Title: 14/15 PSI - Mathematica

Date: Sep 04, 2014 10:30 AM

URL: http://pirsa.org/14090039

Abstract:

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Numerics.nb

Numerical Methods

Introduction to an Introduction

Outline

Below, we solve a differential equation with numerical methods.
That is, we will obtain an approximate solution, employing methods that work for a large variety of differential equations, even when exact methods fail.

We will use two different methods, demonstrating the depth of the field of Numerical Analysis.

Method I

Orthogonal Polynomials

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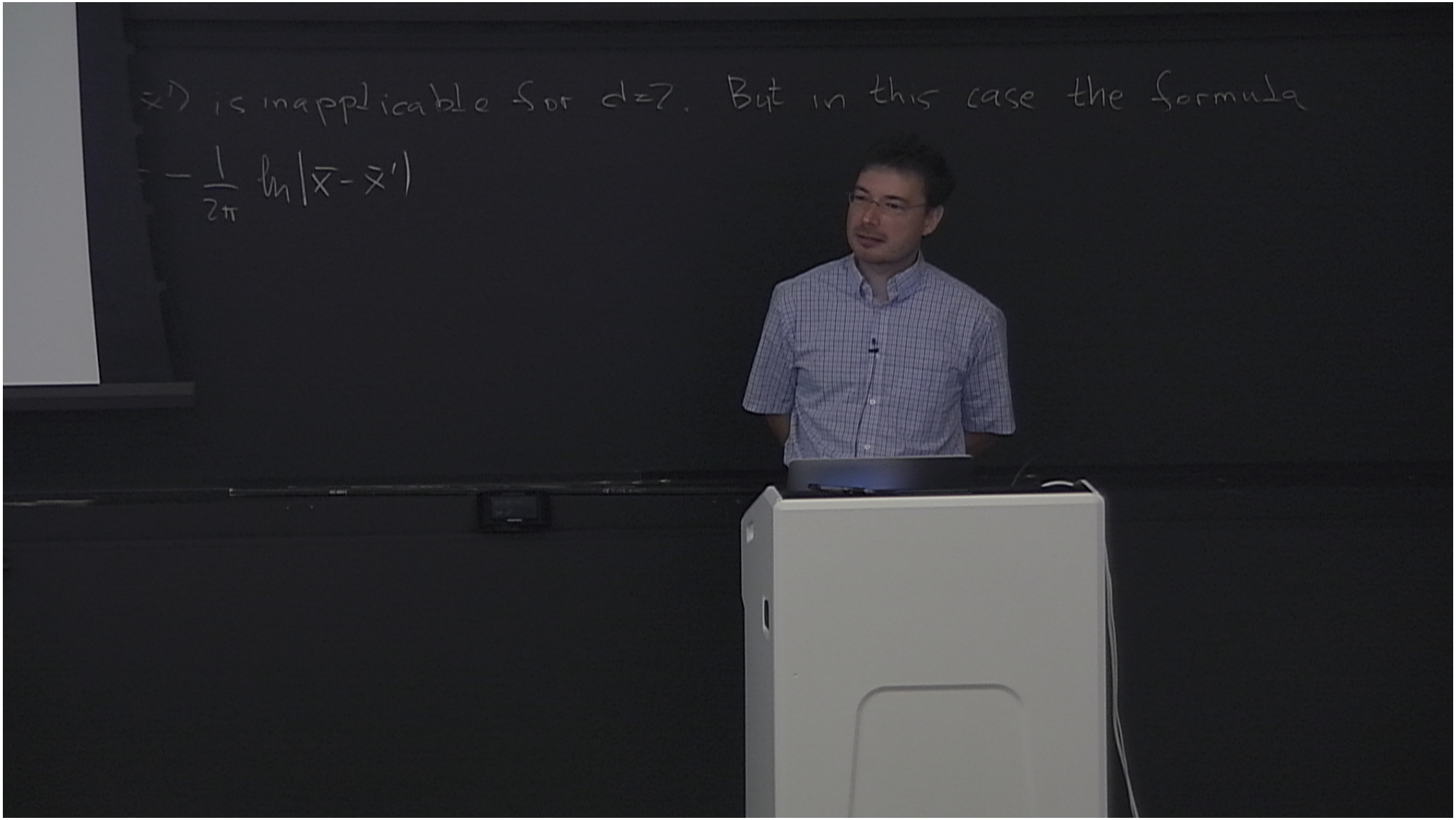
x') is inapplicable for d=2. But in this case the formula
$-\frac{1}{2\pi}\ln|\vec{x}-\vec{x}'|$

Method I

Orthogonal Polynomials

To simplify things, we restrict ourselves to a one-dimensional case.
We choose a domain of [-1;+1].
An appropriate polynomial basis are Legendre or Chebyshev polynomials.

```
Plot[Table[LegendreP[i, x], {i, 0, 4}], {x, -1, 1}]
```

We test orthogonality:
Note: Chebyshev polynomials have a weight function in their dot product!

$$\text{dot}[f\_, g\_, x\_] := \int_{-1}^{1} f\, g\, dx$$

```
MatrixForm[Table[dot[LegendreP[i, x], LegendreP[j, x], x],
  {i, 0, 4}, {j, 0, 4}]]
```

The polynomials are orthogonal, but not orthonormal.

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

$$\text{expand}[f\_, x\_, n\_] := \text{Table}\left[\frac{\text{dot}[f, \text{LegendreP}[i, x], x]}{\ldots}, \right.$$

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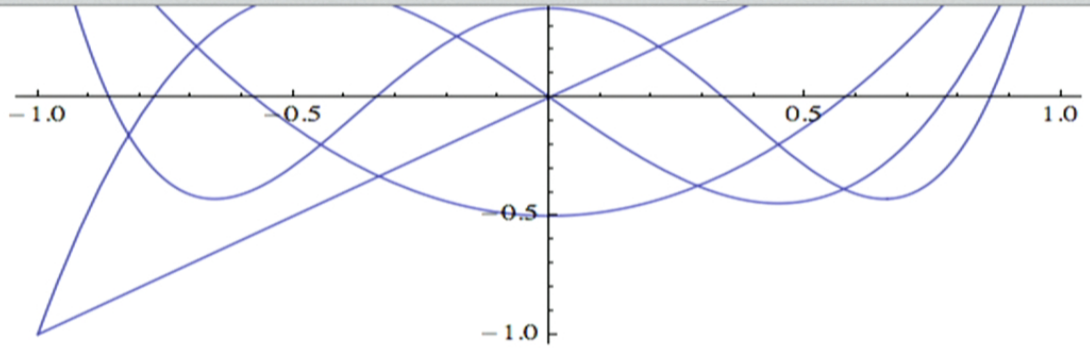
Out[1]=



We test orthogonality:
Note: Chebyshev polynomials have a weight function in their dot product!

```
dot[f_, g_, x_] := ∫_-1^1 f g dx
```

```
MatrixForm[Table[dot[LegendreP[i, x], LegendreP[j, x], x],
  {i, 0, 4}, {j, 0, 4}]]
```

The polynomials are orthogonal, but not orthonormal.

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

```
expand[f_, x_, n_] := Table[dot[f, LegendreP[i, x], x] / dot[LegendreP[i, x], LegendreP[i, x], x],
  {i, 0, n}]
```

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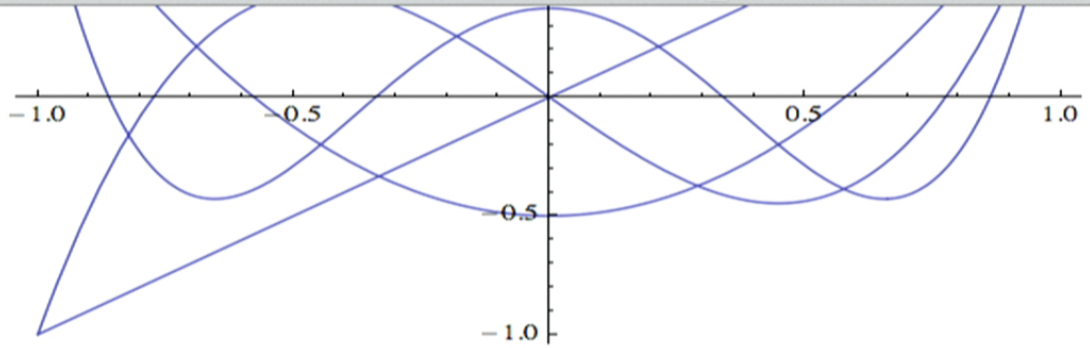
Numerics.nb

Out[1]=



We test orthogonality:
Note: Chebyshev polynomials have a weight function in their dot product!

```
dot[f_, g_, x_] := Integrate[f*g, {x, -1, 1}]

MatrixForm[Table[dot[LegendreP[i, x], LegendreP[j, x], x],
  {i, 0, 4}, {j, 0, 4}]]
```

The polynomials are orthogonal, but not orthonormal.

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

$$\texttt{expand}[f\_, x\_, n\_] := \texttt{Table}\left[\frac{\texttt{dot}[f, \texttt{LegendreP}[i, x], x]}{\texttt{dot}[\texttt{LegendreP}[i, x], \texttt{LegendreP}[i, x], x]}, \{i, 0, n\}\right]$$

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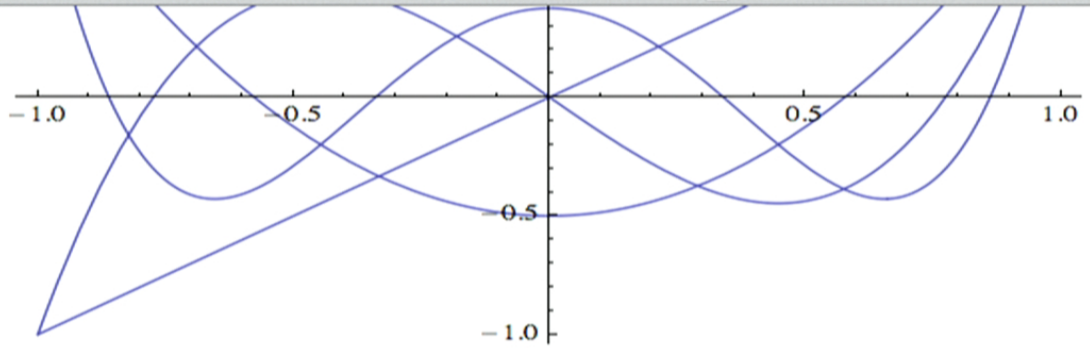
Numerics.nb

Out[1]=



We test orthogonality:

Note: Chebyshev polynomials have a weight function in their dot product!

In[2]:=
```
dot[f_, g_, x_] := ∫_-1^1 f g dx
```

In[3]:=
```
MatrixForm[Table[dot[LegendreP[i, x], LegendreP[j, x], x],
  {i, 0, 4}, {j, 0, 4}]]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

Assuming a matrix | Use as *a list of lists* instead

determinant inverse matrix plot eigenvalues *more...*

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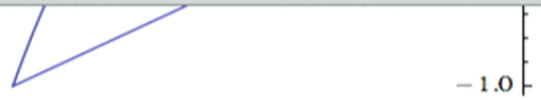
Numerics.nb



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MatrixForm[Table[dot[LegendreP[i, x], LegendreP[j, x], x],
  {i, 0, 4}, {j, 0, 4}]]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

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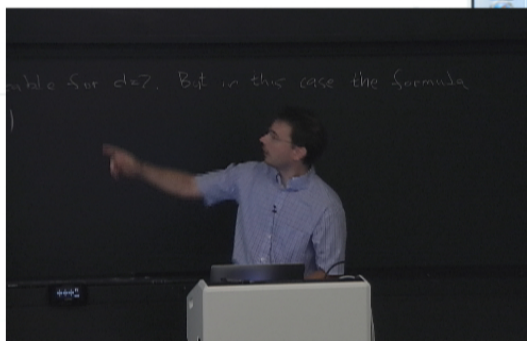
Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:



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$$\begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

The polynomials are orthogonal, but not orthonormal.

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

In[4]:=
```
expand[f_, x_, n_] := Table[dot[f, LegendreP[i, x], x] / dot[LegendreP[i, x], LegendreP[i, x], x],
  {i, 0, n}]
```

In[5]:=
```
recon[cs_, x_] := Sum[cs[[i + 1]] LegendreP[i, x], {i, 0, Length[cs] - 1}]
```

Test this:

```
expand[Cos[Pi x], x, 4]
```

Reconstruct a function from this expansion

```
Simplify[recon[expand[Cos[Pi x], x, 4], x]]
```

Plot various expansion orders

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Numerics.nb

$$\begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

The polynomials are orthogonal, but not orthonormal.

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

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In[4]:= expand[f_, x_, n_] := Table[dot[f, LegendreP[i, x], x] / dot[LegendreP[i, x], LegendreP[i, x], x],
          {i, 0, n}]

In[5]:= recon[cs_, x_] := Sum[cs[[i + 1]] LegendreP[i, x], {i, 0, Length[cs] - 1}]
```

Test this:

```
In[6]:= expand[Cos[Pi x], x, 4]
```

$$\text{Out[6]=} \left\{0, 0, -\frac{15}{\pi^2}, 0, \frac{9\left(210 - 20\,\pi^2\right)}{2\,\pi^4}\right\}$$

total sort mean delete duplicates more...

Reconstruct a function from this expansion

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Numerics.nb

Expansion into Polynomials

Expand a function in Legendre polynomials up to order n:

```
In[4]:= expand[f_, x_, n_] := Table[dot[f, LegendreP[i, x], x] / dot[LegendreP[i, x], LegendreP[i, x], x], {i, 0, n}]

In[5]:= recon[cs_, x_] := Sum[cs[[i + 1]] LegendreP[i, x], {i, 0, Length[cs] - 1}]
```

Test this:

```
In[6]:= expand[Cos[Pi x], x, 4]
```

$$\text{Out[6]}= \left\{0,\ 0,\ -\frac{15}{\pi^2},\ 0,\ \frac{9\left(210-20\,\pi^2\right)}{2\,\pi^4}\right\}$$

Reconstruct a function from this expansion

```
In[7]:= Simplify[recon[expand[Cos[Pi x], x, 4], x]]
```

$$\text{Out[7]}= -\frac{105\left(\pi^2\left(2-24\,x^2+30\,x^4\right)-9\left(3-30\,x^2+35\,x^4\right)\right)}{8\,\pi^4}$$

plot expand factor x derivative more...

Plot various expansion orders

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Numerics.nb

Reconstruct a function from this expansion

In[7]:=
```
Simplify[recon[expand[Cos[Pi x], x, 4], x]]
```

Out[7]=
$$-\frac{105\left(\pi^2\left(2-24\,x^2+30\,x^4\right)-9\left(3-30\,x^2+35\,x^4\right)\right)}{8\,\pi^4}$$

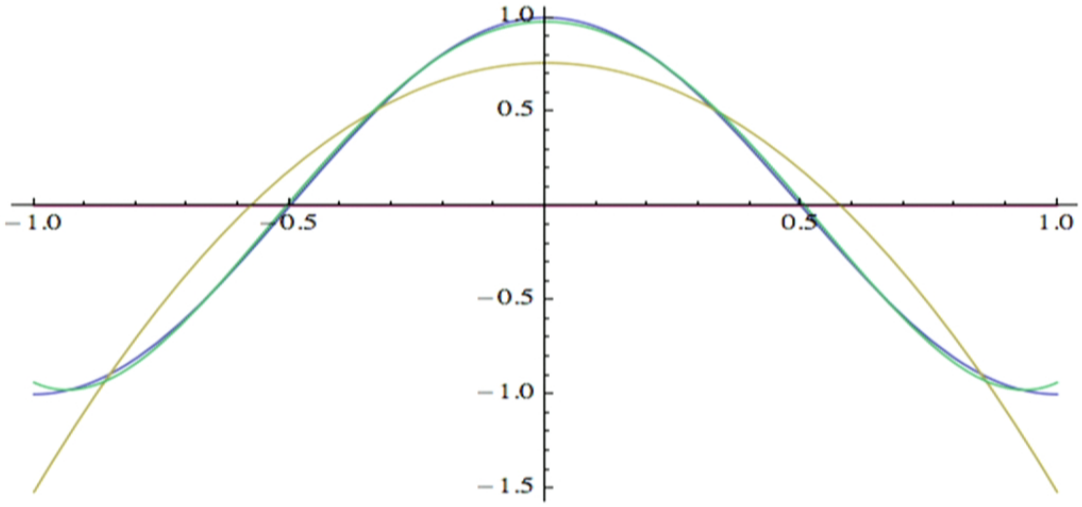
Plot various expansion orders

In[8]:=
```
t = Table[recon[expand[Cos[Pi x], x, i], x], {i, 0, 4, 2}]
```

Out[8]=
$$\left\{0,\ -\frac{15\left(-1+3\,x^2\right)}{2\,\pi^2},\ -\frac{15\left(-1+3\,x^2\right)}{2\,\pi^2}+\frac{9\left(210-20\,\pi^2\right)\left(3-30\,x^2+35\,x^4\right)}{16\,\pi^4}\right\}$$

In[9]:=
```
Plot[{Cos[Pi x], t}, {x, -1, 1}]
```

Out[9]=



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Numerics.nb

Reconstruct a function from this expansion

In[7]:=
```
Simplify[recon[expand[Cos[Pi x], x, 4], x]]
```

Out[7]=
$$-\frac{105\left(\pi^2\left(2-24\,x^2+30\,x^4\right)-9\left(3-30\,x^2+35\,x^4\right)\right)}{8\,\pi^4}$$

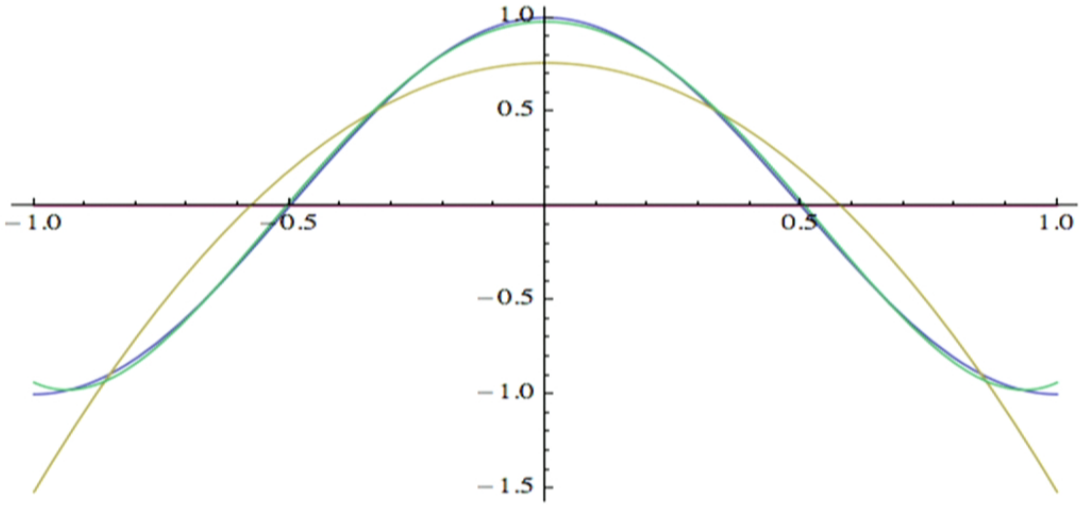
Plot various expansion orders

In[8]:=
```
t = Table[recon[expand[Cos[Pi x], x, i], x], {i, 0, 4, 2}]
```

Out[8]=
$$\left\{0,\ -\frac{15\left(-1+3\,x^2\right)}{2\,\pi^2},\ -\frac{15\left(-1+3\,x^2\right)}{2\,\pi^2}+\frac{9\left(210-20\,\pi^2\right)\left(3-30\,x^2+35\,x^4\right)}{16\,\pi^4}\right\}$$

In[9]:=
```
Plot[{Cos[Pi x], t}, {x, -1, 1}]
```

Out[9]=



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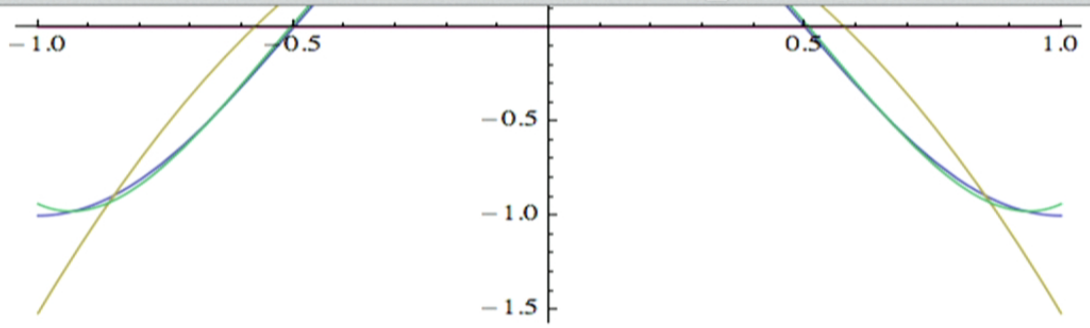
Out[9]=



frame axes image size plot style... more...

