Title: 14/15 PSI Quantum Theory-4

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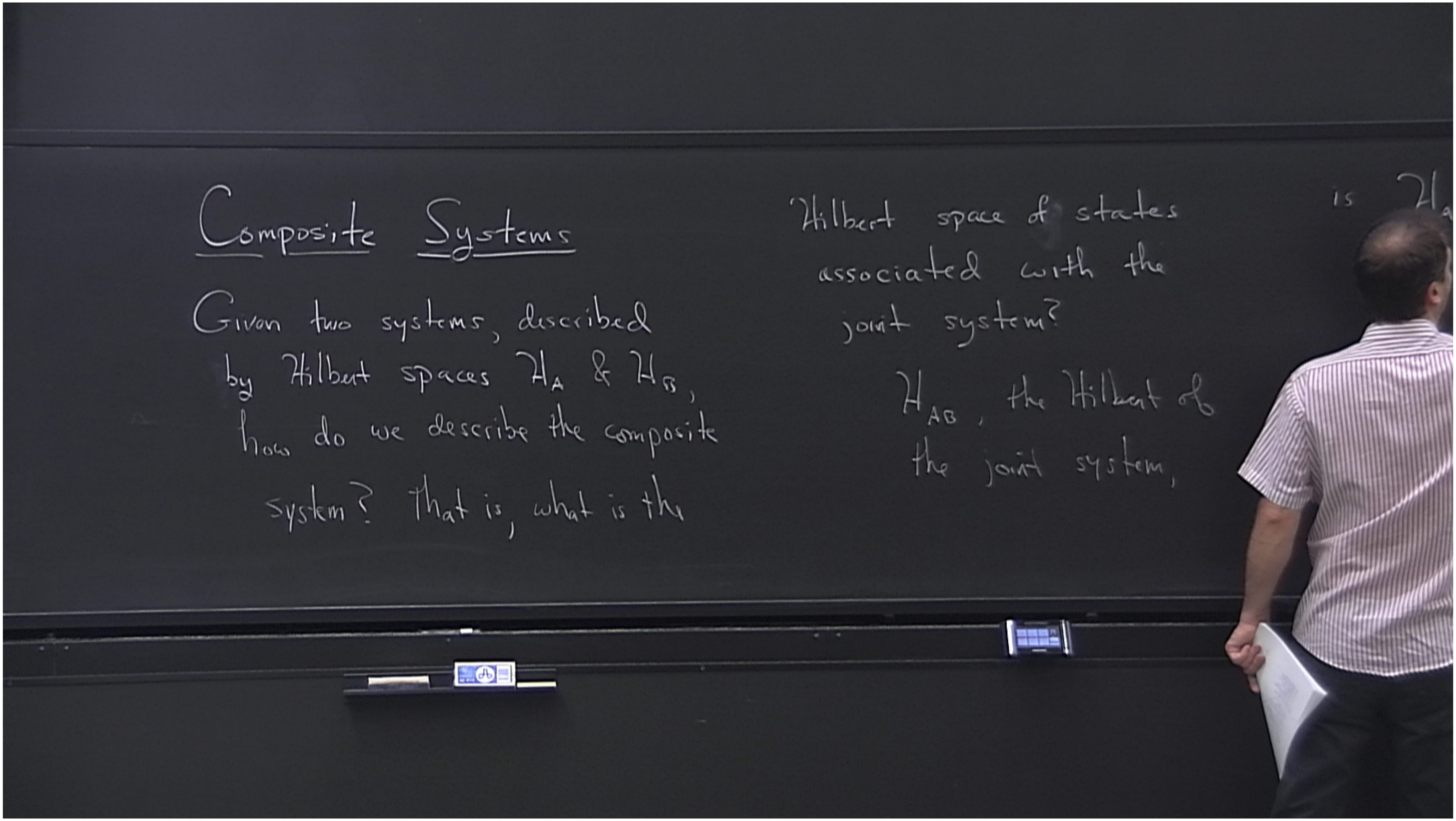
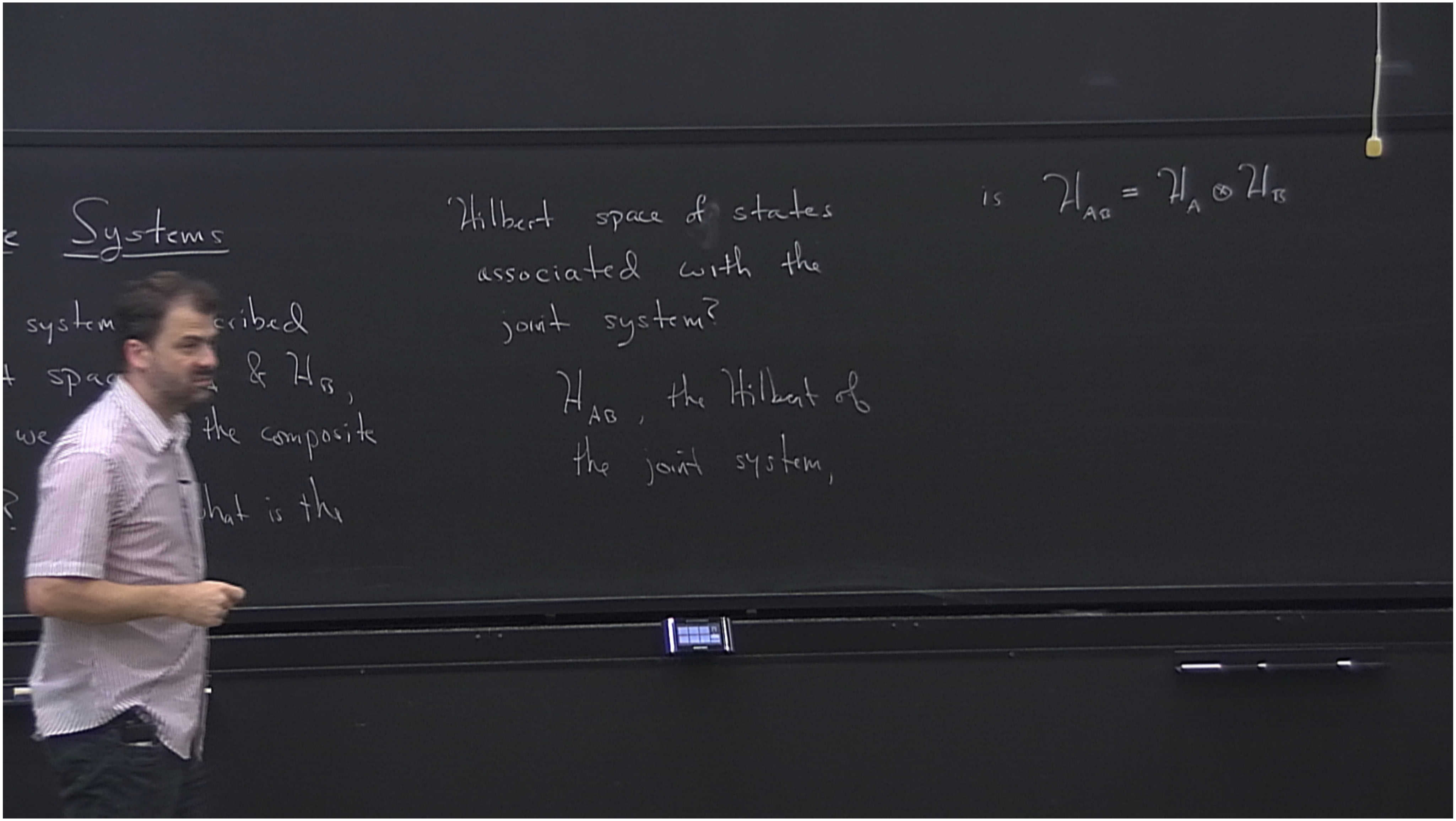
Abstract:


Composite Systems
Given two systems, described
by Hilbert spaces $\mathcal{H}_A$ & $\mathcal{H}_B$,
how do we describe the composite
system? That is, what is the
Hilbert space of states
associated with the
joint system?
$\mathcal{H}_{AB}$, the Hilbert of
the joint system,
is


Systems
system cribed
& $\mathcal{H}_B$,
the composite
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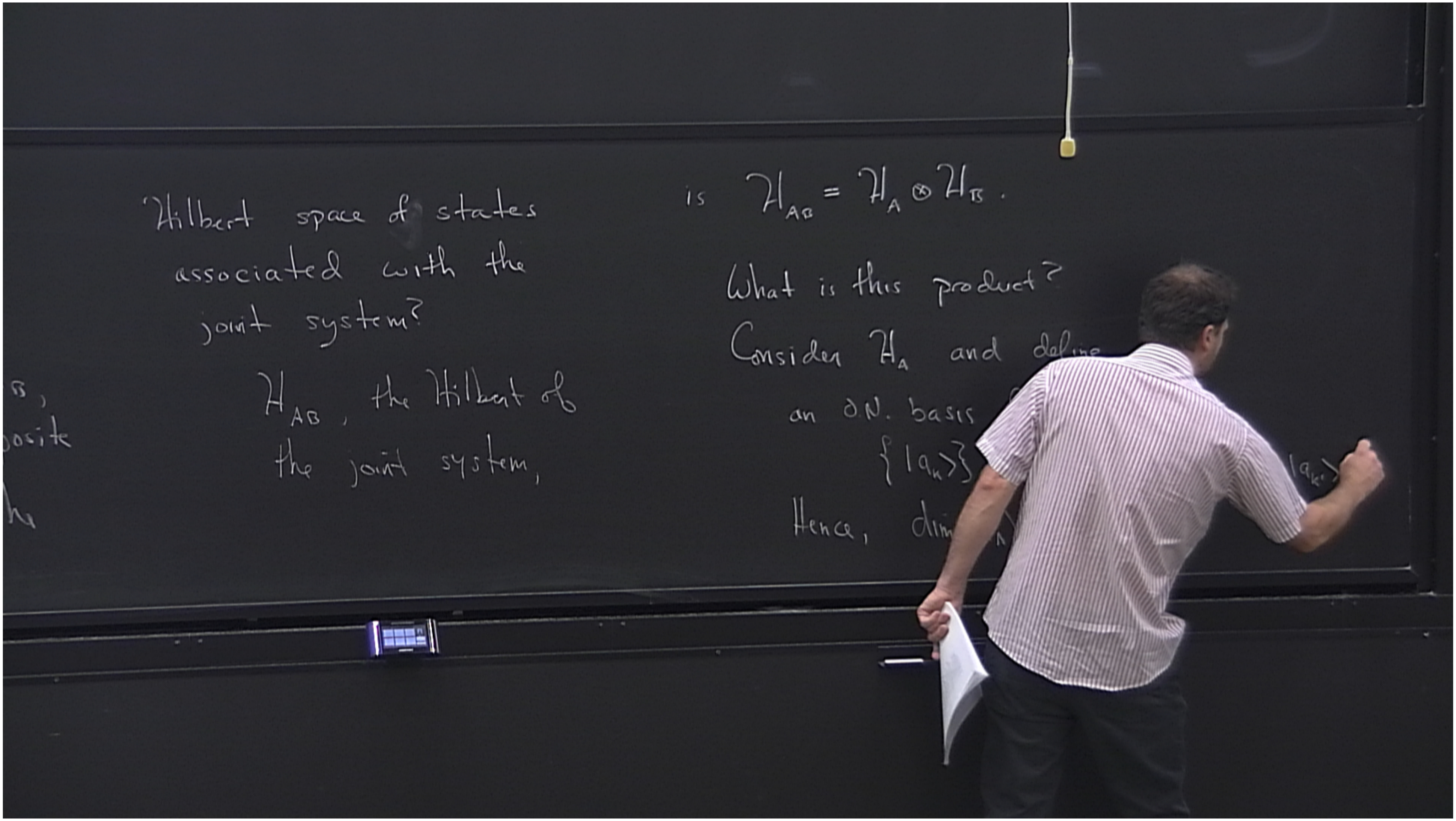
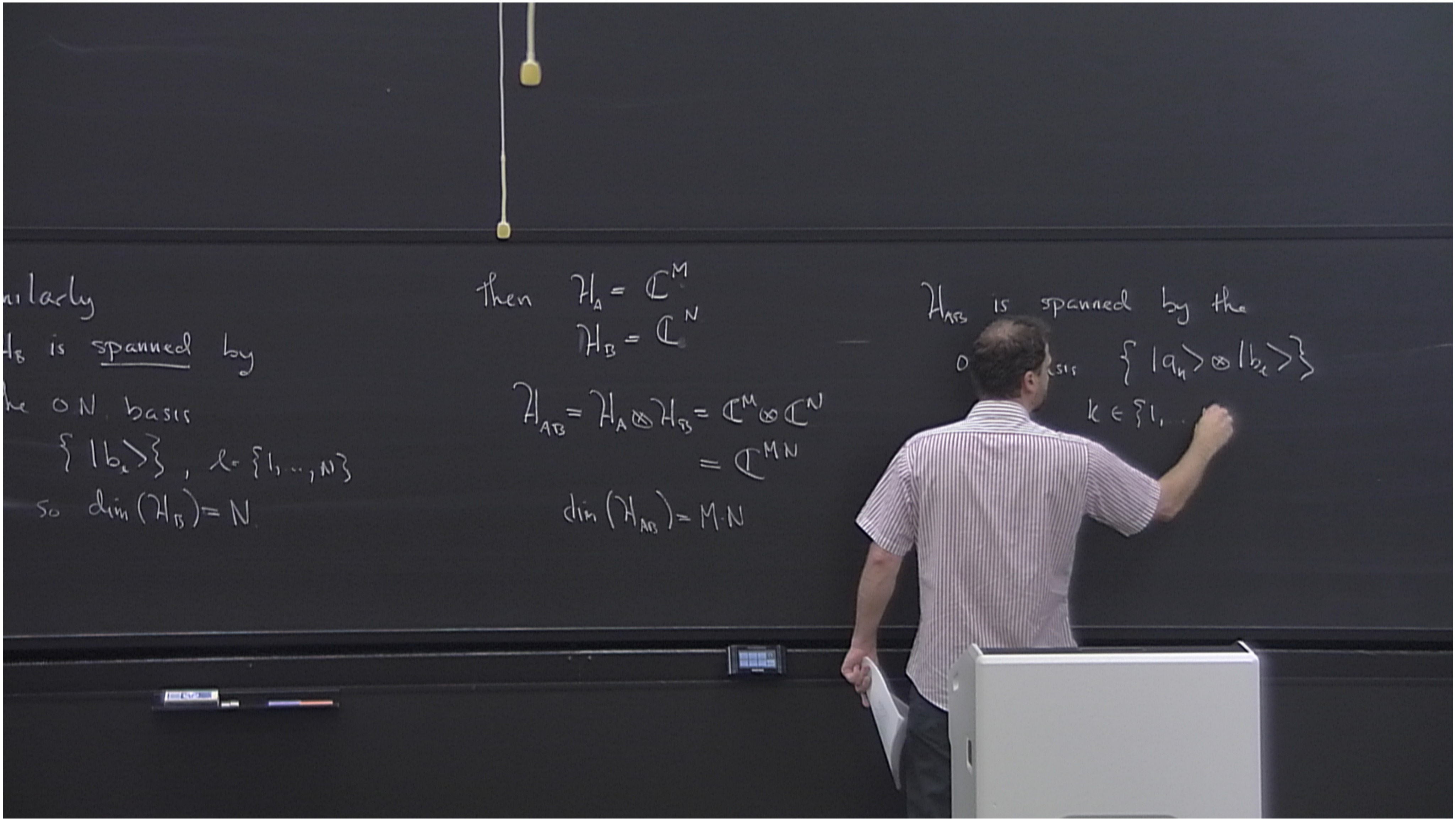
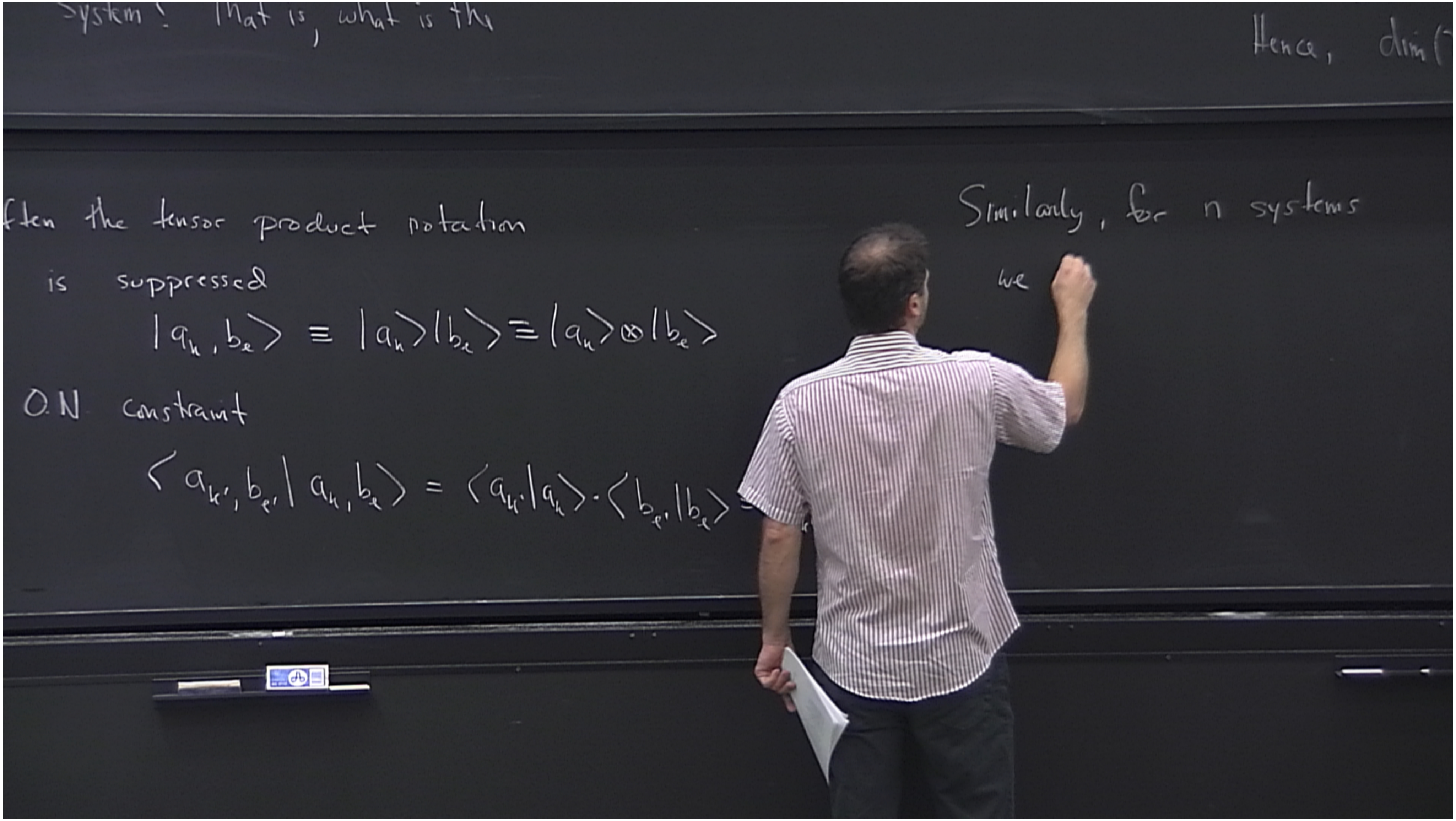
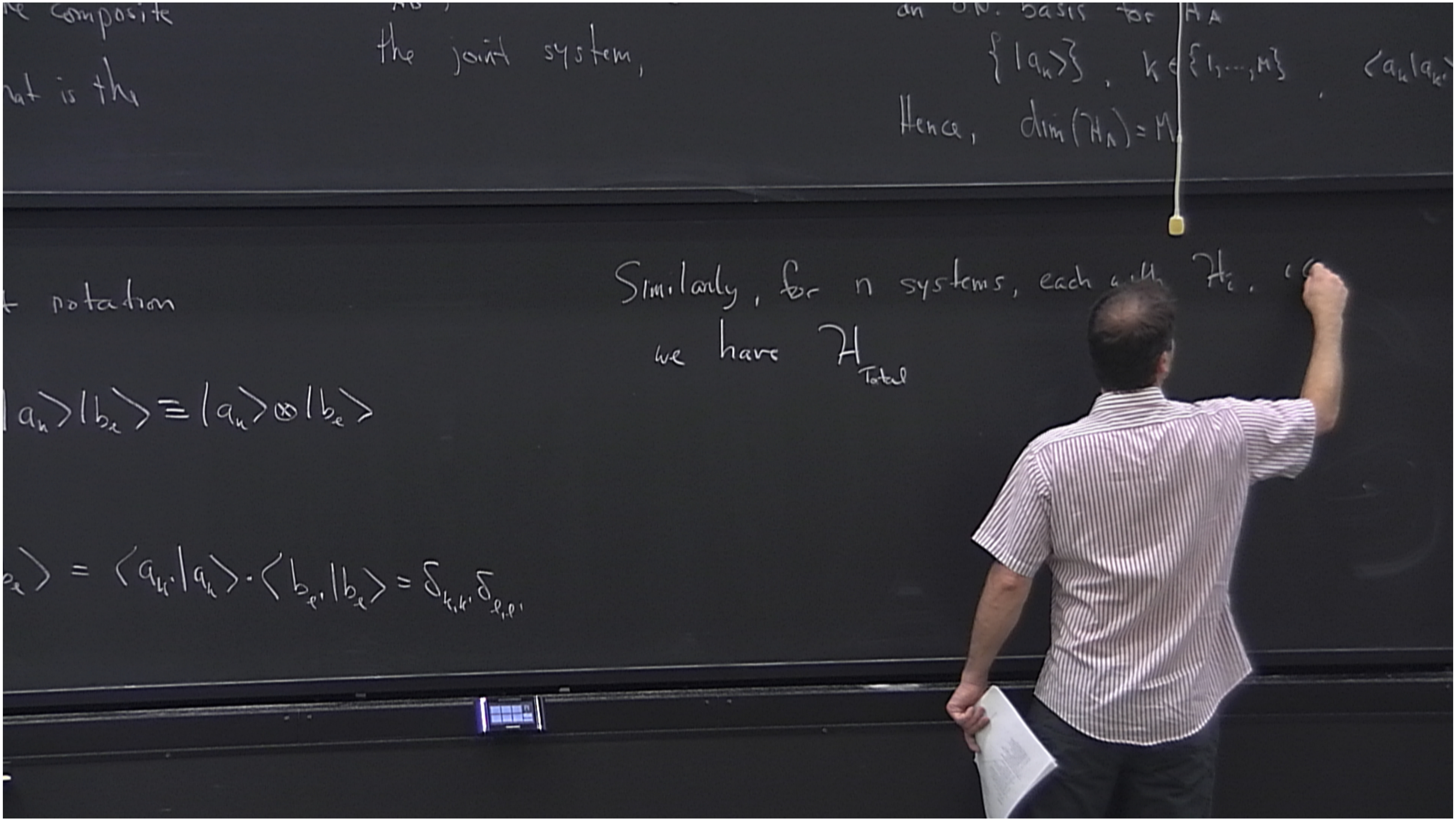
joint system?
$\mathcal{H}_{AB}$, the Hilbert of
the joint system,
is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$


Hilbert space of states
associated with the
joint system?
$\mathcal{H}_{AB}$, the Hilbert of
the joint system,
is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.
What is this product?
Consider $\mathcal{H}_A$ and define
an O.N. basis
$\{|a_n\rangle\}$
Hence,



milarly
$\mathcal{H}_B$ is spanned by
the ON basis
$\{|b_\ell\rangle\}$, $\ell \in \{1, \ldots, N\}$
so $\dim(\mathcal{H}_B) = N$
Then $\mathcal{H}_A = \mathbb{C}^M$
$\mathcal{H}_B = \mathbb{C}^N$
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^M \otimes \mathbb{C}^N$
$= \mathbb{C}^{M \cdot N}$
$\dim(\mathcal{H}_{AB}) = M \cdot N$
$\mathcal{H}_{AB}$ is spanned by the
$\{|a_k\rangle \otimes |b_\ell\rangle\}$
$k \in \{1, \ldots$


system? That is, what is the
Hence, dim(
Often the tensor product notation
is suppressed
$|a_k, b_\ell\rangle \equiv |a_k\rangle|b_\ell\rangle \equiv |a_k\rangle \otimes |b_\ell\rangle$
O.N constraint
$\langle a_{k'}, b_{\ell'} | a_k, b_\ell\rangle = \langle a_{k'}|a_k\rangle \cdot \langle b_{\ell'}|b_\ell\rangle$
Similarly, for n systems
we


composite
the joint system,
an O.N. basis for H A
{|a_k>}, k ∈ {1,...,M}
at is the
Hence, dim(H_A) = M
notation
Similarly, for n systems, each with H_i,
we have H_Total
|a_k>|b_l> ≡ |a_k>⊗|b_l>
= <a_k'|a_k>·<b_l'|b_l> = δ_k,k' δ_l,l'

...e composite ... the joint system, ... an O.N. basis for $\mathcal{H}_A$

...at is the ... $\{|a_k\rangle\}$, $k \in \{1,\dots,M\}$, $\langle a_k | a_{k'} \rangle$ ...

