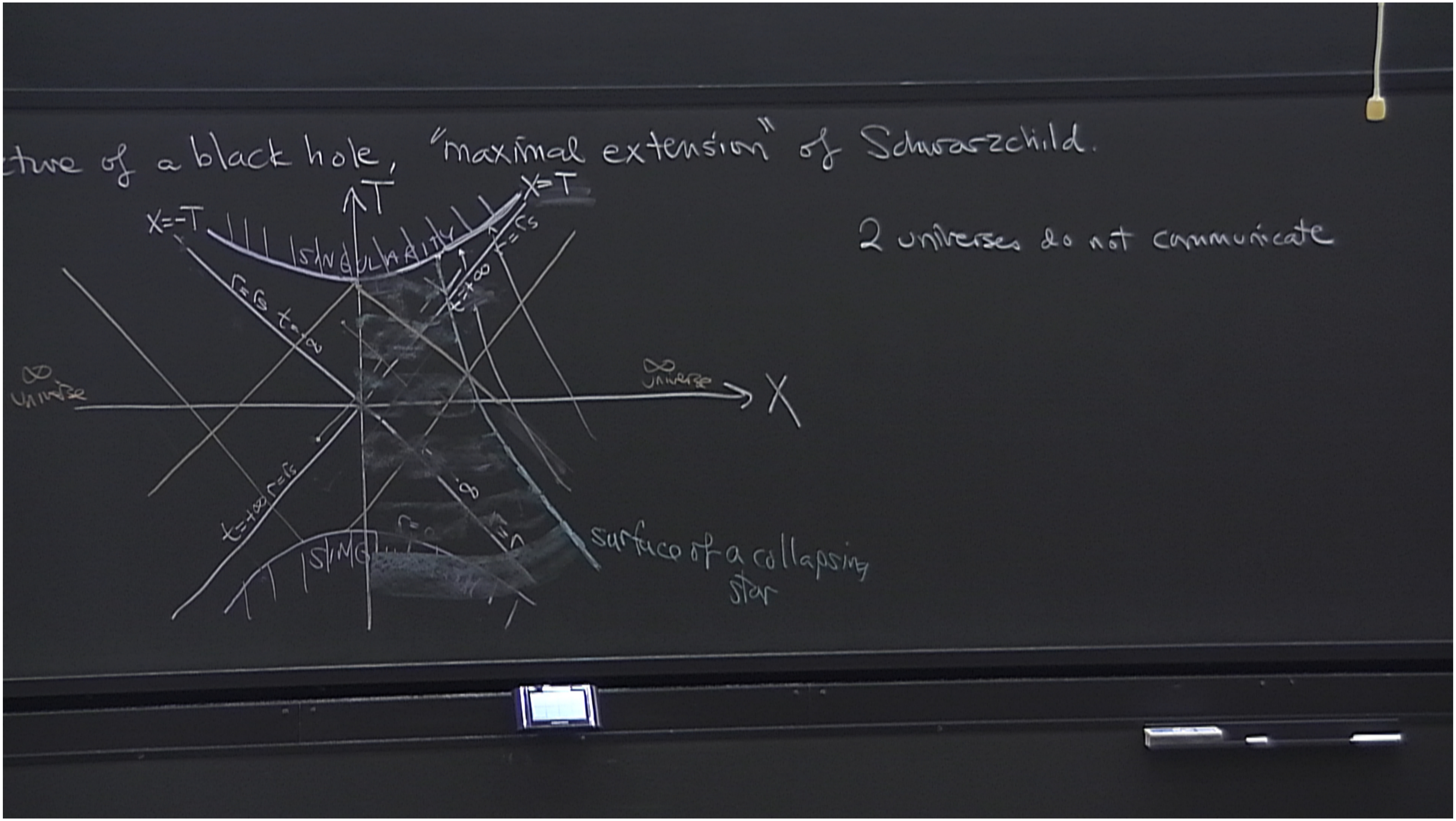
Kruskal picture of a black hole, "maximal extension" of Schwarzschild.




Kruskal picture of a black hole, "maximal extension" of Schwarzchild.
T
X=T
SINGULARITY
r=0
r=rs
t=+∞
t=-∞
∞
universe
X
t=-∞
r=rs
2 universes do not c


ture of a black hole, "maximal extension" of Schwarzschild.
2 universes do not communicate
X=T
T
SINGULARITY
r=0
X
surface of a collapsing star


cture of a black hole, "maximal extension" of Schwarzchild.
2 universes do not communicate
T
X
X=-T
X=T
SINGULARITY
$r=r_s$ $t=-\infty$
$t=+\infty$ $r=r_s$
∞ universe
∞ universe
surface of a collapsing star