A Differential Equation

We solve the Poisson equation with a non-trivial source.
The source is a "spread-out" Dirac delta distribution.

```
$Assumptions = {w > 0}
```

A top-hat function:

```
tophat[w_, x_] := If[Abs[x] < w, 1/(2 w), 0]
```

```
Integrate[tophat[w, x], {x, -Infinity, Infinity}]
```

A Gaussian, another possible approximation for the Dirac delta distribution:

```
gaussian[w_, x_] := Exp[-(x/w)^2]
```

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Running...Numerics.nb

A Differential Equation

We solve the Poisson equation with a non-trivial source.
The source is a "spread-out" Dirac delta distribution.

In[10]:=
```
$Assumptions = {w > 0}
```

Out[10]=
```
{w > 0}
```

A top-hat function:

In[11]:=
```
tophat[w_, x_] := If[Abs[x] < w, 1/(2 w), 0]
```

In[12]:=
```
Integrate[tophat[w, x], {x, -Infinity, Infinity}]
```

Out[12]=
```
1
```

A Gaussian, another possible approximation for the Dirac delta distribution:

In[13]:=
```
gaussian[w_, x_] := Exp[-(x/w)^2] / (Sqrt[π] w)
```

In[14]:=
```
Integrate[gaussian[w, x], {x, -Infinity, Infinity}]
```

```
Plot[{tophat[1/4, x], gaussian[1/4, x]}, {x, -1, 1}, PlotRange → All]
```

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Numerics.nb

In[10]:=
```
$Assumptions = {w > 0}
```
Out[10]=
```
{w > 0}
```

A top-hat function:

In[11]:=
```
tophat[w_, x_] := If[Abs[x] < w, 1/(2 w), 0]
```

In[12]:=
```
Integrate[tophat[w, x], {x, -Infinity, Infinity}]
```
Out[12]=
```
1
```

A Gaussian, another possible approximation for the Dirac delta distribution:

In[13]:=
```
gaussian[w_, x_] := Exp[-(x/w)^2]/(Sqrt[π] w)
```

In[14]:=
```
Integrate[gaussian[w, x], {x, -Infinity, Infinity}]
```
Out[14]=
```
1
```

```
Plot[{tophat[1/4, x], gaussian[1/4, x]}, {x, -1, 1}, PlotRange → All]
```

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the “==0”, we write a-b.

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Numerics.nb

Out[12]= 1

A Gaussian, another possible approximation for the Dirac delta distribution:

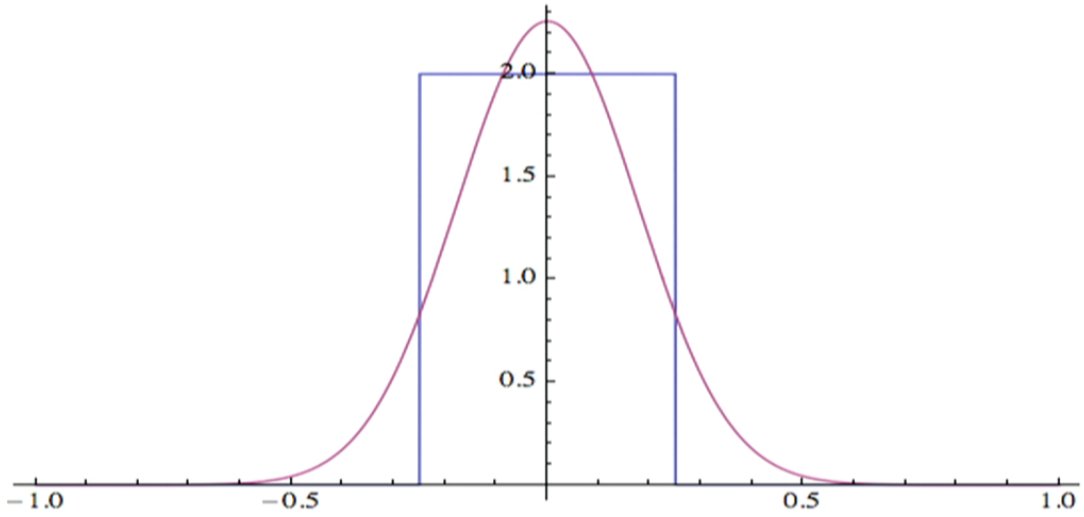
In[13]:=
$$\text{gaussian}[w\_, x\_] := \frac{\text{Exp}\left[-\left(\frac{x}{w}\right)^2\right]}{\sqrt{\pi}\, w}$$

In[14]:=
```
Integrate[gaussian[w, x], {x, -Infinity, Infinity}]
```

Out[14]= 1

In[15]:=
$$\text{Plot}\left[\left\{\text{tophat}\left[\frac{1}{4}, x\right], \text{gaussian}\left[\frac{1}{4}, x\right]\right\}, \{x, -1, 1\}, \text{PlotRange} \to \text{All}\right]$$

Out[15]=



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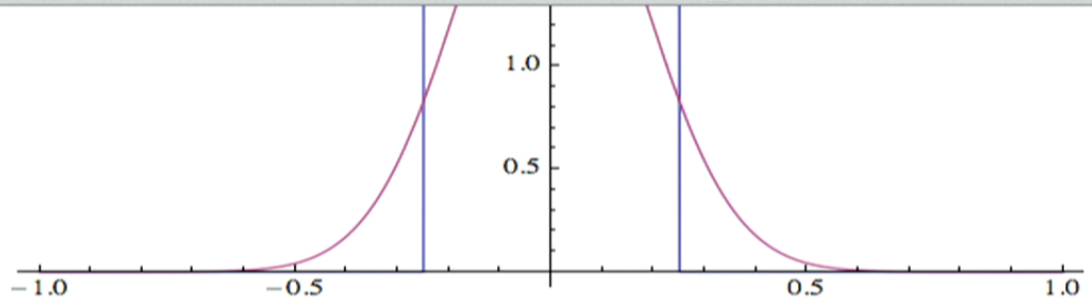
Out[15]=



frame axes image size plot style... more...

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

Below, try both "gaussian" and "tophat" approximations:

```
eqn = ∂x,x f[x] - gaussian[1/4, x]
```

Choose boundary conditions:

Dirichlet boundary conditions:

```
dir[f_, df_, x_] = f
```

Robin boundary conditions:

```
rob[f_, df_, x_] = f + x df
```

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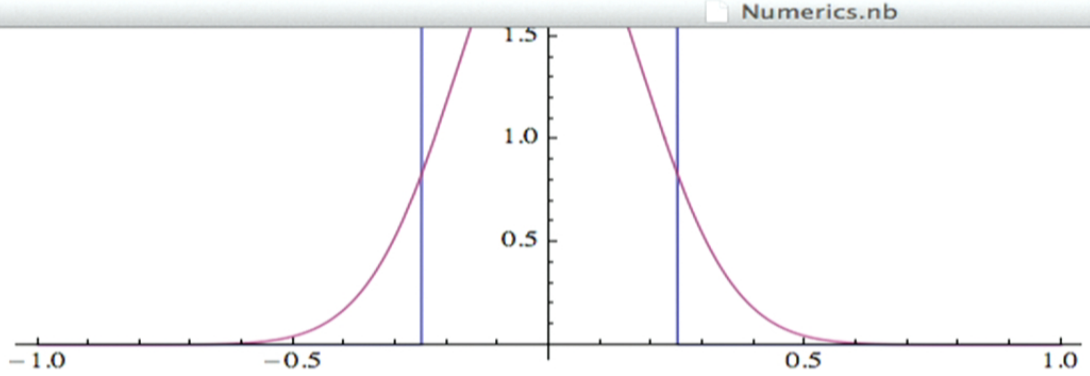
Numerics.nb

Out[15]=



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```

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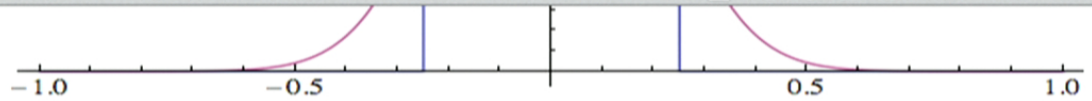

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

Below, try both "gaussian" and "tophat" approximations:

In[16]:=
```
eqn = ∂x,x f[x] - gaussian[1/4, x]
```

Out[16]=
$$-\frac{4\, e^{-16\, x^2}}{\sqrt{\pi}} + f''[x]$$

Choose boundary conditions:

Dirichlet boundary conditions:

```
dir[f_, df_, x_] = f
```

Robin boundary conditions:

```
rob[f_, df_, x_] = f + x df
```

```
bcs = {rob[f[-1], f'[-1], -1], rob[f[1], f'[1], 1]}
```

Cheat by asking Mathematica to solve this equation:

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Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

In[18]:=
```
rob[f_, df_, x_] = f + x df
```

Out[18]=
```
f + df x
```

In[20]:=
```
bcs = {dir[f[-1], f'[-1], -1], dir[f[1], f'[1], 1]}
```

Out[20]=
```
{f[-1], f[1]}
```

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

```
msol = f[x] /. msols[[1]]
```

```
Plot[msol, {x, -1, 1}]
```

Numerical Solution

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Robin boundary conditions:

```
In[18]:= rob[f_, df_, x_] = f + x df
Out[18]= f + df x

In[20]:= bcs = {dir[f[-1], f'[-1], -1], dir[f[1], f'[1], 1]}
Out[20]= {f[-1], f[1]}
```

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

```
In[21]:= msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

$$\text{Out[21]= } \left\{\left\{f[x] \to -\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4\,e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

Assuming a list of rules | Use as *a two-dimensional array* instead

apply rules to variable | apply rules to expr... | convert rules to equations | convert rules to lists

```
msol = f[x] /. msols[[1]]

Plot[msol, {x, -1, 1}]
```

Numerical Solution

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In[21]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

Out[21]=

$$\left\{\left\{f[x] \to -\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

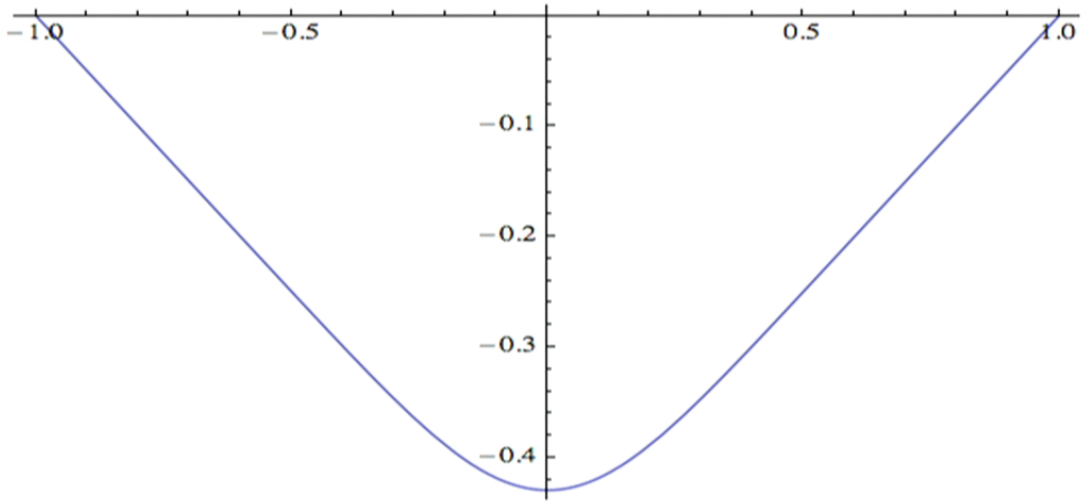
In[22]:=
```
msol = f[x] /. msols[[1]]
```

Out[22]=

$$-\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}$$

In[23]:=
```
Plot[msol, {x, -1, 1}]
```

Out[23]=



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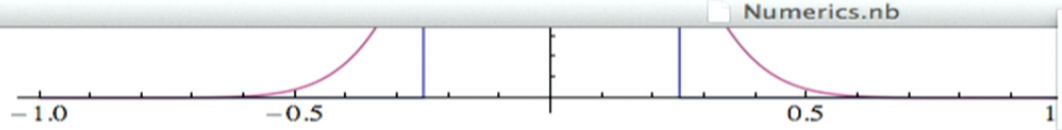


Heather Bruvelaitis
[Perimeter Institute] Re: Broken chair in Alice room

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

Below, try both "gaussian" and "tophat" approximations:

In[16]:=
```
eqn = ∂x,x f[x] - gaussian[1/4, x]
```

Out[16]=
$$-\frac{4\, e^{-16\, x^2}}{\sqrt{\pi}} + f''[x]$$

Choose boundary conditions:

Dirichlet boundary conditions:

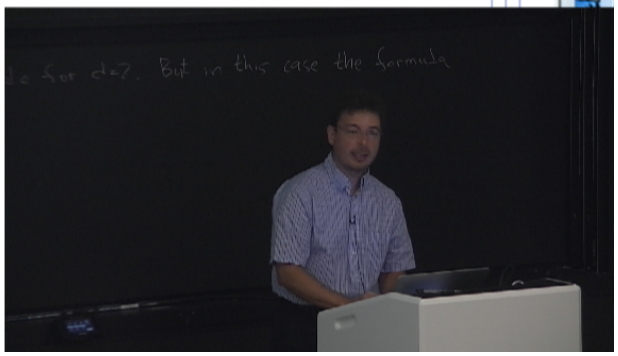
In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

In[18]:=
```
rob[f_, df_, x_] = f + x df
```

Out[18]=
```
f + df x
```



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Numerics.nb

Out[20]= {f[-1], f[1]}

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

In[21]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

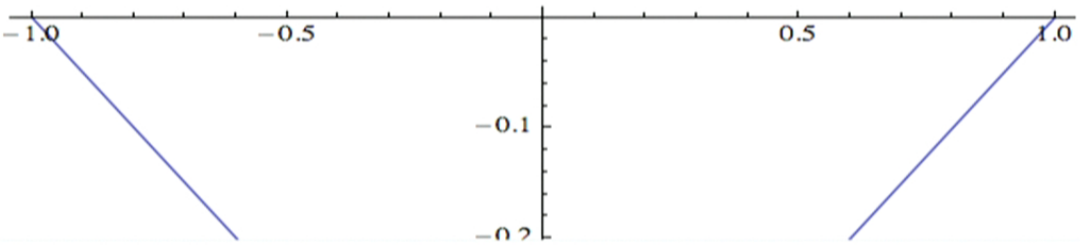
Out[21]=
$$\left\{\left\{f[x] \to -\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

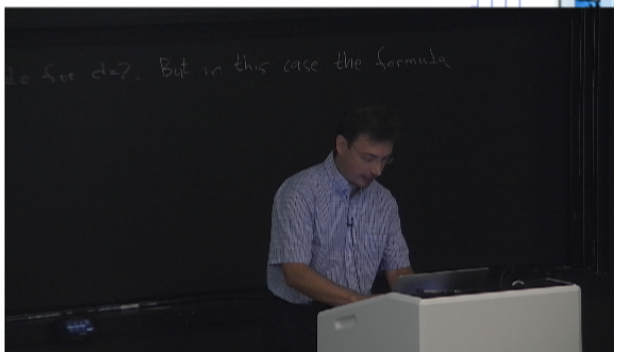
In[22]:=
```
msol = f[x] /. msols[[1]]
```

Out[22]=
$$-\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}$$

```
fp
```

In[23]:=
```
Plot[msol, {x, -1, 1}]
```





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In[22]:=
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msol = f[x] /. msols[[1]]
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Out[22]=
$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}$$

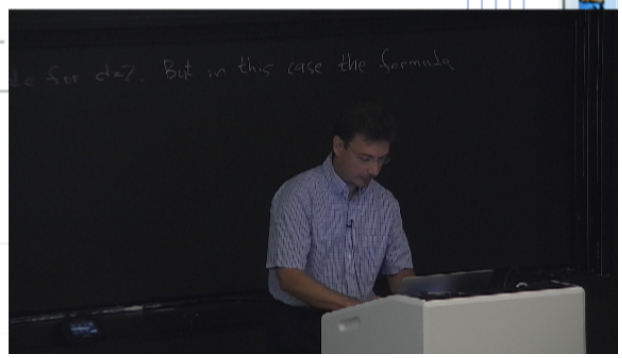
In[24]:=
```
f[x_] = msol
```

Out[24]=
$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4e^{16+16x^2}\sqrt{\pi}\right.$$

Assuming a math expression | Use as *a definition* instead

plot simplify denominator numerator *more...*

In[23]:=
```
Plot[msol, {x, -1, 1}]
```



−1.0 −0.5 0.5 1.0

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

Below, try both "gaussian" and "tophat" approximations:

In[25]:=
```
eqn = ∂x,x f[x] - gaussian[1/4, x]
```

Out[25]=
$$-\frac{4\,e^{-16\,x^2}}{\sqrt{\pi}}+\frac{4\,e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{\sqrt{\pi}}-\frac{128\,e^{-16-16\,x^2}\,x^2\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi}}8\,e^{-16-16\,x^2}\,x\left(-32\,e^{16}\,x+32\,e^{16\,x^2}\,x+128\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4\,x]-128\,e^{16+16\,x^2}\sqrt{\pi}\,x^2\,\text{Erf}[4\,x]\right)-\frac{1}{8\sqrt{\pi}}e^{-16-16\,x^2}\left(-64\,e^{16}+32\,e^{16\,x^2}-1024\,e^{16}\,x^2+1024\,e^{16\,x^2}\,x^2+128\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]+4096\,e^{16+16\,x^2}\sqrt{\pi}\,x^2\,\text{Erf}[4]-\right.$$

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Numerics.nb

−1.0 −0.5 0.5 1.0

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

Below, try both "gaussian" and "tophat" approximations:

```
Cl
Clear
Close
ClearAll
Clip
ClosenessCentrality
```

In[25]:
$$\text{gaussian}\left[\frac{1}{4}, x\right]$$

Out[25]:
$$\frac{4\, e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\, e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\, e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{\sqrt{\pi}} -$$
$$\frac{128\, e^{-16-16\,x^2}\,x^2\left(-e^{16}+e^{16\,x^2}+4\, e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\, e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{\sqrt{\pi}}$$
$$+\frac{1}{\sqrt{\pi}}8\, e^{-16-16\,x^2}\,x\left(-32\, e^{16}\,x+32\, e^{16\,x^2}\,x+128\, e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4]-\right.$$
$$\left.4\, e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4\,x]-128\, e^{16+16\,x^2}\sqrt{\pi}\,x^2\,\text{Erf}[4\,x]\right)-$$
$$\frac{1}{8\sqrt{\pi}}e^{-16-16\,x^2}\left(-64\, e^{16}+32\, e^{16\,x^2}-1024\, e^{16}\,x^2+1024\, e^{16\,x^2}\,x^2+\right.$$

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Numerics.nb

In[20]:= bcs = {dif[f[-1], f'[-1], -1], dif[f[1], f'[1], 1]}

Out[20]= {f[-1], f[1]}

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

In[21]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

Out[21]=

$$\left\{\left\{f[x] \to -\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

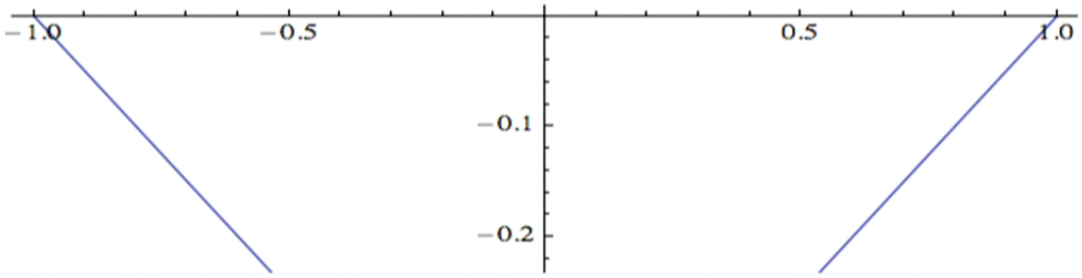
In[28]:=
```
msol = f[x] /. msols[[1]]
```

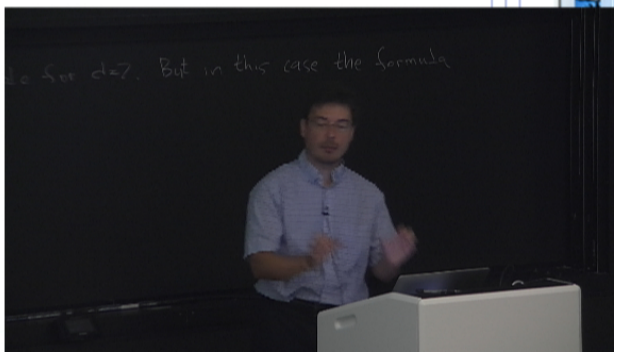
Out[28]=

$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}$$

In[29]:=
```
Plot[msol, {x, -1, 1}]
```

Out[29]=





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Numerics.nb

In[20]:= bcs = {dif[f[-1], f'[-1], -1], dif[f[1], f'[1], 1]}

Out[20]= {f[-1], f[1]}

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

In[21]:= msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]

#|{a, b, c}

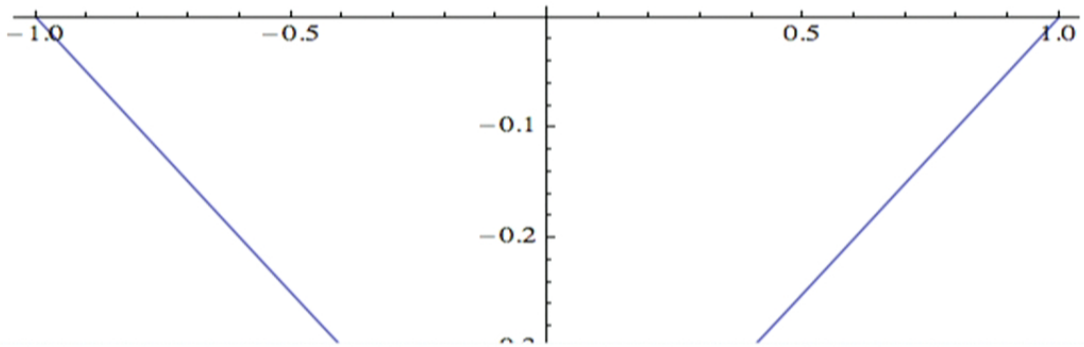
Out[30]= {a, b, c}

In[28]:= msol = f[x] /. msols[[1]]

$$\text{Out[28]}= -\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+4e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}$$

In[29]:= Plot[msol, {x, -1, 1}]

Out[29]=



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Numerics.nb

In[20]:= bcs = {dif[f[-1], f'[-1], -1], dif[f[1], f'[1], 1]}

Out[20]= {f[-1], f[1]}

Cheat by asking Mathematica to solve this equation:
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In[21]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

In[31]:=
```
# == 0 & /@ {a, b, c}
```

Out[31]= {a == 0, b == 0, c == 0}

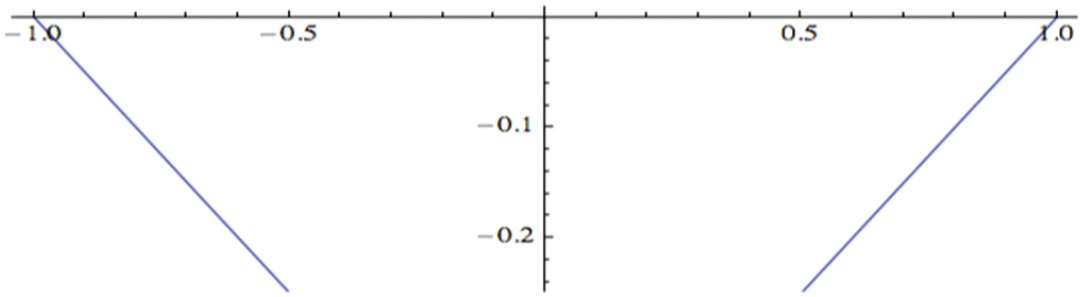
solve for (a, b, c) | find instance | eliminate a | reduce

In[28]:=
```
msol = f[x] /. msols[[1]]
```

Out[28]=
$$-\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+4\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}$$

In[29]:=
```
Plot[msol, {x, -1, 1}]
```

Out[29]=



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Numerics.nb

Numerical Solution

Choose an approximation order

```
order = 4
```

Our ansatz is a set of coefficients...

```
ansatz = Table[c[i], {i, 0, order}]
```

...which leads to a reconstructed function:

```
fn[x_] = recon[ansatz, x]
```

To solve the ODE, we need to determine these coefficients.

With this ansatz, the boundary conditions look as follows:

```
bcs /. f → fn
```

The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.

```
eqn /. f → fn
```

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be

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In[32]:=
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Numerics.nb

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Choose an approximation order

```
In[32]:= order = 4
Out[32]= 4
```

Our ansatz is a set of coefficients...

```
In[33]:= ansatz = Table[c[i], {i, 0, order}]
Out[33]= {c[0], c[1], c[2], c[3], c[4]}
```

reverse | all subsets | permutations | rotate right | more...

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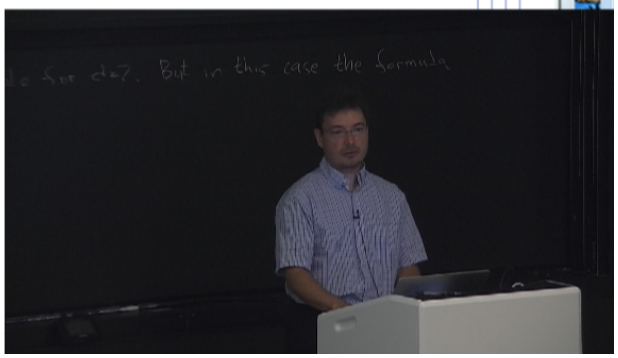

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Numerics.nb

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In[32]:=
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```
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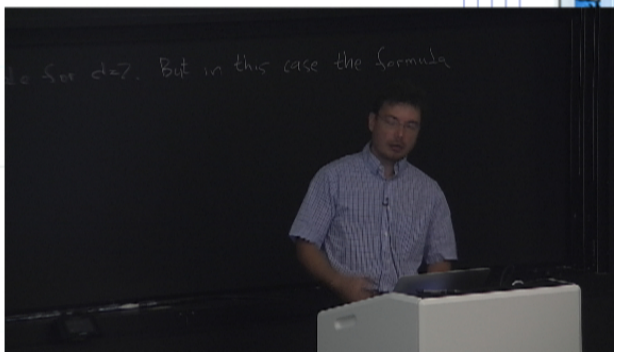
In[34]:=
```
fn[x_] = recon[ansatz, x]
```
Out[34]=
$$c[0] + x\,c[1] + \frac{1}{2}\left(-1 + 3x^2\right)c[2] + \frac{1}{2}\left(-3x + 5x^3\right)c[3] + \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)c[4]$$

Assuming a polynomial | Use as *a definition* instead

plot simplify expand factor more...

To solve the ODE, we need to determine these coefficients.

With this ansatz, the boundary conditions look as follows:



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Numerics.nb

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```
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```

reverse curl div array rules more...

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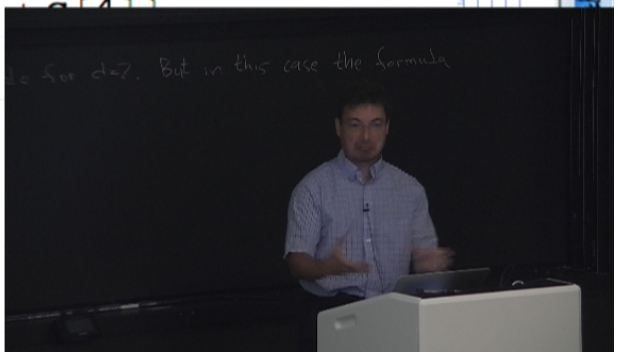

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Numerics.nb

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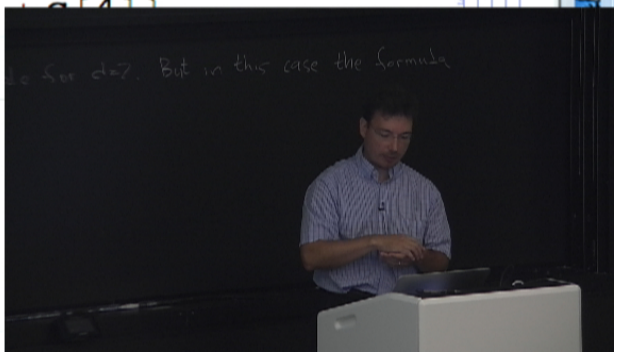

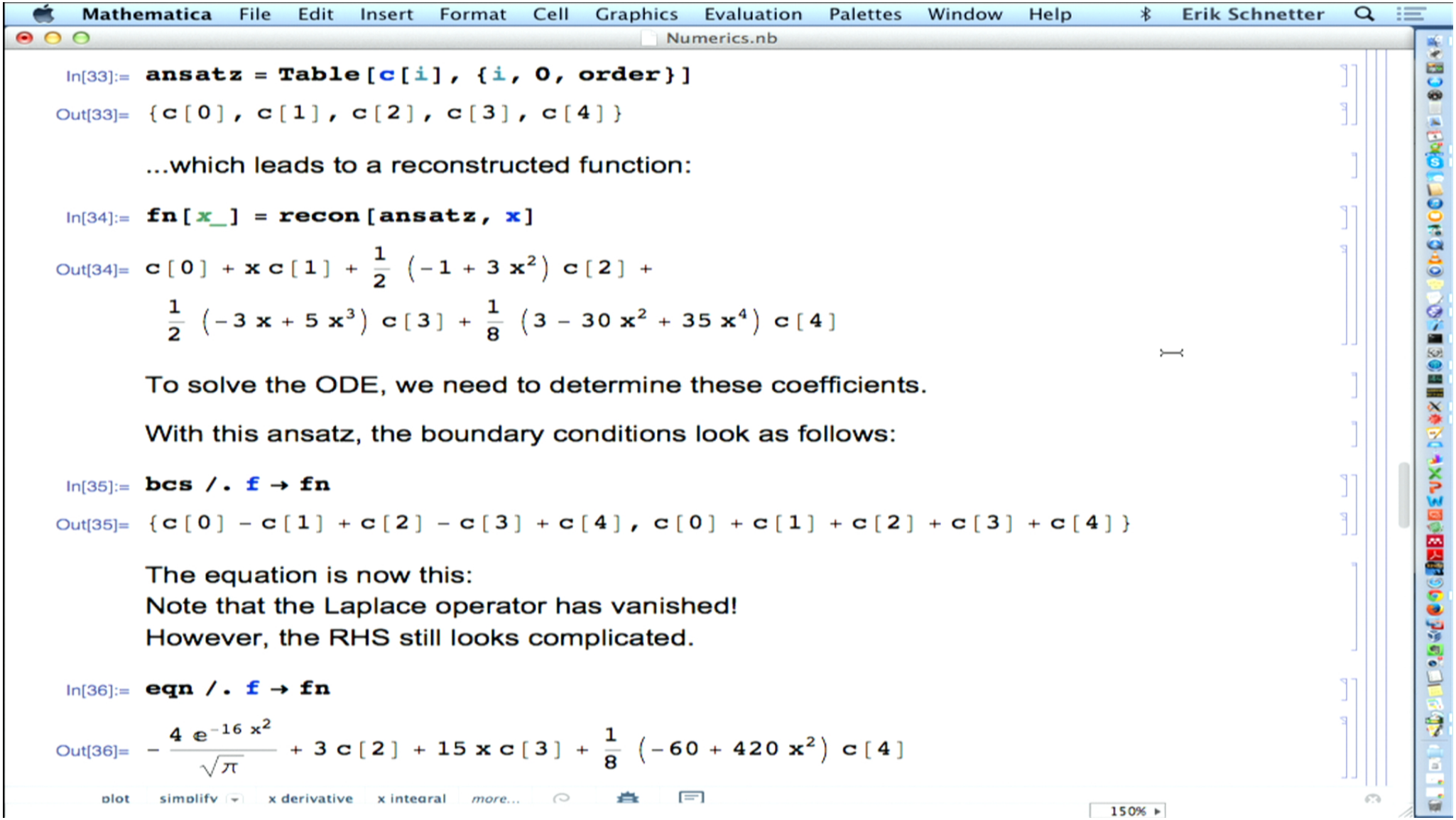
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Numerics.nb
In[33]:= ansatz = Table[c[i], {i, 0, order}]
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...which leads to a reconstructed function:
In[34]:= fn[x_] = recon[ansatz, x]
Out[34]= c[0] + x c[1] + 1/2 (-1 + 3 x^2) c[2] + 1/2 (-3 x + 5 x^3) c[3] + 1/8 (3 - 30 x^2 + 35 x^4) c[4]
To solve the ODE, we need to determine these coefficients.
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In[35]:= bcs /. f → fn
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The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.
In[36]:= eqn /. f → fn
Out[36]= -4 e^(-16 x^2)/√π + 3 c[2] + 15 x c[3] + 1/8 (-60 + 420 x^2) c[4]
plot simplify x derivative x integral more...
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Numerics.nb

Out[34]= $c[0] + x\,c[1] + \frac{1}{2}\left(-1 + 3x^2\right) c[2] +$

$\frac{1}{2}\left(-3x + 5x^3\right) c[3] + \frac{1}{8}\left(3 - 30x^2 + 35x^4\right) c[4]$

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bcs /. f → fn
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The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.

In[36]:=
```
eqn /. f → fn
```

Out[36]= $-\frac{4\,e^{-16x^2}}{\sqrt{\pi}} + 3\,c[2] + 15\,x\,c[3] + \frac{1}{8}\left(-60 + 420x^2\right) c[4]$

plot simplify x derivative x integral more...

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

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Running...Numerics.nb

Out[36]= $-\frac{4\, e^{-16\, x^2}}{\sqrt{\pi}} + 3\, c[2] + 15\, x\, c[3] + \frac{1}{8}\left(-60 + 420\, x^2\right) c[4]$

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

This is one such projection -- note that the dependency on x vanishes:

In[37]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[37]= 6 c[2] + 20 c[4] - Erf[4]

These are all projections up to our approximation order :

In[38]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Note that the last two projections are "not good", since they don't contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the respective coefficients were left out.
That means we are missing two conditions. This is also the reason why we need two additional conditions ("boundary conditions").

