

Title: 14/15 PSI - Green Functions

Date: Sep 03, 2014 09:00 AM

URL: <http://pirsa.org/14090034>

Abstract:

Suppose we have

$$\text{some } \vec{x} \in \mathbb{R}^d$$

$\nearrow$ d-dimensional
real coordinate
space

Fourier transform

$$u(\vec{k}) =$$

$u(\vec{x})$ is an integrand

Fourier transform

$u(\vec{x})$ is an integrable function

$$u(\vec{k}) = \mathcal{F}[u] = \int_{\mathbb{R}^d} u(\vec{x}) e^{i\vec{k} \cdot \vec{x}} d\vec{x}$$

$$u(\vec{x}) = \hat{\mathcal{F}}^{-1}[u] = \int_{\mathbb{R}^d} u(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d}$$

have

$\mathbb{R}^d$

d-dimensional
real coordinate
space

Example. Compute $\mathcal{F}[e^{-\vec{x}^2/2}]$, $e^{-\vec{x}^2/2} = \exp\{-\frac{1}{2} \sum_{i=1}^d x_i^2\}$,

$$\mathcal{F}[e^{-\vec{x}^2/2}] = \prod_{i=1}^d I(k_i)$$

$$[e^{-\vec{x}^2/2}], \quad e^{-\vec{x}^2/2} = \exp\left\{-\frac{1}{2} \sum_{i=1}^d x_i^2\right\}, \quad \vec{k} \cdot \vec{x} = \sum_{i=1}^d k_i x_i$$

$$I(\vec{k}) = \int_{-\infty}^{+\infty} dx_i e^{-x_i^2/2 + i k_i x_i}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n k_i x_i$$

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} dx e^{-ax^2 + bx} = e^{b^2/4a} \sqrt{\frac{\pi}{a}}$$

Reminder

compute $\mathcal{F}[e^{-\vec{x}^2/2}]$, $e^{-\vec{x}^2/2} = \exp\{-\frac{1}{2} \sum_{i=1}^d x_i^2\}$, $\vec{k} \cdot \vec{x} = \sum_{i=1}^d k_i x_i$

$$\prod_{i=1}^d I(k_i), \quad I(k_i) = \int_{-\infty}^{\infty} dx_i e^{-x_i^2/2 + i k_i x_i} = e^{-k_i^2/2} \sqrt{2\pi}$$

$$[e^{-\vec{x}^2/2}] = (2\pi)^{d/2} e^{-\vec{k}^2/2} \Rightarrow \mathcal{F}^{-1}\left[e^{-\vec{k}^2/2} (2\pi)^{d/2}\right] = e^{-\vec{x}^2/2}$$

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1) Linearity

$$\text{If } w(\vec{x}) = u(\vec{x}) + v(\vec{x})$$

$$w(\vec{k}) = u(\vec{k}) + v(\vec{k})$$

Basic Properties of Fourier Transform

2) Translation

If for any $\vec{x}_0$, $v(\vec{x}) = u(\vec{x} - \vec{x}_0)$, then

$$= e^{i\vec{k}\vec{x}_0}$$

5) Derivatives, $D = \frac{\partial}{\partial x_i}$

$$\mathcal{F}[D_1 D_2 \dots D_n u(\vec{x})] = (-ik_1)(-ik_2)\dots(-ik_n)u(\vec{k})$$

Similarly, $\mathcal{F}[(ix_i)u(\vec{x})] = \frac{\partial u(\vec{k})}{\partial k_i} \rightsquigarrow$