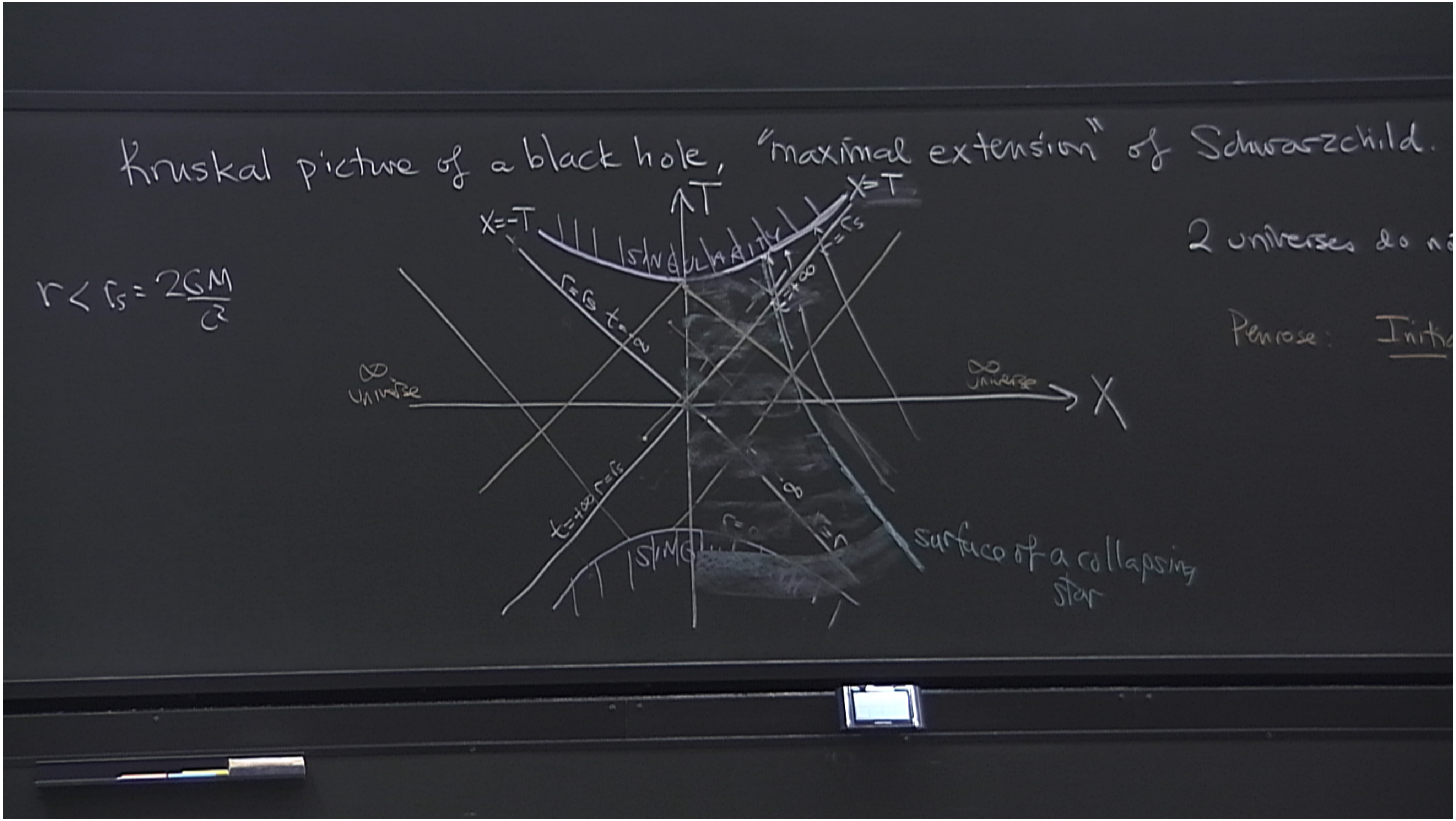


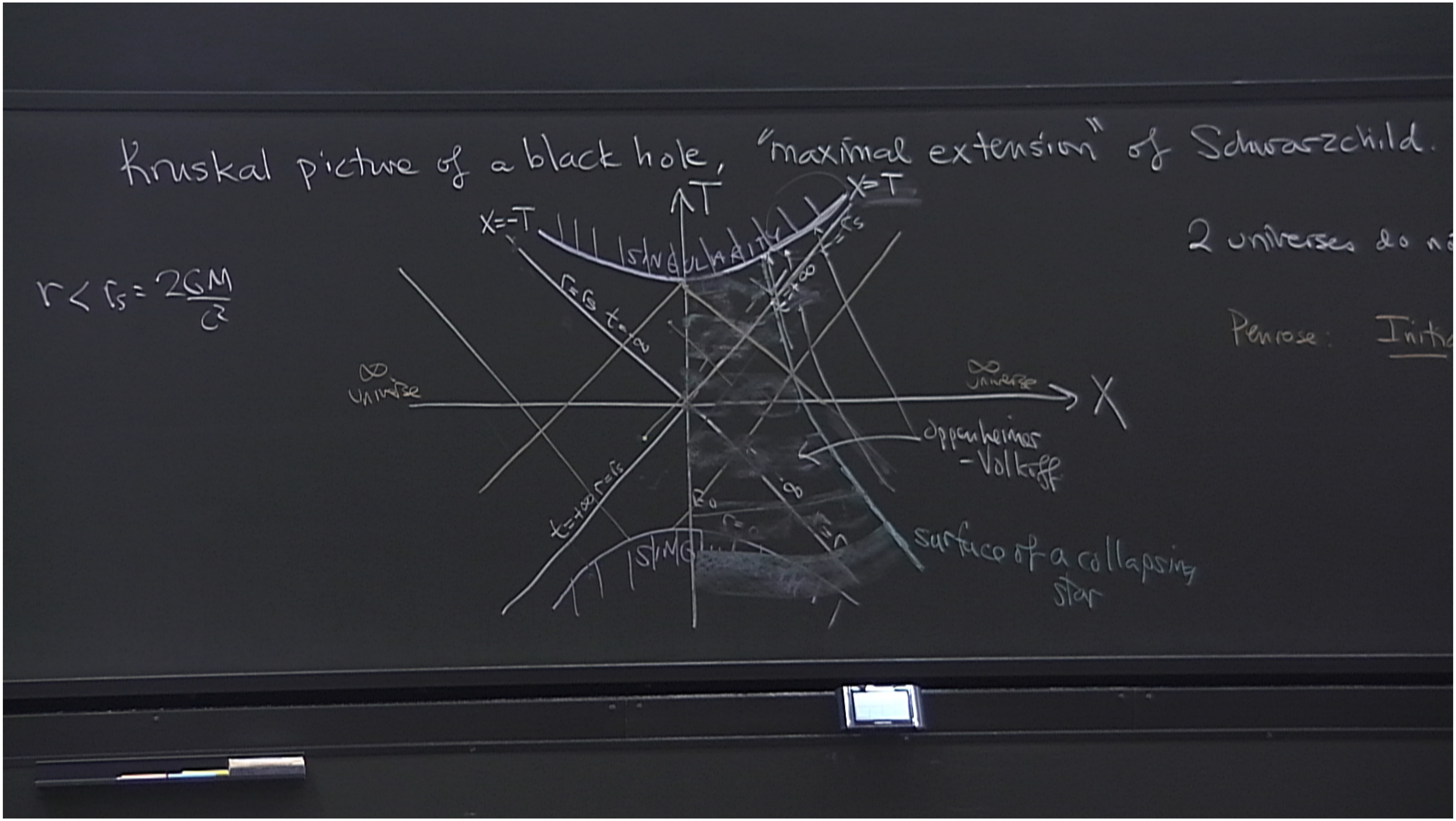
cture of a black hole, "maximal extension" of Schwarzschild.
2 universes do not communicate
T
X
X=T
X=-T
SINGULARITY
SING
∞ universe
∞ universe
surface of a collapsing star

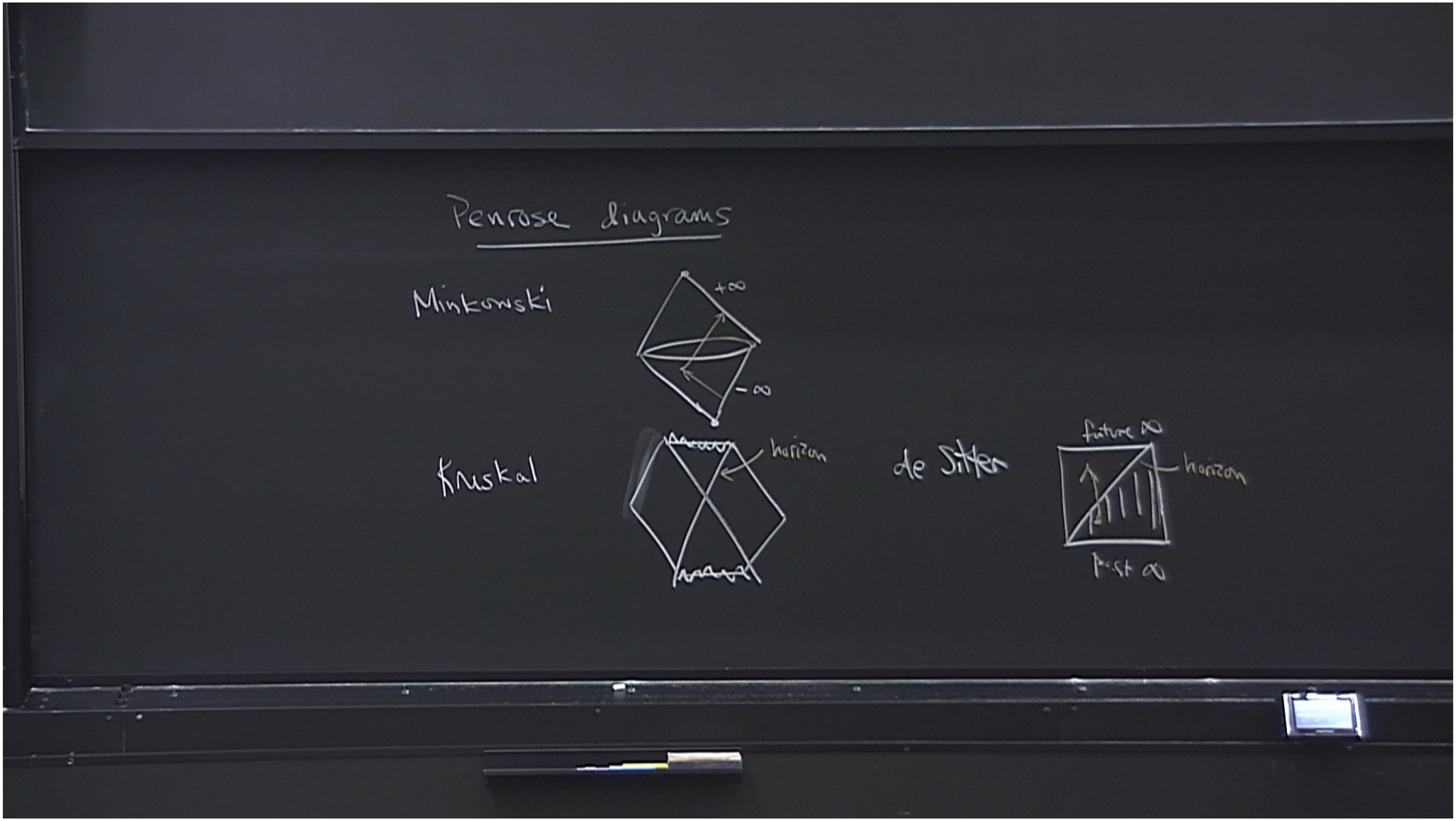
Schwarzschild.