Hence, $\dim(\mathcal{H}_A) = M$

...t notation

$|a_k\rangle|b_\ell\rangle \equiv |a_k\rangle \otimes |b_\ell\rangle$

$\dots b_\ell\rangle = \langle a_{k'}|a_k\rangle \cdot \langle b_{\ell'}|b_\ell\rangle = \delta_{k,k'}\delta_{\ell,\ell'}$

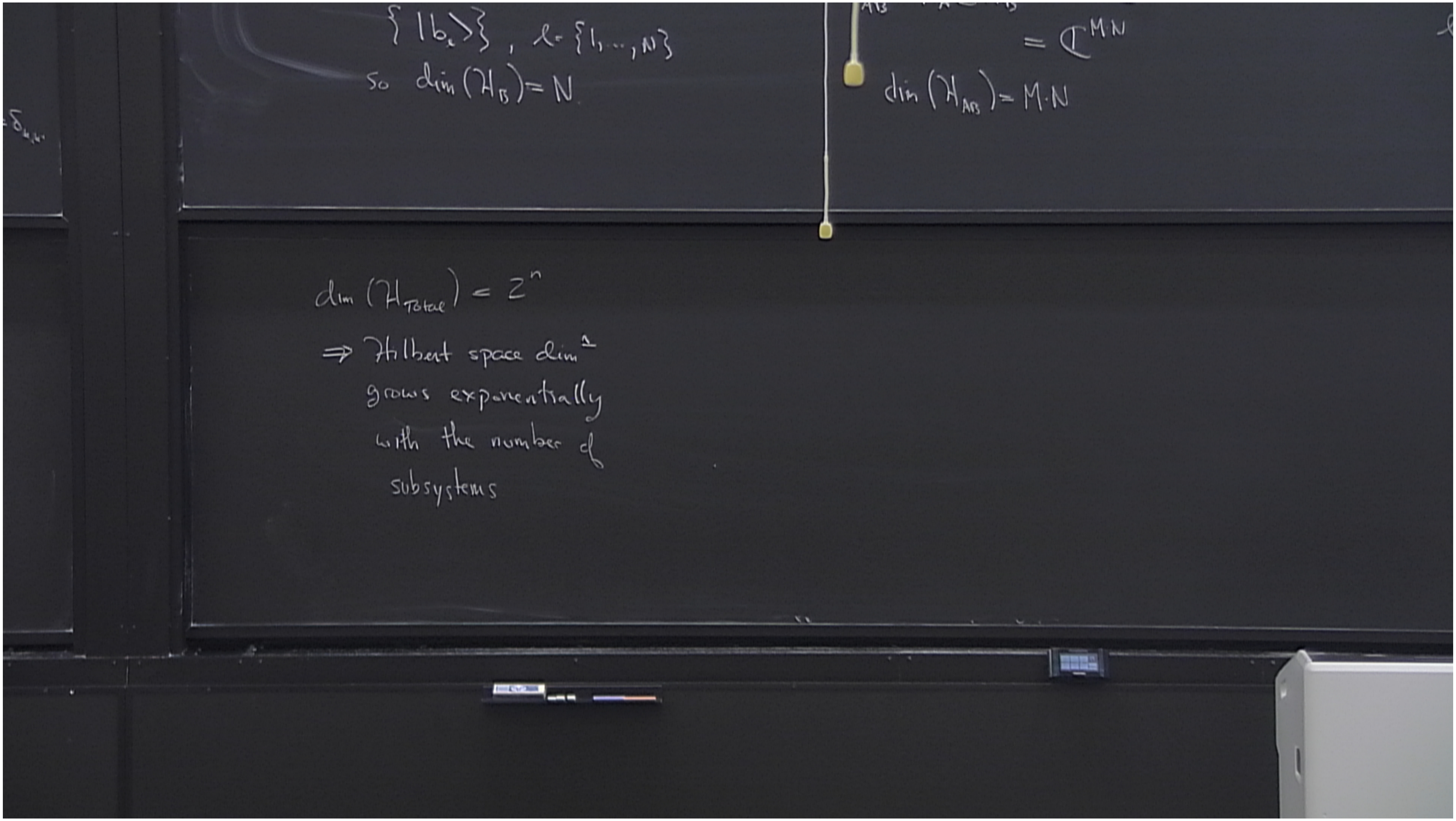
Similarly, for n systems, each with $\mathcal{H}_i$, $i \in \{1,\dots,n\}$

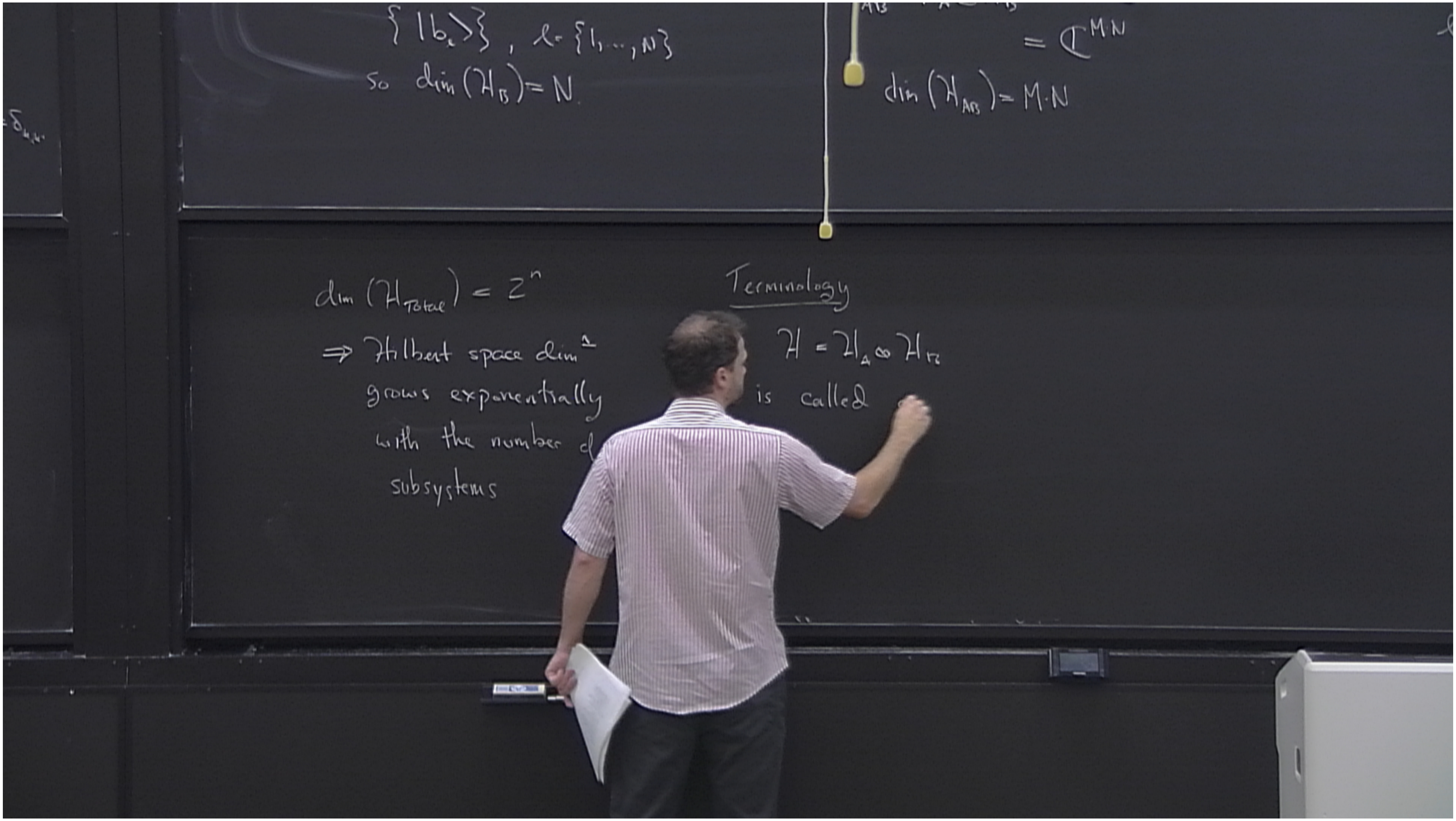
we have $\mathcal{H}_{Total} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$

e.g. a system composed of n spin-1/2 particles,

then $\mathcal{H}_i = \mathbb{C}^2$

$$\mathcal{H}_{Total} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{\text{n-fold product}} \equiv \mathbb{C}^{2^n}$$


{|b_l⟩}, l = {1, ..., N}
so dim(H_B) = N
= C^{M·N}
dim(H_AB) = M·N
dim(H_Total) = 2^n
⇒ Hilbert space dim^n
grows exponentially
with the number of
subsystems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⇒ Hilbert space dim^n
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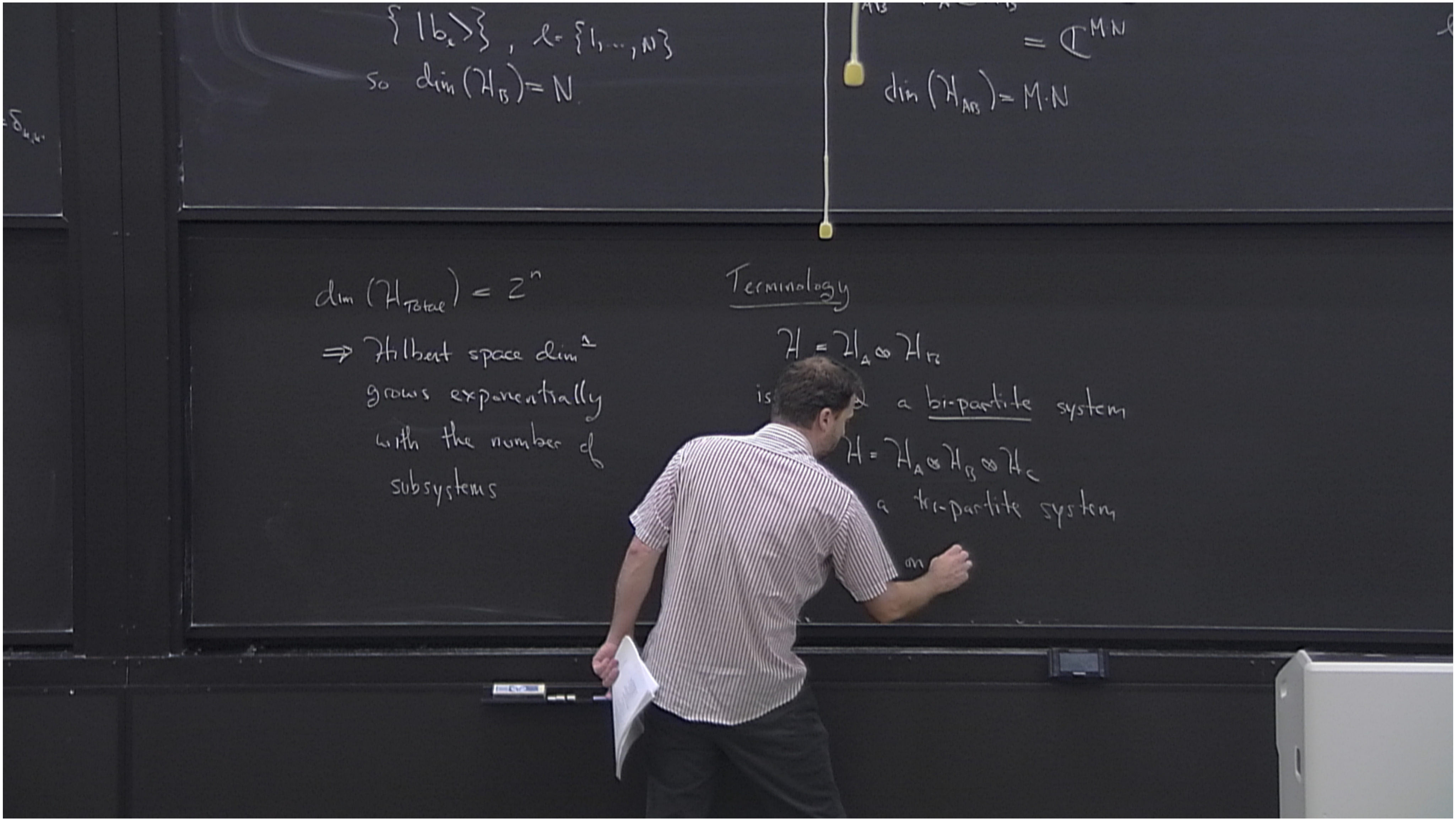
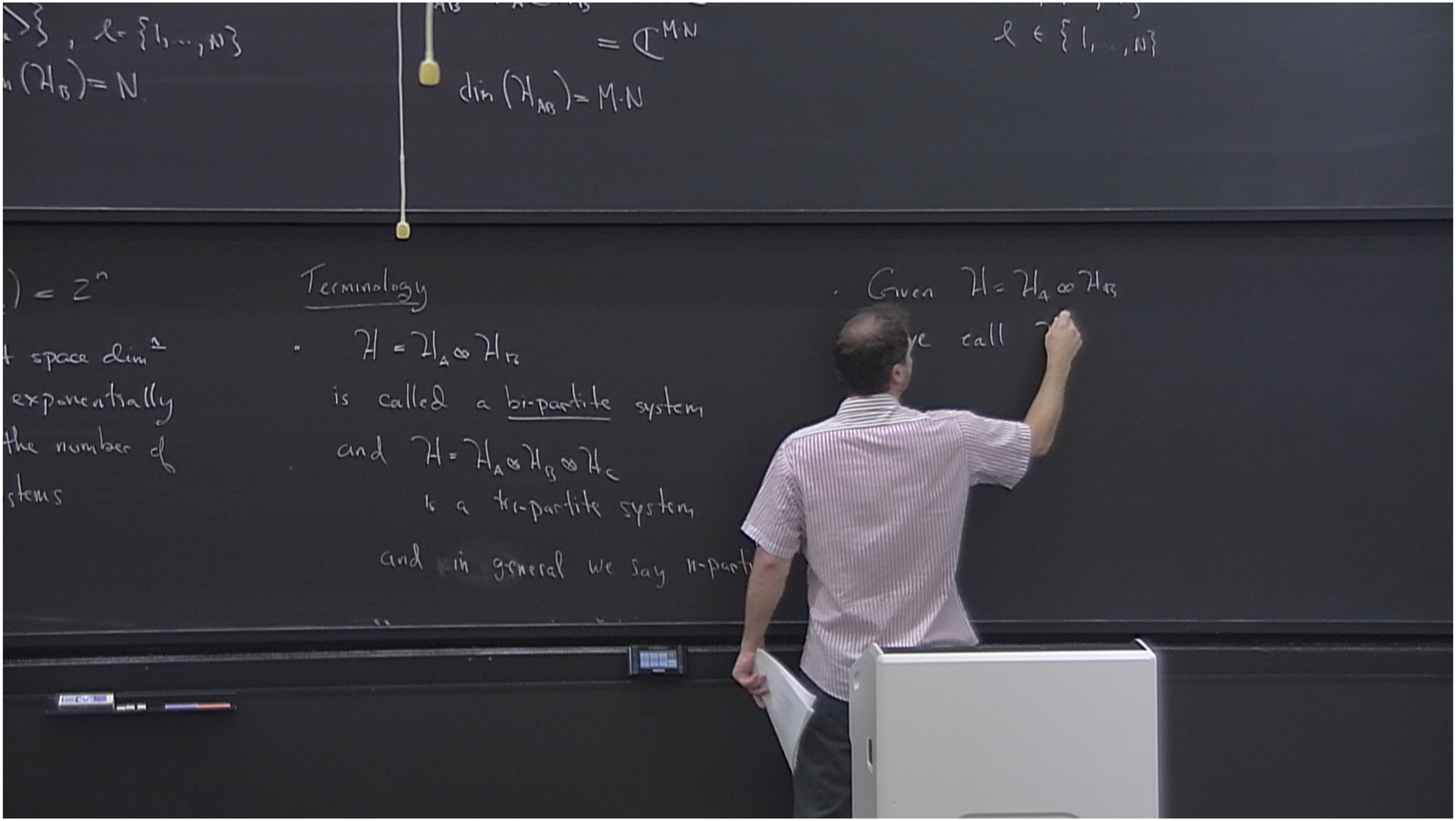
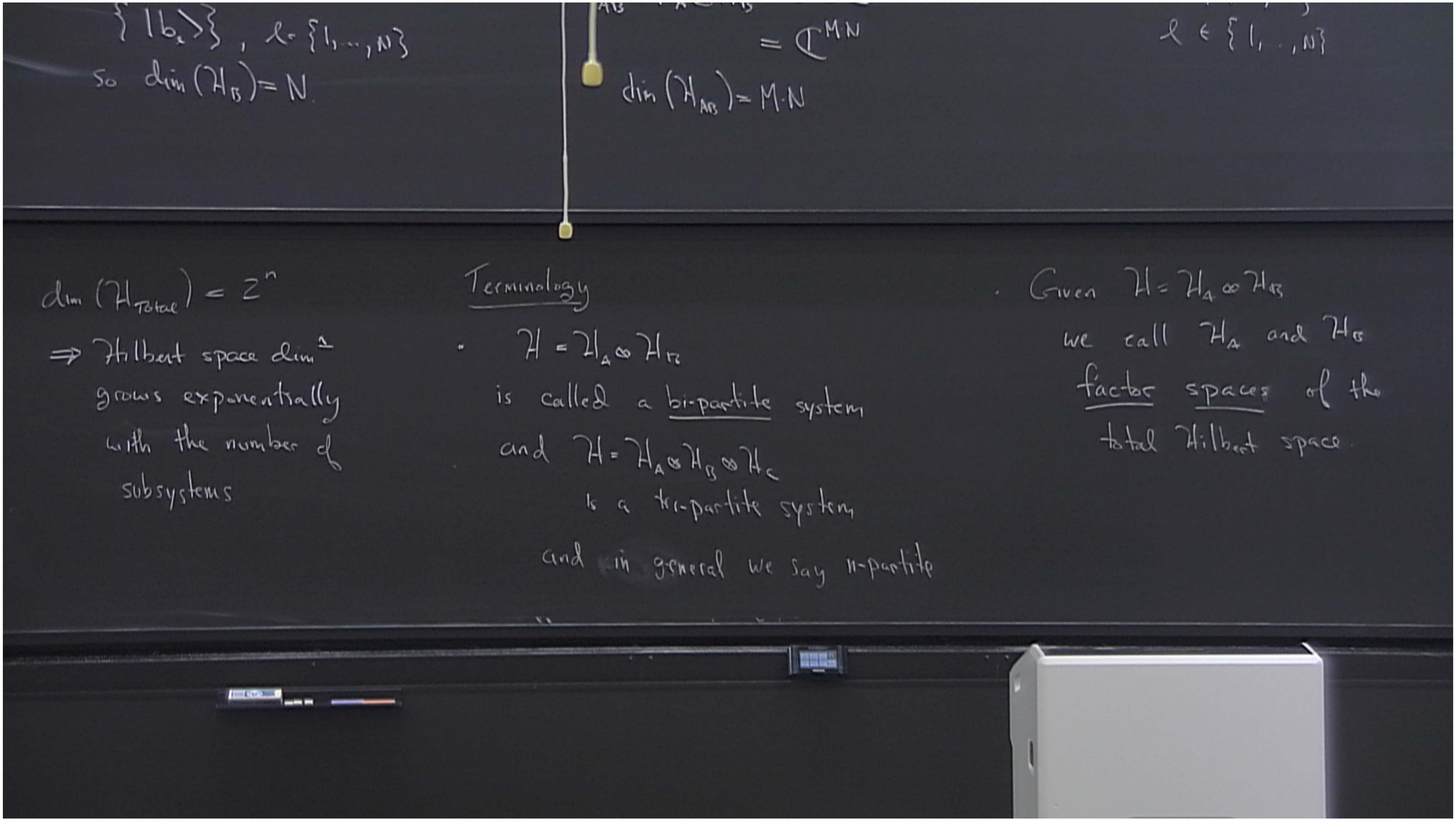
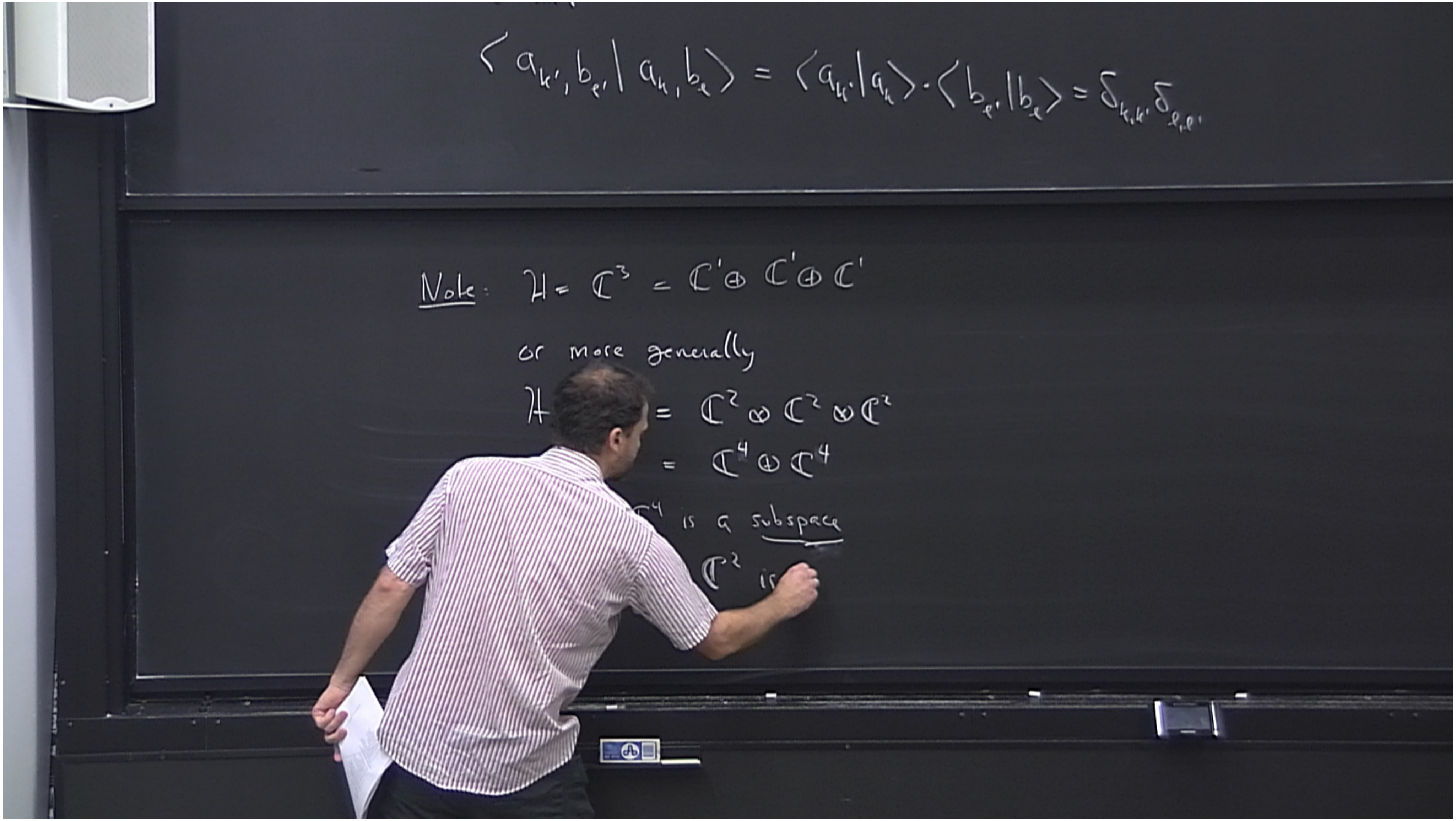
subsystems
Terminology
H = H_A ⊗ H_B
is called


{ |b_l⟩ } , l = {1, ..., N}
so dim(H_B) = N
= C^{MN}
dim(H_AB) = M·N
dim(H_Total) = 2^n
⇒ Hilbert space dim^n
grows exponentially
with the number of
subsystems
Terminology
H = H_A ⊗ H_B
a bi-partite system
H = H_A ⊗ H_B ⊗ H_C
a tri-partite system


dim(H_AB) = M·N
Terminology
H = H_A ⊗ H_B
is called a bi-partite system
and H = H_A ⊗ H_B ⊗ H_C
is a tri-partite system
and in general we say
Given H = H_A ⊗ H_B
call
space dim
exponentially
the number of



$\{|b_\ell\rangle\}$, $\ell = \{1, .., N\}$
so $\dim(\mathcal{H}_B) = N$
$= \mathbb{C}^{MN}$
$\dim(\mathcal{H}_{AB}) = M \cdot N$
$\ell \in \{1, .., N\}$
$\dim(\mathcal{H}_{Total}) = 2^n$
$\Rightarrow$ Hilbert space dim^{n} grows exponentially with the number of subsystems
Terminology
• $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called a bi-partite system
and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is a tri-partite system
and in general we say n-partite
• Given $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ we call $\mathcal{H}_A$ and $\mathcal{H}_B$ factor spaces of the total Hilbert space.