Proof: $ix_i u(\vec{x}) = \int_{\mathbb{R}^n} i$

$$-ik_j u(\vec{k})$$

Proof:
$$ix_j u(\vec{x}) = \int_{\mathbb{R}^d} ix_j u(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d} = \int_{\mathbb{R}^d} u(\vec{k}) \left(-\frac{\partial}{\partial k_j}\right) e^{-i\vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d} =$$

~~~~~



$$-ik_j u(\vec{k})$$

Proof:

$$\begin{aligned} i x_j u(\vec{x}) &= \int_{\mathbb{R}^d} i x_j u(\vec{k}) e^{-i \vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d} = \int_{\mathbb{R}^d} u(\vec{k}) \left( -\frac{\partial}{\partial k_j} \right) e^{-i \vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d} = \\ &= \int_{\mathbb{R}^d} \frac{\partial u(\vec{k})}{\partial k_j} e^{-i \vec{k} \cdot \vec{x}} \frac{d\vec{k}}{(2\pi)^d} \end{aligned}$$

6) Convolution property

$$w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{x} - \vec{y}) v(\vec{y}) d\vec{y}$$



on property

$$w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{x} - \vec{y}) v(\vec{y}) d\vec{y} \quad \leftarrow \text{convolution}$$

6) Convolution property

$$w(\vec{x}) = (u * v)(\vec{x}) = \int_{\mathbb{R}^d} u(\vec{x} - \vec{y}) v(\vec{y}) d\vec{y}$$

$$w(\vec{k}) = u(\vec{k}) v(\vec{k})$$

7) Parseval's equality: For two functions  $u(\vec{x})$  and  $v^*(\vec{x})$

$$\int_{\mathbb{R}^d} u(\vec{x}) v^*(\vec{x}) d\vec{x} = \int_{\mathbb{R}^d} u(\vec{k}) v^*(\vec{k}) \frac{d\vec{k}}{(2\pi)^d}$$



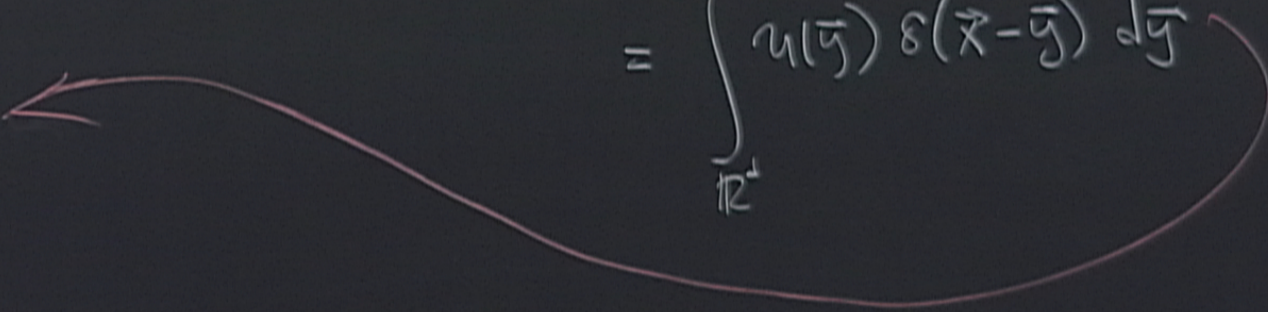
Fourier representation of a  $\delta$ -function

$$u(\vec{x}) = \int_{\mathbb{R}^d} u(t)$$



$$\begin{aligned}
 u(\vec{x}) &= \int_{\mathbb{R}^d} u(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^d} = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} u(\vec{y}) e^{i\vec{k}\cdot\vec{y}} d\vec{y} \right) e^{-i\vec{k}\cdot\vec{x}} \frac{d\vec{k}}{(2\pi)^d} = \\