```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
  ansatz]
```

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Numerics.nb

Out[36]= $-\frac{4\, e^{-16\, x^2}}{\sqrt{\pi}} + 3\, c[2] + 15\, x\, c[3] + \frac{1}{8}\left(-60 + 420\, x^2\right) c[4]$

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

This is one such projection -- note that the dependency on x vanishes:

In[37]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[37]=
```
6 c[2] + 20 c[4] - Erf[4]
```

These are all projections up to our approximation order :

In[38]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[38]= $\left\{\frac{1}{2}\,(6\, c[2] + 20\, c[4] - \text{Erf}[4]),\ 15\, c[3],\right.$

$\left.\frac{5}{2}\left(\frac{3}{8\, e^{16}\sqrt{\pi}} + 14\, c[4] + \frac{29\, \text{Erf}[4]}{64}\right),\ 0,\ \frac{9\left(\frac{2120}{e^{16}\sqrt{\pi}} - 2217\, \text{Erf}[4]\right)}{16\,384}\right\}$

curl div sort reverse more...

Note that the last two projections are "not good", since they don't contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the

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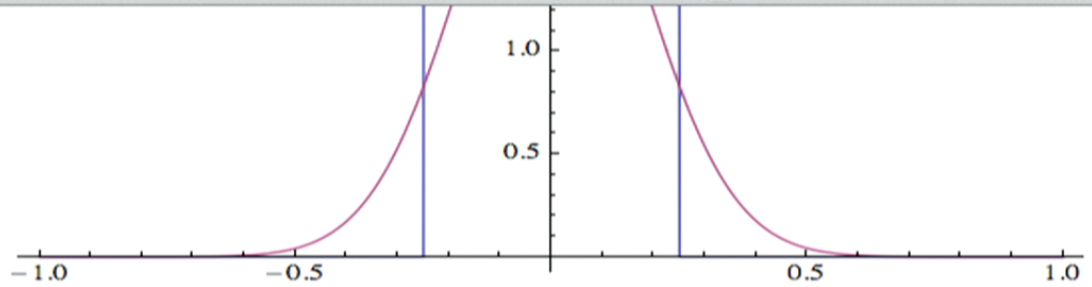
Numerics.nb

Out[15]=



The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

In[43]:=
```
eqn = ∂x,x f[x] - tophat[1/4, x]
```

Out[43]=
$$-\text{If}\left[\text{Abs}[x] < \frac{1}{4},\ \frac{1}{\frac{2}{4}},\ 0\right] + f''[x]$$

Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

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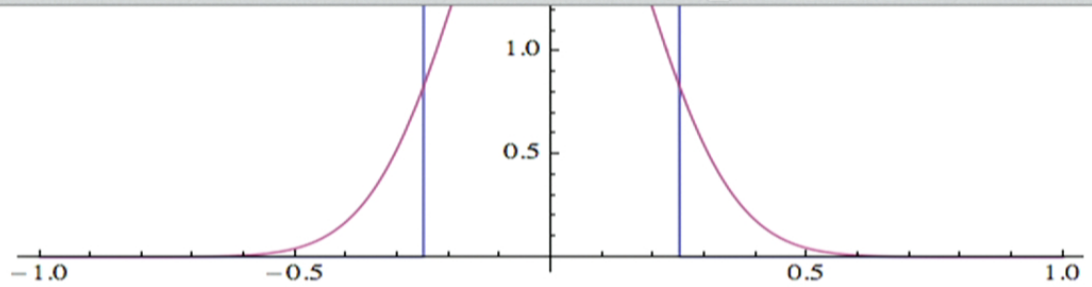
Numerics.nb

Out[15]=



The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

In[43]:=
```
eqn = ∂x,x f[x] - tophat[1/4, x]
```

Out[43]=
```
-If[Abs[x] < 1/4, 1/(2/4), 0] + f''[x]
```

Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

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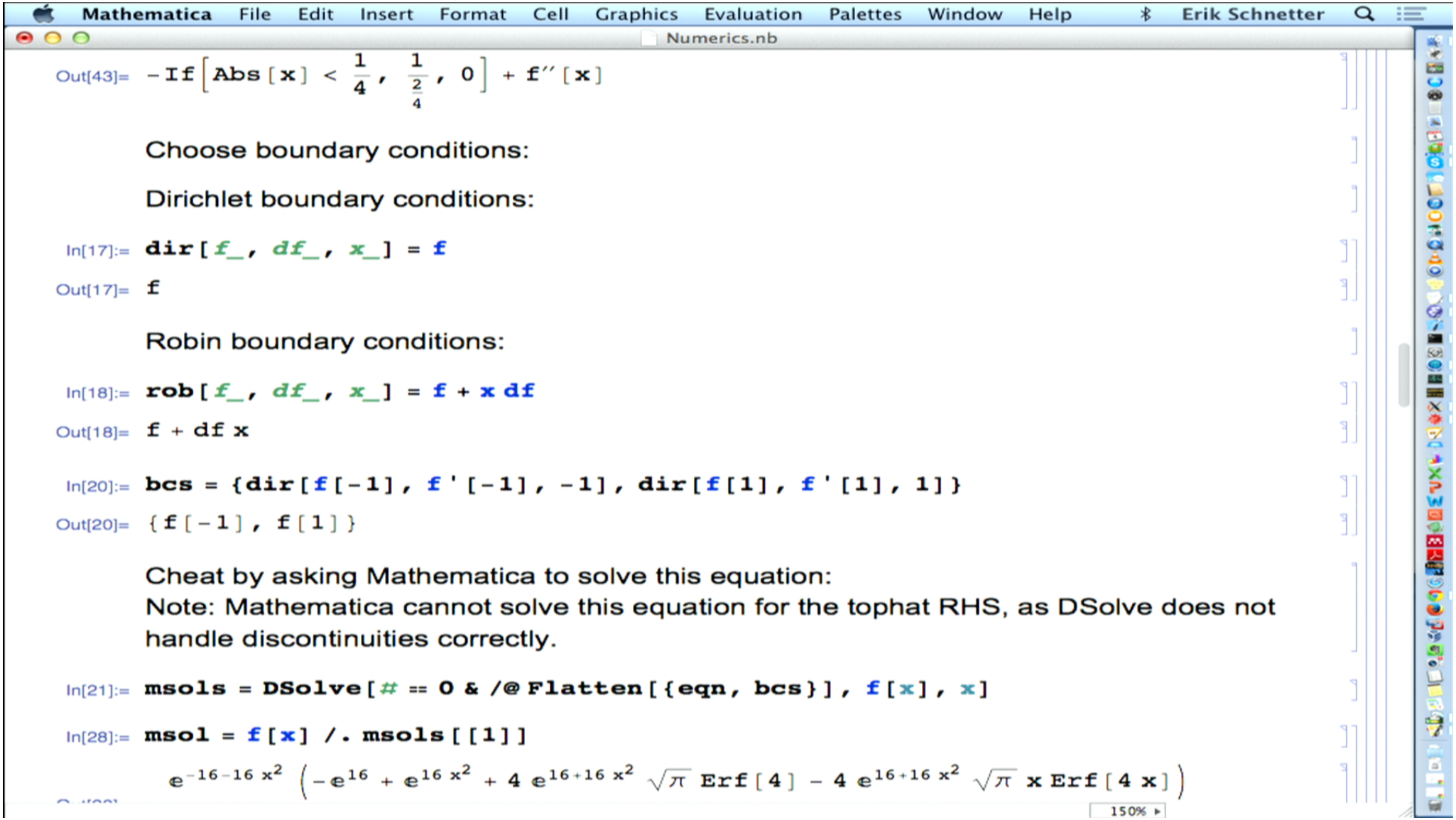
Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help Erik Schnetter
Numerics.nb
Out[43]= -If[Abs[x] < 1/4, 1/(2/4), 0] + f''[x]
Choose boundary conditions:
Dirichlet boundary conditions:
In[17]:= dir[f_, df_, x_] = f
Out[17]= f
Robin boundary conditions:
In[18]:= rob[f_, df_, x_] = f + x df
Out[18]= f + df x
In[20]:= bcs = {dir[f[-1], f'[-1], -1], dir[f[1], f'[1], 1]}
Out[20]= {f[-1], f[1]}
Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.
In[21]:= msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
In[28]:= msol = f[x] /. msols[[1]]
e^(-16-16 x^2) (-e^16 + e^(16 x^2) + 4 e^(16+16 x^2) Sqrt[π] Erf[4] - 4 e^(16+16 x^2) Sqrt[π] x Erf[4 x])
150%

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Numerics.nb

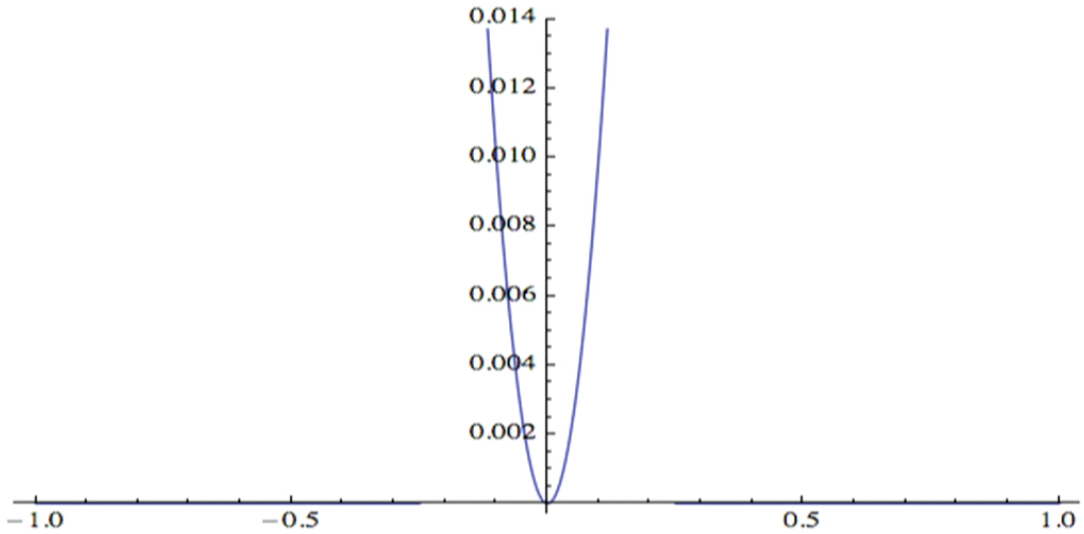
Out[44]=
$$\left\{\left\{f[x] \to \begin{cases} x^2 & \text{Abs}[x] < \frac{1}{4} \\ 0 & \text{True} \end{cases}\right\}\right\}$$

In[45]:=
```
msol = f[x] /. msols[[1]]
```

Out[45]=
$$\begin{cases} x^2 & \text{Abs}[x] < \frac{1}{4} \\ 0 & \text{True} \end{cases}$$

In[46]:=
```
Plot[msol, {x, -1, 1}]
```

Out[46]=



frame axes image size plot style... more...

Numerical Solution

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Running...Numerics.nb

However, the RHS still looks complicated.

In[48]:=
```
eqn /. f → fn
```

Out[48]=
$$3\,c[2] + 15\,x\,c[3] + \frac{1}{8}\left(-60 + 420\,x^2\right) c[4] - \text{If}\left[\text{Abs}[x] < \frac{1}{4},\ \frac{1}{\frac{2}{4}},\ 0\right]$$

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

This is one such projection -- note that the dependency on x vanishes:

In[49]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[49]=
$$-1 + 6\,c[2] + 20\,c[4]$$

These are all projections up to our approximation order :

In[50]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[38]=
$$\left\{\frac{1}{2}\left(6\,c[2] + 20\,c[4] - \text{Erf}[4]\right),\ 15\,c[3],\right.$$
$$\left.\frac{5}{2}\left(\frac{3}{8\,e^{16}\sqrt{\pi}} + 14\,c[4] + \frac{29\,\text{Erf}[4]}{64}\right),\ 0,\ \frac{9\left(\frac{2120}{e^{16}\sqrt{\pi}} - 2217\,\text{Erf}[4]\right)}{16\,384}\right\}$$

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Running...Numerics.nb

In[50]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[50]=
$$\left\{\frac{1}{2}\,(-1+6\,c[2]+20\,c[4]),\ 15\,c[3],\ \frac{5}{64}\,(15+448\,c[4]),\ 0,\ -\frac{5535}{4096}\right\}$$

Note that the last two projections are “not good”, since they don’t contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the respective coefficients were left out.
That means we are missing two conditions. This is also the reason why we need two additional conditions (“boundary conditions”).

In[51]:=
```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
  ansatz]
```

Out[51]=
$$\left\{\left\{c[0]\to -\frac{47}{192},\ c[1]\to 0,\ c[2]\to \frac{187}{672},\ c[3]\to 0,\ c[4]\to -\frac{15}{448}\right\}\right\}$$

In[52]:=
```
nsol = fn[x] /. nsols[[1]]
```

Out[52]=
$$-\frac{47}{192}+\frac{187\left(-1+3\,x^2\right)}{1344}-\frac{15\left(3-30\,x^2+35\,x^4\right)}{3584}$$

In[53]:=
```
nsol // N // Simplify
```

Out[53]=
$$-0.395682+0.532269\,x^2-0.136682\,x^4$$

Compare Mathematica’s and our numerical solution

In[42]:=
```
Plot[{msol, nsol}, {x, -1, 1}]
```

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Numerics.nb

In[52]:= `nsol = fn[x] /. nsols[[1]]`

Out[52]= $-\frac{47}{192} + \frac{187\left(-1+3x^2\right)}{1344} - \frac{15\left(3-30x^2+35x^4\right)}{3584}$

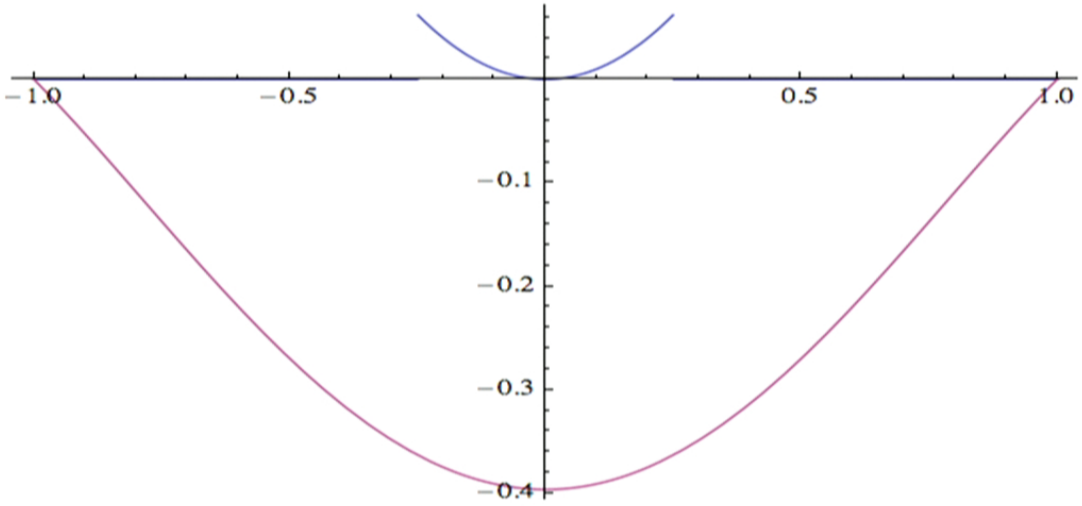
In[53]:= `nsol // N // Simplify`

Out[53]= $-0.396484 + 0.542969\,x^2 - 0.146484\,x^4$

Compare Mathematica's and our numerical solution

In[54]:= `Plot[{msol, nsol}, {x, -1, 1}]`

Out[54]=



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Numerics.nb

In[52]:= `nsol = fn[x] /. nsols[[1]]`

Out[52]= $-\frac{47}{192} + \frac{187\left(-1+3x^2\right)}{1344} - \frac{15\left(3-30x^2+35x^4\right)}{3584}$

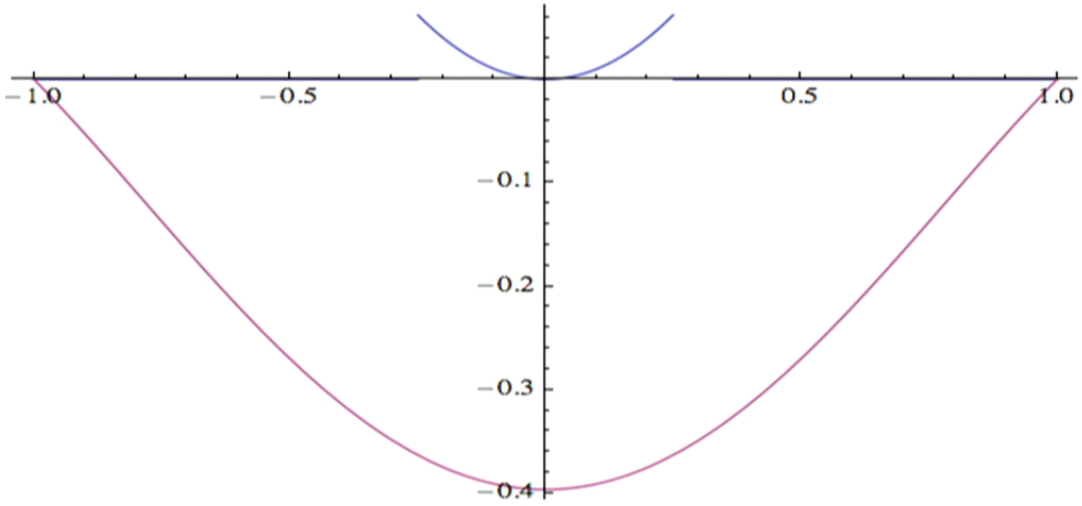
In[53]:= `nsol // N // Simplify`

Out[53]= $-0.396484 + 0.542969\,x^2 - 0.146484\,x^4$

Compare Mathematica's and our numerical solution

In[54]:= `Plot[{msol, nsol}, {x, -1, 1}]`

Out[54]=



frame axes image size plot style... more...

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much faster. However, this won't work in general, and out of laziness, we don't do that here either.

This is one such projection -- note that the dependency on x vanishes:

In[49]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[49]=
$$-1 + 6\,c[2] + 20\,c[4]$$

These are all projections up to our approximation order :

In[50]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[50]=
$$\left\{\frac{1}{2}\,(-1 + 6\,c[2] + 20\,c[4]),\ 15\,c[3],\ \frac{5}{64}\,(15 + 448\,c[4]),\ 0,\ -\frac{5535}{4096}\right\}$$

Note that the last two projections are "not good", since they don't contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the respective coefficients were left out.
That means we are missing two conditions. This is also the reason why we need two additional conditions ("boundary conditions").

In[51]:=
```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
   ansatz]
```

Out[51]=
$$\left\{\left\{c[0] \to -\frac{47}{192},\ c[1] \to 0,\ c[2] \to \frac{187}{672},\ c[3] \to 0,\ c[4] \to -\frac{15}{448}\right\}\right\}$$

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To solve the ODE, we need to determine these coefficients.

With this ansatz, the boundary conditions look as follows:

In[47]:=
```
bcs /. f → fn
```

Out[47]=
```
{c[0] - c[1] + c[2] - c[3] + c[4], c[0] + c[1] + c[2] + c[3] + c[4]}
```

The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.

In[48]:=
```
eqn /. f → fn
```

Out[48]=
$$3\,c[2] + 15\,x\,c[3] + \frac{1}{8}\left(-60 + 420\,x^2\right) c[4] - \text{If}\left[\text{Abs}[x] < \frac{1}{4},\ \frac{1}{\frac{2}{4}},\ 0\right]$$

To separate the cofficients, here we choose to project the equation onto our basis functions.
Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

This is one such projection -- note that the dependency on x vanishes:

In[49]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[49]= -1 + 6 c[2] + 20 c[4]

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Numerics.nb

Note: Since our equation is linear, it would suffice to project the RHS, which would be much faster. However, this won't work in general, and out of laziness, we don't do that here either.

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In[50]:=
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conds = expand[eqn /. f → fn, x, order]
```

Out[50]=
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In[51]:=
```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
   ansatz]
```

Out[51]=
$$\left\{\left\{c[0] \to -\frac{47}{192},\ c[1] \to 0,\ c[2] \to \frac{187}{672},\ c[3] \to 0,\ c[4] \to -\frac{15}{448}\right\}\right\}$$

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Out[49]=
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These are all projections up to our approximation order :

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conds = expand[eqn /. f → fn, x, order]
```

Out[50]=
$$\left\{\frac{1}{2}\,(-1 + 6\,c[2] + 20\,c[4]),\ 15\,c[3],\ \frac{5}{64}\,(15 + 448\,c[4]),\ 0,\ -\frac{5535}{4096}\right\}$$

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In[51]:=
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nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
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```

Out[51]=
$$\left\{\left\{c[0] \to -\frac{47}{192},\ c[1] \to 0,\ c[2] \to \frac{187}{672},\ c[3] \to 0,\ c[4] \to -\frac{15}{448}\right\}\right\}$$

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Numerics.nb

Method 2

Piecewise Polynomial Ansatz

Location of the junctions

```
xlocs[n_] := Range[-1, 1, 2/n]
```

```
xlocs[4]
```

Expand by sampling the function

```
ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
```

```
ppexpand[f[x], x, 4]
```

Reconstruct by interpolation, using Mathematica's interpolation function

Here we use second order (linear) interpolation; could easily use a higher order, and would definitively do so in practice