2 universes do not communicate

Penrose: Initial singularities should have $C_{\mu\nu\rho\lambda} \to 0$


Kruskal picture of a black hole, "maximal extension" of Schwarzschild.
$r < r_s = \frac{2GM}{c^2}$
2 universes do n
Penrose: Initi
X=-T
X=T
T
X
SINGULARITY
SING
∞ universe
∞ universe
$r=r_s$ $t=-\infty$
$t=+\infty$ $r=r_s$
surface of a collapsing star


Kruskal picture of a black hole, "maximal extension" of Schwarzschild.
r < rs = 2GM/c²
T
X
X=-T
X=T
SINGULARITY
r=rs t=-∞
t=+∞ r=rs
∞ universe
∞ universe
Oppenheimer-Volkoff
surface of a collapsing star
SING
2 universes do
Penrose:


Penrose diagrams
Minkowski
+∞
−∞
Kruskal
horizon
de Sitter
future ∞
horizon
past ∞

the universe is homogeneous and isotropic (to a first approx^n):

forces the metric into a certain form

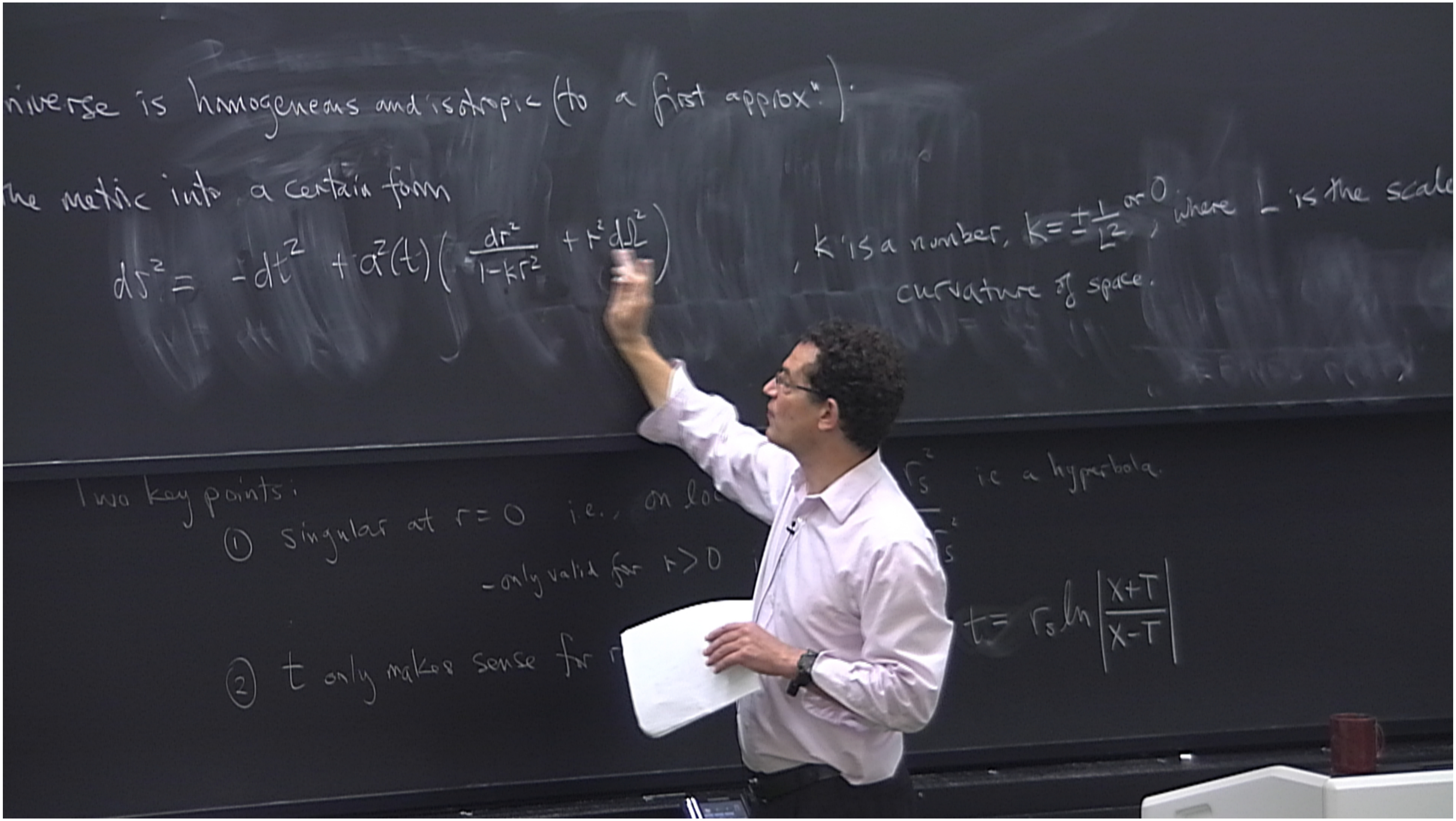
(FRW metric) $$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$, k is a number, $k = \frac{1}{L^2}$, where L is t

