Title: 14/15 PSI - Green Functions

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Abstract:

Today: * Wave equation

* Green's function for wave equation

* Maxwell's equations and retarded potentials

The Laplace operator $L = -\vec{\nabla}^2$ is self-adjoint with respect to the inner product $\int_\Omega \phi^* \chi \, d\vec{x}$ $\quad \phi^* L \chi - \chi L \phi^*$

Proof. $\displaystyle \boxed{\int_\Omega \left[\phi^* \left(-\vec{\nabla}^2 \chi\right) - \chi \left(-\vec{\nabla}^2 \phi^*\right)\right] d\vec{x} = -\int_{\partial\Omega} \left[\phi^* \vec{\nabla}\chi - \chi \vec{\nabla}\phi^*\right] \cdot \vec{n}\, dS} = -\int_\Omega \vec{\nabla}\left[\phi^* (\vec{\nabla}\chi) - \chi \vec{\nabla}\phi^*\right] d\vec{x}$

Green's identity

$$\nabla^2 \varphi(\bar{x},t) - \frac{1}{c^2}\frac{\partial^2 \varphi(\bar{x},t)}{\partial t^2} = f(\bar{x},t)$$

$\{\underbrace{\text{Wave equation}}\}$

Hyperbolic types and requires Cauchy BC

Consider homogeneous wave equation and assume $\bar{x} \in \mathbb{R}^d$. Assume initial conditions

$$\varphi(\bar{x}, 0) = u_0(\bar{x})$$

$$\frac{\partial \varphi(\bar{x}, 0)}{\partial t} = v_0(\bar{x})$$

Wave equation

$$\nabla^2 \varphi(\bar{x},t) - \frac{1}{c^2}\frac{\partial^2 \varphi(\bar{x},t)}{\partial t^2} = f(\bar{x},t)$$

Hyperbolic types and requires Cauchy BC

Fourier transform

$$\begin{pmatrix} \varphi(\bar{x},t) \\ u_0(\bar{x}) \\ v_0(\bar{x}) \end{pmatrix} = \int_{\mathbb{R}^d} \begin{pmatrix} \varphi(\bar{k},t) \\ u_0(\bar{k}) \\ v_0(\bar{k}) \end{pmatrix} e^{-i\bar{k}\cdot\bar{x}} \frac{d\bar{k}}{(2\pi)^d}$$

general solution is $\varphi(\vec{k},t) = a(\vec{k})\, e^{ickt} + b(\vec{k})\, e^{-ickt}$

$k = |\vec{k}|, \; \omega_k = c|\vec{k}|$

$$\frac{1}{c^2}\frac{\partial^2 \varphi(\vec{k},t)}{\partial t^2} + \vec{k}^2 \varphi(\vec{k},t) = 0$$

$\Rightarrow$ The most general solution is $\varphi($

$$\begin{cases} \varphi(\vec{k},0) = a(\vec{k}) + b(\vec{k}) = u_0(\vec{k}) \\ \dfrac{\partial \varphi}{\partial t}(\vec{k},0) = ick\, a(\vec{k}) - ick\, b(\vec{k}) = v_0(\vec{k}) \end{cases} \Rightarrow \boxed{\begin{aligned} a(\vec{k}) &= \frac{1}{2}\left[u_0(\vec{k}) + \frac{1}{ick} v_0(\vec{k})\right] \\ b(\vec{k}) &= \frac{1}{2}\left[u_0(\vec{k}) - \frac{1}{ick} v_0(\vec{k})\right] \end{aligned}}$$

$$\varphi(\bar{x}, t) = \frac{1}{2} \int_{\mathbb{R}^d} \left[u_0(\bar{k}) + \frac{v_0(\bar{k})}{ick} \right] e^{ickt - i\bar{k}\cdot\bar{x}} \frac{d\bar{k}}{(2\pi)^d} + \frac{1}{2} \int_{\mathbb{R}^d} \left[u_0(\bar{k}) - \frac{v_0(\bar{k}}{ick} \right.$$

$$\left[u_0(\vec{k}) - \frac{v_0(\vec{k})}{ick} \right] e^{-ickt - i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^d}$$ – the most general solution satisfying initial conditions

$$\varphi(\vec{x},t) = \frac{1}{2}\int_{\mathbb{R}^d} \left[u_0(\vec{k}) + \frac{v_0(\vec{k})}{ick} \right] e^{ickt - i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^d} + \frac{1}{2}\int_{\mathbb{R}^d} \left[u_0(\vec{k}) - \frac{v_0(\vec{k})}{ick} \right.$$