```
pprecon[cs_, x_] :=
 Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
   InterpolationOrder → 3][x]
```

```
ppexpand[Cos[Pi x], x, 4]
```

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Numerics.nb

Method 2

Piecewise Polynomial Ansatz

Location of the junctions

In[55]:=
```
xlocs[n_] := Range[-1, 1, 2/n]
```

In[56]:=
```
xlocs[4]
```

Out[56]=
$$\left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

total mean max reverse more...

Expand by sampling the function

```
ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
```

```
ppexpand[f[x], x, 4]
```

Reconstruct by interpolation, using Mathematica's interpolation function

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Numerics.nb

Method 2

Piecewise Polynomial Ansatz

Location of the junctions

```
In[55]:= xlocs[n_] := Range[-1, 1, 2/n]

In[56]:= xlocs[4]
```

$$\text{Out[56]=} \quad \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

Expand by sampling the function

```
In[57]:= ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]

In[58]:= ppexpand[f[x], x, 4]
```

$$\text{Out[58]=} \quad \left\{f[-1], f\left[-\frac{1}{2}\right], f[0], f\left[\frac{1}{2}\right], f[1]\right\}$$

reverse all subsets permutations rotate right more...

Reconstruct by interpolation, using Mathematica's interpolation function

Here we use second order (linear) interpolation; could easily use a higher order, and

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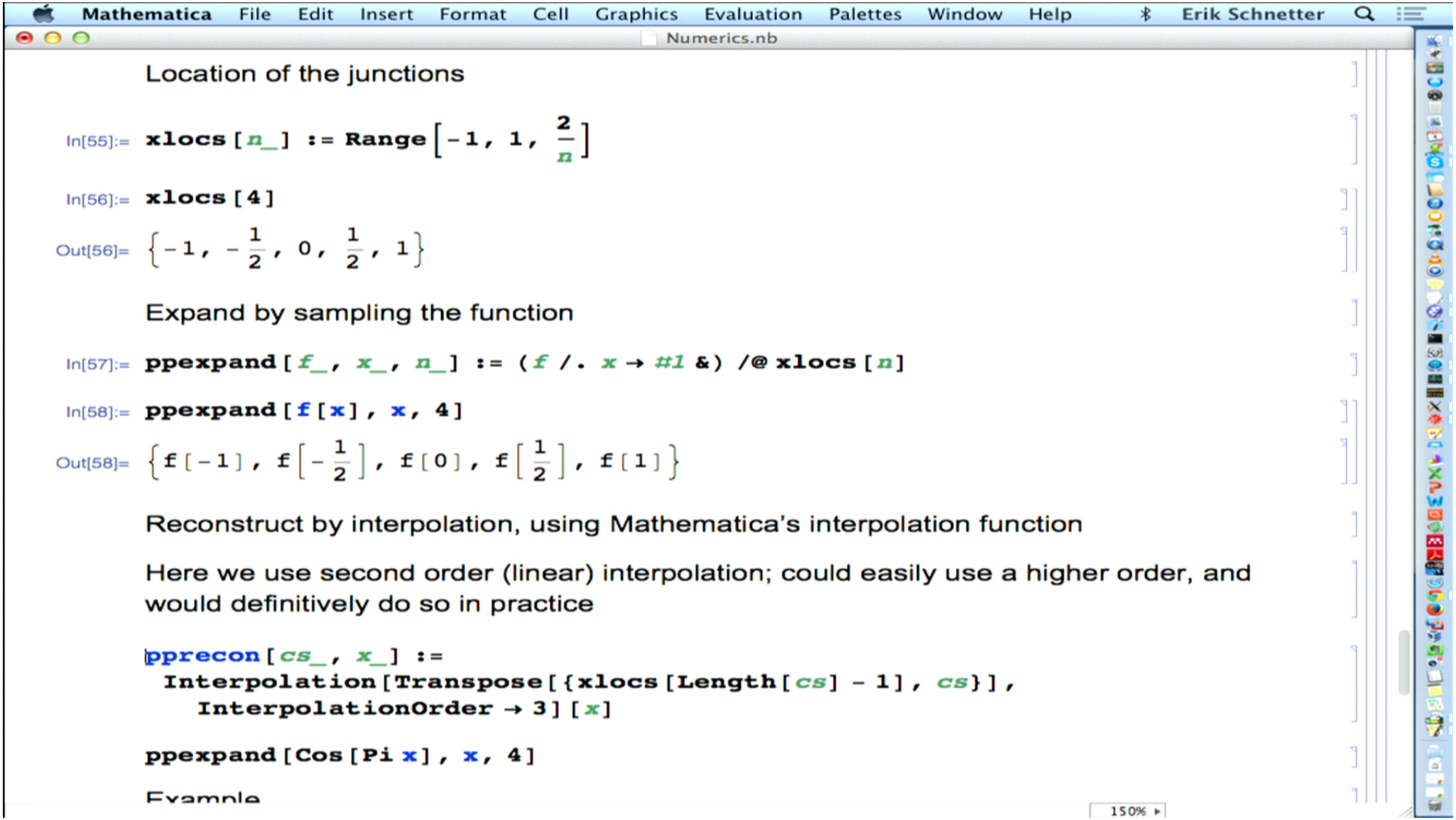
Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help Erik Schnetter
Numerics.nb
Location of the junctions
In[55]:= xlocs[n_] := Range[-1, 1, 2/n]
In[56]:= xlocs[4]
Out[56]= {-1, -1/2, 0, 1/2, 1}
Expand by sampling the function
In[57]:= ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
In[58]:= ppexpand[f[x], x, 4]
Out[58]= {f[-1], f[-1/2], f[0], f[1/2], f[1]}
Reconstruct by interpolation, using Mathematica's interpolation function
Here we use second order (linear) interpolation; could easily use a higher order, and would definitively do so in practice
pprecon[cs_, x_] :=
Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
InterpolationOrder → 3][x]
ppexpand[Cos[Pi x], x, 4]
Example
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Numerics.nb

In[55]:=
```
xlocs[n_] := Range[-1, 1, 2/n]
```

In[56]:=
```
xlocs[4]
```

Out[56]=
$$\left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

Expand by sampling the function

In[57]:=
```
ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
```

In[58]:=
```
ppexpand[f[x], x, 4]
```

Out[58]=
$$\left\{f[-1], f\left[-\frac{1}{2}\right], f[0], f\left[\frac{1}{2}\right], f[1]\right\}$$

Reconstruct by interpolation, using Mathematica's interpolation function

Here we use second order (linear) interpolation; could easily use a higher order, and would definitively do so in practice

In[59]:=
```
pprecon[cs_, x_] :=
 Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
   InterpolationOrder → 3][x]

ppexpand[Cos[Pi x], x, 4]
```

Example

```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

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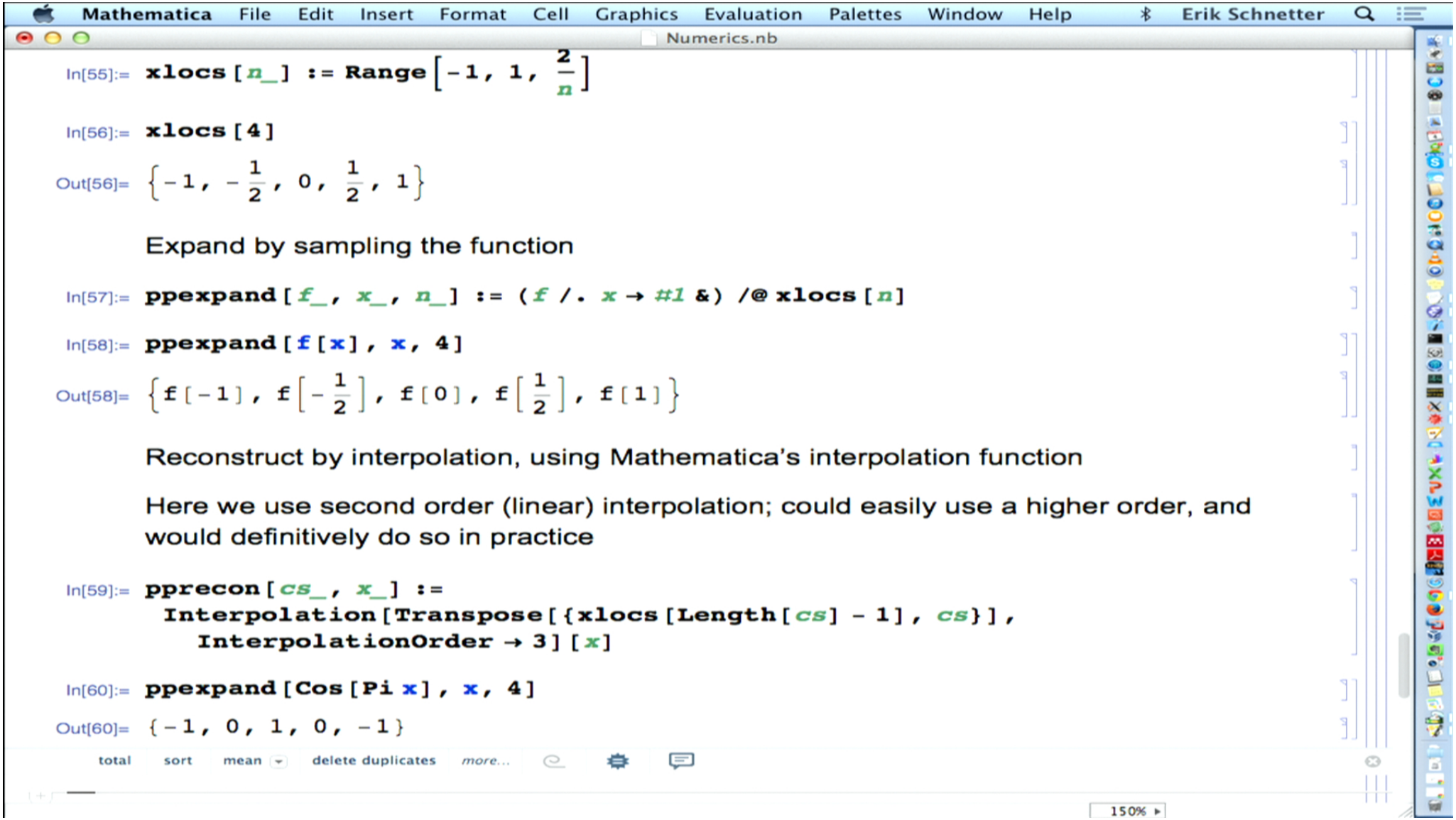
Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help Erik Schnetter
Numerics.nb
In[55]:= xlocs[n_] := Range[-1, 1, 2/n]
In[56]:= xlocs[4]
Out[56]= {-1, -1/2, 0, 1/2, 1}
Expand by sampling the function
In[57]:= ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
In[58]:= ppexpand[f[x], x, 4]
Out[58]= {f[-1], f[-1/2], f[0], f[1/2], f[1]}
Reconstruct by interpolation, using Mathematica's interpolation function
Here we use second order (linear) interpolation; could easily use a higher order, and would definitively do so in practice
In[59]:= pprecon[cs_, x_] :=
Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
InterpolationOrder → 3][x]
In[60]:= ppexpand[Cos[Pi x], x, 4]
Out[60]= {-1, 0, 1, 0, -1}
total sort mean delete duplicates more...
150%

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Numerics.nb

```
    Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
      InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

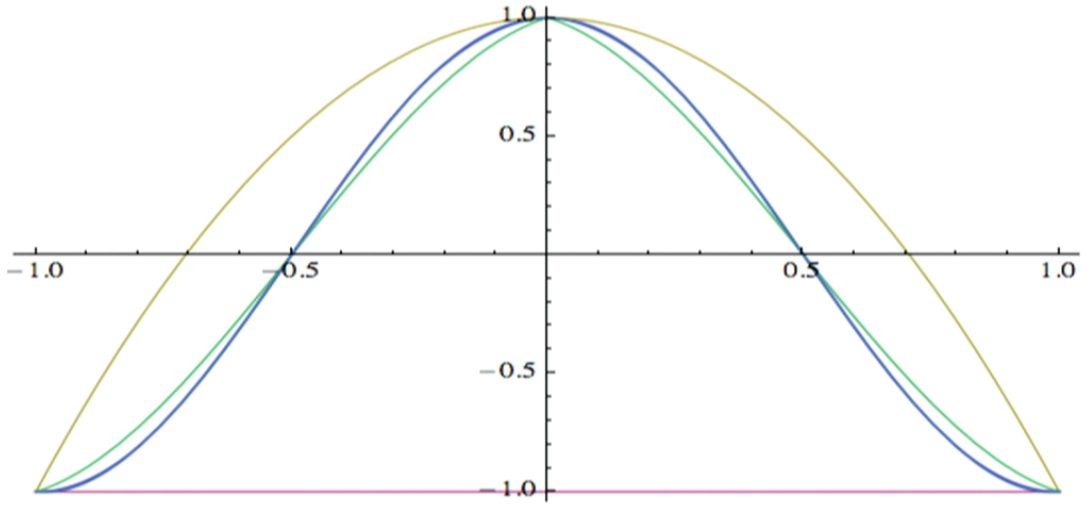
Example

In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

```
Interpolation::inhr : Requested order is too high; order has been reduced to {1}. ≫
Interpolation::inhr : Requested order is too high; order has been reduced to {2}. ≫
```

In[62]:=
```
Plot[{Cos[Pi x], t}, {x, -1, 1}]
```

Out[62]=



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Numerics.nb

```
    Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
      InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

Example

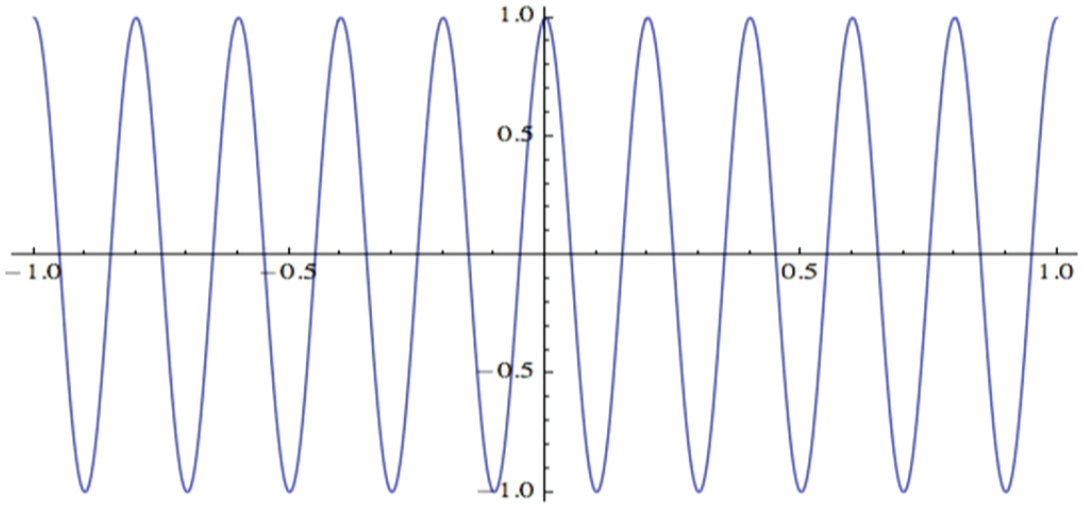
In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

Interpolation::inhr : Requested order is too high; order has been reduced to {1}. ≫

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. ≫

In[63]:=
```
Plot[Cos[10 Pi x], {x, -1, 1}]
```

Out[63]=



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Numerics.nb

```
    Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
      InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

Example

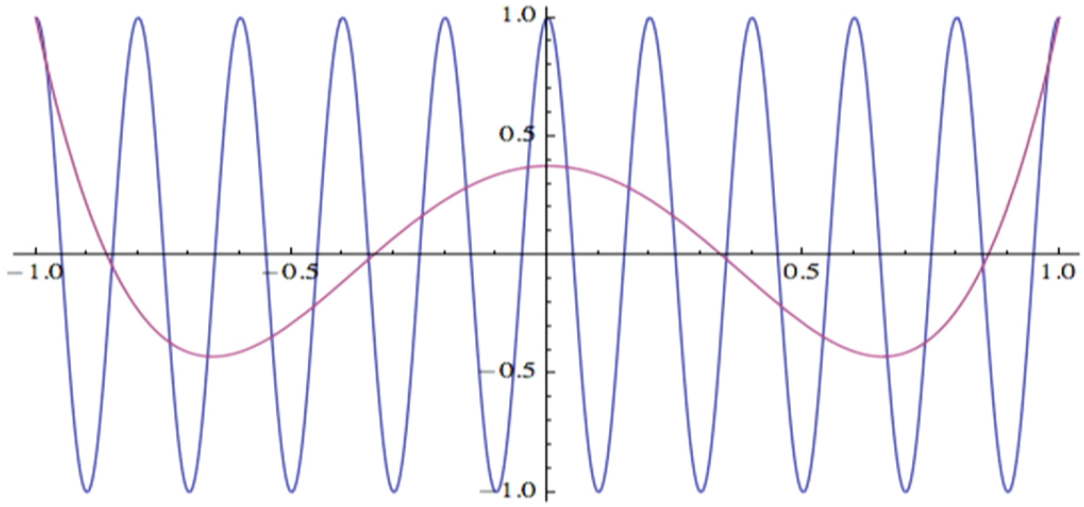
In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

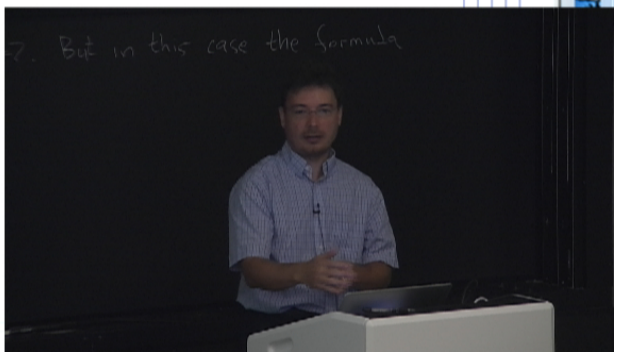
Interpolation::inhr : Requested order is too high; order has been reduced to {1}. »

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. »

In[64]:=
```
Plot[{Cos[10 Pi x], LegendreP[4, x]}, {x, -1, 1}]
```

Out[64]=





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Numerics.nb

```
    Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
      InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