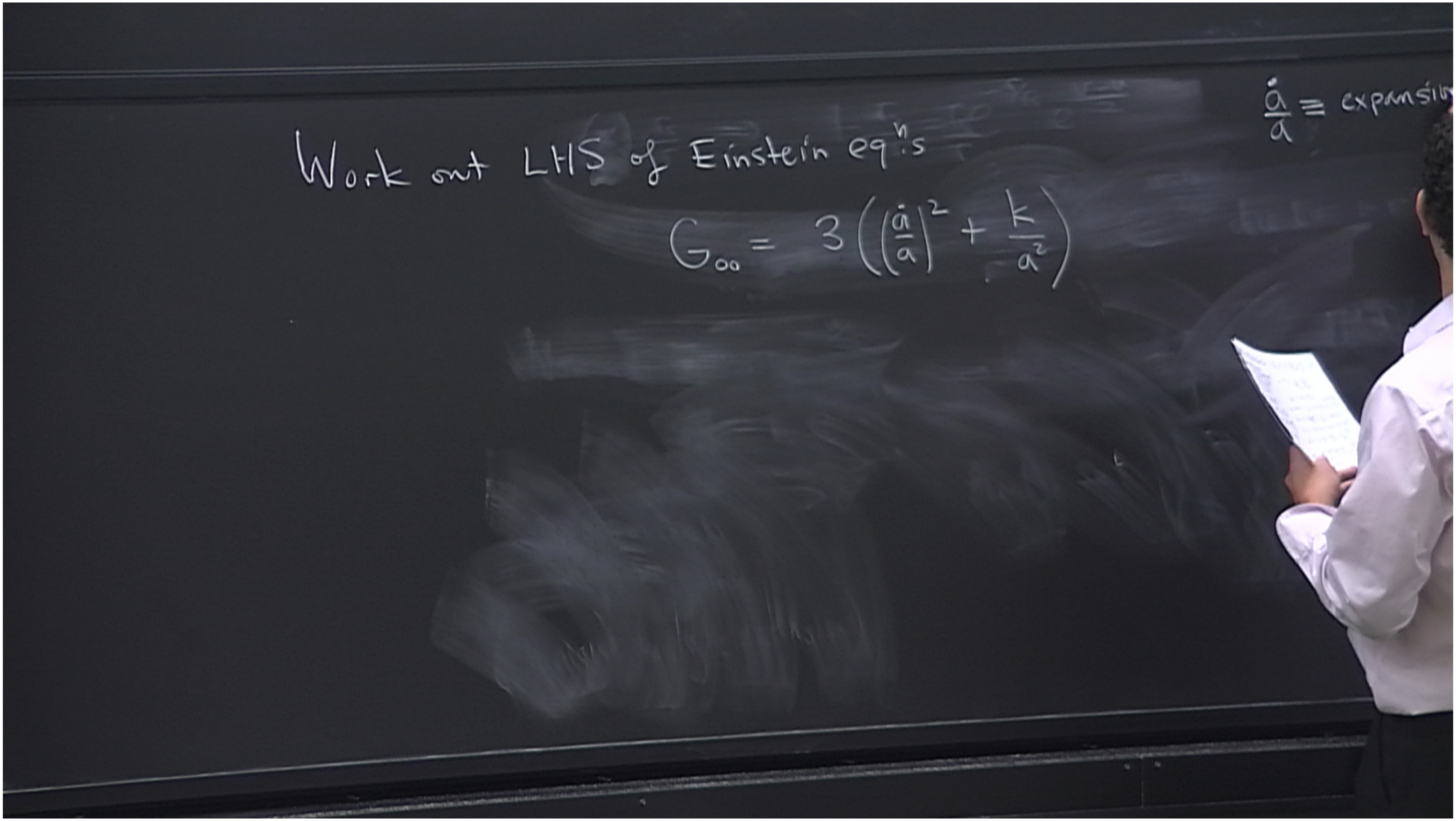
Two key points:

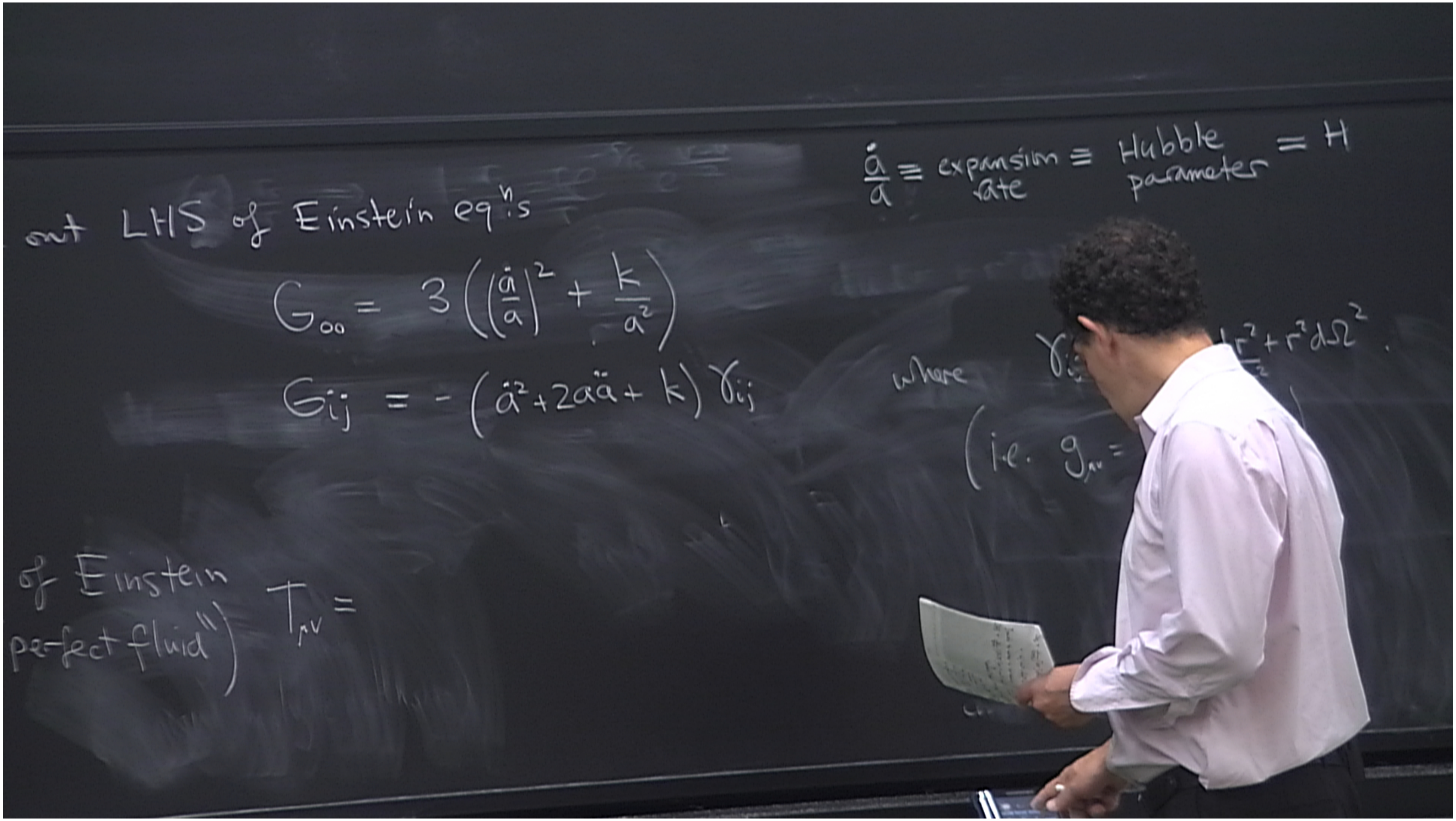
① Singular at $r=0$ i.e., on locus $T^2 - X^2 = r_s^2$ i.e. a hyperbola