 &= \int_{\mathbb{R}^d} u(\vec{y}) \delta(\vec{x}-\vec{y}) d\vec{y}
 \end{aligned}$$



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# Fourier representation of a $\delta$ -function

$$u(\vec{x}) =$$

$$\int_{\mathbb{R}^d} e^{i\vec{k}\cdot\vec{x}} d\vec{x} = (2\pi)^d \delta(\vec{x})$$

$$\delta(\vec{x}-\vec{y}) = \int_{\mathbb{R}^d} \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$



## Fourier representation of a $\delta$ -function

$$u(\vec{x}) =$$

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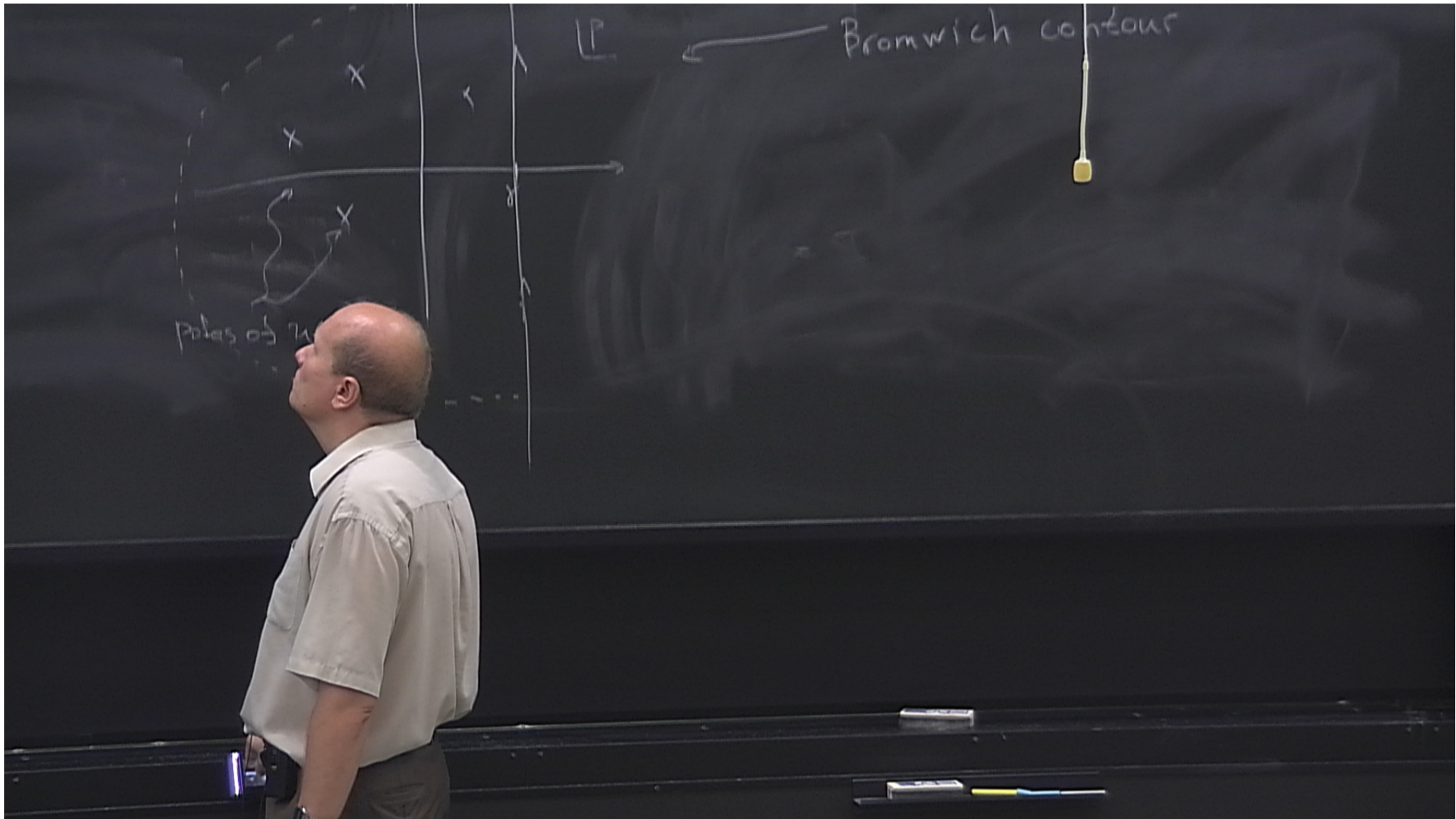
Note Other relevant transforms:

(i) Laplace transform For  $u(x)$

$$u(p) = \mathcal{L}(u) = \int_0^{\infty} u(x) e^{-px} dx$$

$$u(x) = \mathcal{L}^{-1}(u) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} u(p) e^{px} dp$$







(ii) Mellin transform

For  $u(x)$  ,  $u(s) = \mathcal{M}(u) = \int_0^\infty u(x) x^{s-1} dx$

$u(x) = \mathcal{M}^{-1}(u)$



For  $u(x)$  ,  $u(s) = \mathcal{M}(u) = \int_0^\infty u(x) x^{s-1} dx,$

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For  $u(x)$  ,  $u(\sigma) = \mathcal{M}(u) = \int_0^\infty u(x) x^{\sigma-1} dx,$

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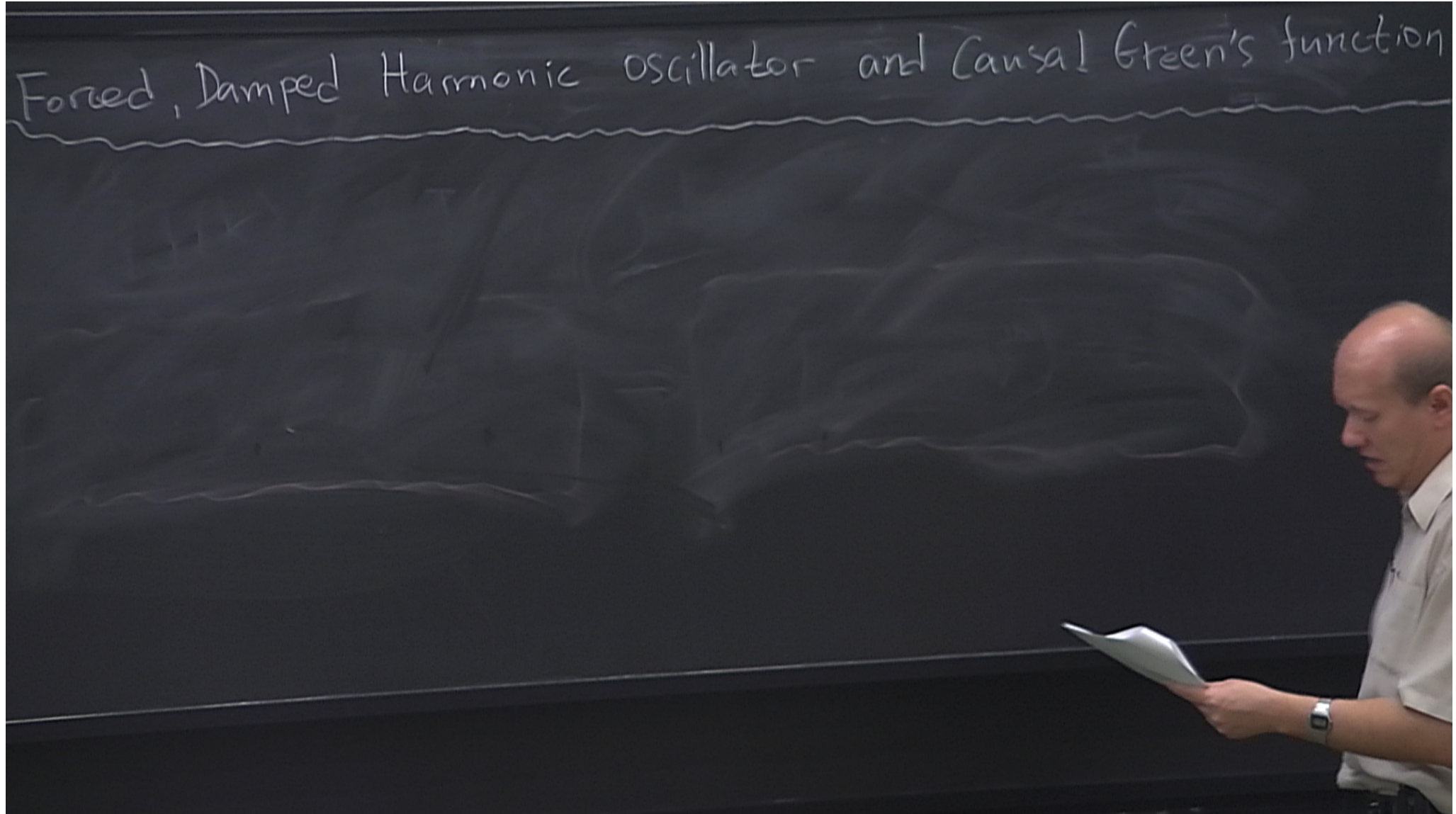
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Forced, Damped Harmonic oscillator and Causal Green's function







Forced, Damped Harmonic oscillator and Causal Green's function

$$\ddot{x} + 2\gamma\dot{x} + (\Omega^2 + \gamma^2)x = F(t), \text{ What is } x(t)?$$



# Forced, Damped Harmonic oscillator and Causal Green's function

$$\ddot{x} + 2\gamma\dot{x} + (\Omega^2 + \gamma^2)x = F(t), \text{ What is } x(t)?$$

$$L_t X = F(t), \quad L_t = \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + (\Omega^2 + \gamma^2)$$