Example

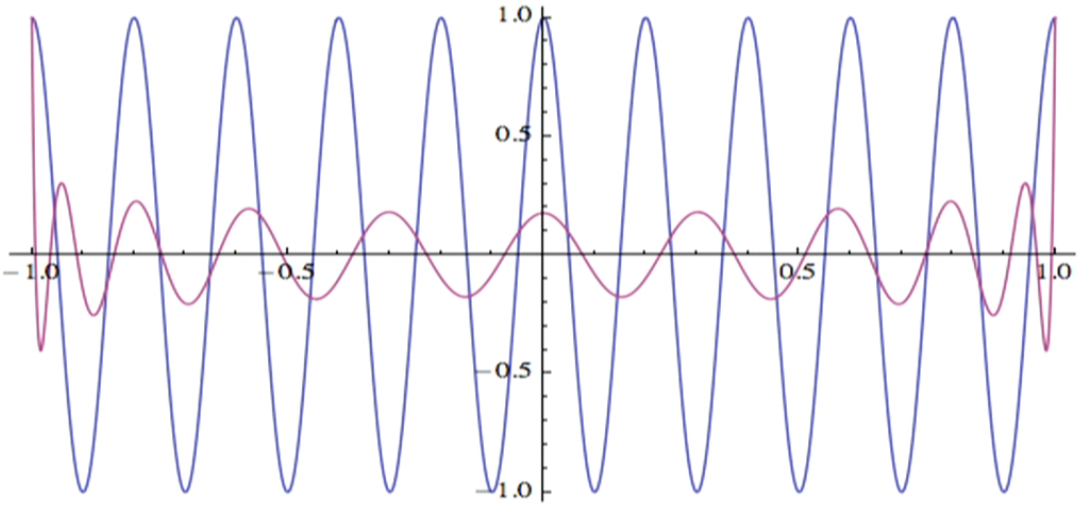
In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

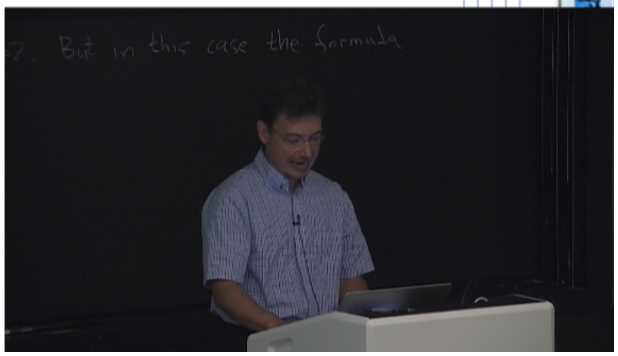
Interpolation::inhr : Requested order is too high; order has been reduced to {1}. »

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. »

In[66]:=
```
Plot[{Cos[10 Pi x], LegendreP[20, x]}, {x, -1, 1}]
```

Out[66]=





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Numerics.nb

```
    Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
      InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

Example

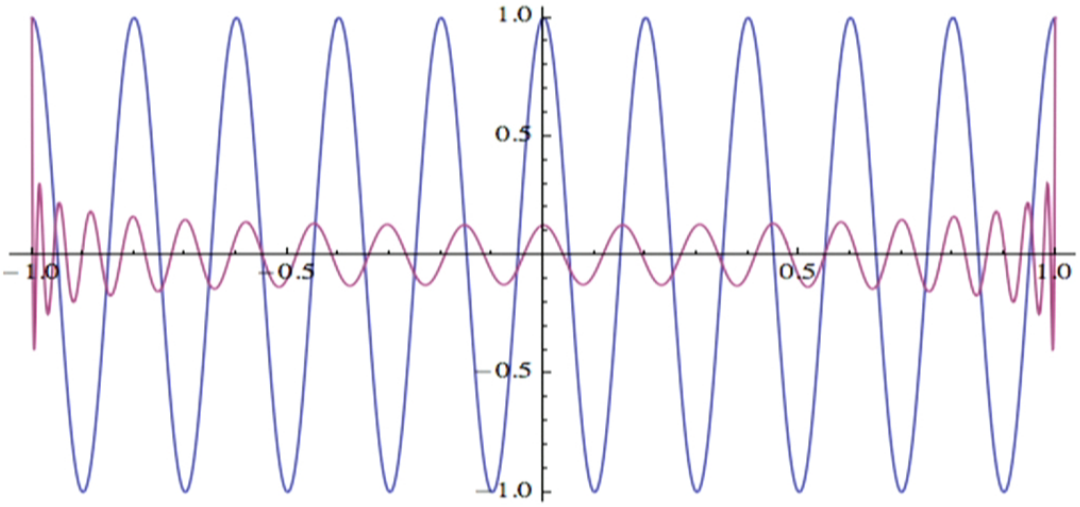
In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

Interpolation::inhr : Requested order is too high; order has been reduced to {1}. »

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. »

In[67]:=
```
Plot[{Cos[10 Pi x], LegendreP[40, x]}, {x, -1, 1}]
```

Out[67]=



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Numerics.nb

Example

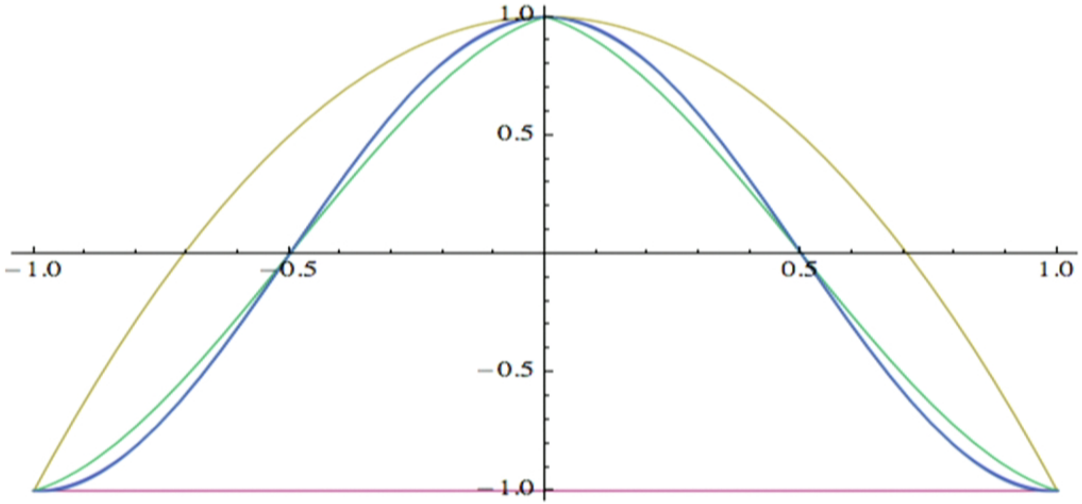
In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

Interpolation::inhr : Requested order is too high; order has been reduced to {1}. »

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. »

In[68]:=
```
Plot[{Cos[Pi x], t}, {x, -1, 1}]
```

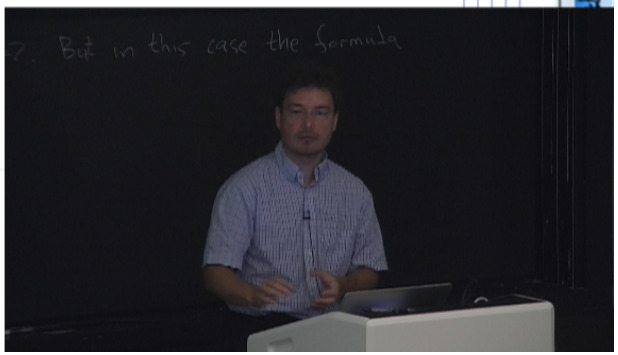
Out[68]=



frame axes image size plot style... more...

Solve

Use 8 pieces -- could easily use many more



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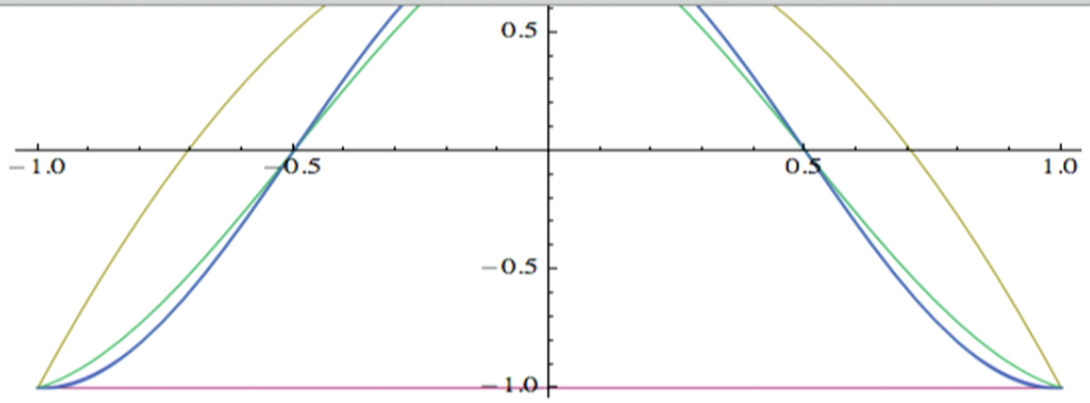
Numerics.nb

Out[68]=



Solve

Use 8 pieces -- could easily use many more

In[69]:=
```
ppcount = 8
```

Out[69]= 8

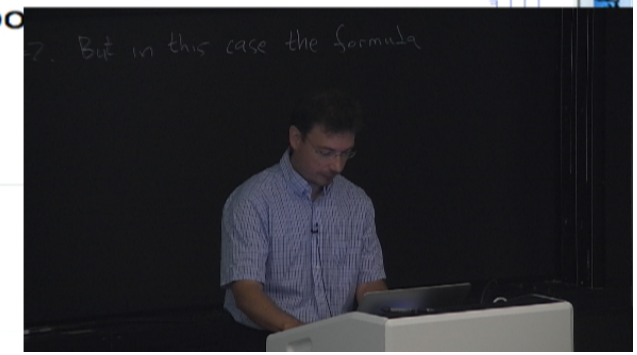
What follows is the same as for projecting onto basis functions abo

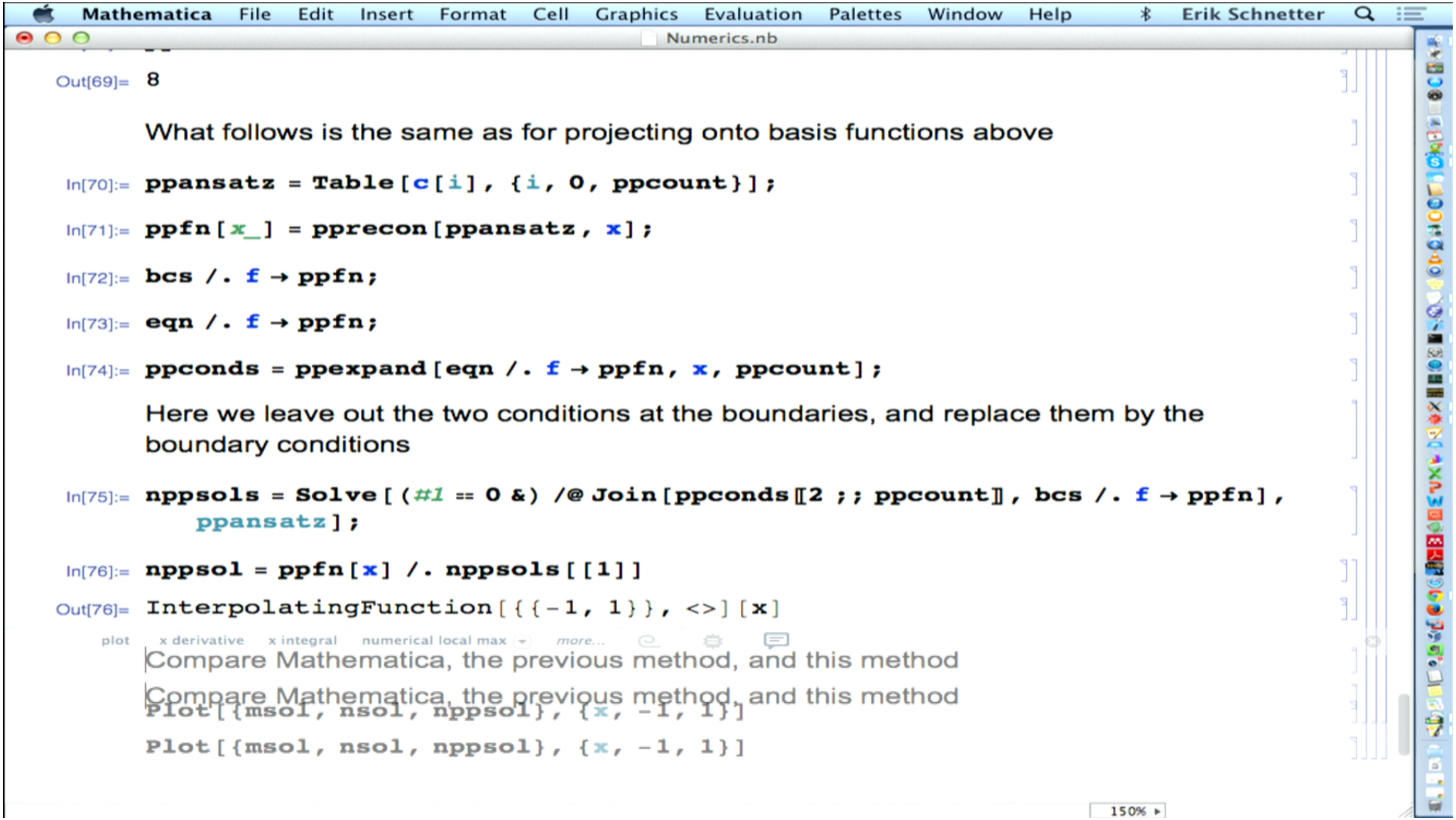
In[70]:=
```
ppansatz = Table[c[i], {i, 0, ppcount}];
```

In[71]:=
```
ppfn[x_] = pprecon[ppansatz, x];
```

```
bcs /. f → ppfn;
```

```
eqn /. f → ppfn;
```




Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help Erik Schnetter
Numerics.nb
Out[69]= 8
What follows is the same as for projecting onto basis functions above
In[70]:= ppansatz = Table[c[i], {i, 0, ppcount}];
In[71]:= ppfn[x_] = pprecon[ppansatz, x];
In[72]:= bcs /. f → ppfn;
In[73]:= eqn /. f → ppfn;
In[74]:= ppconds = ppexpand[eqn /. f → ppfn, x, ppcount];
Here we leave out the two conditions at the boundaries, and replace them by the boundary conditions
In[75]:= nppsols = Solve[(#1 == 0 &) /@ Join[ppconds[[2 ;; ppcount]], bcs /. f → ppfn], ppansatz];
In[76]:= nppsol = ppfn[x] /. nppsols[[1]]
Out[76]= InterpolatingFunction[{{-1, 1}}, <>][x]
plot x derivative x integral numerical local max more...
Compare Mathematica, the previous method, and this method
Compare Mathematica, the previous method, and this method
Plot[{msol, nsol, nppsol}, {x, -1, 1}]
Plot[{msol, nsol, nppsol}, {x, -1, 1}]
150%

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Numerics.nb

In[74]:=
```
ppconds = ppexpand[eqn /. f → ppfn, x, ppcount];
```

Here we leave out the two conditions at the boundaries, and replace them by the boundary conditions

In[75]:=
```
nppsols = Solve[(#1 == 0 &) /@ Join[ppconds[[2 ;; ppcount]], bcs /. f → ppfn],
    ppansatz];
```

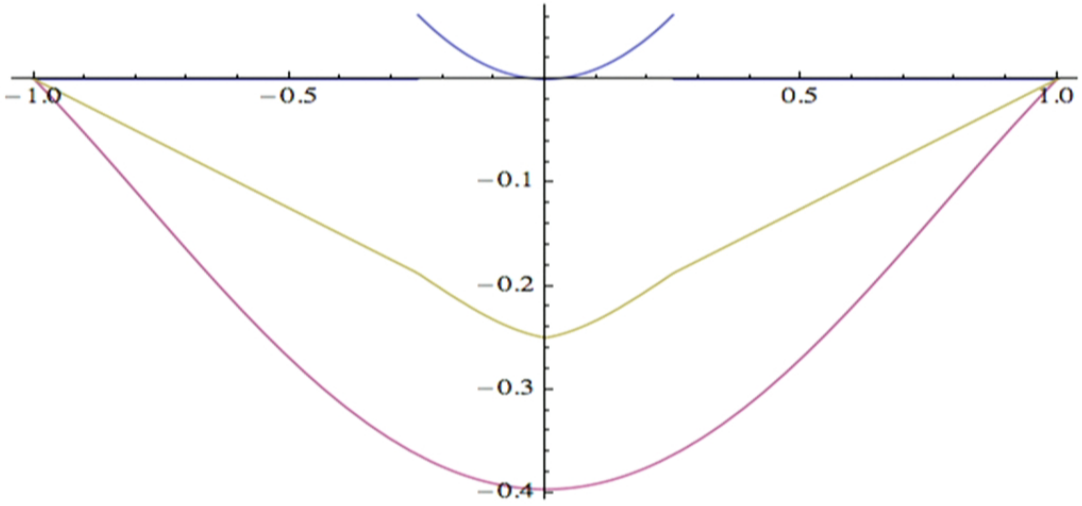
In[76]:=
```
nppsol = ppfn[x] /. nppsols[[1]]
```

Out[76]=
```
InterpolatingFunction[{{-1, 1}}, <>][x]
```

Compare Mathematica, the previous method, and this method

In[77]:=
```
Plot[{msol, nsol, nppsol}, {x, -1, 1}]
```

Out[77]=



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Running...Numerics.nb

In[57]:=
```
ppexpand[f_, x_, n_] := (f /. x → #1 &) /@ xlocs[n]
```

In[58]:=
```
ppexpand[f[x], x, 4]
```

Out[58]=
$$\left\{f[-1], f\left[-\frac{1}{2}\right], f[0], f\left[\frac{1}{2}\right], f[1]\right\}$$

Reconstruct by interpolation, using Mathematica's interpolation function

Here we use second order (linear) interpolation; could easily use a higher order, and would definitively do so in practice

In[59]:=
```
pprecon[cs_, x_] :=
 Interpolation[Transpose[{xlocs[Length[cs] - 1], cs}],
   InterpolationOrder → 3][x]
```

In[60]:=
```
ppexpand[Cos[Pi x], x, 4]
```

Out[60]=
```
{-1, 0, 1, 0, -1}
```

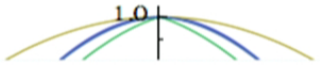
Example

In[61]:=
```
t = Table[pprecon[ppexpand[Cos[Pi x], x, i], x], {i, {1, 2, 4, 8}}];
```

Interpolation::inhr : Requested order is too high; order has been reduced to {1}. »

Interpolation::inhr : Requested order is too high; order has been reduced to {2}. »

In[68]:=
```
Plot[{Cos[Pi x], t}, {x, -1, 1}]
```



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Numerics.nb

In[16]:= $\partial_{x,x} f[x] - \text{tophat}\left[\frac{1}{4}, x\right]$

Out[16]= $-\text{If}\left[\text{Abs}[x] < \frac{1}{4}, \frac{1}{\frac{2}{4}}, 0\right] + f''[x]$

Choose boundary conditions:

Dirichlet boundary conditions:

```
In[17]:= dir[f_, df_, x_] = f
Out[17]= f
```

Robin boundary conditions:

```
In[18]:= rob[f_, df_, x_] = f + x df
Out[18]= f + df x
```

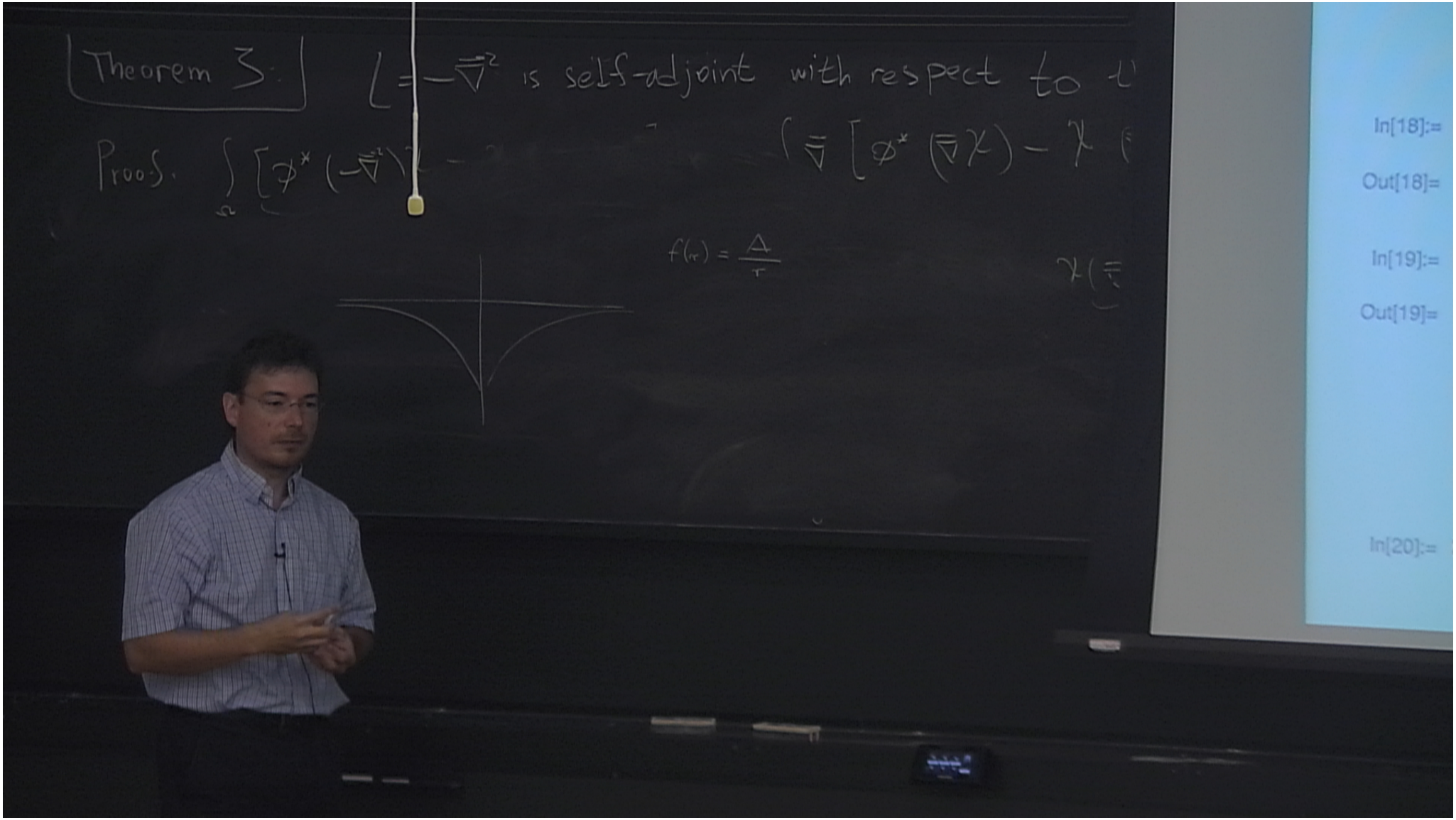
```
In[19]:= bcs = {dir[f[-1], f'[-1], -1], dir[f[1], f'[1], 1]}
Out[19]= {f[-1], f[1]}
```

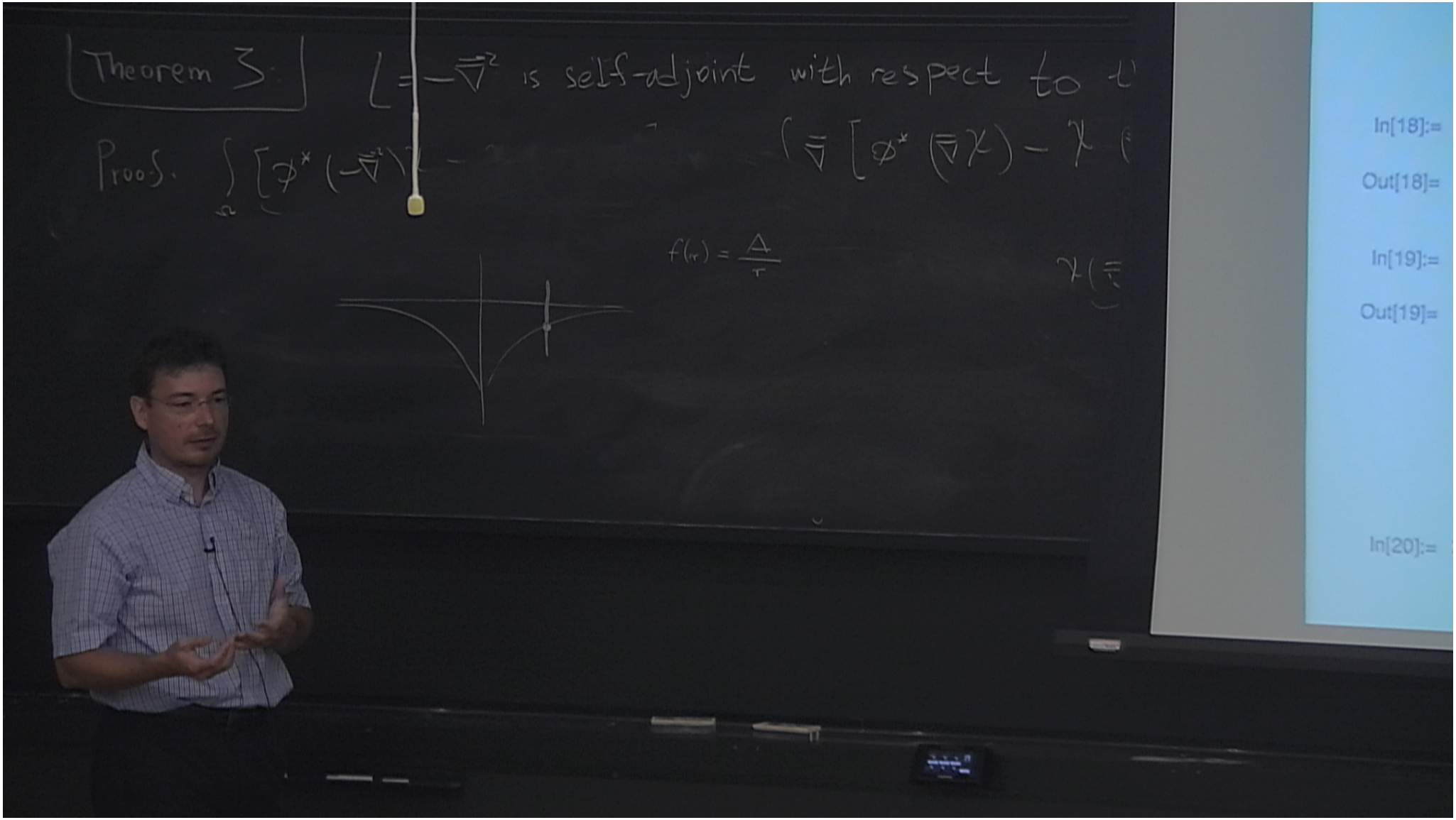
Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

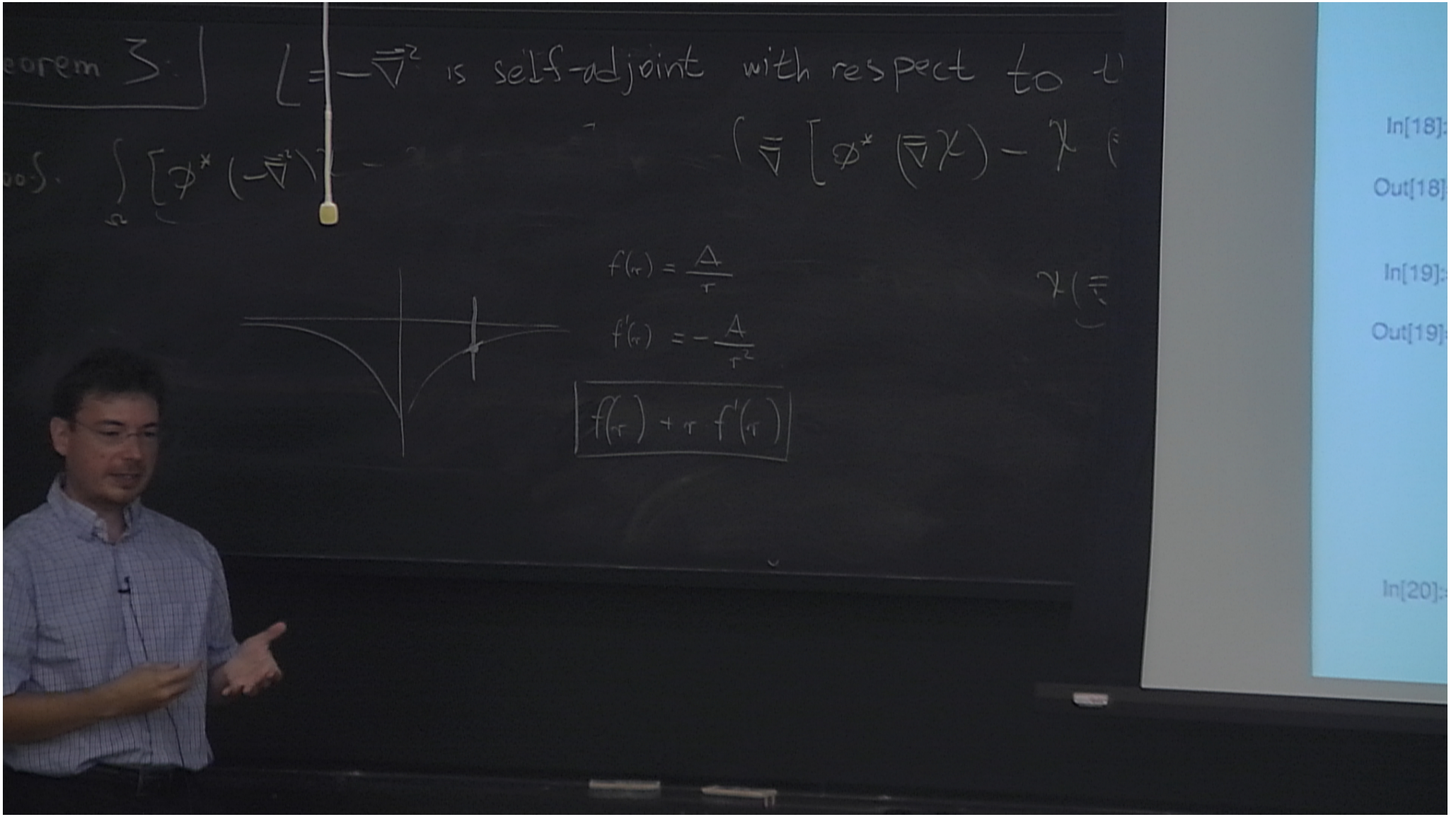
```
In[20]:= msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

$\left\{\left\{f[x] \to \left[x^2 \quad \text{Abs}[x] < \frac{1}{\ldots}\right.\right.\right.$

150%


Theorem 3:
$L = -\vec{\nabla}^2$ is self-adjoint with respect to
Proof.
$f(r) = \frac{A}{r}$
In[18]:=
Out[18]=
In[19]:=
Out[19]=
In[20]:=


Theorem 3
Proof.
In[18]:=
Out[18]=
In[19]:=
Out[19]=
In[20]:=


orem 3
$L = -\vec{\nabla}^2$ is self-adjoint with respect to
$\int_\Omega [\phi^* (-\vec{\nabla}^2)$
$(\vec{\nabla} [\phi^* (\vec{\nabla}\chi) - \chi$
$f(r) = \frac{A}{r}$
$f'(r) = -\frac{A}{r^2}$
$f(r) + r f'(r)$
$\chi($
In[18]:
Out[18]
In[19]:
Out[19]
In[20]:

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Numerics.nb

In[16]:=
$$\text{eqn} = \partial_{x,x} f[x] - \text{tophat}\left[\frac{1}{4}, x\right]$$

Out[16]=
$$-\text{If}\left[\text{Abs}[x] < \frac{1}{4}, \frac{1}{\frac{2}{4}}, 0\right] + f''[x]$$

Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

In[18]:=
```
rob[f_, df_, x_] = f + x df
```

Out[18]=
```
f + df x
```

In[19]:=
```
bcs = {dir[f[-1], f'[-1], -1], dir[f[1], f'[1], 1]}
```

Out[19]=
```
{f[-1], f[1]}
```

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

In[20]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

$$\left\{\left\{f[x] \to \begin{cases} x^2 & \text{Abs}[x] < \frac{1}{4} \\ \end{cases}\right.\right.$$

150%

The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

In[16]:=
```
eqn = ∂x,x f[x] - tophat[1/4, x]
```

Out[16]=
$$-\text{If}\left[\text{Abs}[x] < \frac{1}{4},\ \frac{1}{\frac{2}{4}},\ 0\right] + f''[x]$$

Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

Robin boundary conditions:

In[18]:=
```
rob[f_, df_, x_] = f + x df
```

Out[18]=
```
f + df x
```

In[69]:=
```
bcs = {rob[f[-1], f'[-1], -1], rob[f[1], f'[1], 1]}
```

Out[69]=
$$\{f[-1] - f'[-1],\ f[1] + f'[1]\}$$

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Numerics.nb

```
In[12]:= Integrate[tophat[w, x], {x, -Infinity, Infinity}]
Out[12]= 1
```

1

A Gaussian, another possible approximation for the Dirac delta distribution:

A Gaussian, another possible approximation for the Dirac delta distribution:

$$\text{In[13]:= gaussian}[w\_, x\_] := \frac{\text{Exp}\left[-\left(\frac{x}{w}\right)^2\right]}{\sqrt{\pi}\, w}$$

In[13]:= gaussian[w_, x_] := Exp[-(x/w)^2] / (Sqrt[π] w)

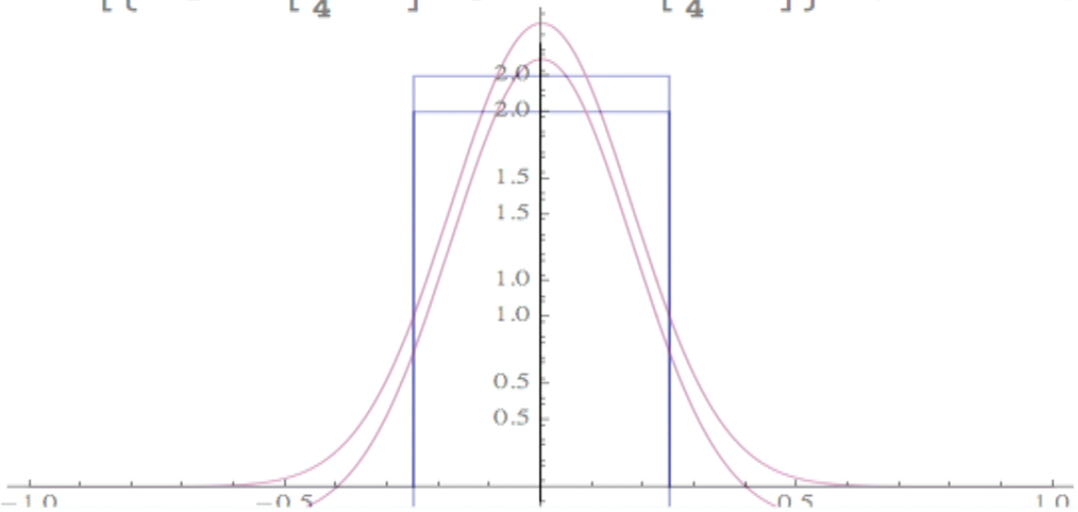
```
In[14]:= Integrate[gaussian[w, x], {x, -Infinity, Infinity}]
Out[14]= 1
```

1

```
In[15]:= Plot[{tophat[1/4, x], gaussian[1/4, x]}, {x, -1, 1}, PlotRange → All]
```

In[15]:= Plot[{tophat[1/4, x], gaussian[1/4, x]}, {x, -1, 1}, PlotRange → All]

Out[15]=



Out[15]=

150%

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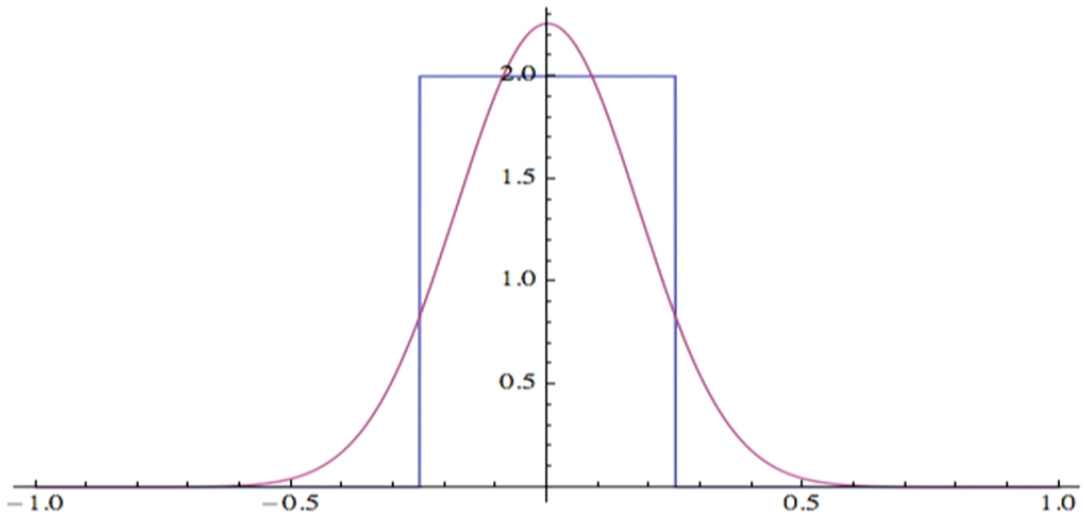
Numerics.nb

In[15]:= Plot[{tophat[1/4, x], gaussian[1/4, x]}, {x, -1, 1}, PlotRange → All]

Out[15]=



The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

```
eqn = ∂x,x f[x] - gau[1/., x]
```
gaussian

Out[16]= $-\text{If}\left[\text{Abs}[x] < \frac{1}{4}, \frac{1}{\frac{2}{4}}, 0\right] + f''[x]$

Choose boundary conditions:

150%

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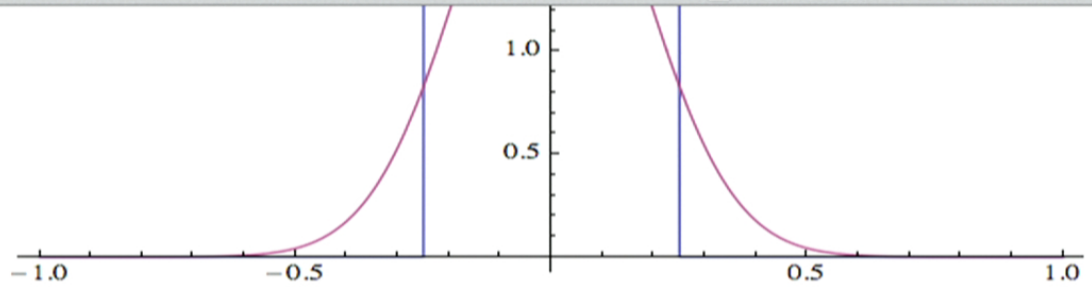
Numerics.nb

Out[15]=



The equation:

We write the equation and boundary conditions as expressions that should be zero. That is, an equation a==b is written as a-b==0, and after omitting the "==0", we write a-b.