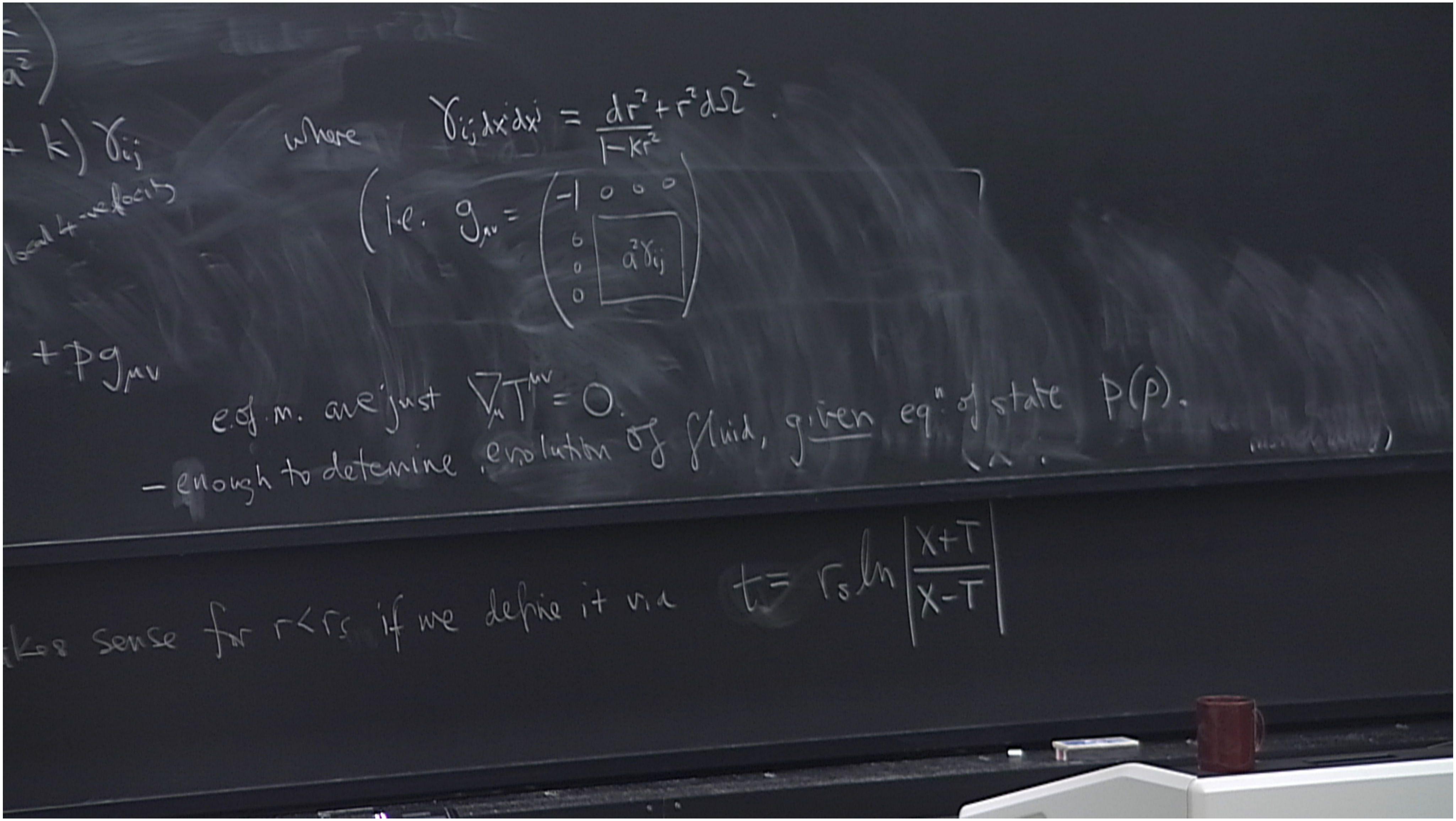
– only valid for $r>0$ i.e. $T^2 - X^2 < r_s^2$

② t only makes sense for $r<r_s$ if we define it via $t = r_s \ln\left|\frac{X+T}{X-T}\right|$

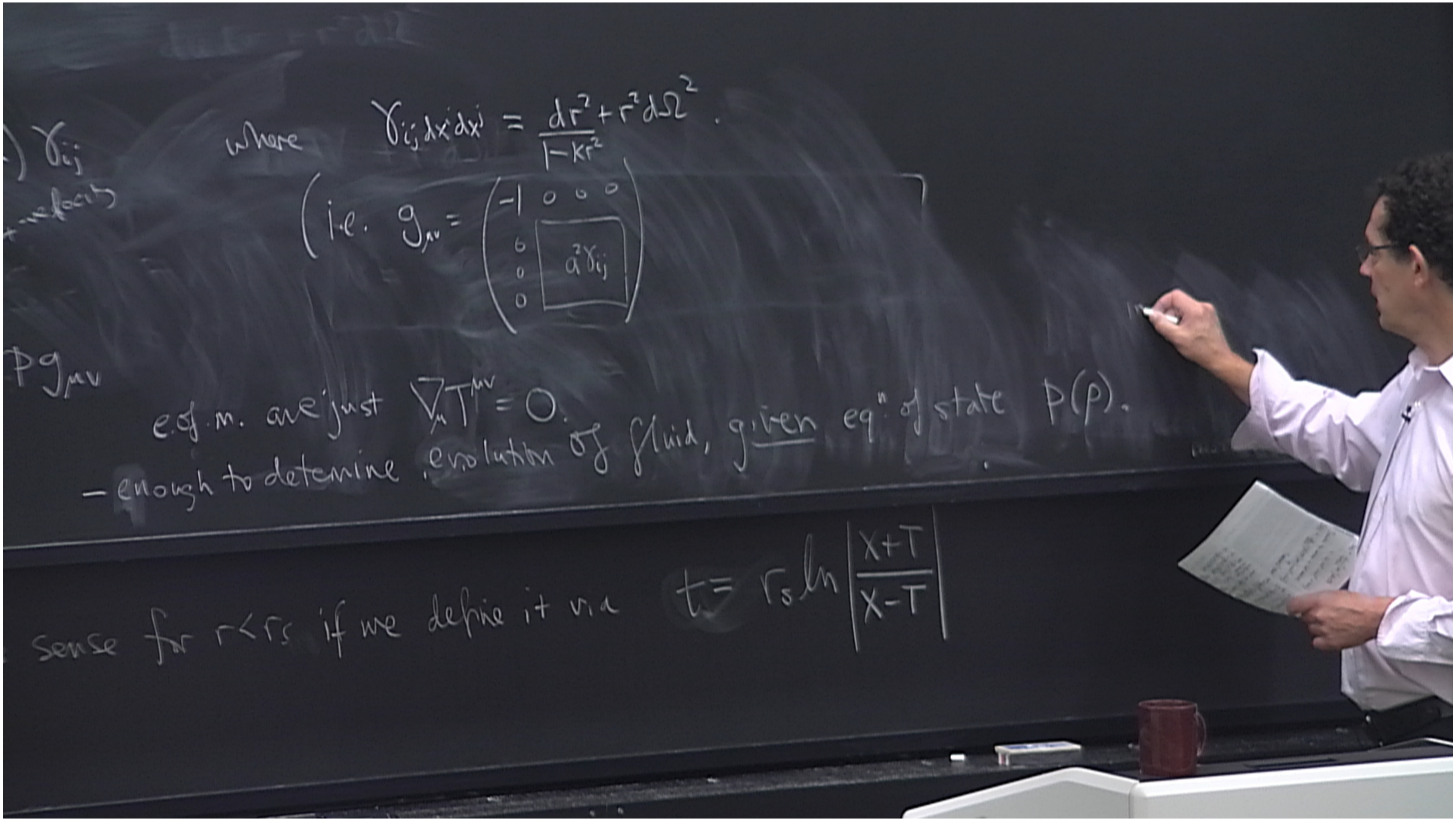

niverse is homogeneous and isotropic (to a first approxn):
he metric into a certain form
ds² = -dt² + a²(t)(dr²/(1-kr²) + r²dΩ²)
, k is a number, k = ± 1/L² or 0, where L is the scale
curvature of space.
Two key points:
① singular at r = 0
- only valid for r > 0
② t only makes sense for r
ie a hyperbola.
t = rs ln |(X+T)/(X-T)|