Exercise: Show that in $d=1$

$$\varphi(\vec{x},t) = \frac{1}{2}\left\{ u_0(x+ct) + u_0(x-ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,d\xi$$

$$\varphi(\bar{x},t) = \frac{1}{2}\int_{\mathbb{R}^d} \left[u_0(\bar{k}) + \frac{v_0(\bar{k})}{ick}\right] e^{ickt - i\bar{k}\cdot\bar{x}} \frac{d\bar{k}}{(2\pi)^d} + \frac{1}{2}\int_{\mathbb{R}^d}\left[u_0(\bar{k}) - \frac{v_0(\bar{k})}{ick}\right] e^{-ickt - i\bar{k}\cdot\bar{x}}\frac{d\bar{k}}{(2\pi)^d}$$

– the sa

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d'Alembert solution

Green's identity

Consider homogeneous wave equation and assume $\bar{x} \in \mathbb{R}^d$. Assume initial conditions

$$\varphi(\bar{x}, 0) = u_0(\bar{x})$$

$$\frac{\partial \varphi(\bar{x}, 0)}{\partial t} = v_0(\bar{x})$$

Green's function for wave eq

$$\vec{\nabla}_x^2 G(\bar{x}-\bar{\xi}, t-\tau) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G(\bar{x}-\bar{\xi}, t-\tau) = \delta(\bar{x}-\bar{\xi})\,\delta(t-\tau)$$

$\Rightarrow$ Fou

for wave equation

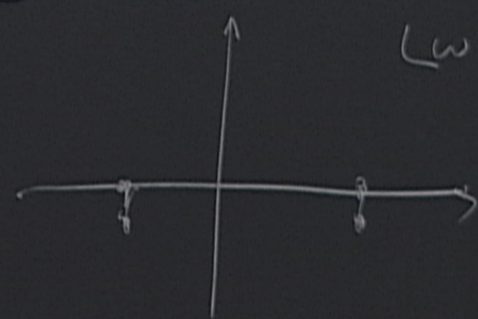
$\Rightarrow$ Fourier expand $$G(\vec{x}-\vec{\xi}, t-\tau) = \int_{\mathbb{R}^d} \frac{d\vec{k}}{(2\pi)^d} \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)} G(\vec{k},\omega)\, e^{-i\vec{k}\cdot(\vec{x}-\vec{\xi})}\, e^{-i\omega(t-\tau)}$$

Green's function for wave equation

$$\vec{\nabla}_x^2 G(\bar{x}-\xi, t-\tau) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G(\bar{x}-\xi, t-\tau) = \delta(\bar{x}-\xi)\,\delta(t-\tau)$$ $\Rightarrow$ Fourier e

$$\Rightarrow \quad \left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right) G(\vec{k}, \omega) = 1 \quad \Rightarrow \quad \boxed{G(\vec{k},\omega) = \frac{c^2}{\omega^2 - c^2 \vec{k}^2}}$$

To e



$$(\omega + i\varepsilon)^2 = c^2 k^2$$

$$\omega = \pm ck - i\varepsilon$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{c^2}{(\omega+i\varepsilon)^2-c^2k^2} e^{-i\omega(t-\tau)} = -\frac{ic^2}{2ck}\left[e^{-ick(t-\tau)} - e^{ick(t-\tau)} \right] \theta(t-\tau)$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{c^2}{(\omega+i\varepsilon)^2 - c^2 k^2} e^{-i\omega(t-\tau)} = -\frac{ic^2}{2ck}\left[e^{-ick(t-\tau)} - e^{ick(t-\tau)} \right] \theta(t-\tau) = -$$