In[70]:=
```
eqn = ∂x,x f[x] - gaussian[1/4, x]
```

Out[70]=
$$-\frac{4\, e^{-16\, x^2}}{\sqrt{\pi}} + f''[x]$$

together x derivative x integral exp to trig more...

Choose boundary conditions:

Dirichlet boundary conditions:

In[17]:=
```
dir[f_, df_, x_] = f
```

Out[17]=
```
f
```

150%

Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help Erik Schnetter

Numerics.nb

Choose boundary conditions:

Dirichlet boundary conditions:

```
In[17]:= dir[f_, df_, x_] = f
Out[17]= f
```

Robin boundary conditions:

```
In[18]:= rob[f_, df_, x_] = f + x df
Out[18]= f + df x
```

```
In[69]:= bcs = {rob[f[-1], f'[-1], -1], rob[f[1], f'[1], 1]}
Out[69]= {f[-1] - f'[-1], f[1] + f'[1]}
```

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

```
In[20]:= msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
Out[20]= {{f[x] → {x^2   Abs[x] < 1/4
                   0     True}}}
```

```
In[21]:= msol = f[x] /. msols[[1]]
         {x^2   Abs[x] < 1/4
```

150%

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Numerics.nb

In[69]:=
```
bcs = {rob[f[-1], f'[-1], -1], rob[f[1], f'[1], 1]}
```

Out[69]=
```
{f[-1] - f'[-1], f[1] + f'[1]}
```

Cheat by asking Mathematica to solve this equation:
Note: Mathematica cannot solve this equation for the tophat RHS, as DSolve does not handle discontinuities correctly.

In[71]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

Out[71]=

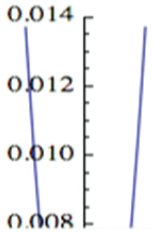
$$\left\{\left\{f[x] \to -\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+8\,e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

In[72]:=
```
msol = f[x] /. msols[[1]]
```

Out[72]=

$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+8\,e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}$$

In[22]:=
```
Plot[msol, {x, -1, 1}]
```



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Numerics.nb

In[71]:=
```
msols = DSolve[# == 0 & /@ Flatten[{eqn, bcs}], f[x], x]
```

Out[71]=

$$\left\{\left\{f[x] \to -\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+8\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}\right\}\right\}$$

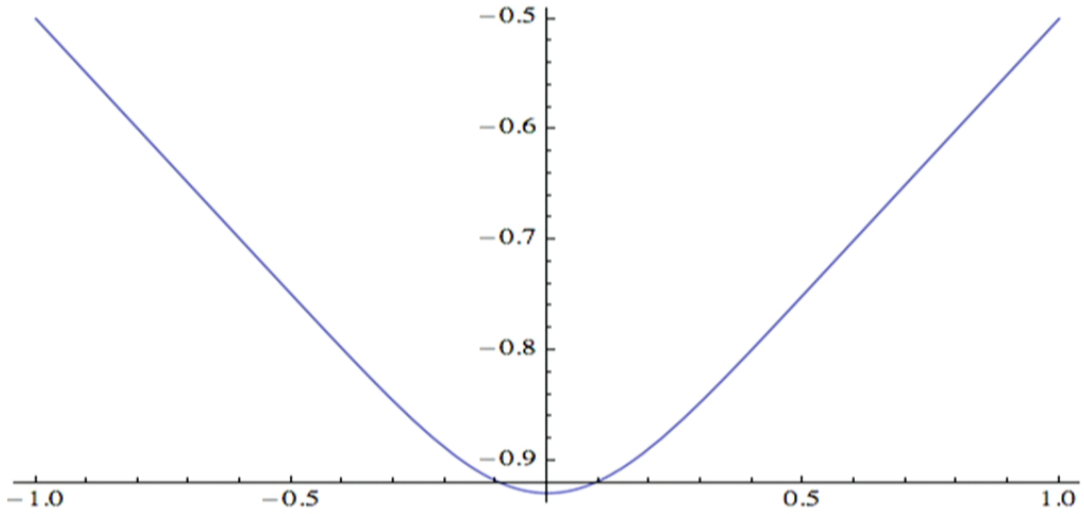
In[72]:=
```
msol = f[x] /. msols[[1]]
```

Out[72]=

$$-\frac{e^{-16-16\,x^2}\left(-e^{16}+e^{16\,x^2}+8\,e^{16+16\,x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16\,x^2}\sqrt{\pi}\,x\,\text{Erf}[4\,x]\right)}{8\sqrt{\pi}}$$

In[73]:=
```
Plot[msol, {x, -1, 1}]
```

Out[73]=



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Numerics.nb

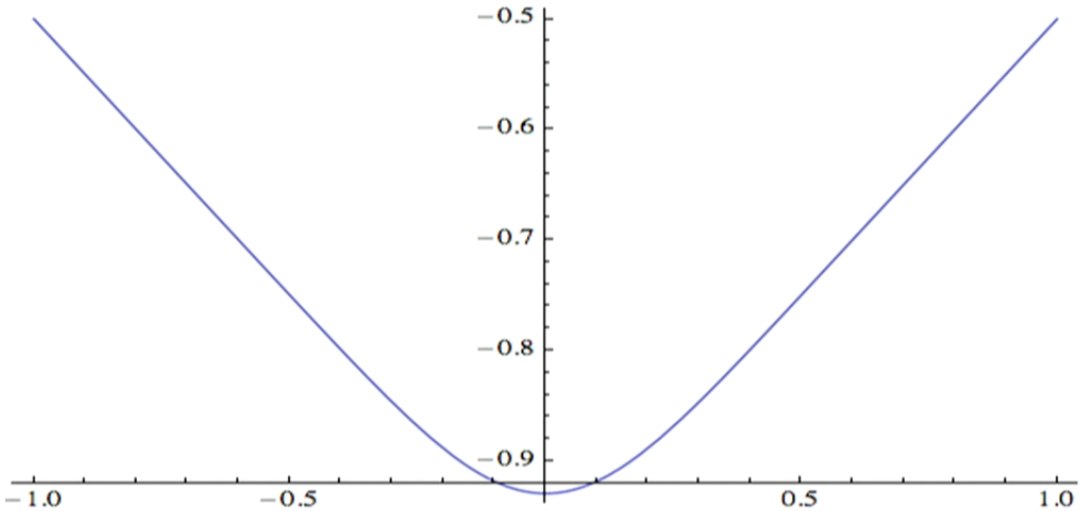
$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+8\,e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}\}\}$$

In[72]:=
```
msol = f[x] /. msols[[1]]
```

Out[72]=
$$-\frac{e^{-16-16x^2}\left(-e^{16}+e^{16x^2}+8\,e^{16+16x^2}\sqrt{\pi}\,\text{Erf}[4]-4\,e^{16+16x^2}\sqrt{\pi}\,x\,\text{Erf}[4x]\right)}{8\sqrt{\pi}}$$

In[73]:=
```
Plot[msol, {x, -1, 1}]
```

Out[73]=



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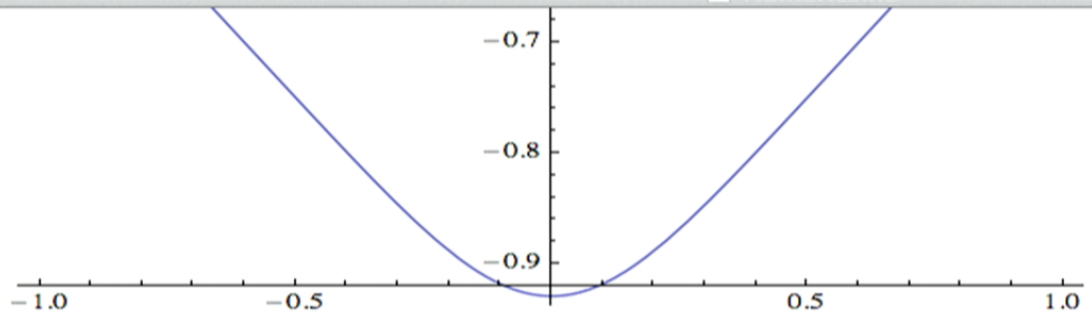
Numerics.nb

Out[73]=



Numerical Solution

Choose an approximation order

In[23]:=
```
order = 4
```

Out[23]= 4

Our ansatz is a set of coefficients...

In[24]:=
```
ansatz = Table[c[i], {i, 0, order}]
```

Out[24]= {c[0], c[1], c[2], c[3], c[4]}

...which leads to a reconstructed function:

In[25]:=
```
fn[x_] = recon[ansatz, x]
```

Out[25]= $c[0] + x\,c[1] + \frac{1}{2}\left(-1 + 3\,x^2\right) c[2] +$

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Numerics.nb

In[74]:=
```
ansatz = Table[c[i], {i, 0, order}]
```

Out[74]=
```
{c[0], c[1], c[2], c[3], c[4]}
```

...which leads to a reconstructed function:

In[25]:=
```
fn[x_] = recon[ansatz, x]
```

Out[25]=
$$c[0] + x\,c[1] + \frac{1}{2}\left(-1 + 3x^2\right)c[2] + \frac{1}{2}\left(-3x + 5x^3\right)c[3] + \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)c[4]$$

To solve the ODE, we need to determine these coefficients.

With this ansatz, the boundary conditions look as follows:

In[26]:=
```
bcs /. f → fn
```

Out[26]=
```
{c[0] - c[1] + c[2] - c[3] + c[4], c[0] + c[1] + c[2] + c[3] + c[4]}
```

The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.

In[27]:=
```
eqn /. f → fn
```

Out[27]=
$$3\,c[2] + 15\,x\,c[3] + \frac{1}{8}\left(-60 + 420x^2\right)c[4] - \text{If}\left[\text{Abs}[x] < \frac{1}{4}, \frac{1}{\frac{2}{4}}, 0\right]$$

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Numerics.nb

In[75]:=
```
ansatz = Table[c[i], {i, 0, order}]
```

Out[75]=
```
{c[0], c[1], c[2], c[3], c[4]}
```

...which leads to a reconstructed function:

In[76]:=
```
fn[x_] = recon[ansatz, x]
```

Out[76]=
$$c[0] + x\,c[1] + \frac{1}{2}\left(-1 + 3x^2\right)c[2] + \frac{1}{2}\left(-3x + 5x^3\right)c[3] + \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)c[4]$$

To solve the ODE, we need to determine these coefficients.

With this ansatz, the boundary conditions look as follows:

In[77]:=
```
bcs /. f → fn
```

Out[77]=
```
{c[0] - 2 c[1] + 4 c[2] - 7 c[3] + 11 c[4],
 c[0] + 2 c[1] + 4 c[2] + 7 c[3] + 11 c[4]}
```

reverse curl div array rules more...

The equation is now this:
Note that the Laplace operator has vanished!
However, the RHS still looks complicated.

In[27]:=
```
eqn /. f → fn
```

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Running...Numerics.nb

In[79]:=
```
dot[LegendreP[0, x], eqn /. f → fn, x]
```

Out[79]=
$$6\,c[2] + 20\,c[4] - \text{Erf}[4]$$

These are all projections up to our approximation order :

In[80]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[29]=
$$\left\{\frac{1}{2}\,(-1 + 6\,c[2] + 20\,c[4]),\ 15\,c[3],\ \frac{5}{64}\,(15 + 448\,c[4]),\ 0,\ -\frac{5535}{4096}\right\}$$

Note that the last two projections are “not good”, since they don’t contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the respective coefficients were left out.
That means we are missing two conditions. This is also the reason why we need two additional conditions (“boundary conditions”).

In[30]:=
```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
   ansatz]
```

Out[30]=
$$\left\{\left\{c[0] \to -\frac{47}{192},\ c[1] \to 0,\ c[2] \to \frac{187}{672},\ c[3] \to 0,\ c[4] \to -\frac{15}{448}\right\}\right\}$$

In[31]:=
```
nsol = fn[x] /. nsols[[1]]
```

Out[31]=
$$-\frac{47}{192} + \frac{187\,\left(-1 + 3\,x^2\right)}{1344} - \frac{15\,\left(3 - 30\,x^2 + 35\,x^4\right)}{3584}$$

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These are all projections up to our approximation order.

In[80]:=
```
conds = expand[eqn /. f → fn, x, order]
```

Out[29]=
$$\left\{\frac{1}{2}\,(-1+6\,c[2]+20\,c[4]),\ 15\,c[3],\ \frac{5}{64}\,(15+448\,c[4]),\ 0,\ -\frac{5535}{4096}\right\}$$

Note that the last two projections are “not good”, since they don’t contain any coefficients c[i]. Even worse, the last cannot be satisfied at all.
This is to be expected, since we artificially cut off our ansatz at this order, so the respective coefficients were left out.
That means we are missing two conditions. This is also the reason why we need two additional conditions (“boundary conditions”).

In[30]:=
```
nsols = Solve[(#1 == 0 &) /@ Join[conds[[1 ;; order - 1]], bcs /. f → fn],
   ansatz]
```

Out[30]=
$$\left\{\left\{c[0]\to -\frac{47}{192},\ c[1]\to 0,\ c[2]\to \frac{187}{672},\ c[3]\to 0,\ c[4]\to -\frac{15}{448}\right\}\right\}$$

In[31]:=
```
nsol = fn[x] /. nsols[[1]]
```

Out[31]=
$$-\frac{47}{192}+\frac{187\left(-1+3\,x^2\right)}{1344}-\frac{15\left(3-30\,x^2+35\,x^4\right)}{3584}$$

In[32]:=
```
nsol // N // Simplify
```

Out[32]=
$$-0.396484+0.542969\,x^2-0.146484\,x^4$$

Compare Mathematica’s and our numerical solution

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Numerics.nb

$$\frac{1}{2}\left(-1+3x^2\right)\left(\frac{5}{56\,e^{16}\sqrt{\pi}}+\frac{123\,\text{Erf}[4]}{448}\right)-\frac{95\,\text{Erf}[4]}{128}$$

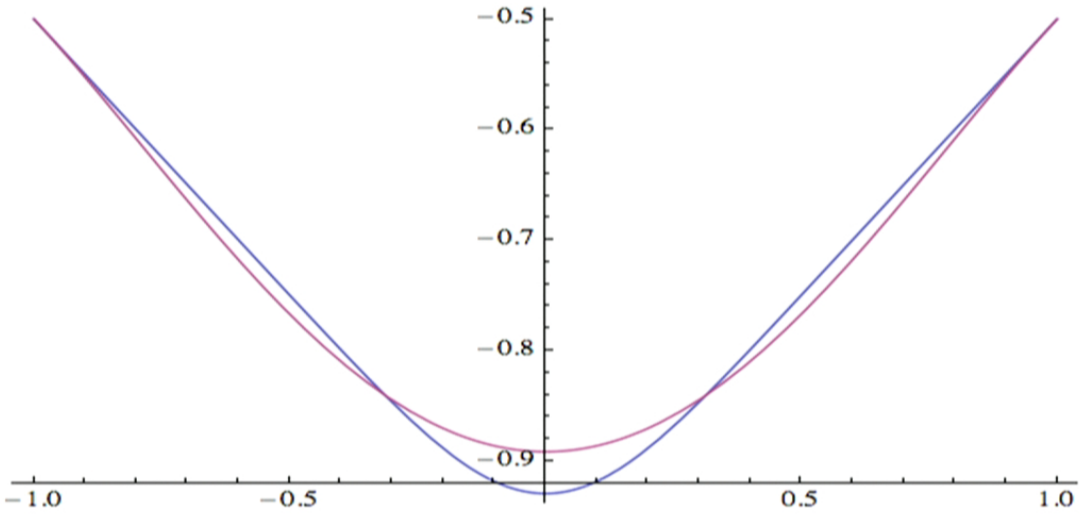
In[83]:=
```
nsol // N // Simplify
```

Out[83]=
$$-0.891602+0.533203\,x^2-0.141602\,x^4$$

Compare Mathematica's and our numerical solution

In[84]:=
```
Plot[{msol, nsol}, {x, -1, 1}]
```

Out[84]=



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Numerics.nb

In[98]:=
```
ppconds = ppexpand[eqn /. f → ppfn, x, ppcount];
```

Here we leave out the two conditions at the boundaries, and replace them by the boundary conditions

In[99]:=
```
nppsols = Solve[(#1 == 0 &) /@ Join[ppconds[[2 ;; ppcount]], bcs /. f → ppfn],
    ppansatz];
```

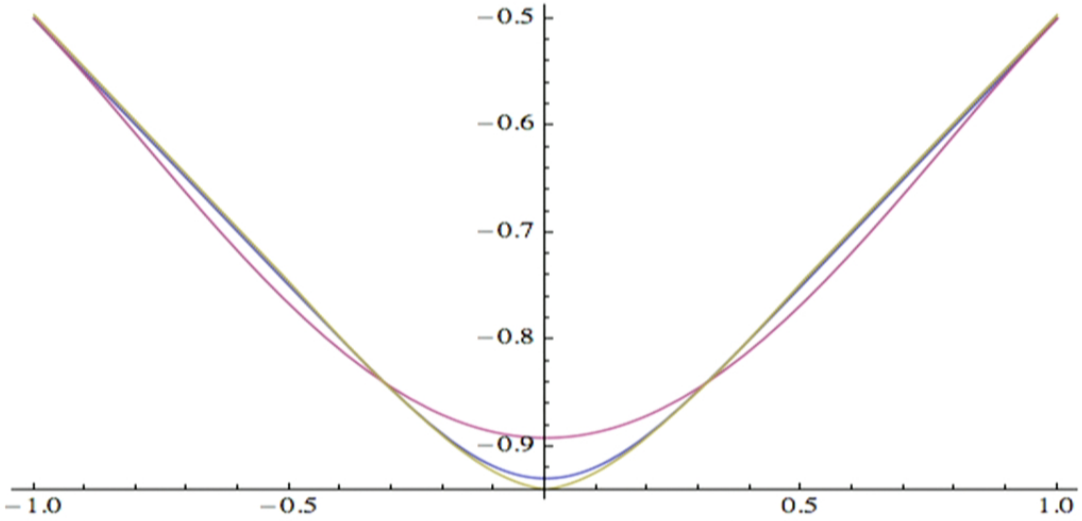
In[100]:=
```
nppsol = ppfn[x] /. nppsols[[1]]
```

Out[100]=
```
InterpolatingFunction[{{-1, 1}}, <>][x]
```

Compare Mathematica, the previous method, and this method

In[101]:=
```
Plot[{msol, nsol, nppsol}, {x, -1, 1}]
```

Out[101]=



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