Work out LHS of Einstein eq^n's
$G_{00} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$
$\frac{\dot{a}}{a} \equiv$ expansio


out LHS of Einstein eqⁿs
G₀₀ = 3((ȧ/a)² + k/a²)
Gij = −(ȧ² + 2aä + k) γij
of Einstein
perfect fluid")
Tμν =
ȧ/a ≡ expansion rate ≡ Hubble parameter = H
where
(i.e. gμν =



where $\gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$.
(i.e. $g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & a^2\gamma_{ij} & \\ 0 & & & \end{pmatrix}$
$+ k)\gamma_{ij}$
local 4-velocity
$+ p g_{\mu\nu}$
e.o.m. are just $\nabla_\mu T^{\mu\nu} = 0$.
– enough to determine evolution of fluid, given eq^n of state $p(\rho)$.
makes sense for $r < r_s$ if we define it via $t = r_s \ln\left|\frac{X+T}{X-T}\right|$


where $\gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$.
i.e. $g_{\mu\nu} =$
$a^2\gamma_{ij}$
$p\, g_{\mu\nu}$
e.o.m. are just $\nabla_\mu T^{\mu\nu} = 0$.
– enough to determine evolution of fluid, given eq^n of state $p(\rho)$.
sense for $r < r_s$ if we define it via $t = r_s \ln\left|\frac{X+T}{X-T}\right|$

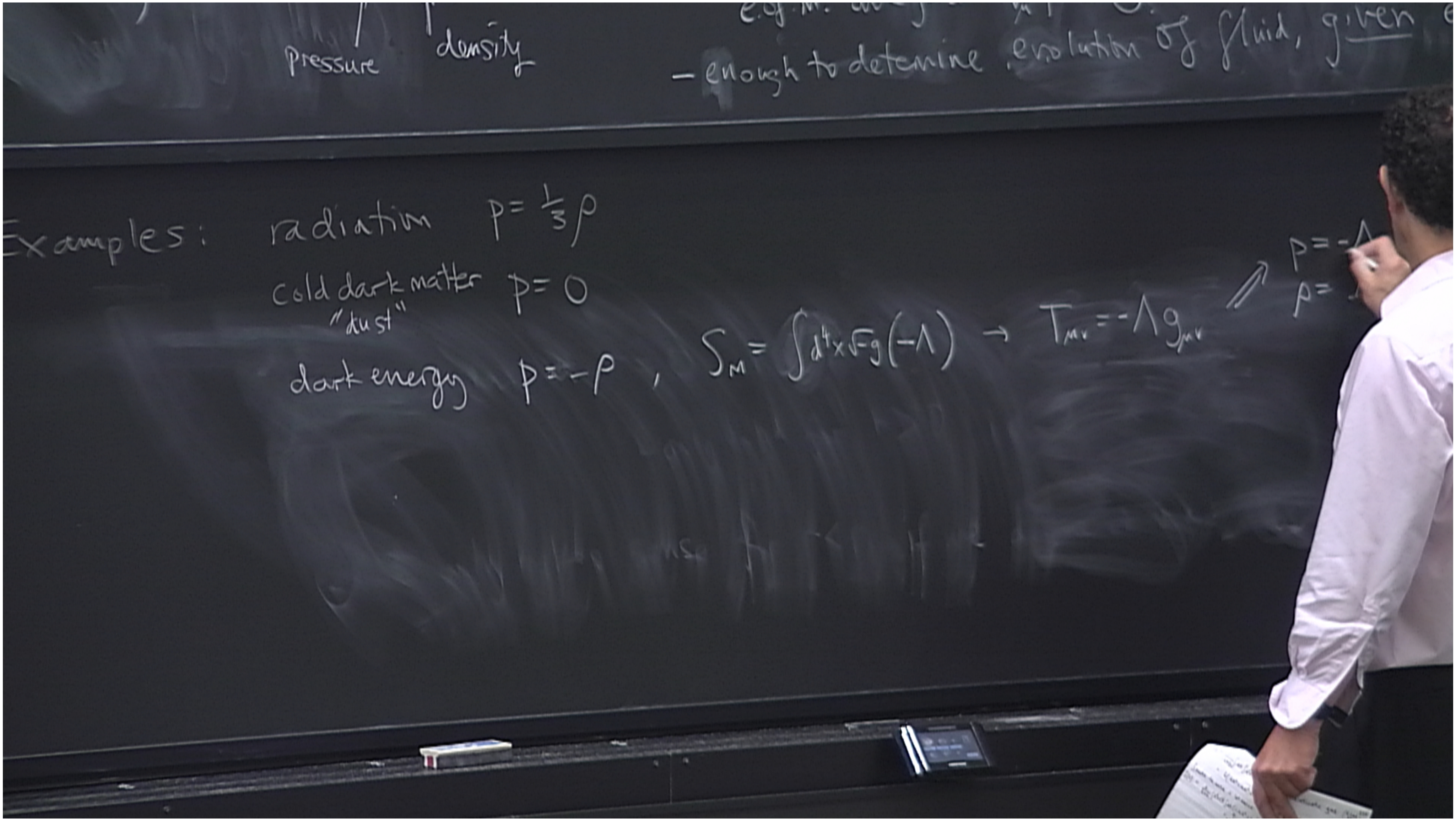
pressure density

– enough to determine e

Examples: radiation $p = \frac{1}{3}\rho$

cold dark matter

"dust"


pressure
density
– enough to detemine evolution of fluid, given
Examples: radiation $p = \frac{1}{3}\rho$
cold dark matter $p = 0$
"dust"
dark energy $p = -\rho$, $S_M = \int d^4x \sqrt{-g}\,(-\Lambda) \rightarrow T_{\mu\nu} = -\Lambda g_{\mu\nu}$