# Forced, Damped Harmonic Oscillator and Causal Green's function

$$\ddot{x} + 2\gamma\dot{x} + (\Omega^2 + \gamma^2)x = F(t), \text{ What is } x(t)?$$

$$L_t X = F(t), \quad L_t = \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + (\Omega^2 + \gamma^2) \Rightarrow L_t G(t) = \delta(t),$$



$$[-\omega^2 - 2i\gamma\omega + (\Omega^2 + \gamma^2)] G(\omega) = 1 \Rightarrow G(\omega) = \frac{1}{-\omega^2 - 2i\gamma\omega + (\Omega^2 + \gamma^2)}$$

$$[-\omega^2 - 2i\gamma\omega + (\Omega^2 + \gamma^2)] G(\omega) = 1 \Rightarrow G(\omega) = \frac{1}{\Omega^2 - (\omega + i\gamma)^2}$$



$$[-\omega^2 - 2i\gamma\omega + (\Omega^2 + \gamma^2)] \quad G(\omega) = 1 \quad \Rightarrow \quad G(\omega) = \frac{1}{\Omega^2 - (\omega + i\gamma)^2} = \frac{1}{2\Omega}$$

$$\Rightarrow G(\omega) = \frac{1}{\Omega^2 - (\omega + i\delta)^2} = \frac{1}{2\Omega} \left[ \frac{1}{\Omega - \omega - i\delta} + \frac{1}{\Omega + \omega + i\delta} \right]$$

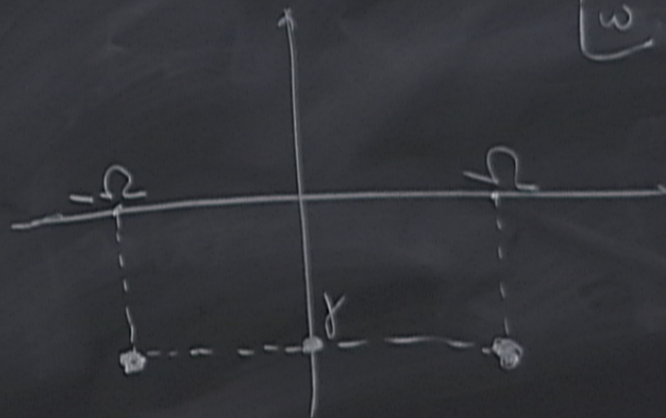


$$[-\omega^2 - 2i\gamma\omega + (\Omega^2 + \gamma^2)] \quad G(\omega) = 1 \Rightarrow G(\omega) = \frac{1}{\Omega^2 - (\omega + i\gamma)^2}$$

The poles of  $G(\omega)$  are

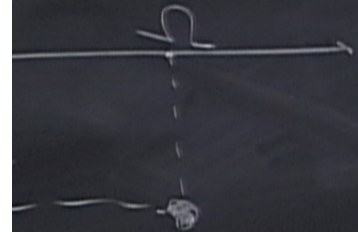
$$\omega_1 = \Omega - i\gamma$$

$$\omega_2 = -\Omega - i\gamma$$



$$= 1 \Rightarrow G(\omega) = \frac{1}{\Omega^2 - (\omega + i\delta)^2} = \frac{1}{2\Omega} \left[ \frac{1}{\Omega - \omega - i\delta} + \frac{1}{\Omega + \omega + i\delta} \right]$$

$$G(t) = \int_{-\infty}^{+\infty} G(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} = \begin{cases} 0, & t < 0 \end{cases}$$





$$= 1 \Rightarrow G(\omega) = \frac{1}{\Omega^2 - (\omega + i\delta)^2} = \frac{1}{2\Omega} \left[ \frac{1}{\Omega - \omega - i\delta} + \frac{1}{\Omega + \omega + i\delta} \right]$$