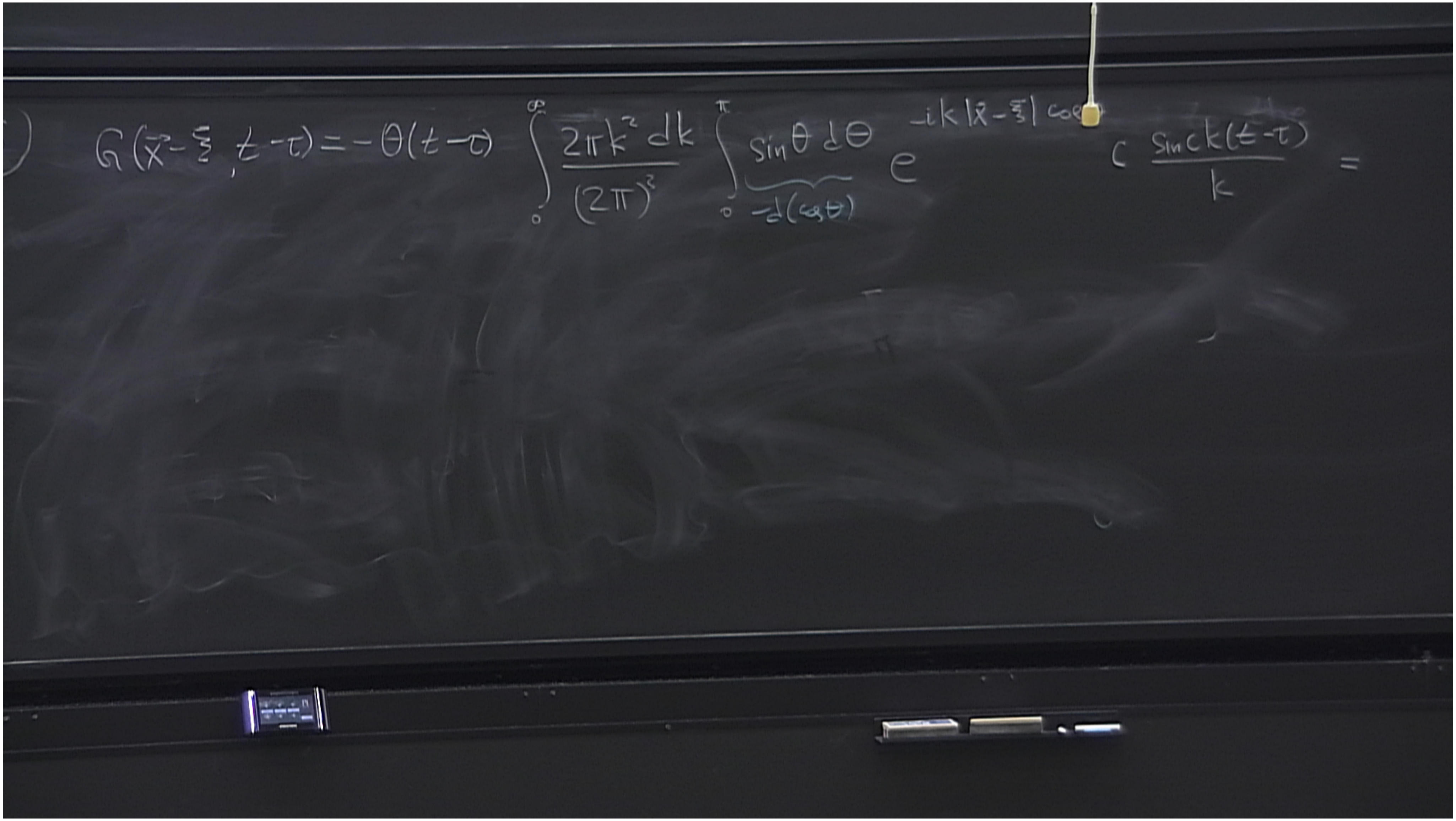
$$\boxed{G(\bar{x}-\bar{\xi}, t-\tau) = -\theta(t-\tau) \int_{\mathbb{R}^d} \frac{d\bar{k}}{(2\pi)^d} e^{-i\bar{k}\cdot(\bar{x}-\bar{\xi})} c\, \frac{\sin ck(t-\tau)}{k}}$$

$$\ldots ck(t-\tau)\Big]\,\theta(t-\tau) = -\,\theta(t-\tau)\, c\, \frac{\sin ck(t-\tau)}{k}$$

$$\boxed{\ldots\; c\,\frac{\sin ck(t-\tau)}{k}}$$

causal Green's function for d-dimensional wave equation

$$G(\vec{x}-\vec{\xi}, t-\tau) = -\Theta(t-\tau)\int_0^\infty \frac{2\pi k^2\,dk}{(2\pi)^3}\int_0^\pi \sin\theta\, d\theta\; e^{-ik|\vec{x}-\vec{\xi}|\cos\theta}\; c\,\frac{\sin ck(t-\tau)}{k}$$