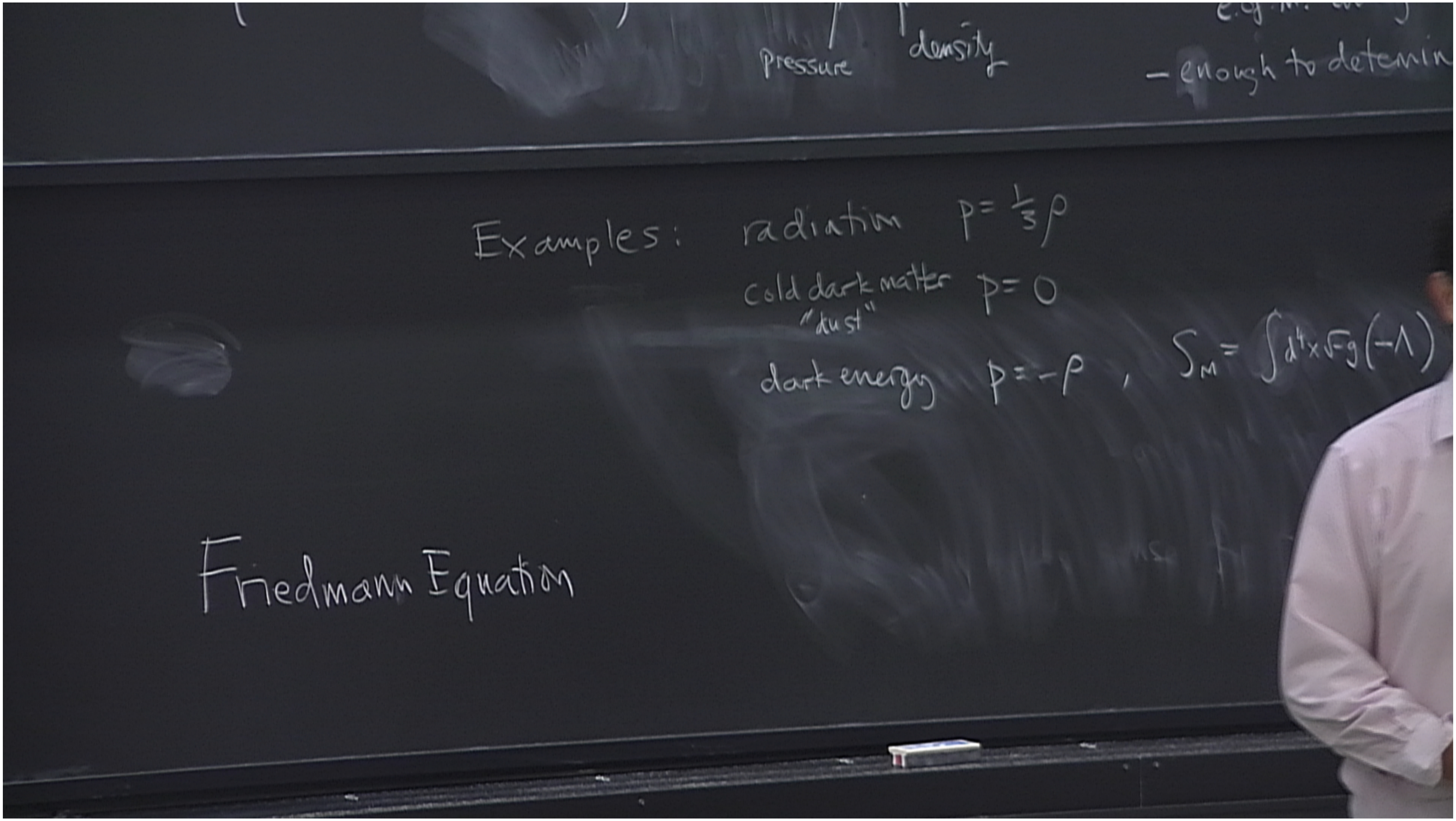
pressure density

— enough to determine evolution of fluid, given eq. of state.

s: radiation $p = \frac{1}{3}\rho$

cold dark matter "dust" $p = 0$

dark energy $p = -\rho$, $S_M = \int d^4x \sqrt{-g}\,(-\Lambda) \to T_{\mu\nu} = -\Lambda g_{\mu\nu} \Rightarrow$ $p = -\Lambda$, $\rho = \Lambda$


pressure
density
— enough to determin
Examples: radiation $p = \frac{1}{3}\rho$
cold dark matter $p = 0$
"dust"
dark energy $p = -\rho$, $S_M = \int d^4x \sqrt{-g}\,(-\Lambda)$
Friedmann Equation

pressure density

e.g. ...

– enough to determine evolution of fluid, giv...

Examples: radiation $p = \frac{1}{3}\rho$

cold dark matter "dust" $p = 0$

dark energy $p = -\rho$, $S_M = \int d^4x \sqrt{-g}\,(-\Lambda) \rightarrow T_{\mu\nu} = -\Lambda g_{\mu\nu} \Rightarrow p = -\Lambda$, $\rho_\Lambda = \Lambda$

Equation $G_{00} = 8\pi G T_{00}$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\Big(\rho_\Lambda + \underbrace{\frac{c_m}{a^3}}_{\rho_{matter}} + \underbrace{\frac{c_r}{a^4}}_{\rho_{radiation}}\Big) - \underbrace{\frac{k}{a^2}}_{\text{space curvature term}}$$