⟨a_k', b_l' | a_k, b_l⟩ = ⟨a_k' | a_k⟩ · ⟨b_l' | b_l⟩ = δ_k,k' δ_l,l'
Note: H = C^3 = C^1 ⊕ C^1 ⊕ C^1
or more generally
H = C^2 ⊗ C^2 ⊗ C^2
= C^4 ⊕ C^4
is a subspace
C^2 is

$$\langle a_{k'}, b_{\ell'} | a_k, b_\ell \rangle = \langle a_{k'} | a_k \rangle \cdot \langle b_{\ell'} | b_\ell \rangle = \delta_{k,k'} \delta_{\ell,\ell'}$$

Note: $\mathcal{H} = \mathbb{C}^3 = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$

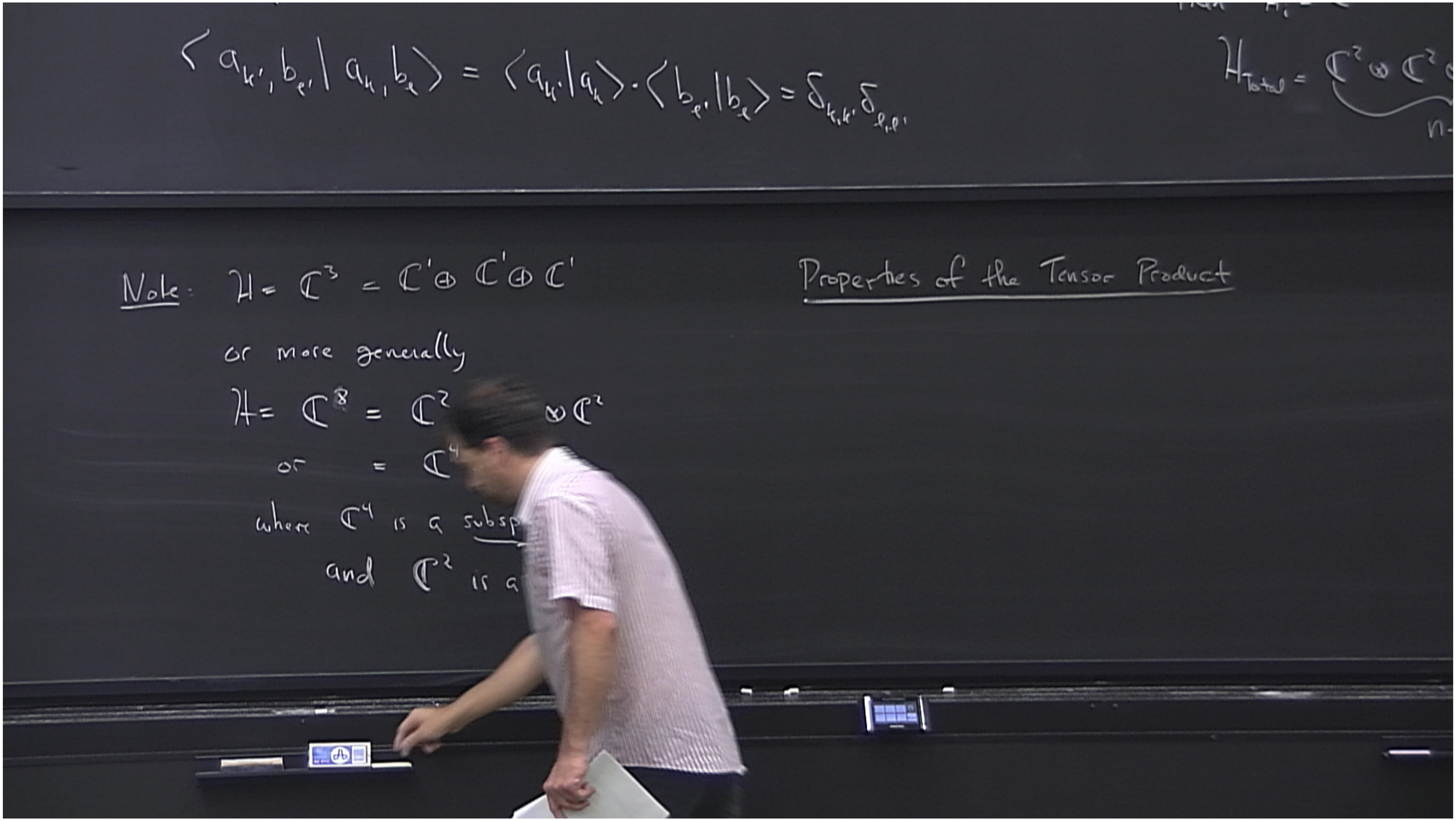
or more generally

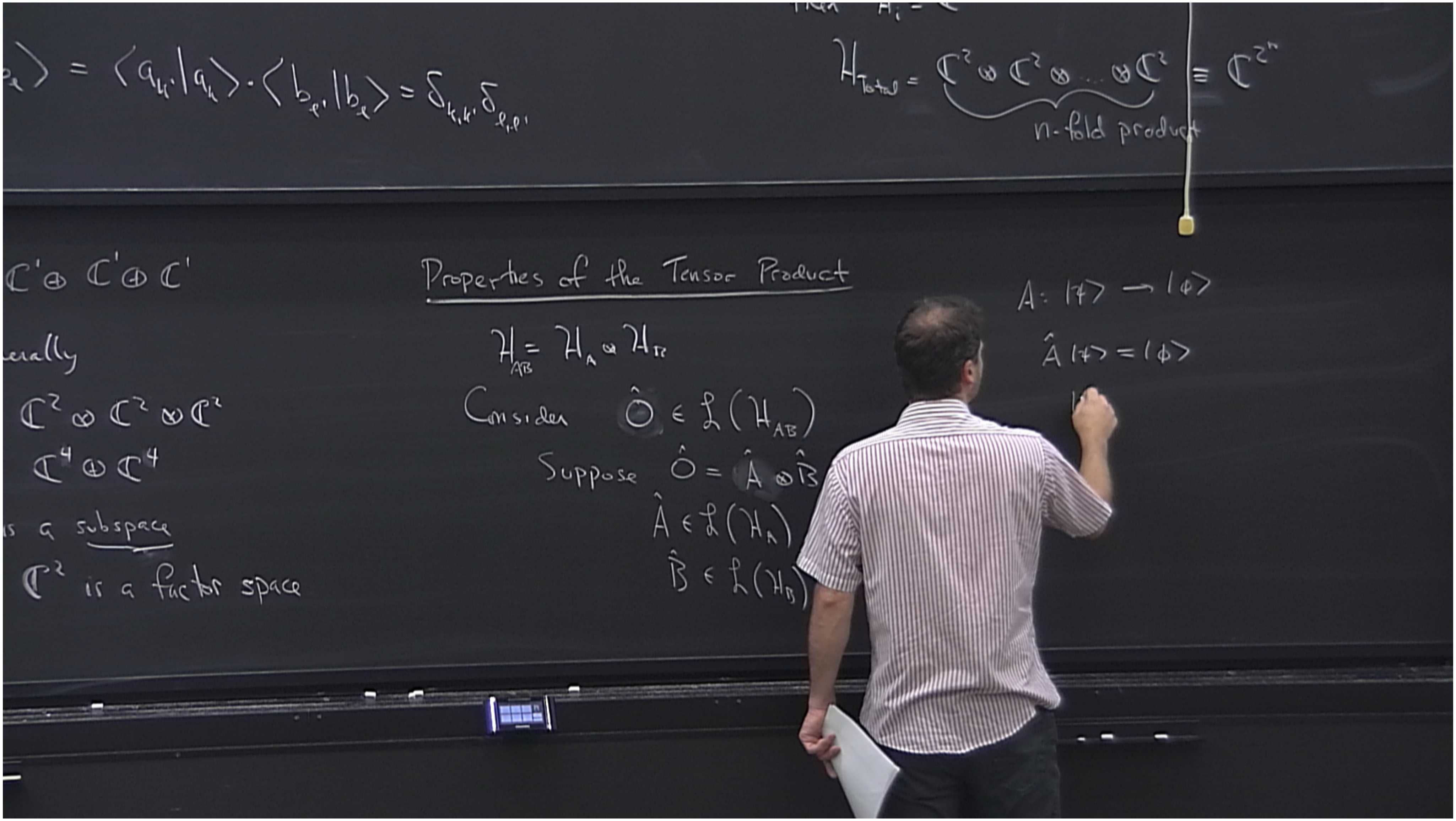
$$\mathcal{H} = \mathbb{C}^8 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

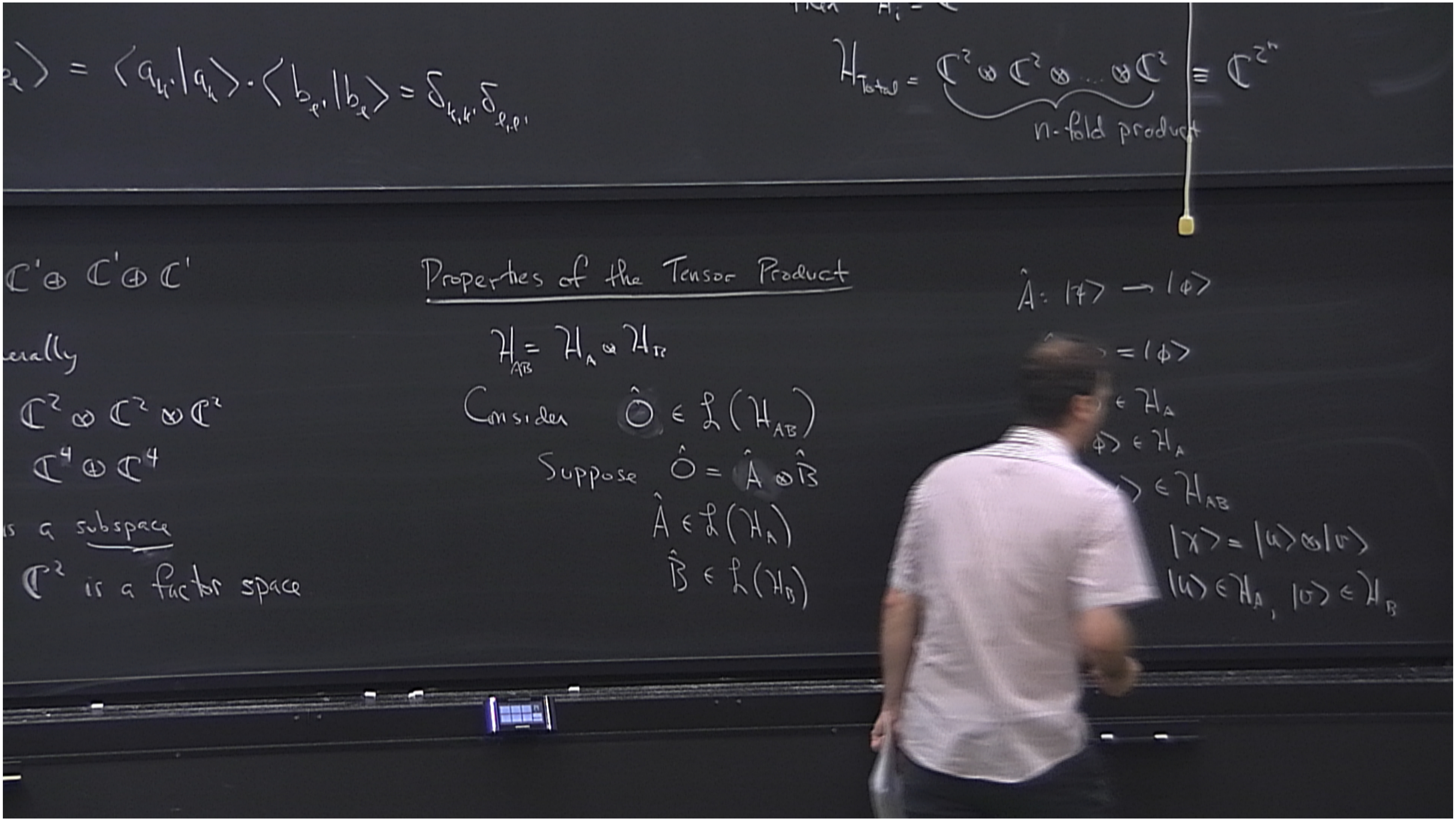
$$\text{or} \quad = \mathbb{C}^4 \oplus \mathbb{C}^4$$

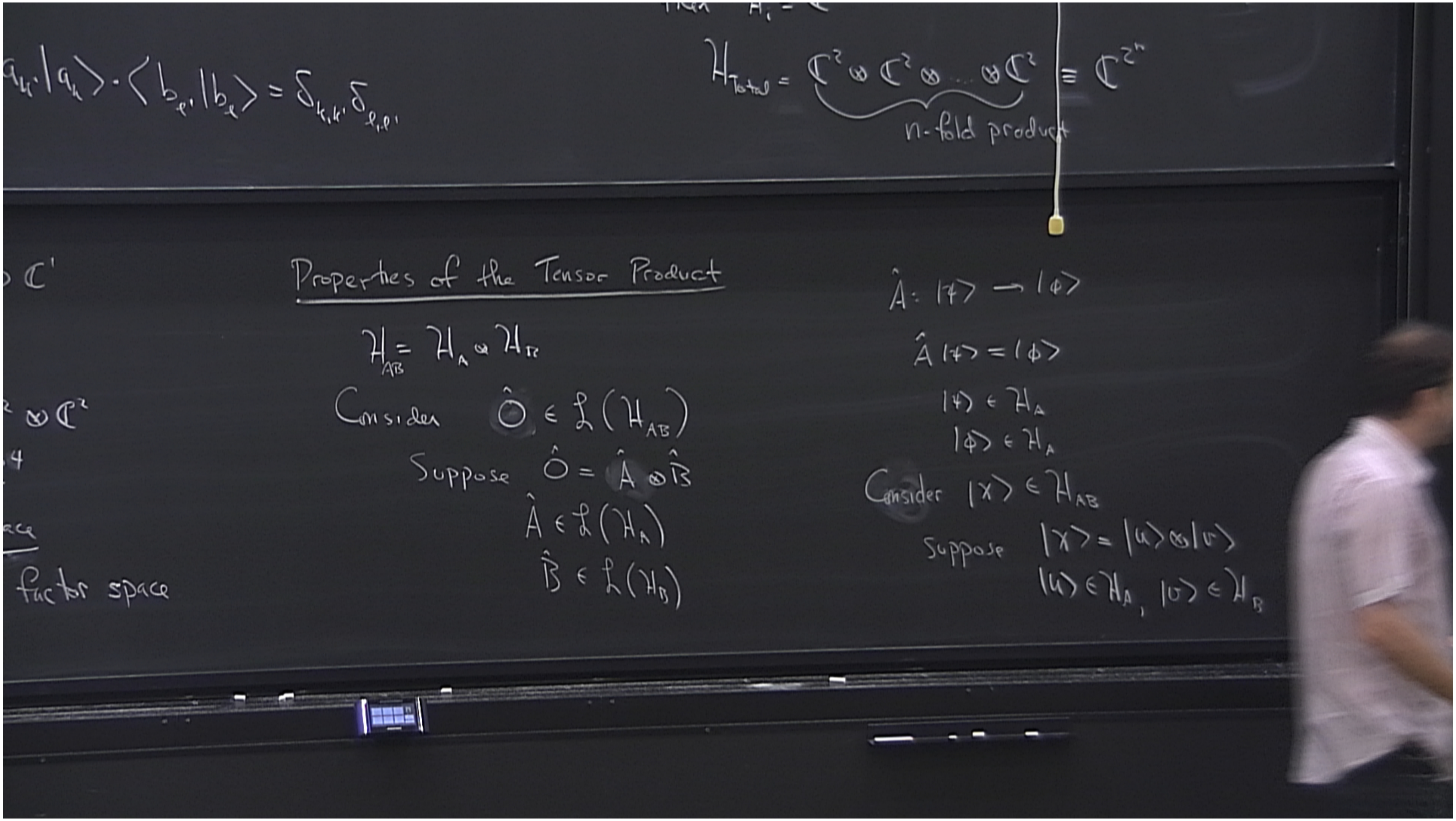
where $\mathbb{C}^4$ is a subspace

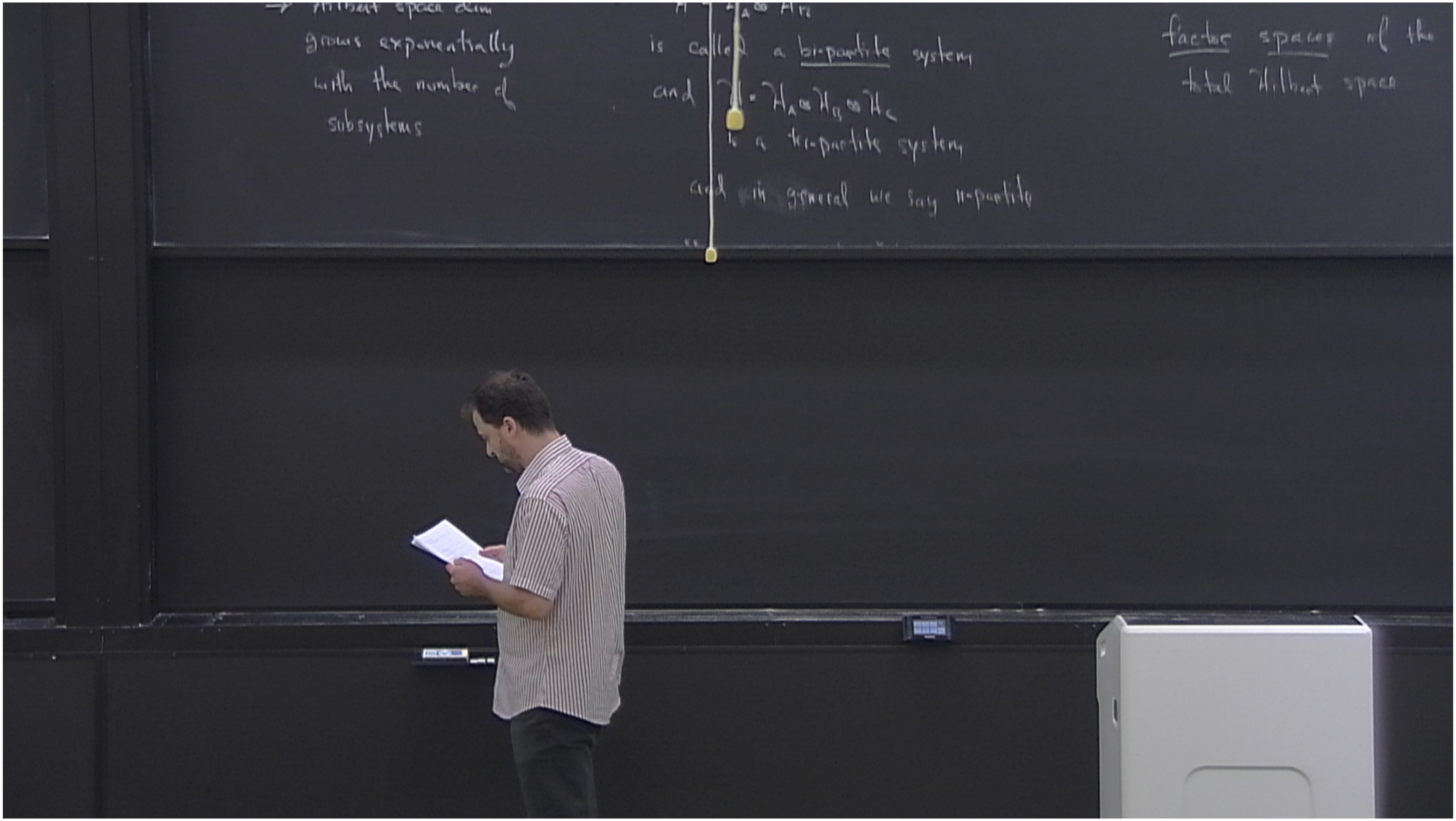
and $\mathbb{C}^2$ is a factor space

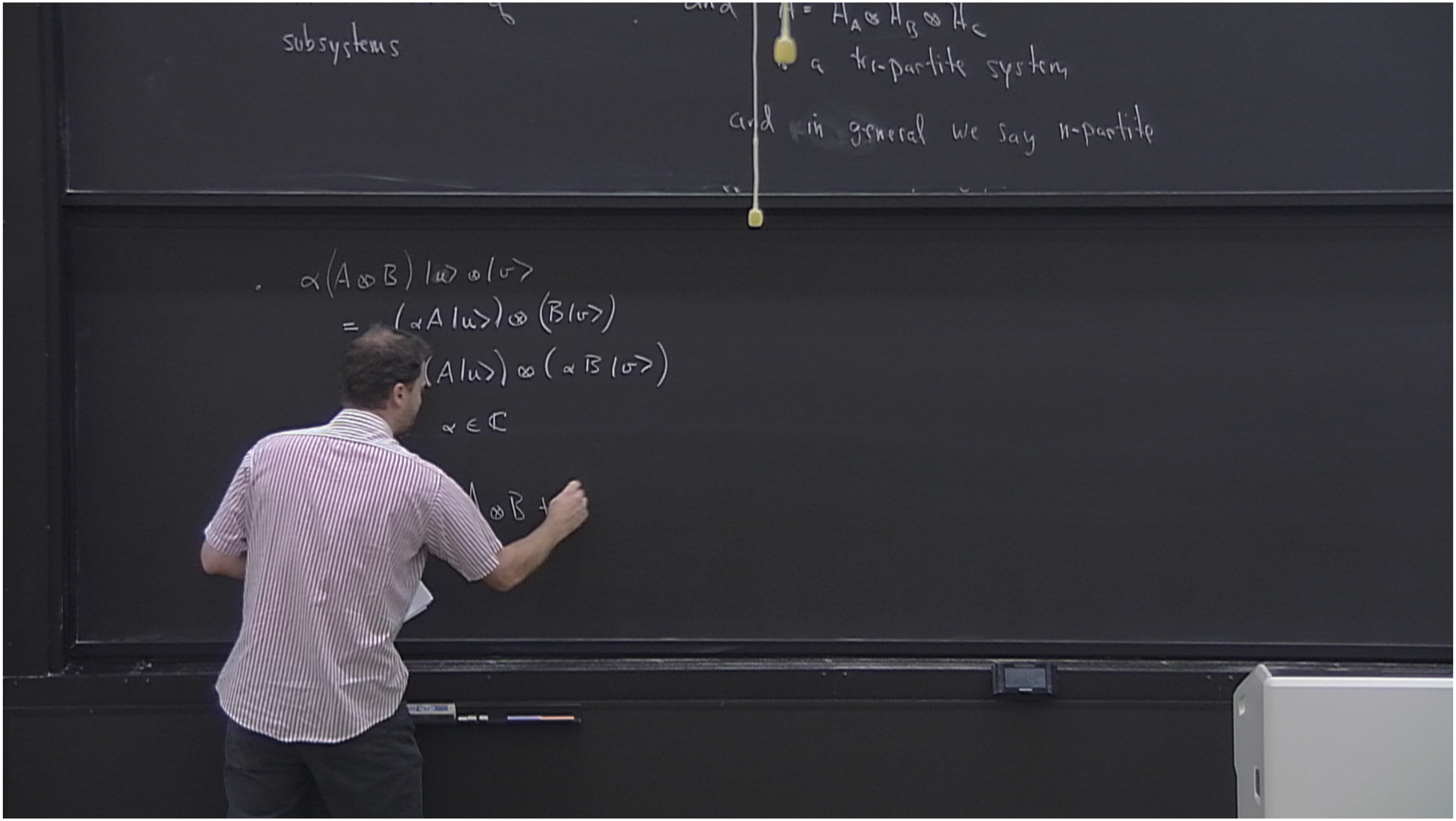

⟨a_k', b_l' | a_k, b_l⟩ = ⟨a_k' | a_k⟩ · ⟨b_l' | b_l⟩ = δ_k,k' δ_l,l'
H_Total = C² ⊗ C²
Note: H = C³ = C¹ ⊕ C¹ ⊕ C¹
or more generally
H = C⁸ = C²
⊗ C²
or = C⁴
where C⁴ is a subsp
and C² is a
Properties of the Tensor Product