$G(\vec{x}-\vec{\xi}, t-\tau) = -\theta(t-\tau) \int_0^\infty \frac{2\pi k^2 dk}{(2\pi)^3} \int_0^\pi \underbrace{\sin\theta \, d\theta}_{-d(\cos\theta)} e^{-ik|\vec{x}-\vec{\xi}|\cos\theta} c \frac{\sin ck(t-\tau)}{k} =$

Green's function in three dimensions

$$G(\vec{x}-\vec{\xi}, t-\tau) = -\Theta(t-\tau)$$

$$= -\Theta(t-\tau) \int_0^\infty \frac{k^2\,dk}{4\pi^2} \left(\frac{1}{-ik|\vec{x}-\vec{\xi}|} \right) \left[e^{-ik|\vec{x}-\vec{\xi}|} - e^{ik|\vec{x}-\vec{\xi}|} \right] c\, \frac{\sin ck(t-\tau)}{k} =$$

$$= - \frac{\Theta(t-\tau)}{|\vec{x}-\vec{\xi}|} \int_0^\infty \frac{dk}{2\pi^2}\, c \sin k|\vec{x}-\vec{\xi}| \cdot \sin ck(t-\tau)$$

$$\int_0^\infty \sin k|\vec{x}-\vec{\xi}| \cdot \sin ck(t-\tau)\, dk = \frac{1}{2}\int_{-\infty}^{+\infty} \sin k|\vec{x}-\vec{\xi}| \cdot \sin ck(t-\tau)\, dk =$$

$$= \frac{1}{4}\int_{-\infty}^{+\infty} \left\{ \cos\left[k|\vec{x}-\vec{\xi}| - ck(t-\tau)\right] - \cos\left[k|\vec{x}-\vec{\xi}| + ck(t-\tau)\right] \right\} dk =$$

$$= \frac{\pi}{2}\left\{ \delta\left(|\vec{x}-\vec{\xi}| - c(t-\tau)\right) - \delta\left(|\vec{x}-\vec{\xi}| + c(t-\tau)\right) \right\}$$

will be 0 because $t-\tau>0$

$$G(\vec{x}-\vec{\xi}, t-\tau) = -\theta(t-\tau)\int_0^\infty \frac{2\pi k^2\,dk}{(2\pi)^3}\int_0^\pi \underbrace{\sin\theta\, d\theta}_{-d(\cos\theta)}\, e^{-ik|\vec{x}-\vec{\xi}|\cos\theta}\, c\,\frac{\sin ck(t-\tau)}{k} =$$

$$\Big]\; c\,\frac{\sin ck(t-\tau)}{k} =$$

Gathering all together

$$\boxed{G(\vec{x}-\vec{\xi}, t-\tau) = -\frac{\theta(t-\tau)\,c}{4\pi|\vec{x}-\vec{\xi}|}\,\delta\big(|\vec{x}-\vec{\xi}| - c(t-\tau)\big)}$$

Green's function in $d=3$

$$ick$$

$$\boxed{b(\underline{k}) = \frac{1}{2}\left[u_0(\underline{k}) - \frac{1}{ick}v_0(\underline{k})\right]}$$

$$\varphi(\vec{x},t) = -\int_{\mathbb{R}^d} d\vec{\xi} \int_{-\infty}^{t} d\tau \; \delta\left(|\vec{x}-\vec{\xi}| - c(t-\tau)\right) \frac{c\, f(\vec{\xi},\tau)}{4\pi |\vec{x}-\vec{\xi}|}$$