...nsity

– enough to determine evolution of fluid, given eq. of state

$p = \frac{1}{3}\rho$

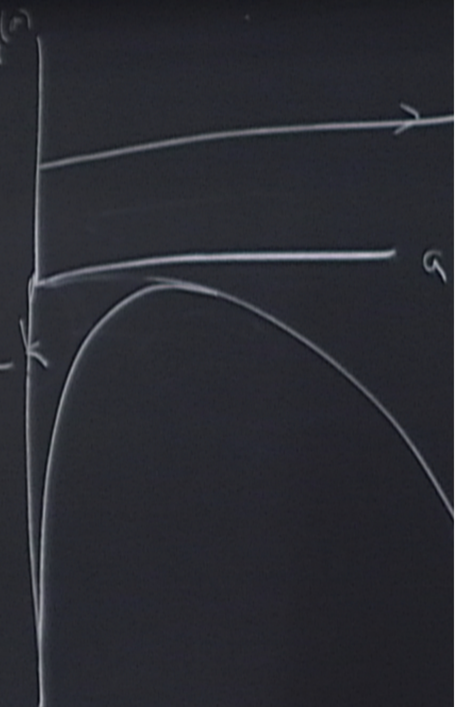
$p = 0$

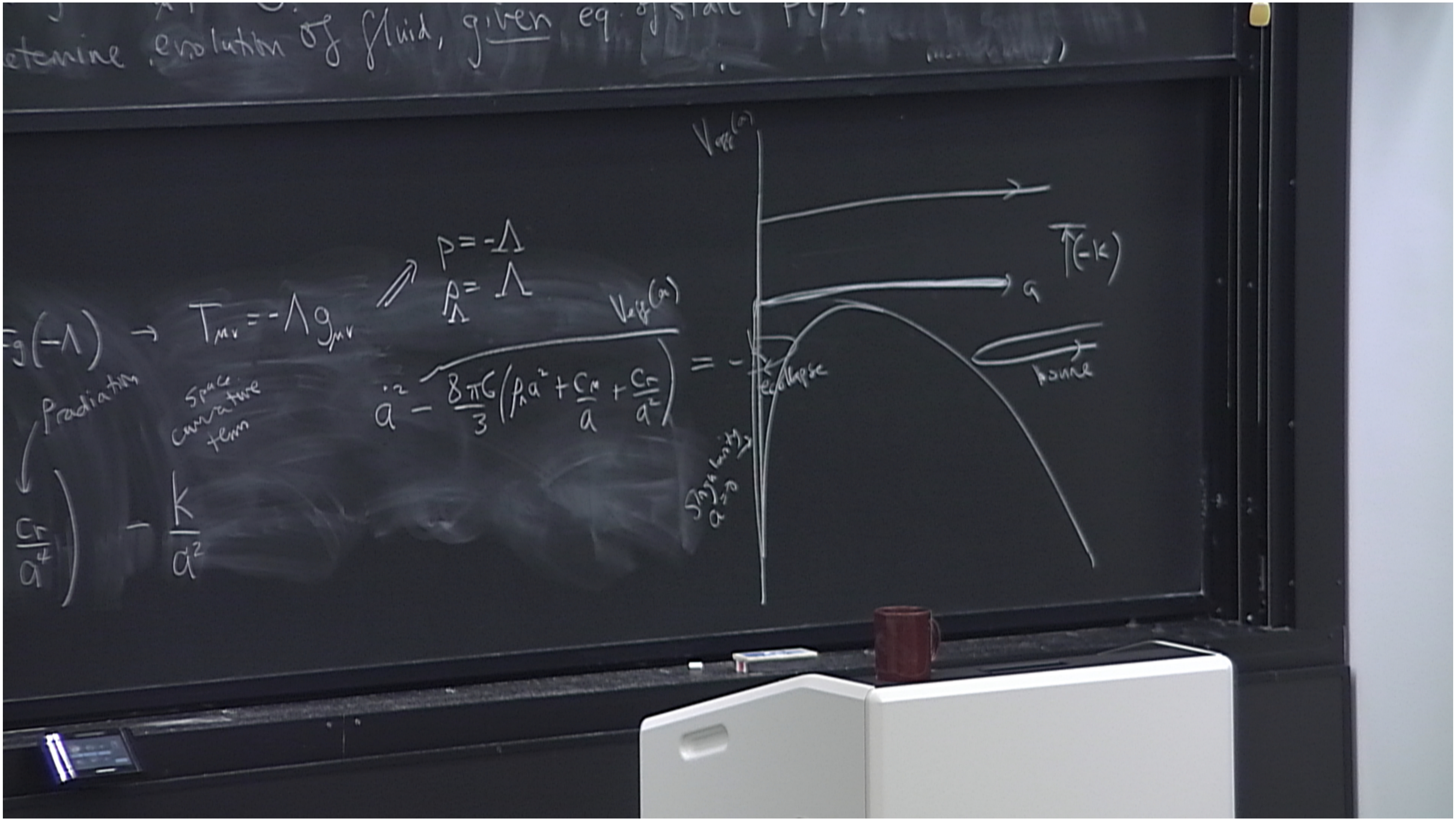
$p = -\rho$, $S_M = \int d^4x \sqrt{-g}\,(-\Lambda) \rightarrow T_{\mu\nu} = -\Lambda g_{\mu\nu} \nearrow \begin{array}{l} p = -\Lambda \\ \rho_\Lambda = \Lambda \end{array}$

ρ_{matter} $\rho_{radiation}$ space curvature term

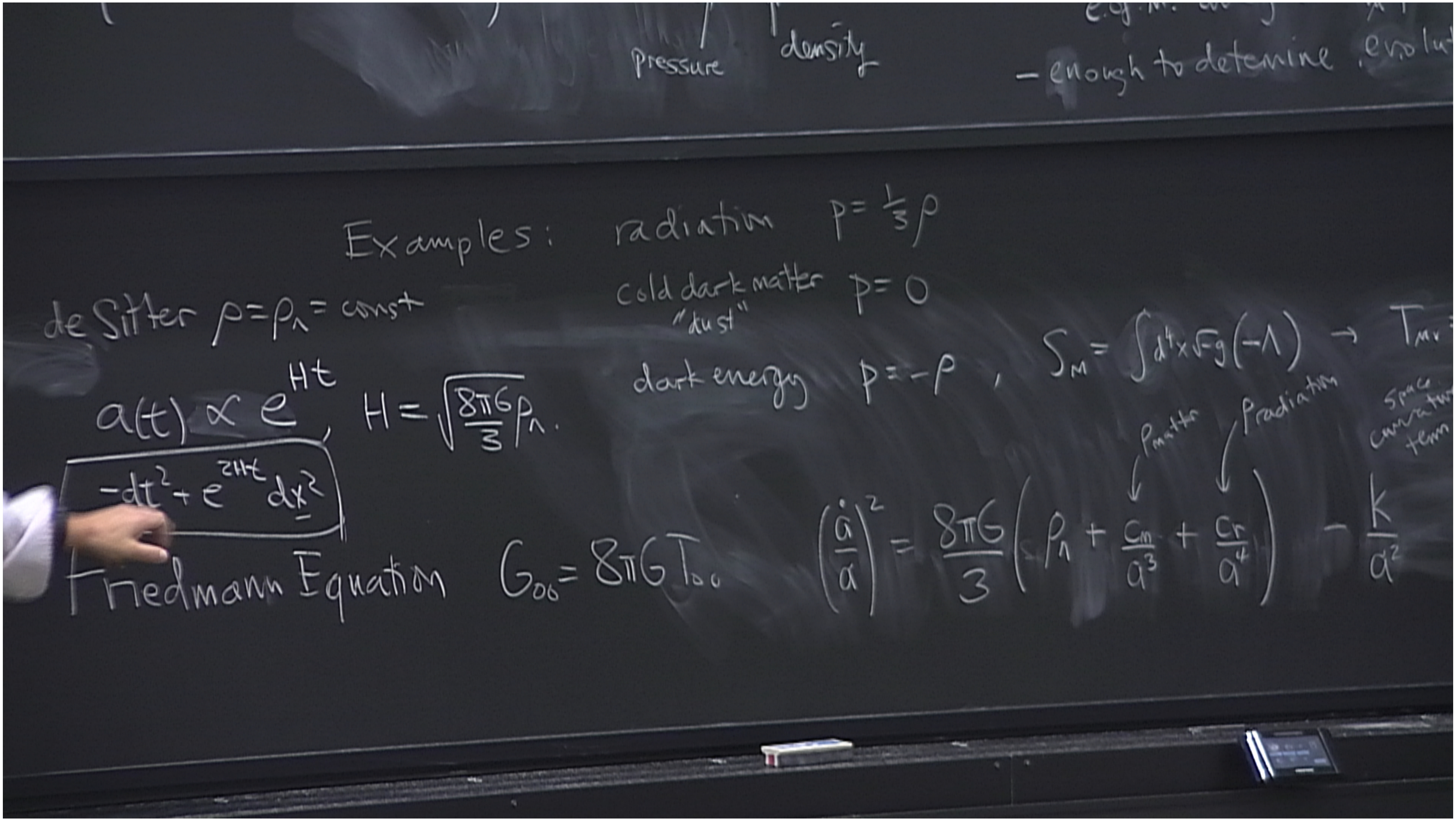
$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_\Lambda + \frac{c_m}{a^3} + \frac{c_r}{a^4}\right) - \frac{k}{a^2}$$

$$\dot a^2 \overbrace{- \frac{8\pi G}{3}\left(\rho_\Lambda a^2 + \frac{c_m}{a} + \frac{c_r}{a^2}\right)}^{V_{eff}(a)} = -k$$





determine evolution of fluid, given eq. of state
p = -Λ
ρ = Λ
T_{μν} = -Λ g_{μν}
radiation
space curvature term
V_eff(a)
collapse
bounce
singularity a=0


pressure
density
- enough to determine evolution
Examples: radiation $p = \frac{1}{3}\rho$
cold dark matter "dust" $p = 0$
dark energy $p = -\rho$
de Sitter $\rho = \rho_\Lambda$ = const
$a(t) \propto e^{Ht}$, $H = \sqrt{\frac{8\pi G}{3}\rho_\Lambda}$.
$-dt^2 + e^{2Ht}\,d\underline{x}^2$
$S_M = \int d^4x \sqrt{-g}\,(-\Lambda) \rightarrow T_{\mu\nu}$
Friedmann Equation $G_{00} = 8\pi G T_{00}$
ρ_{matter}
$\rho_{radiation}$
space curvature term
$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_\Lambda + \frac{c_m}{a^3} + \frac{c_r}{a^4}\right) - \frac{k}{a^2}$