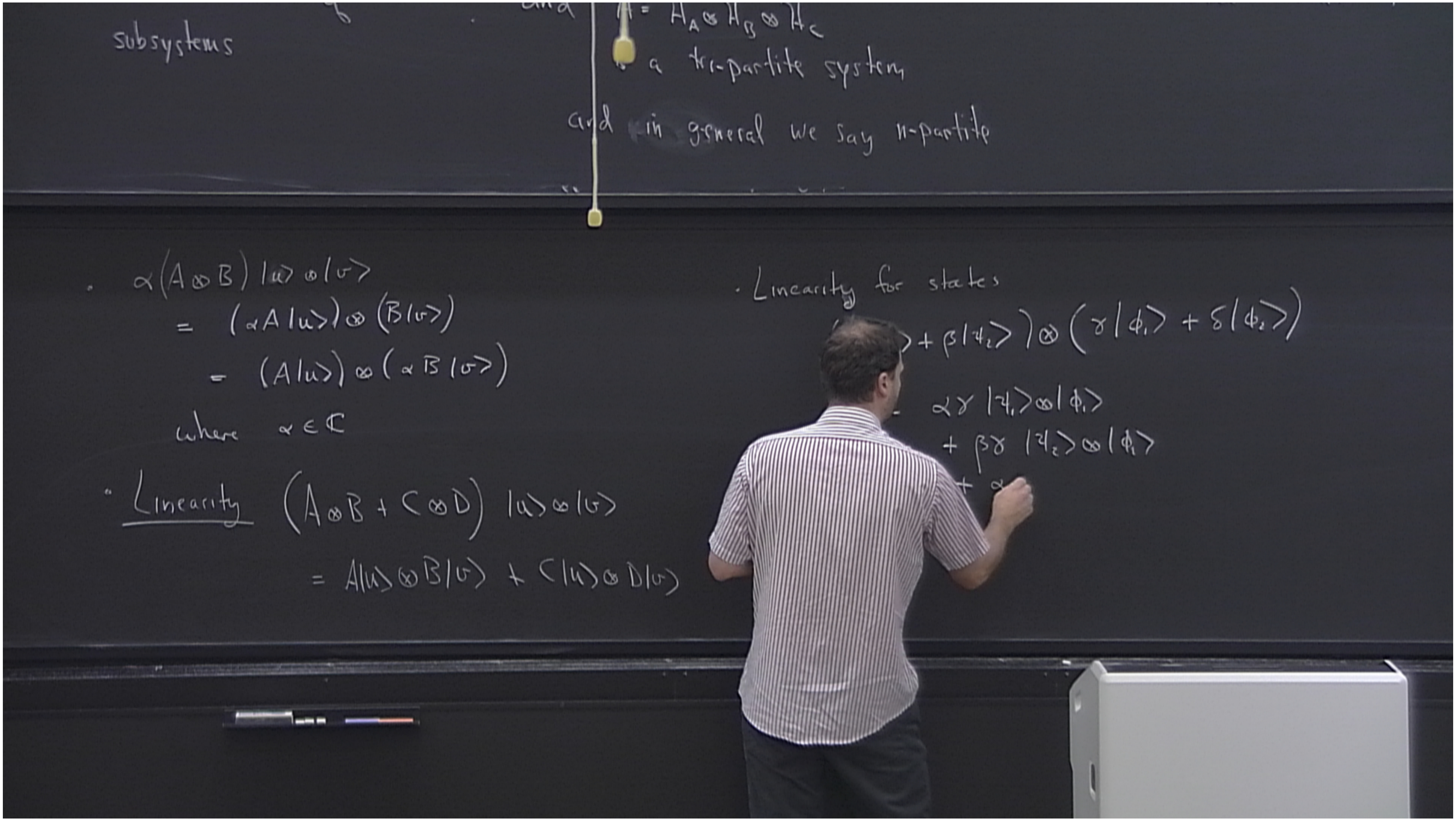

n-fold product
Properties of the Tensor Product
Consider
Suppose
is a subspace
is a factor space

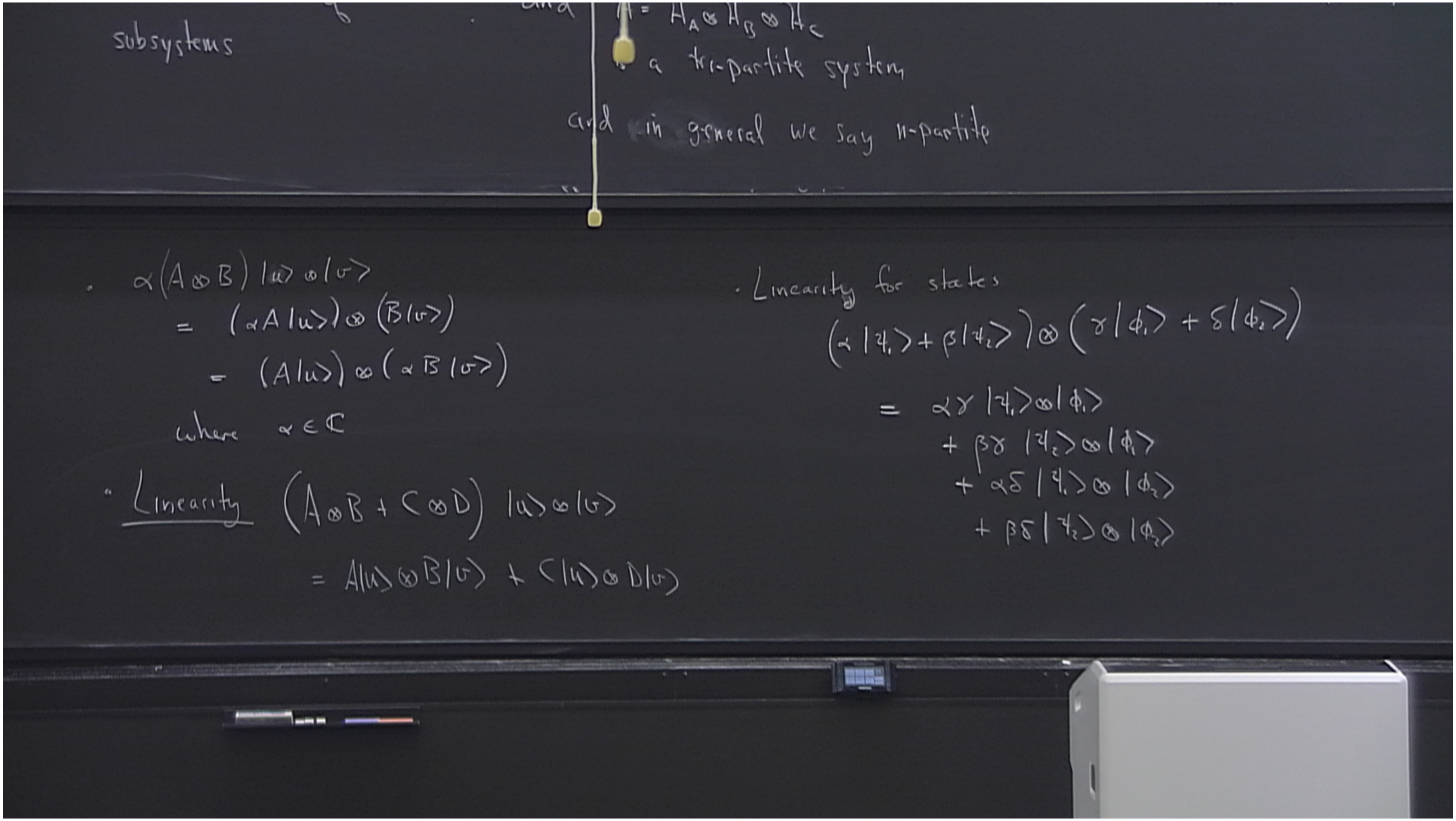

n-fold product
Properties of the Tensor Product
Consider
Suppose
is a subspace
is a factor space

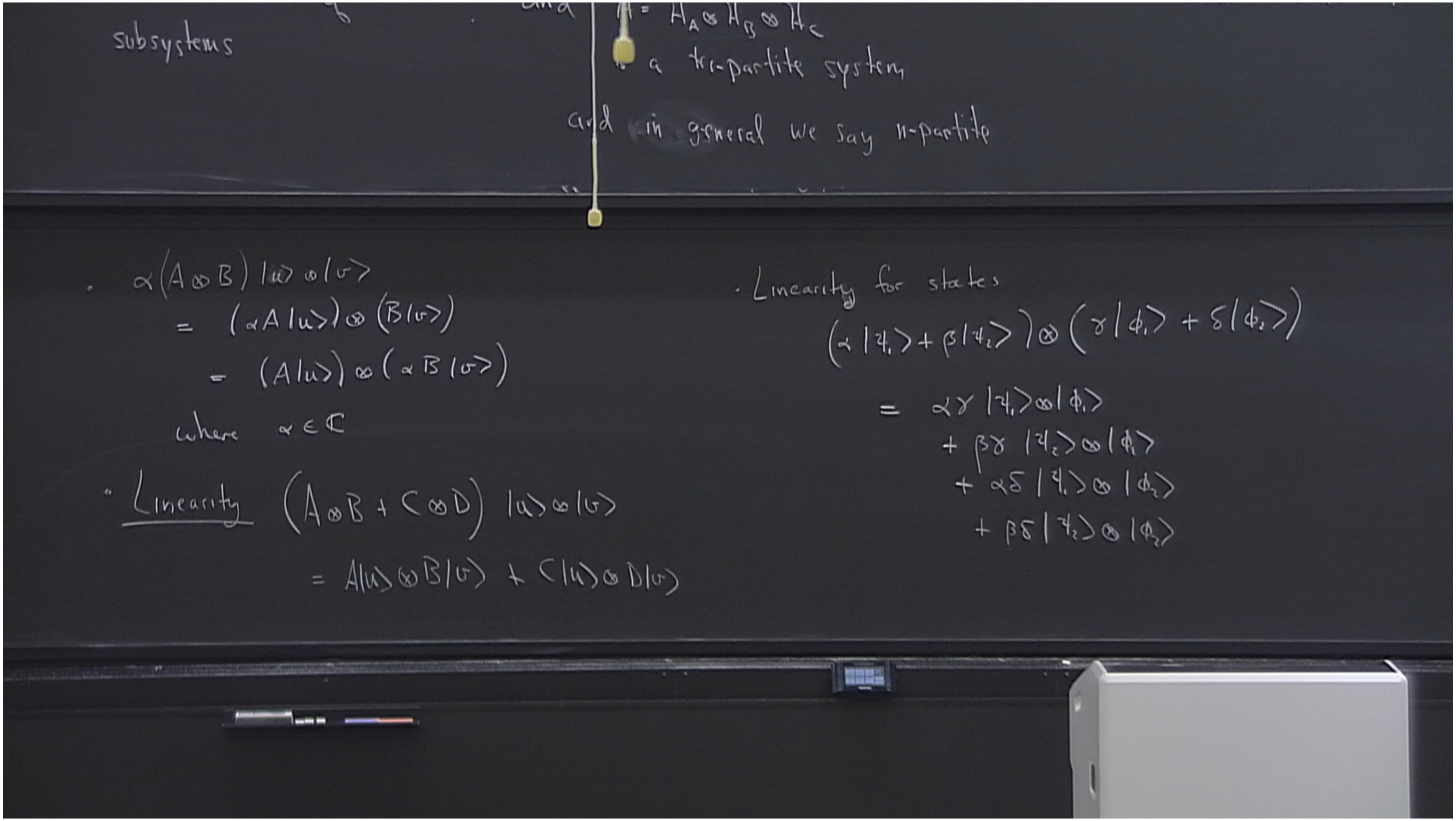

$\langle a_{k'}|a_k\rangle \cdot \langle b_{\ell'}|b_\ell\rangle = \delta_{k,k'}\delta_{\ell,\ell'}$
$\mathcal{H}_{Total} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \equiv \mathbb{C}^{2^n}$
n-fold product
Properties of the Tensor Product
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
Consider $\hat{O} \in \mathcal{L}(\mathcal{H}_{AB})$
Suppose $\hat{O} = \hat{A} \otimes \hat{B}$
$\hat{A} \in \mathcal{L}(\mathcal{H}_A)$
$\hat{B} \in \mathcal{L}(\mathcal{H}_B)$
$\hat{A} : |\psi\rangle \to |\phi\rangle$
$\hat{A}|\psi\rangle = |\phi\rangle$
$|\psi\rangle \in \mathcal{H}_A$
$|\phi\rangle \in \mathcal{H}_A$
Consider $|\chi\rangle \in \mathcal{H}_{AB}$
Suppose $|\chi\rangle = |u\rangle \otimes |v\rangle$
$|u\rangle \in \mathcal{H}_A,\ |v\rangle \in \mathcal{H}_B$
factor space


grows exponentially
with the number of
subsystems
is called a bi-partite system
and H = H_A ⊗ H_B ⊗ H_C
is a tri-partite system
and in general we say n-partite
factor spaces of the
total Hilbert space

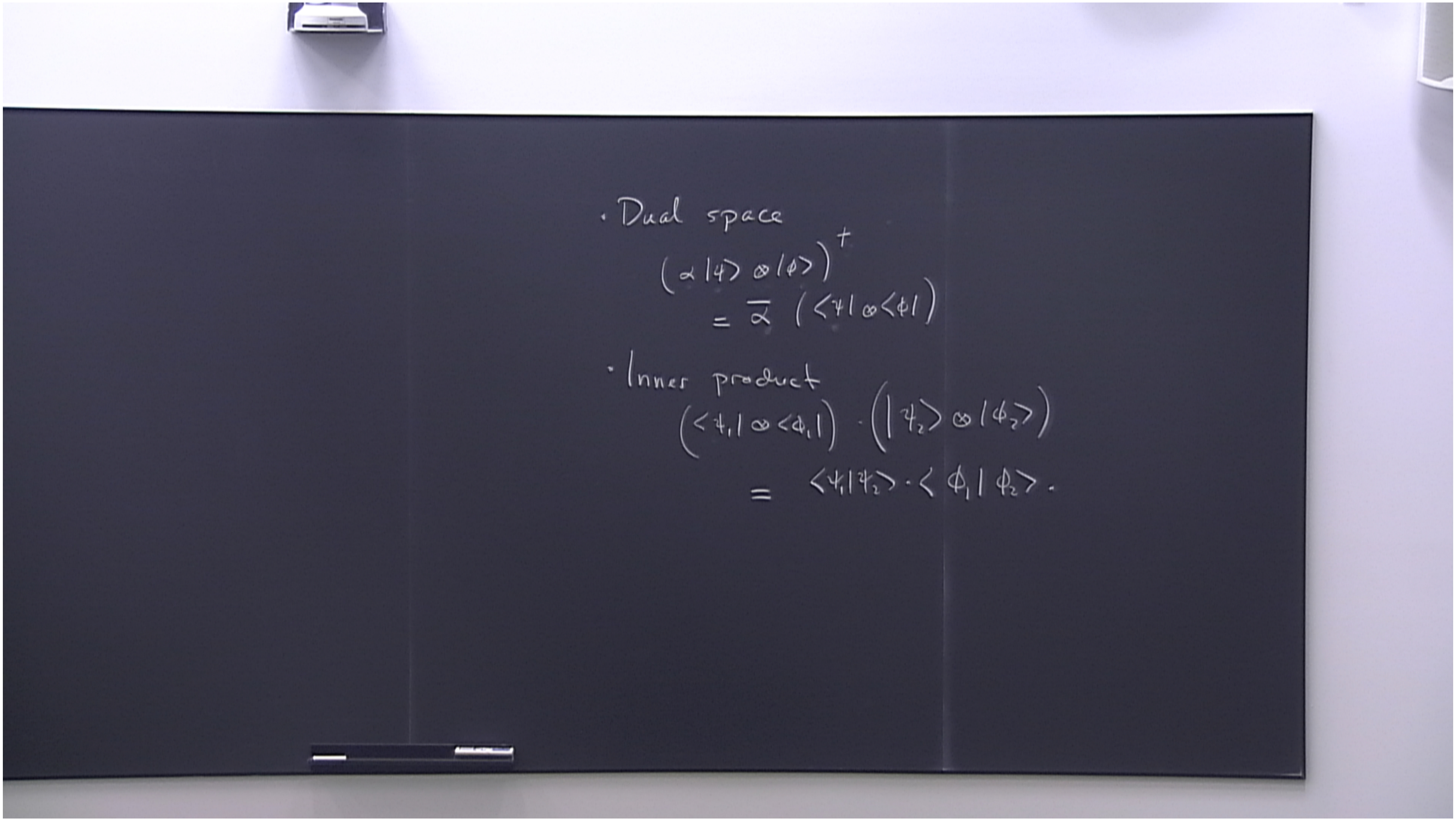

subsystems
a tri-partite system
and in general we say n-partite
α(A⊗B)|u⟩⊗|v⟩
= (αA|u⟩)⊗(B|v⟩)
(A|u⟩)⊗(αB|v⟩)
α ∈ ℂ


subsystems
H = H_A ⊗ H_B ⊗ H_C
a tri-partite system
and in general we say n-partite
α(A⊗B)|u>⊗|v>
= (αA|u>)⊗(B|v>)
= (A|u>)⊗(αB|v>)
where α ∈ ℂ
Linearity (A⊗B + C⊗D)|u>⊗|v>
= A|u>⊗B|v> + C|u>⊗D|v>
Linearity for states
+ β|ψ₂>)⊗(γ|φ₁> + δ|φ₂>)
= αγ|ψ₁>⊗|φ₁>
+ βγ|ψ₂>⊗|φ₁>
+ α

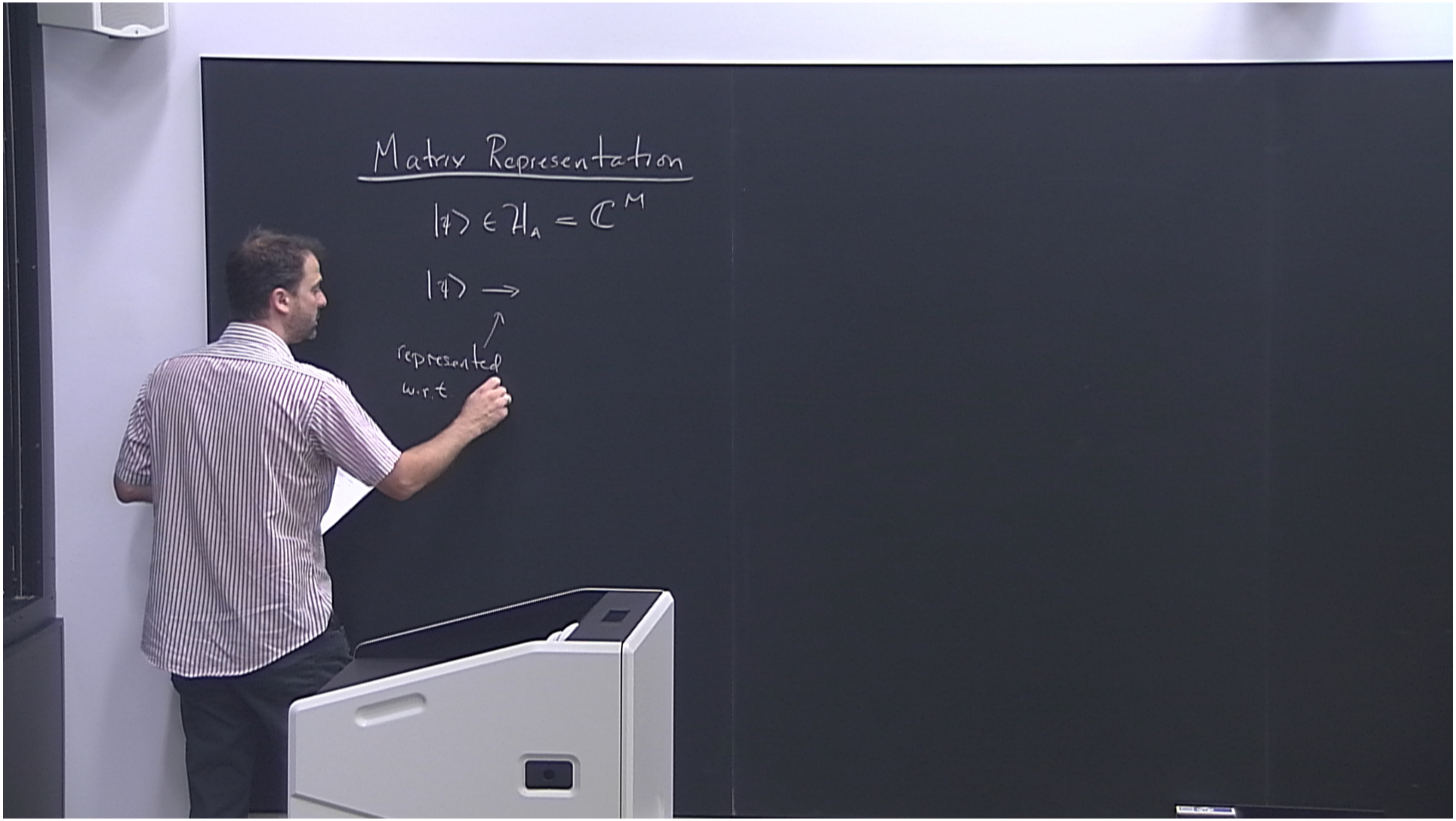

subsystems
and H = H_A ⊗ H_B ⊗ H_C
a tri-partite system
and in general we say n-partite
· α(A ⊗ B) |u⟩ ⊗ |v⟩
= (αA |u⟩) ⊗ (B |v⟩)
= (A |u⟩) ⊗ (αB |v⟩)
where α ∈ ℂ
· Linearity (A ⊗ B + C ⊗ D) |u⟩ ⊗ |v⟩
= A|u⟩ ⊗ B|v⟩ + C|u⟩ ⊗ D|v⟩
· Linearity for states
(α |ψ₁⟩ + β |ψ₂⟩) ⊗ (γ |φ₁⟩ + δ |φ₂⟩)
= αγ |ψ₁⟩ ⊗ |φ₁⟩
+ βγ |ψ₂⟩ ⊗ |φ₁⟩
+ αδ |ψ₁⟩ ⊗ |φ₂⟩
+ βδ |ψ₂⟩ ⊗ |φ₂⟩