$$\varphi(\bar{x},t) = -\int_{\mathbb{R}^d} d\bar{\xi} \int_{-\infty}^{t} d\tau \; \delta\left(|\bar{x}-\bar{\xi}| - c(t-\tau)\right) \frac{c f(\bar{\xi},\tau)}{4\pi |\bar{x}-\bar{\xi}|} = -\frac{1}{4\pi}\int_{\mathbb{R}^d} d\bar{\xi} \; \frac{f(\bar{\xi},t_r)}{|\bar{x}-\bar{\xi}|}\Bigg|_{t_r=}$$

$$t-\tau = \frac{|\bar{x}-\bar{\xi}|}{c}$$

$$\tau \longrightarrow t_r = t - \frac{|\bar{x}-\bar{\xi}|}{c}$$

$$\cdots\frac{f(\bar{\xi},\tau)}{|\bar{x}-\bar{\xi}|} = -\frac{1}{4\pi}\int_{\mathbb{R}^3} d\bar{\xi}\,\frac{f(\bar{\xi},t_r)}{|\bar{x}-\bar{\xi}|}\Bigg|_{t_r = t-\frac{|\bar{x}-\bar{\xi}|}{c}}$$

$$\bar{\xi} \int_{-\infty}^{t} d\tau \; \delta\left(|\bar{x}-\bar{\xi}| - c(t-\tau)\right) \frac{c\, f(\bar{\xi},\tau)}{4\pi |\bar{x}-\bar{\xi}|} = -\frac{c}{4\pi} \int_{\mathbb{R}^3} d\bar{\xi} \; \frac{f(\bar{\xi}, t_r)}{|\bar{x}-\bar{\xi}|} \Bigg|_{t_r = t - \frac{|\bar{x}-\bar{\xi}|}{c}}$$

Maxwell's equations

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J} \quad (1)$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (2)$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \quad (3)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (4)$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{j} \qquad (1)$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (2)$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \qquad (3)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (4)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi$$

$\vec{B} = \vec{\nabla} \times \bar{A}$ $\vec{\nabla} \cdot (\vec{\nabla} \times \bar{A}) = 0$ $\bar{A}, \phi$ - vector and scalar potentials

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\vec{\nabla} \phi$$

$\vec{B}$ and $\vec{E}$ are invariant under $\phi \rightarrow \phi + \frac{\partial \Lambda}{\partial t}$, $\bar{A} \rightarrow \bar{A} + \vec{\nabla}\Lambda$

Lorentz gauge $\rightarrow \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \bar{A} = 0$

er $\phi \longrightarrow \phi + \frac{\partial \Lambda}{\partial t}$, $\bar{A} \longrightarrow \bar{A} + \bar{\nabla}\Lambda$

; Substituting into (3) $\bar{\nabla}\cdot\left(-\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}\right) = 4\pi\rho \Rightarrow$ since $\nabla\cdot\frac{\partial \bar{A}}{\partial t} = -\frac{\partial^2\phi}{\partial t^2}$

$$\bar{\nabla}^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$$

will be 0 because $t-\tau>0$

$\phi \longrightarrow \phi + \frac{\partial\Lambda}{\partial t}$, $\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda$

; Substituting into (3) $\vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = 4\pi\rho \Rightarrow$ since $\nabla\cdot\frac{\partial\vec{A}}{\partial t} = -\frac{\partial^2\phi}{\partial t^2}$

$$\boxed{\vec{\nabla}^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi\rho} \quad (I)$$

$-e \;\; \ldots) \, \theta(t-\tau) = -\theta(t-\tau)\, c \, \frac{\sin ck(t-\tau)}{k}$

Ex: Using $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$ show that

$$\vec{\nabla}^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi \vec{J}$$

Ex. Using $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$ show that

$$\vec{\nabla}^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} = -4\pi\vec{J}$$

$$\Rightarrow \begin{Bmatrix} \varphi(\bar{x},t) \\ \bar{A}(\bar{x},t) \end{Bmatrix} = C \int_{\mathbb{R}^3} \frac{d\bar{\xi}}{|\bar{x}-\bar{\xi}|} \begin{Bmatrix} \rho(\bar{x},t_r) \\ \bar{J}(\bar{x},t_r) \end{Bmatrix} \Bigg|_{t_r = t - \frac{|\bar{x}-\bar{\xi}|}{c}}$$