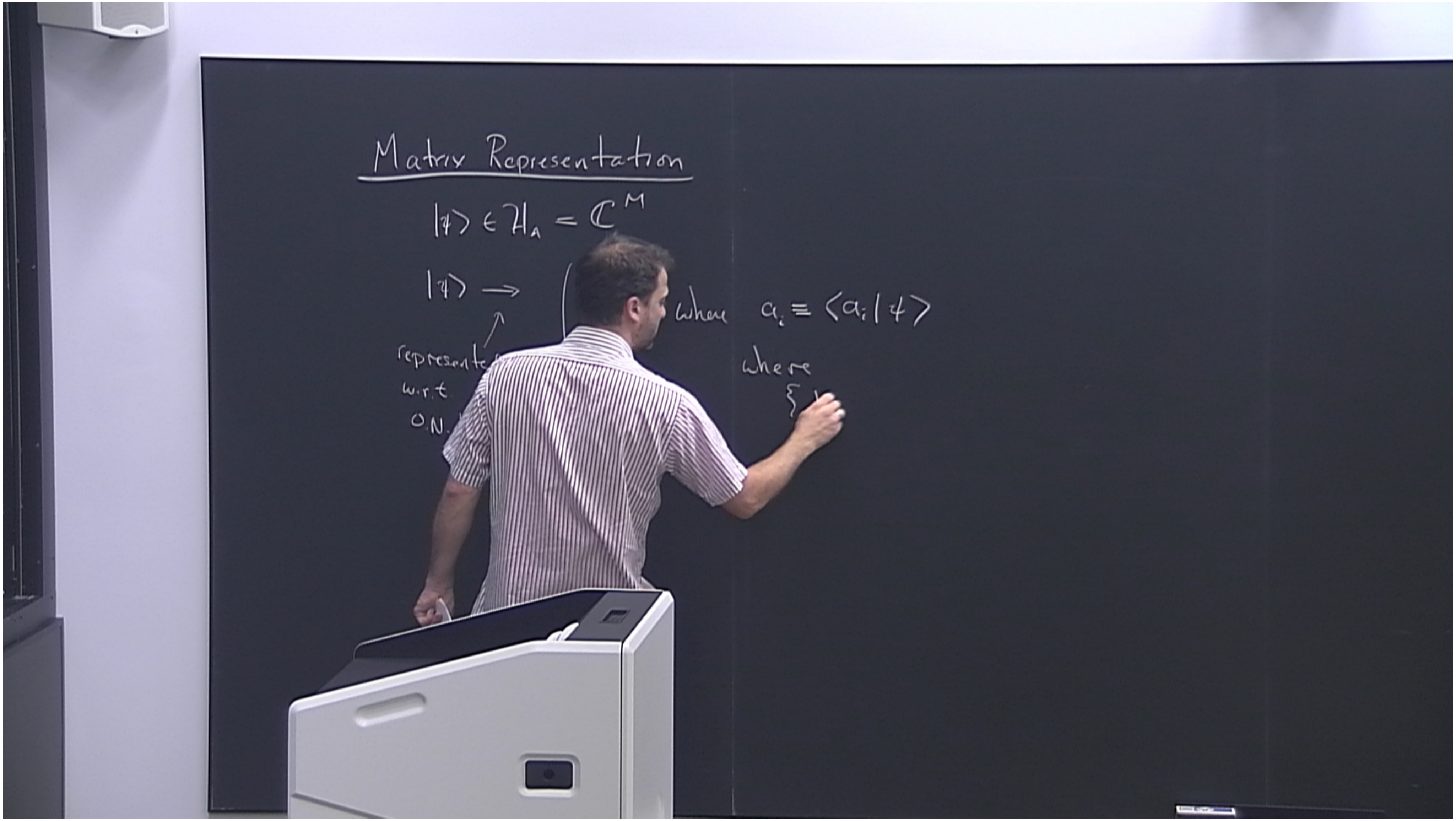


subsystems
and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$
a tri-partite system
and in general we say n-partite
$\alpha (A \otimes B) |u\rangle \otimes |v\rangle$
$= (\alpha A |u\rangle) \otimes (B|v\rangle)$
$= (A|u\rangle) \otimes (\alpha B |v\rangle)$
where $\alpha \in \mathbb{C}$
Linearity $(A \otimes B + C \otimes D) |u\rangle \otimes |v\rangle$
$= A|u\rangle \otimes B|v\rangle + C|u\rangle \otimes D|v\rangle$
Linearity for states
$(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) \otimes (\gamma |\phi_1\rangle + \delta |\phi_2\rangle)$
$= \alpha\gamma |\psi_1\rangle \otimes |\phi_1\rangle$
$+ \beta\gamma |\psi_2\rangle \otimes |\phi_1\rangle$
$+ \alpha\delta |\psi_1\rangle \otimes |\phi_2\rangle$
$+ \beta\delta |\psi_2\rangle \otimes |\phi_2\rangle$

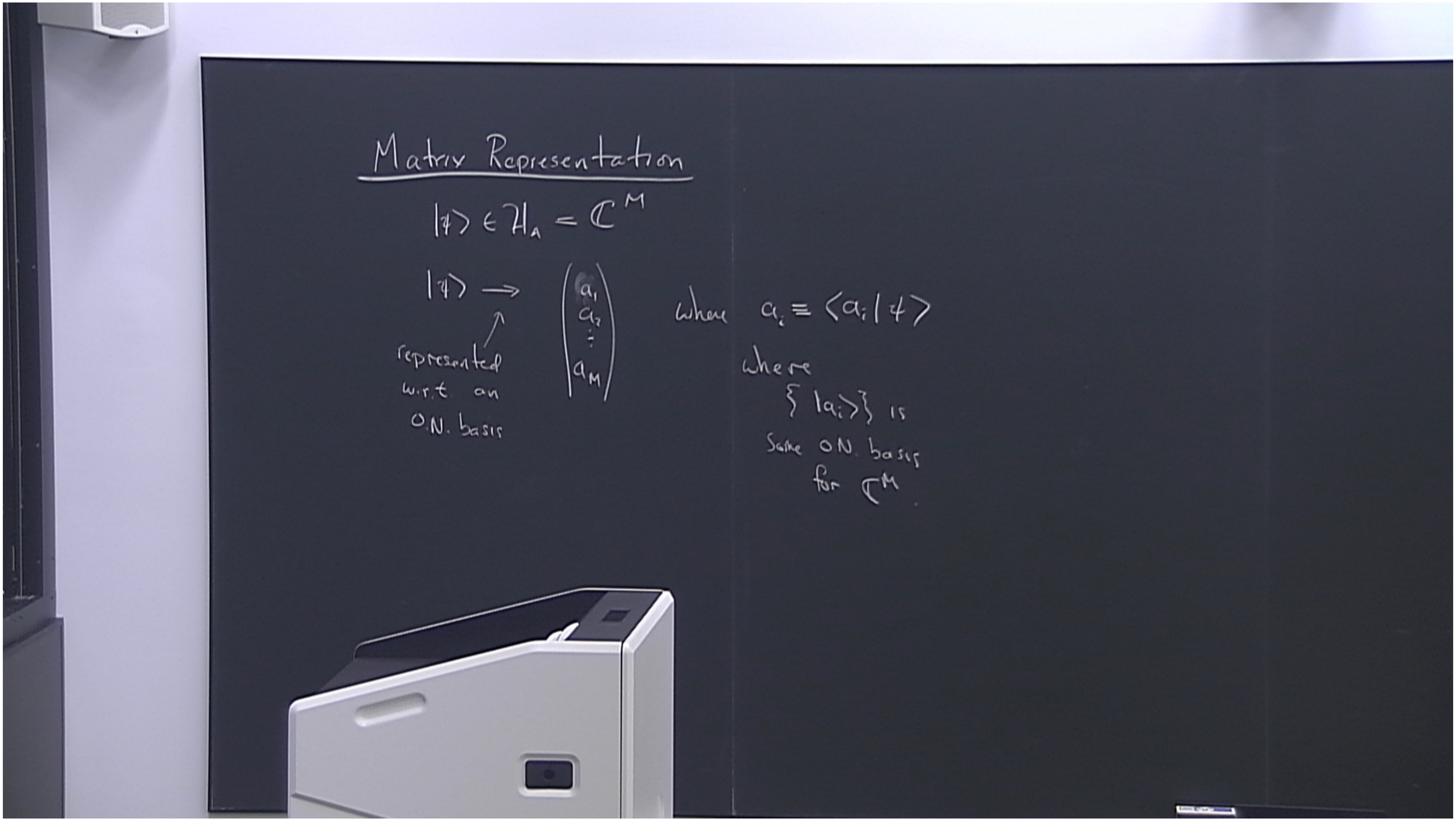


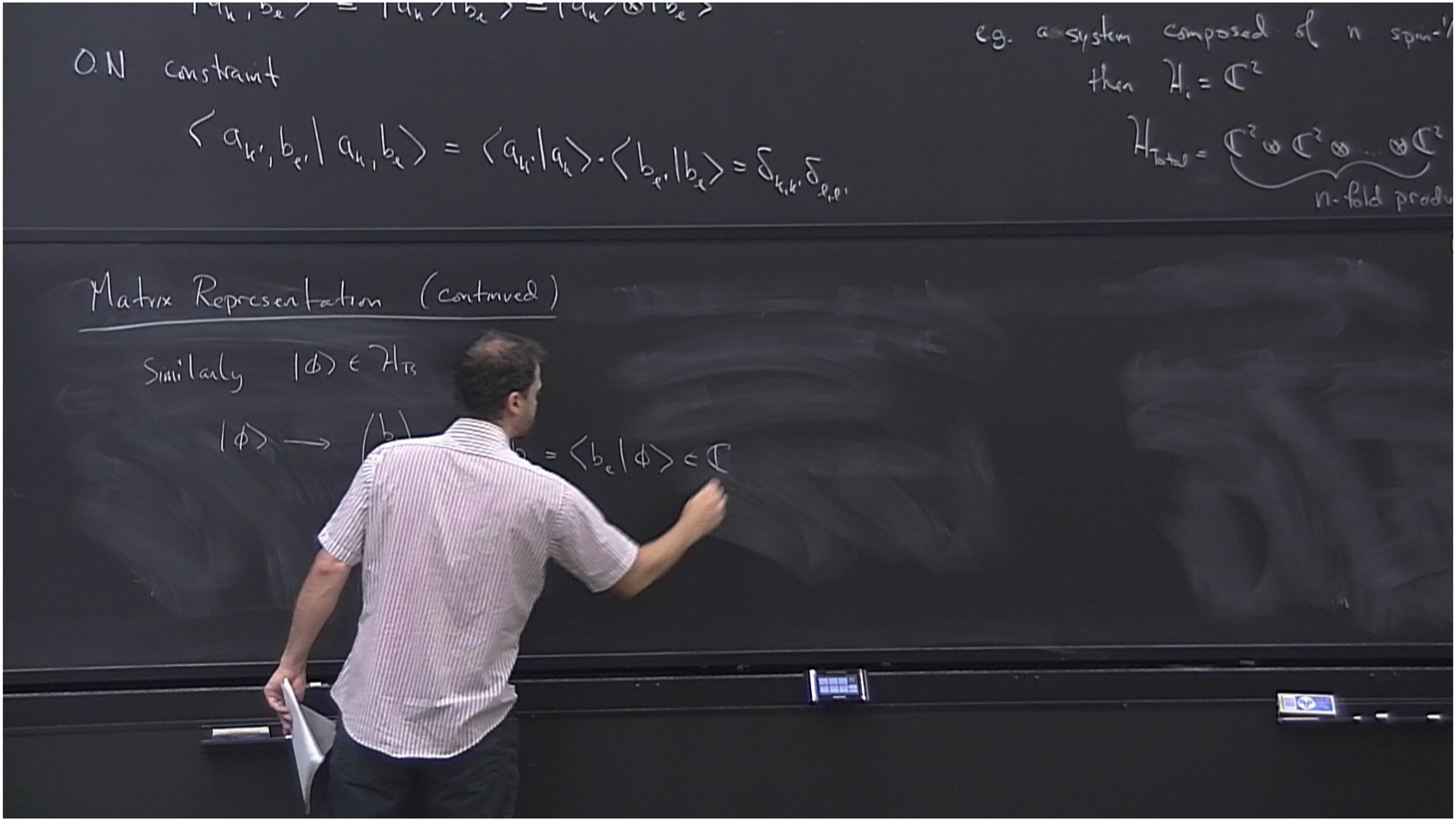
· Dual space
$(\alpha|\psi\rangle \otimes |\phi\rangle)^{\dagger}$
$= \bar{\alpha}\,(\langle\psi| \otimes \langle\phi|)$
· Inner product
$(\langle\psi_1| \otimes \langle\phi_1|)\cdot(|\psi_2\rangle \otimes |\phi_2\rangle)$
$= \langle\psi_1|\psi_2\rangle \cdot \langle\phi_1|\phi_2\rangle .$

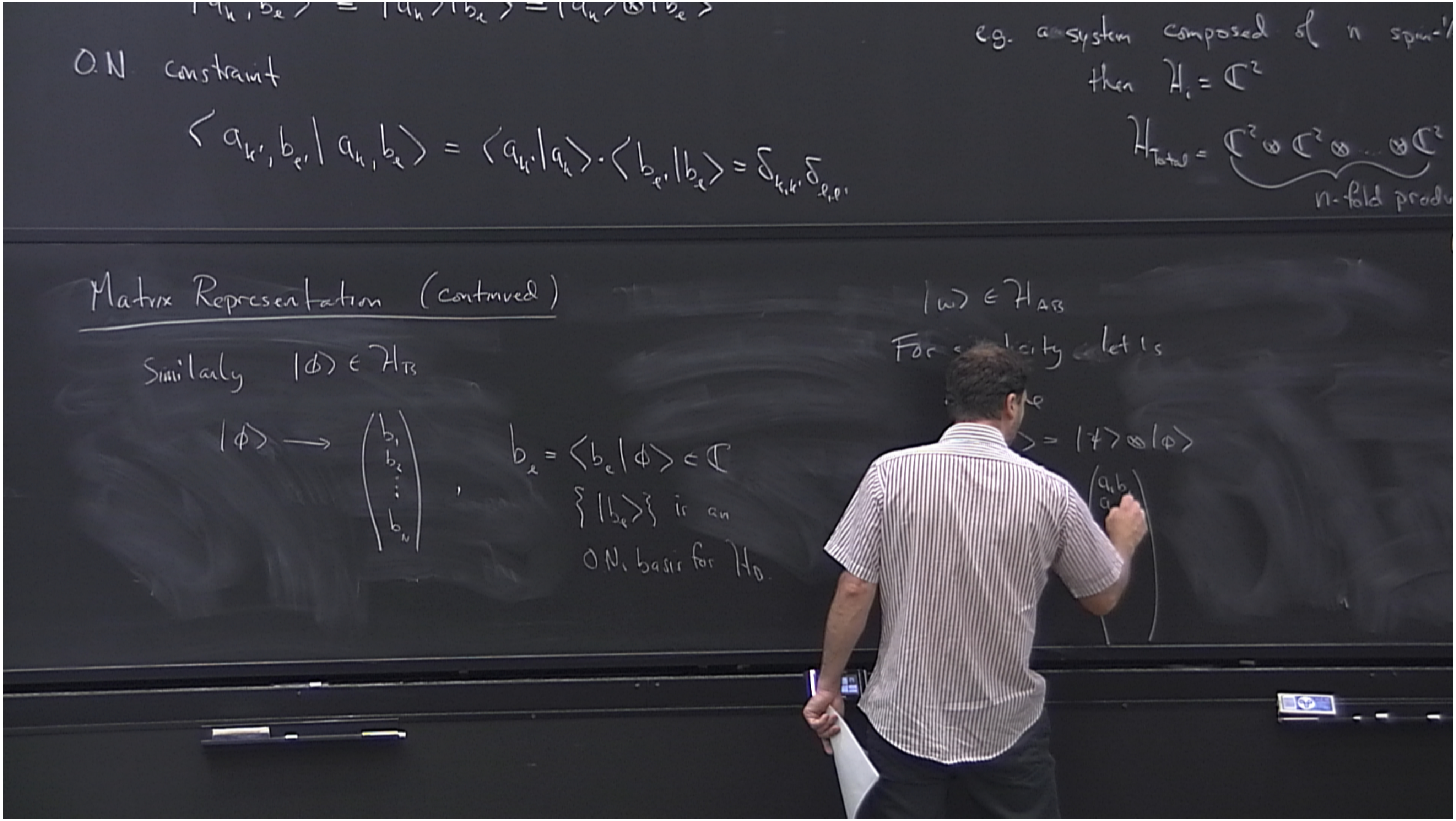

Matrix Representation
$|\psi\rangle \in \mathcal{H}_A = \mathbb{C}^M$
$|\psi\rangle \rightarrow$
represented
w.r.t

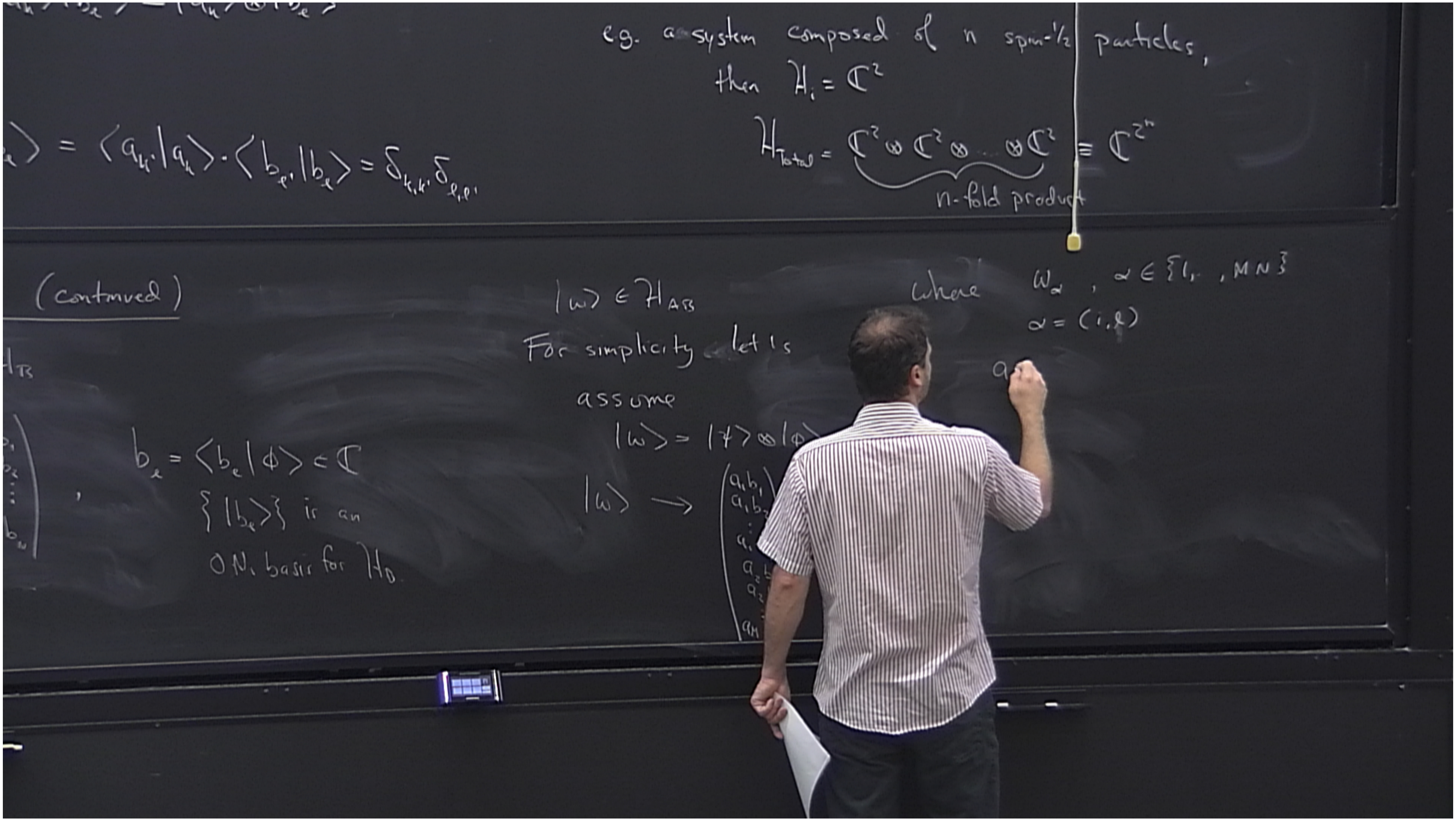


Matrix Representation
$|\psi\rangle \in \mathcal{H}_A = \mathbb{C}^M$
$|\psi\rangle \longrightarrow$
where $a_i \equiv \langle a_i | \psi \rangle$
w.r.t.
O.N.
where

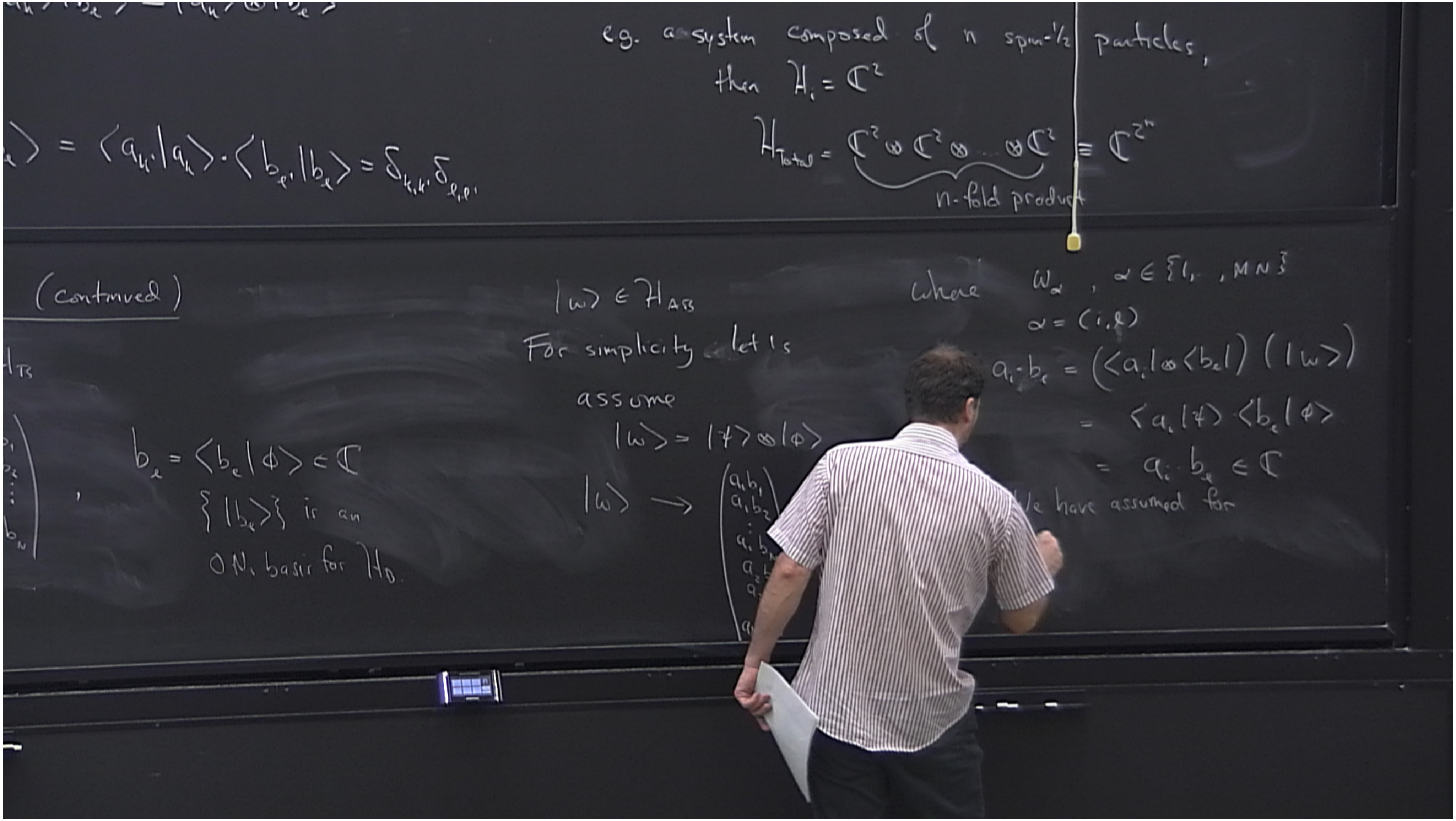

Matrix Representation
|ψ⟩ ∈ H_A = ℂ^M
|ψ⟩ → (a_1, a_2, ⋮, a_M)
where a_i ≡ ⟨a_i|ψ⟩
represented w.r.t. an O.N. basis
where {|a_i⟩} is same O.N. basis for ℂ^M.

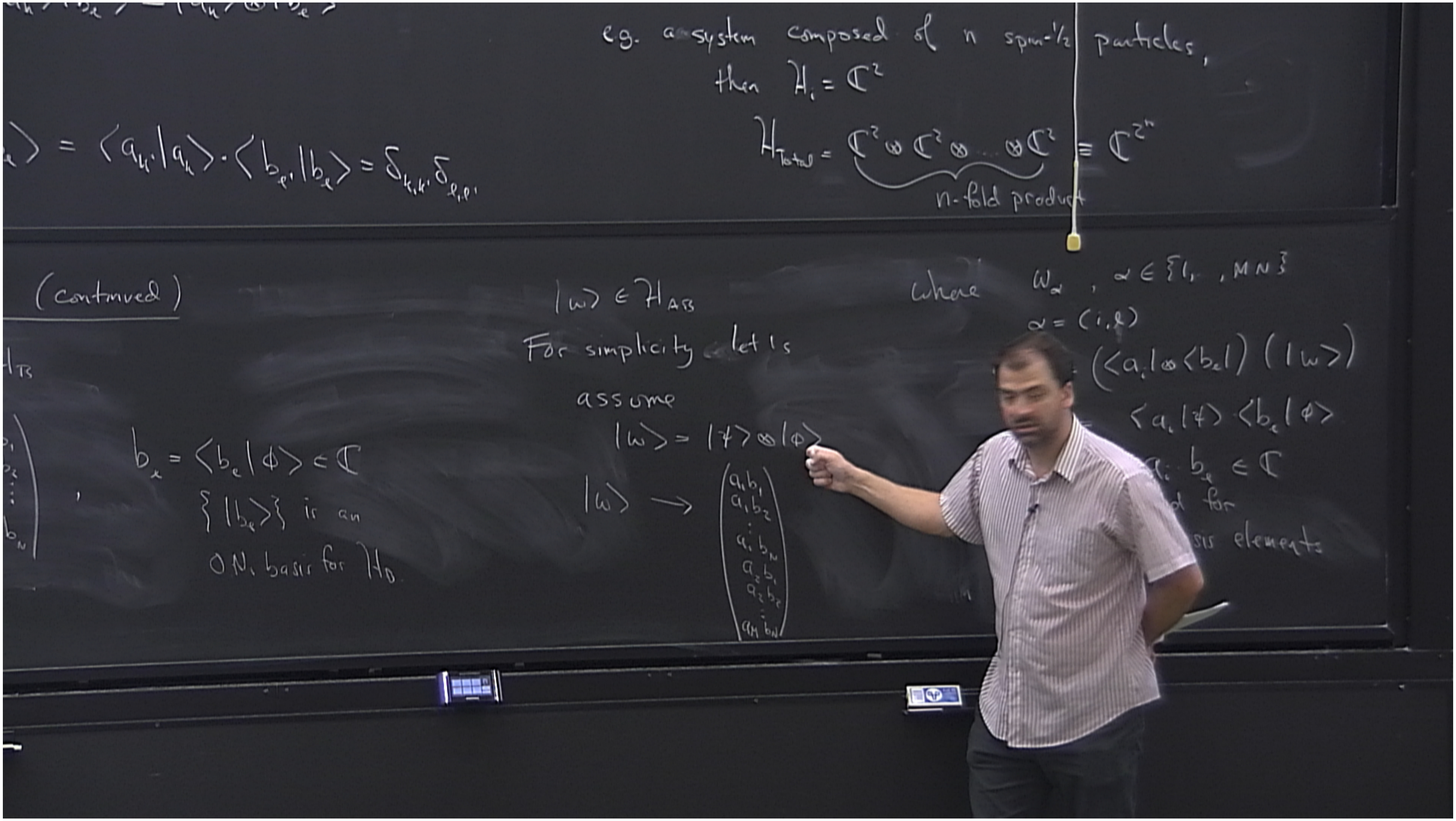

O.N constraint
⟨a_k', b_l' | a_k, b_l⟩ = ⟨a_k'|a_k⟩·⟨b_l'|b_l⟩ = δ_k,k' δ_l,l'
e.g. a system composed of n spin-
then H_i = ℂ²
H_Total = ℂ² ⊗ ℂ² ⊗ ... ⊗ ℂ²
n-fold produ
Matrix Representation (continued)
Similarly |φ⟩ ∈ H_B
|φ⟩ ⟶
= ⟨b_l|φ⟩ ∈ ℂ

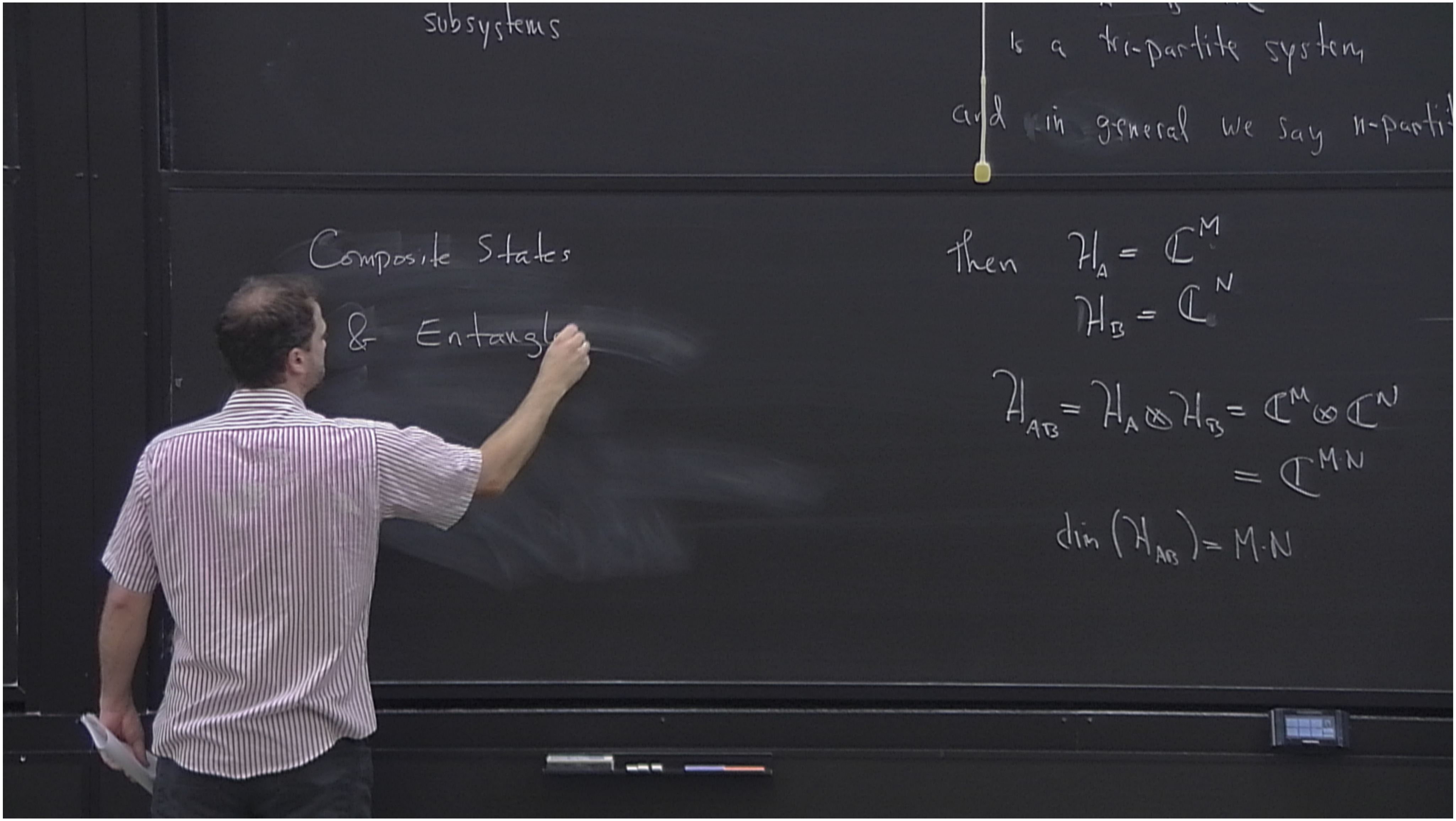

O.N constraint
⟨a_k', b_l' | a_k, b_l⟩ = ⟨a_k' | a_k⟩ · ⟨b_l' | b_l⟩ = δ_k,k' δ_l,l'
e.g. a system composed of n spin-1/2
then H_i = ℂ²
H_Total = ℂ² ⊗ ℂ² ⊗ ... ⊗ ℂ²
n-fold product
Matrix Representation (continued)
Similarly |φ⟩ ∈ H_B
|φ⟩ ⟶ (b_1, b_2, ⋮, b_N)
b_l = ⟨b_l | φ⟩ ∈ ℂ
{|b_l⟩} is an
O.N. basis for H_B.
|ω⟩ ∈ H_AB
|ψ⟩ ⊗ |φ⟩

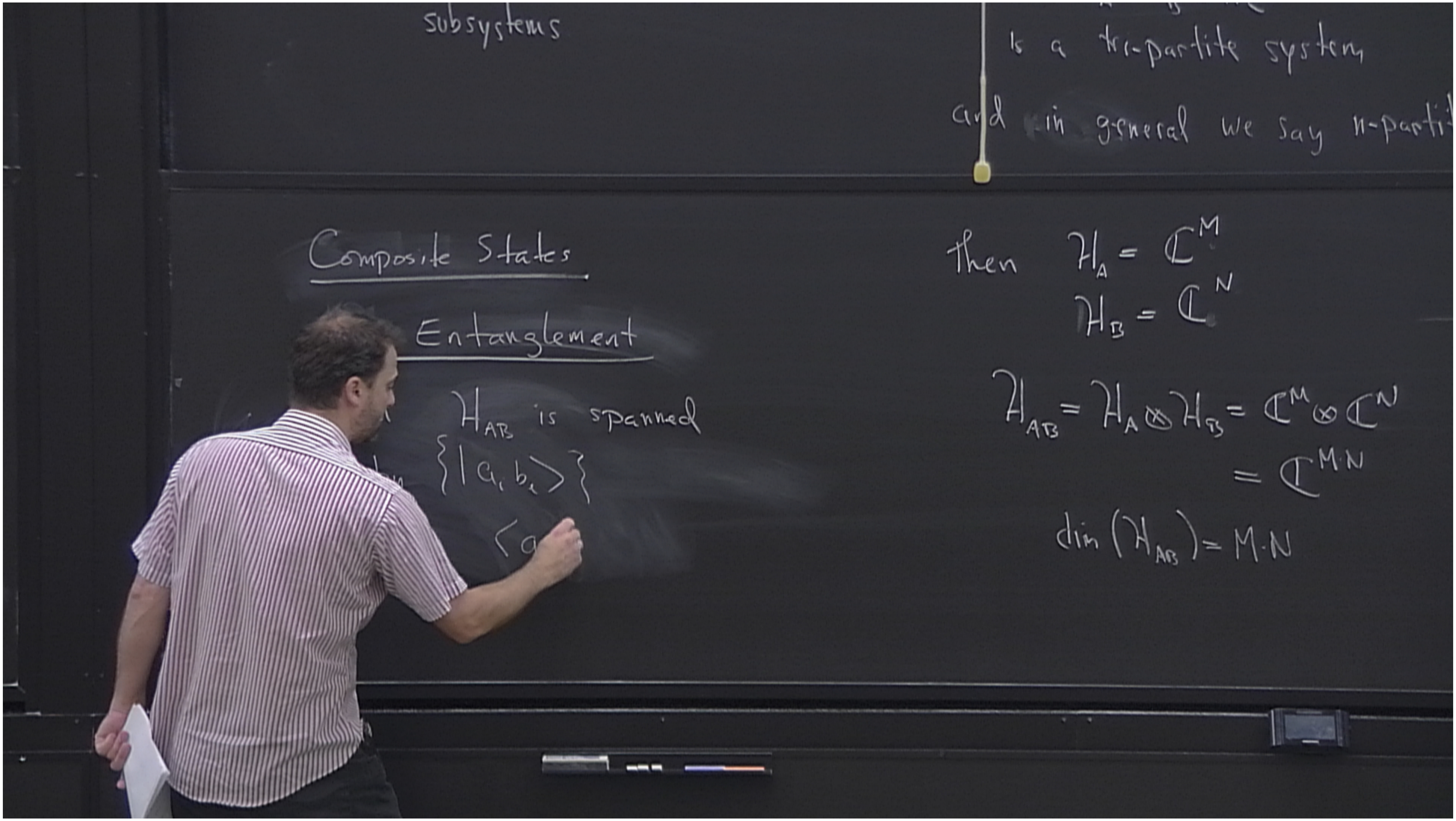


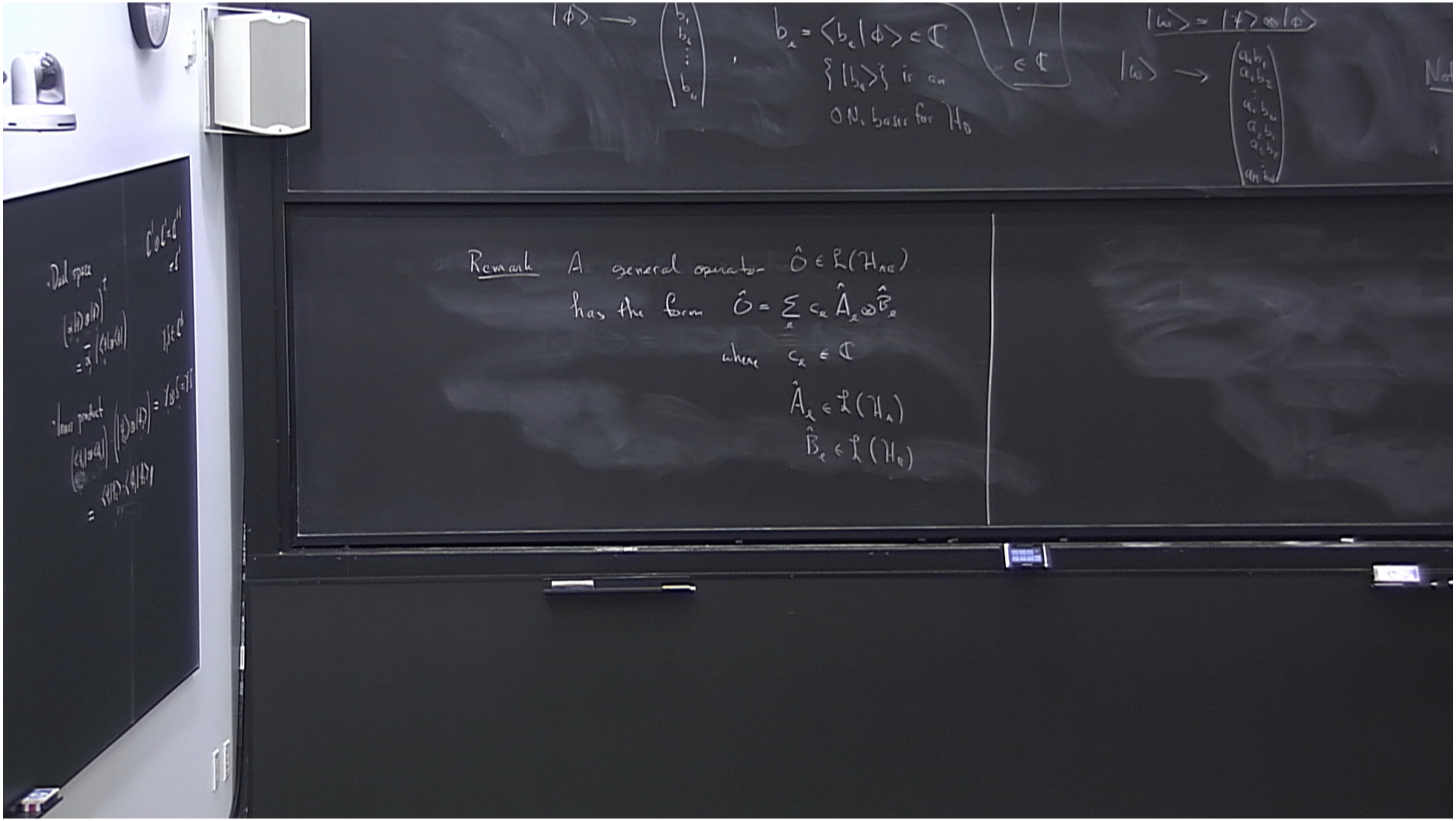
e.g. a system composed of n spin-½ particles,
then
n-fold product
(continued)
For simplicity let's
assume
where
is an
ON basis for


e.g. a system composed of n spin-1/2 particles,
then H_i = C^2
H_Total = C^2 ⊗ C^2 ⊗ ... ⊗ C^2 ≡ C^{2^n}
n-fold product
= ⟨a_k'|a_k⟩·⟨b_ℓ'|b_ℓ⟩ = δ_k,k' δ_ℓ,ℓ'
(continued)
|ω⟩ ∈ H_AB
For simplicity let's
assume
|ω⟩ = |ψ⟩ ⊗ |φ⟩
|ω⟩ →
where ω_α , α ∈ {1, ..., MN}
α = (i, ℓ)
a_i·b_ℓ = (⟨a_i| ⊗ ⟨b_ℓ|)(|ω⟩)
= ⟨a_i|ψ⟩ ⟨b_ℓ|φ⟩
= a_i · b_ℓ ∈ C
We have assumed for
b_ℓ = ⟨b_ℓ|φ⟩ ∈ C
{|b_ℓ⟩} is an
O.N. basis for H_B.

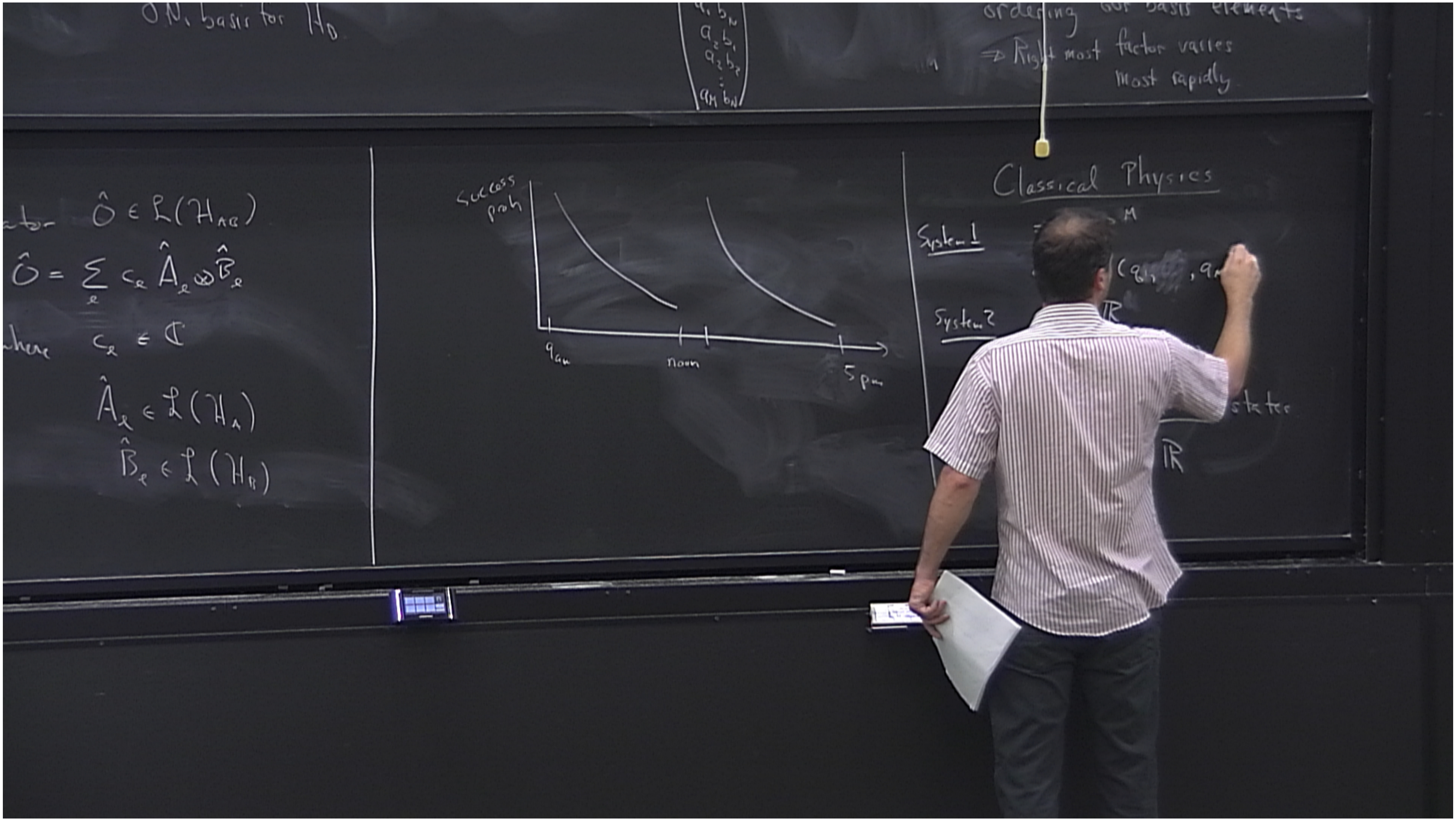

e.g. a system composed of n spin-1/2 particles,
then H_i = C^2
H_Total = C^2 ⊗ C^2 ⊗ ... ⊗ C^2 ≡ C^(2^n)
n-fold product
(continued)
|ω> ∈ H_AB
For simplicity let's assume
|ω> = |ψ> ⊗ |φ>
b_ℓ = <b_ℓ|φ> ∈ C
{|b_ℓ>} is an O.N. basis for H_B
where ω_α , α ∈ {1, ..., MN}
α = (i,ℓ)
(<a_i| ⊗ <b_ℓ|) (|ω>)
<a_i|ψ> <b_ℓ|φ>
basis elements

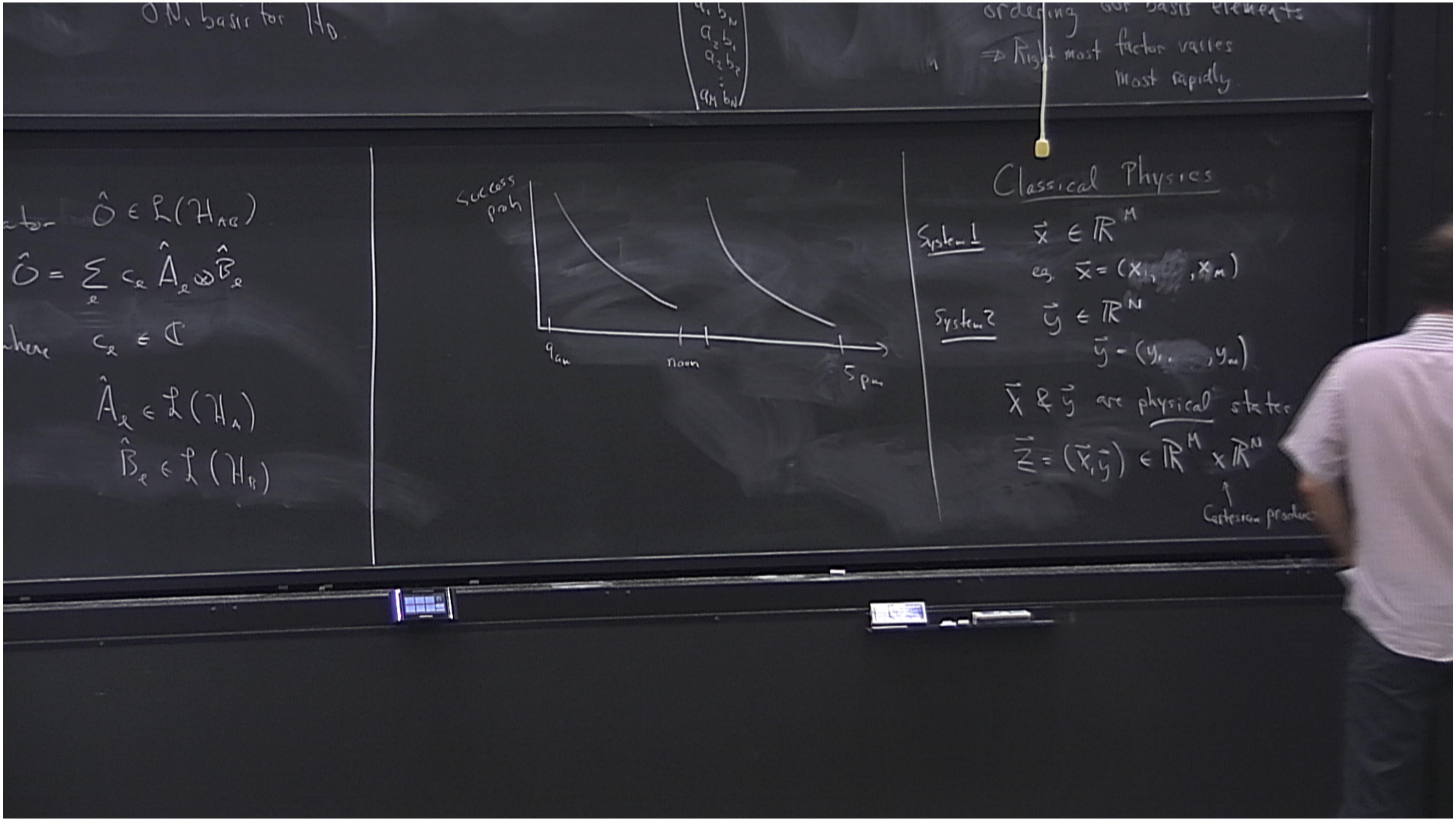

subsystems
is a tri-partite system
and in general we say
Composite States
then $\mathcal{H}_A = \mathbb{C}^M$
$\mathcal{H}_B = \mathbb{C}^N$
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^M \otimes \mathbb{C}^N$
$= \mathbb{C}^{M \cdot N}$
$\dim(\mathcal{H}_{AB}) = M \cdot N$

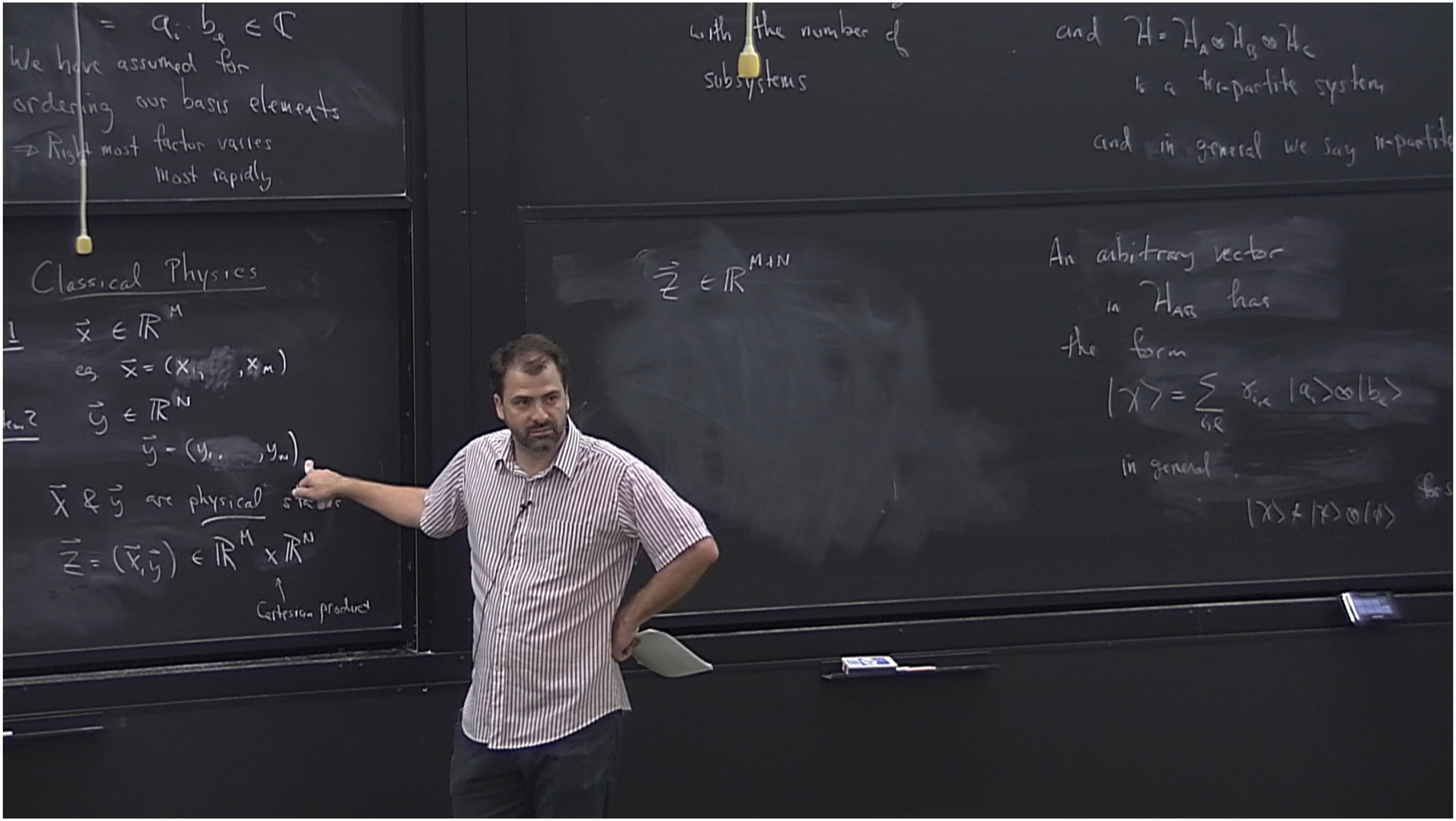

subsystems
is a tri-partite system
and in general we say n-partit
Composite States
Entanglement
$\mathcal{H}_{AB}$ is spanned
$\{|a_i b_j\rangle\}$
then $\mathcal{H}_A = \mathbb{C}^M$
$\mathcal{H}_B = \mathbb{C}^N$
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^M \otimes \mathbb{C}^N$
$= \mathbb{C}^{M \cdot N}$
$\dim(\mathcal{H}_{AB}) = M \cdot N$

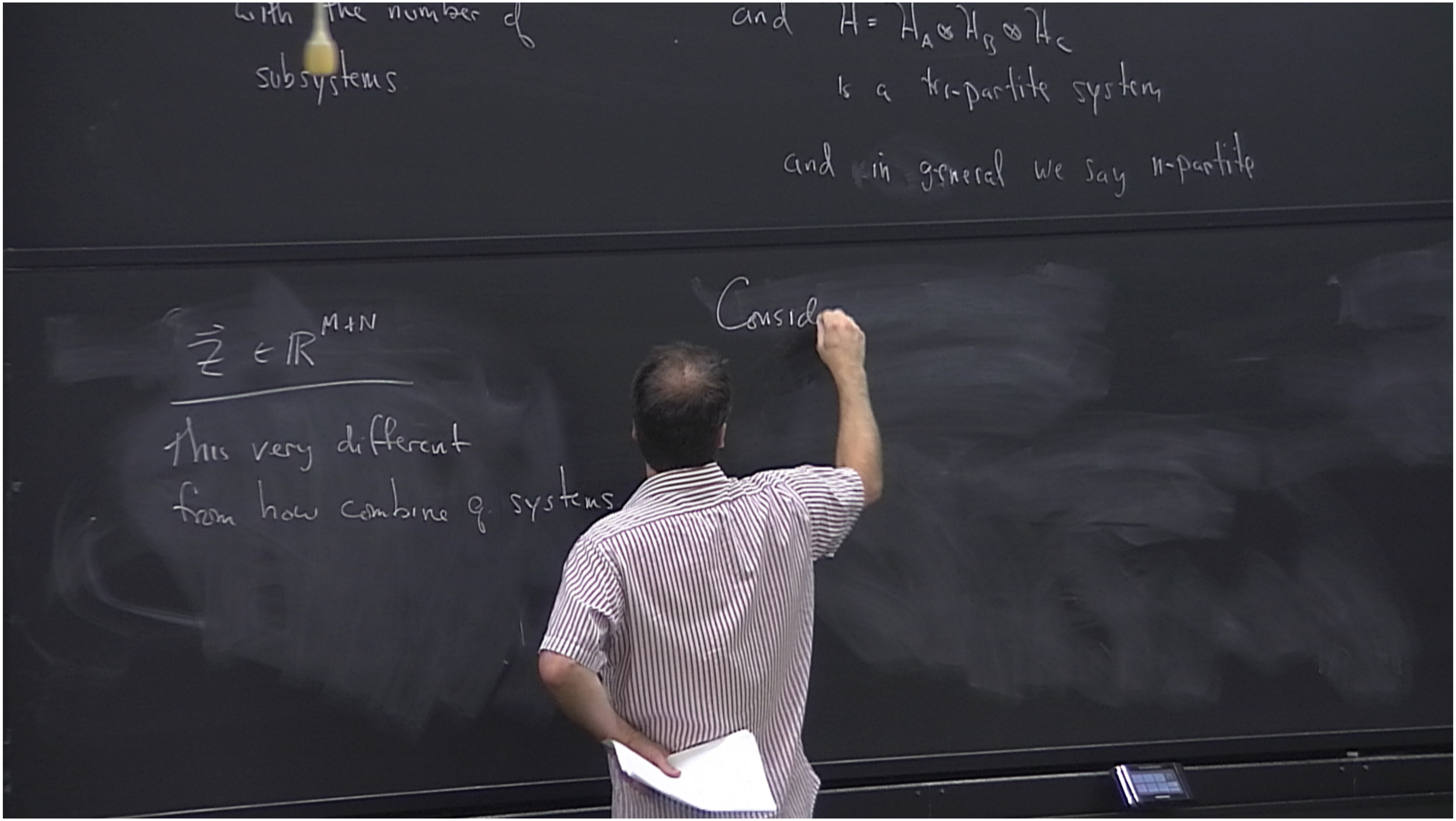

Remark A general operator $\hat{O} \in \mathcal{L}(\mathcal{H}_{AB})$
has the form $\hat{O} = \sum_{\ell} c_{\ell} \hat{A}_{\ell} \otimes \hat{B}_{\ell}$
where $c_{\ell} \in \mathbb{C}$
$\hat{A}_{\ell} \in \mathcal{L}(\mathcal{H}_A)$
$\hat{B}_{\ell} \in \mathcal{L}(\mathcal{H}_B)$
$b_{\ell} = \langle b_{\ell} | \phi \rangle \in \mathbb{C}$
$\{|b_{\ell}\rangle\}$ is an
O.N. basis for $\mathcal{H}_B$
$|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$

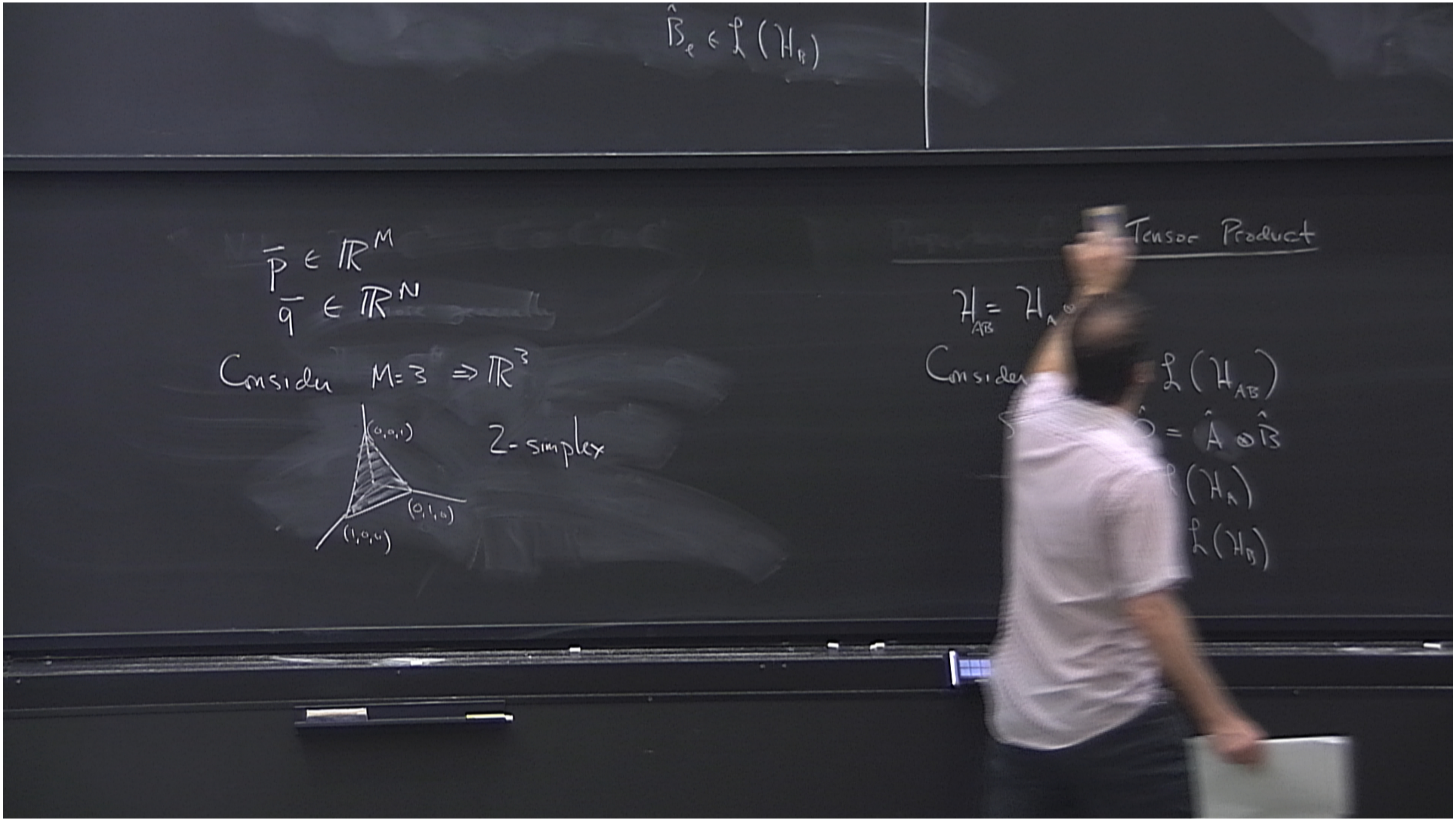

ON. basis for H_B
Classical Physics
$\vec{x} \in \mathbb{R}^2$
$\hat{O} \in \mathcal{L}(\mathcal{H}_{AB})$
$\hat{O} = \sum_l c_l \hat{A}_l \otimes \hat{B}_l$
where $c_l \in \mathbb{C}$
success prob
9am
noon
5pm
⇒ Right most factor varies most rapidly

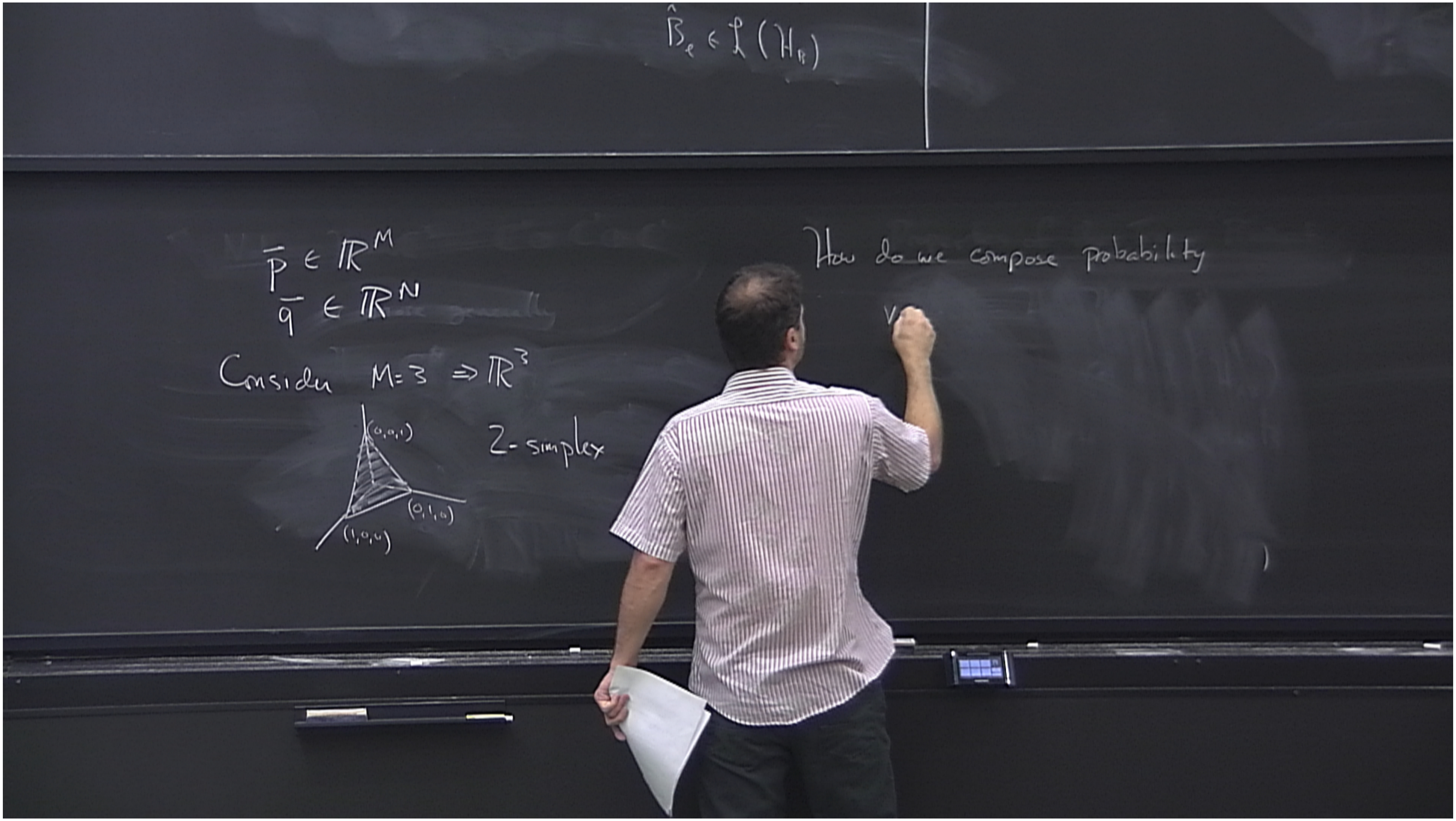

O.N. basis for H_B
ordering our basis elements
⇒ Right most factor varies
most rapidly
Classical Physics
System 1
System 2
success
prob
9am
noon
5pm
states

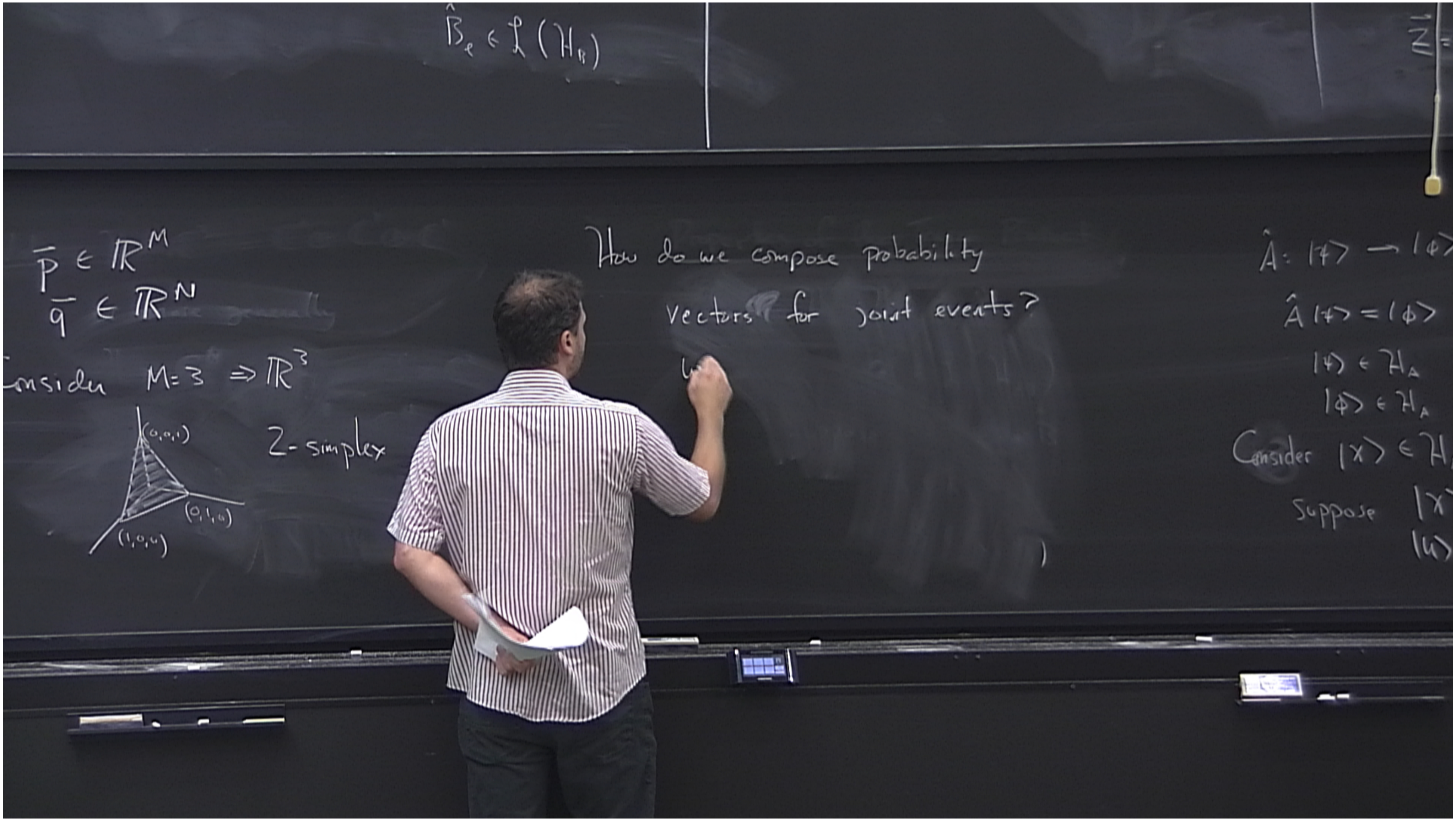

O.N. basis for H_B
ordering our basis elements
⇒ Right most factor varies most rapidly
success prob
9am
noon
5pm
Classical Physics
System 1
System 2
x⃗ & y⃗ are physical states
Cartesian product

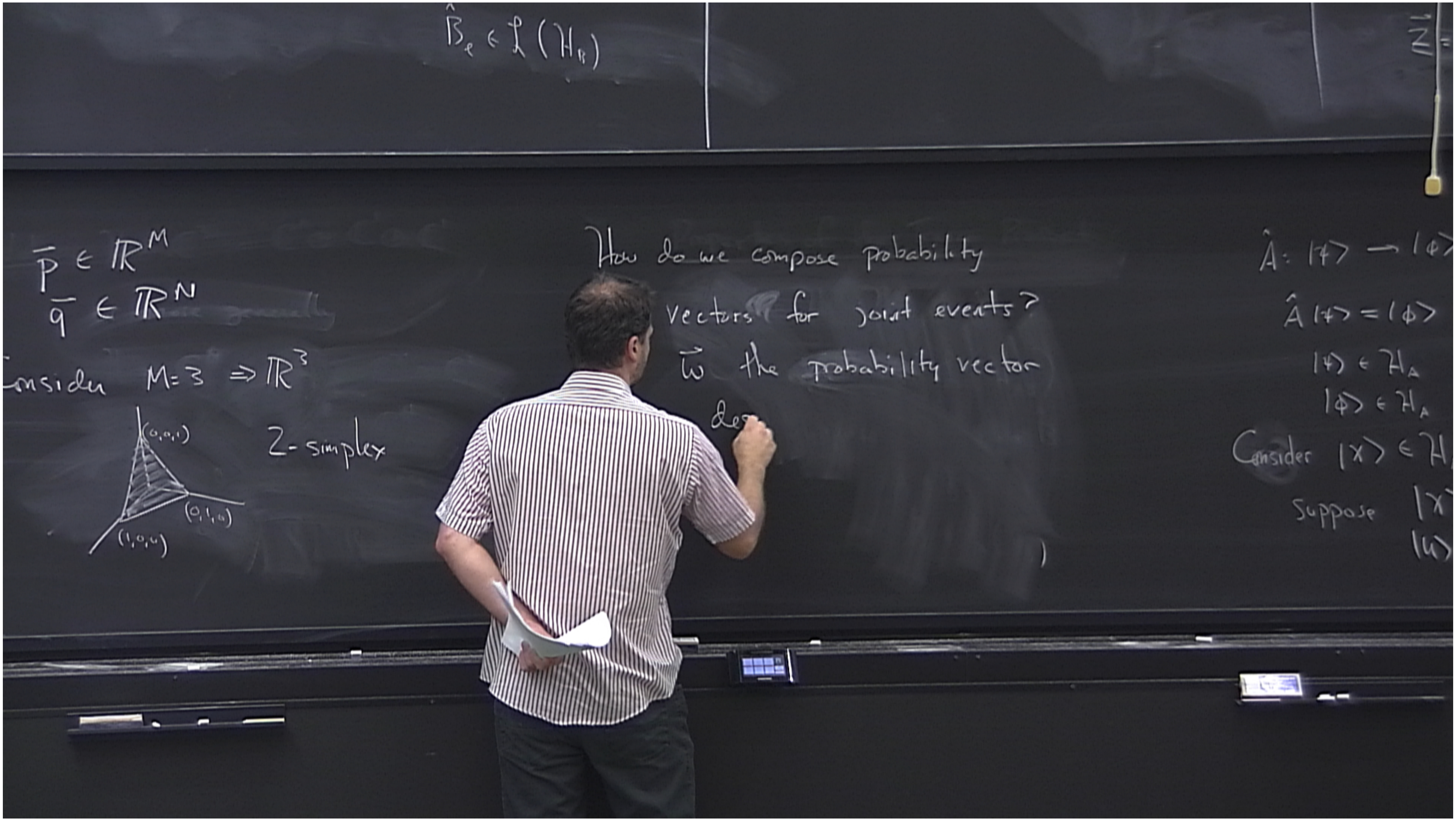

$= a_i \cdot b_\ell \in \mathbb{C}$
We have assumed for ordering our basis elements
$\Rightarrow$ Right most factor varies most rapidly
with the number of subsystems
and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$
is a tri-partite system
and in general we say n-partite
Classical Physics
$\vec{x} \in \mathbb{R}^M$
eg. $\vec{x} = (x_1, \ldots, x_M)$
$\vec{y} \in \mathbb{R}^N$
$\vec{y} = (y_1, \ldots, y_N)$
$\vec{x}$ & $\vec{y}$ are physical
$\vec{z} = (\vec{x}, \vec{y}) \in \mathbb{R}^M \times \mathbb{R}^N$
Cartesian product
$\vec{z} \in \mathbb{R}^{M+N}$
An arbitrary vector in $\mathcal{H}_{AB}$ has the form
$|\chi\rangle = \sum_{i,\ell} \gamma_{i\ell} |a_i\rangle \otimes |b_\ell\rangle$
in general
$|\chi\rangle \neq |\psi\rangle \otimes |\phi\rangle$

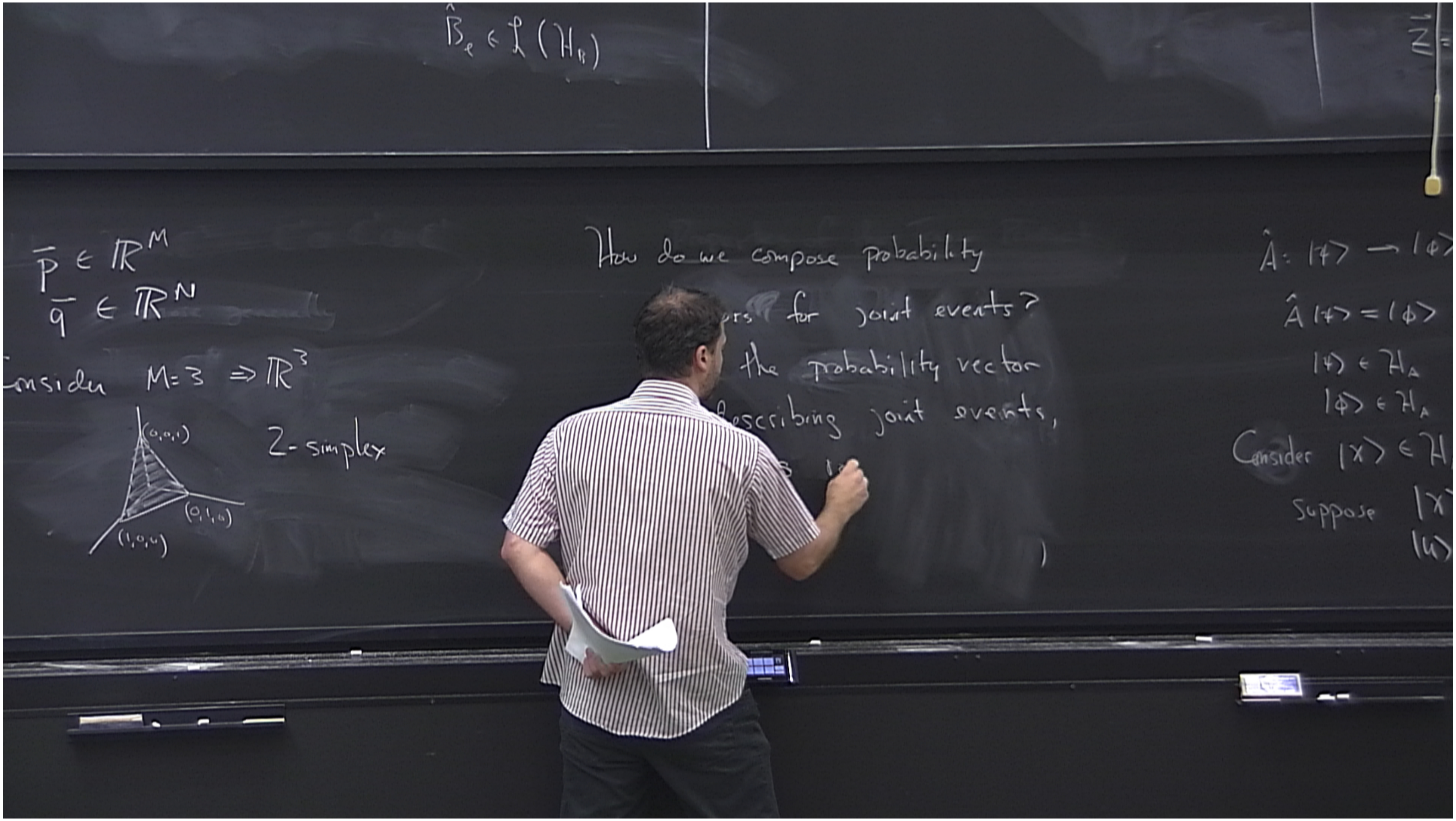

with the number of
subsystems
and H = H_A ⊗ H_B ⊗ H_C
is a tri-partite system
and in general we say n-partite
Z ∈ R^{M+N}
this very different
from how combine q. systems
Consid

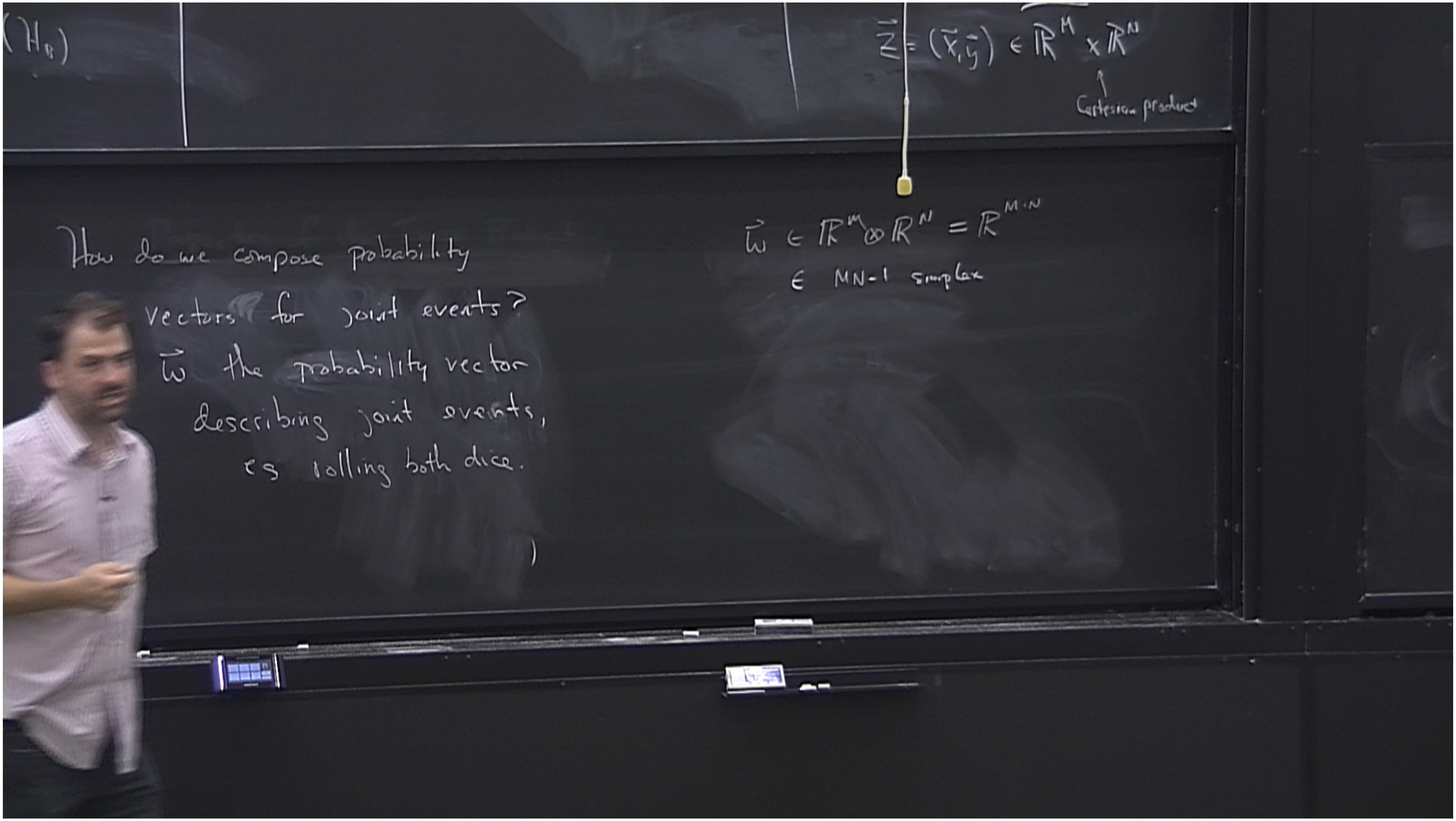

$\hat{B}_e \in \mathcal{L}(\mathcal{H}_B)$
$\bar{p} \in \mathbb{R}^M$
$\bar{q} \in \mathbb{R}^N$
Consider $M = 3 \Rightarrow \mathbb{R}^3$
(0,0,1)
(0,1,0)
(1,0,0)
2-simplex
Tensor Product
$\mathcal{H}_{AB} = \mathcal{H}_A$
Consider
$\mathcal{L}(\mathcal{H}_{AB})$
$= \hat{A} \otimes \hat{B}$
$(\mathcal{H}_A)$
$\mathcal{L}(\mathcal{H}_B)$

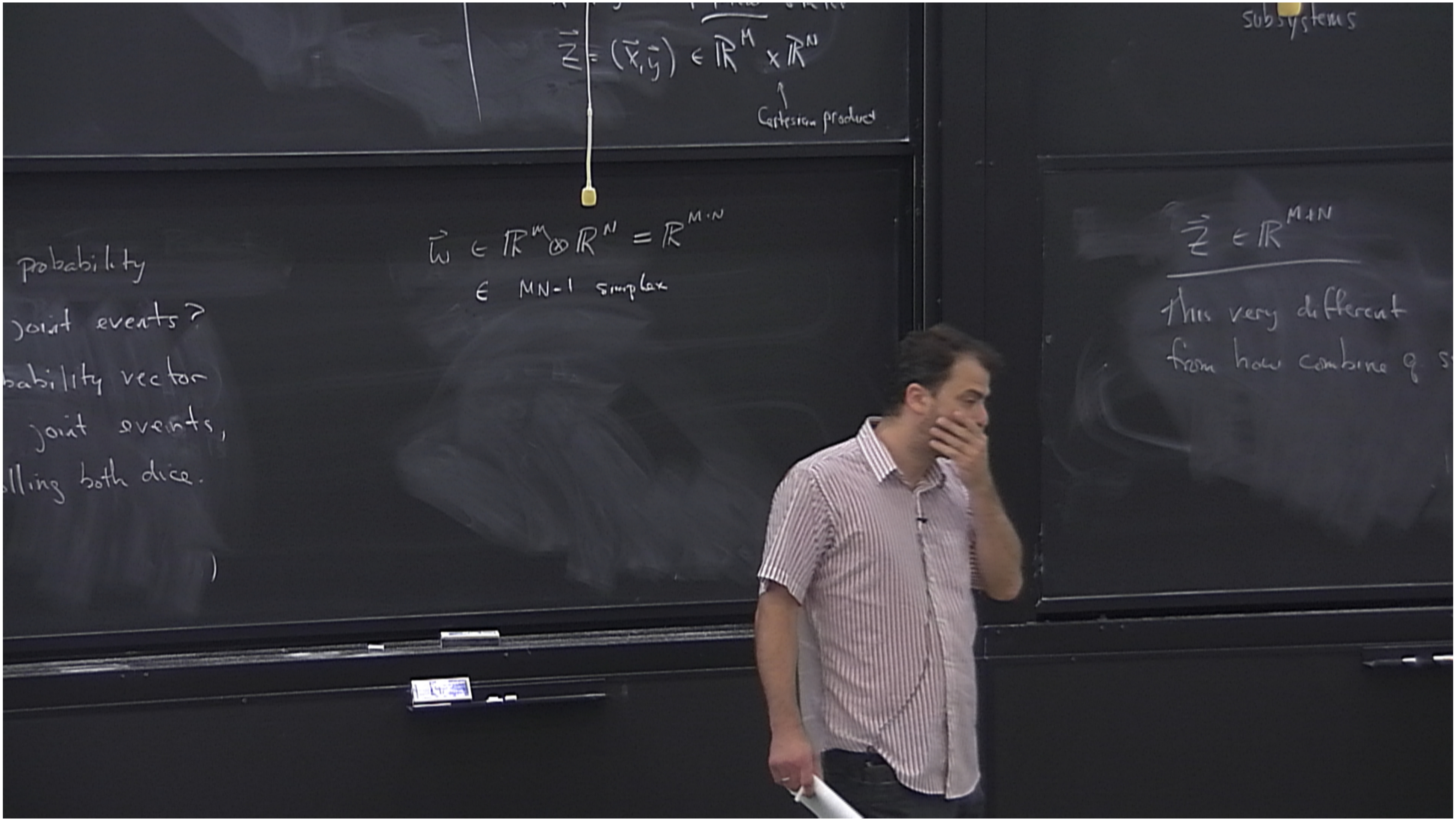

p̄ ∈ ℝ^M
q̄ ∈ ℝ^N
Consider M=3 ⇒ ℝ^3
(0,0,1)
(0,1,0)
(1,0,0)
2-simplex
How do we compose probability


How do we compose probability
vectors for joint events?
2-simplex
Consider
Suppose

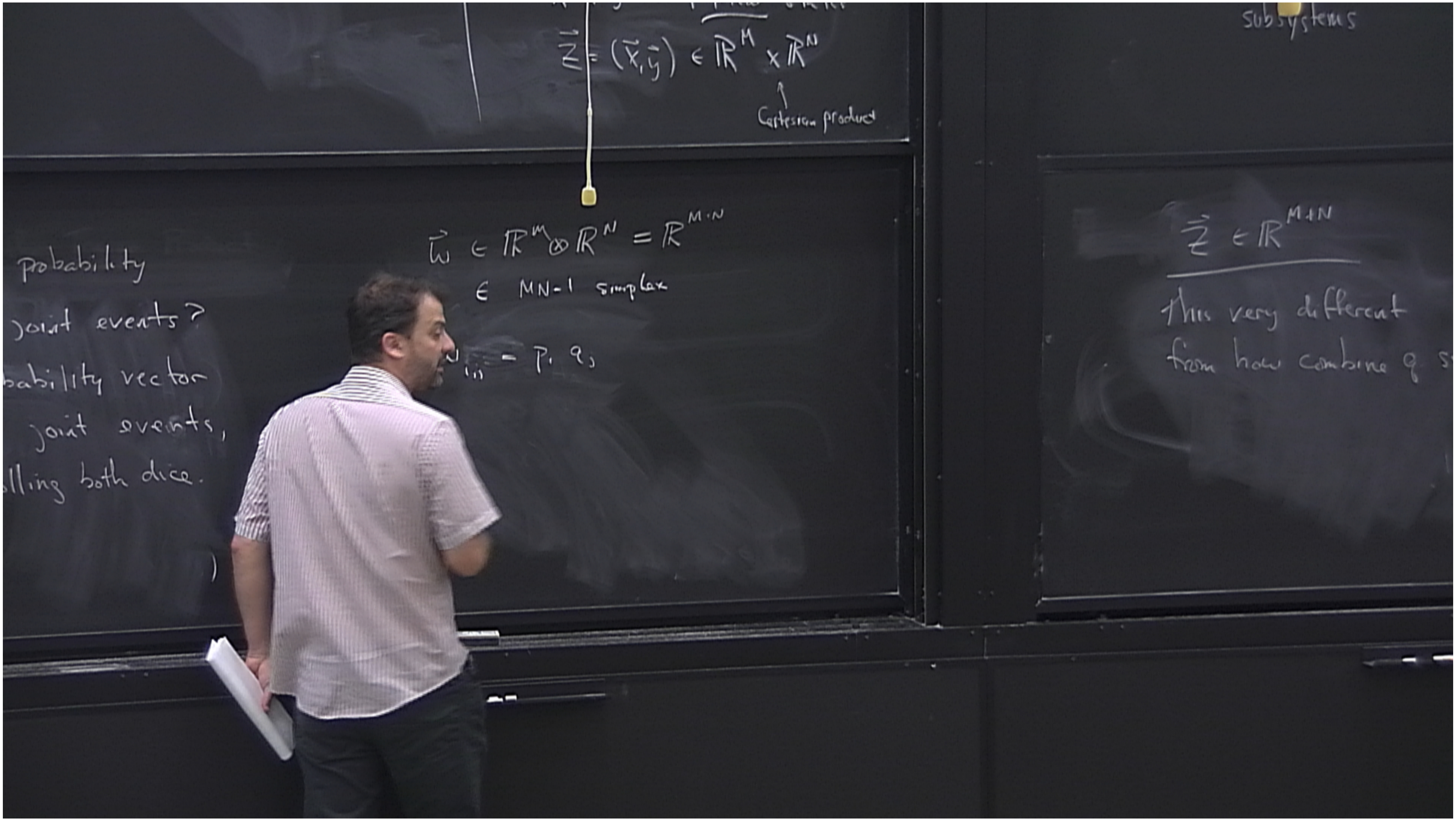

How do we compose probability
vectors for joint events?
the probability vector
Consider M=3 ⇒ ℝ³
2-simplex
(0,0,1)
(0,1,0)
(1,0,0)
Consider
Suppose


$\hat{B}_e \in \mathcal{L}(\mathcal{H}_B)$
$\bar{p} \in \mathbb{R}^M$
$\bar{q} \in \mathbb{R}^N$
Consider $M=3 \Rightarrow \mathbb{R}^3$
(0,0,1)
(0,1,0)
(1,0,0)
2-simplex
How do we compose probability
for joint events?
the probability vector
describing joint events,
$\hat{A}: |\psi\rangle \to |\phi\rangle$
$\hat{A}|\psi\rangle = |\phi\rangle$
$|\psi\rangle \in \mathcal{H}_A$
$|\phi\rangle \in \mathcal{H}_A$
Consider $|\chi\rangle \in \mathcal{H}$
Suppose


$\vec{z} = (\vec{x}, \vec{y}) \in \mathbb{R}^M \times \mathbb{R}^N$
Cartesian product
How do we compose probability vectors for joint events?
$\vec{w}$ the probability vector describing joint events, eg rolling both dice.
$\vec{w} \in \mathbb{R}^M \otimes \mathbb{R}^N = \mathbb{R}^{M \cdot N}$
$\in$ MN-1 simplex



subsystems
Cartesian product
probability
joint events?
bability vector
joint events,
lling both dice.
∈ MN-1 simplex
this very different
from how combine q s


subsystems
Cartesian product
probability
joint events?
bability vector
joint events,
lling both dice.
∈ MN-1 simplex
this very different
from how combine q s