$$G(t) = \int_{-\infty}^{+\infty} G(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} = \begin{cases} 0, & t < 0 \end{cases}$$



$$G(t) = -2\pi i \frac{1}{2\Omega} \left[ -e^{-i\Omega t} e^{-\gamma t} + e^{i\Omega t} e^{-\gamma t} \right] \frac{1}{2\pi} =$$



$$\begin{aligned}
 G(t) &= -2\pi i \frac{1}{2\Omega} \left[ -e^{-i\Omega t} e^{-\gamma t} + e^{i\Omega t} e^{-\gamma t} \right] \frac{1}{2\pi} = \\
 &= \frac{e^{-\gamma t}}{2i\Omega} \left[ e^{i\Omega t} - e^{-i\Omega t} \right] = \frac{e^{-\gamma t}}{\Omega} \sin \Omega t
 \end{aligned}$$

$$G(t) = \theta(t) \frac{e^{-\gamma t}}{\Omega} \sin \Omega t \quad \leftarrow \text{Causal Green's function}$$

$$x(t) = \int_{-\infty}^t \frac{e^{-\gamma(t-\tau)}}{\Omega} \sin \Omega(t-\tau) F(\tau) d\tau$$



Green's functions for PDE's

$L =$

for PDE'S {

$$L = a \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x \partial y} + c \frac{\partial^2}{\partial y^2} + d \frac{\partial}{\partial x} + e \frac{\partial}{\partial y} + f$$



for PDE'S {

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$D = ac - b^2$  ,  $D < 0$  - hyperbolic  
 $D = 0$  - parabolic  
 $D > 0$  - elliptic



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$$L = a \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x \partial y} + c \frac{\partial^2}{\partial y^2} + d \frac{\partial}{\partial x} + e \frac{\partial}{\partial y} + f$$

$$D = ac - b^2$$

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the following examples

(1) Heat (diffusion) equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \gamma \frac{\partial \varphi}{\partial t} = f(x, t) \quad \leftarrow \text{Parabolic type}$$



1) Linearity

$$\text{If } w(\vec{x}) = u(\vec{x}) +$$

$$v(\vec{x})$$

{ Basic Properties of Fourier Trans

when  $\vec{x} =$



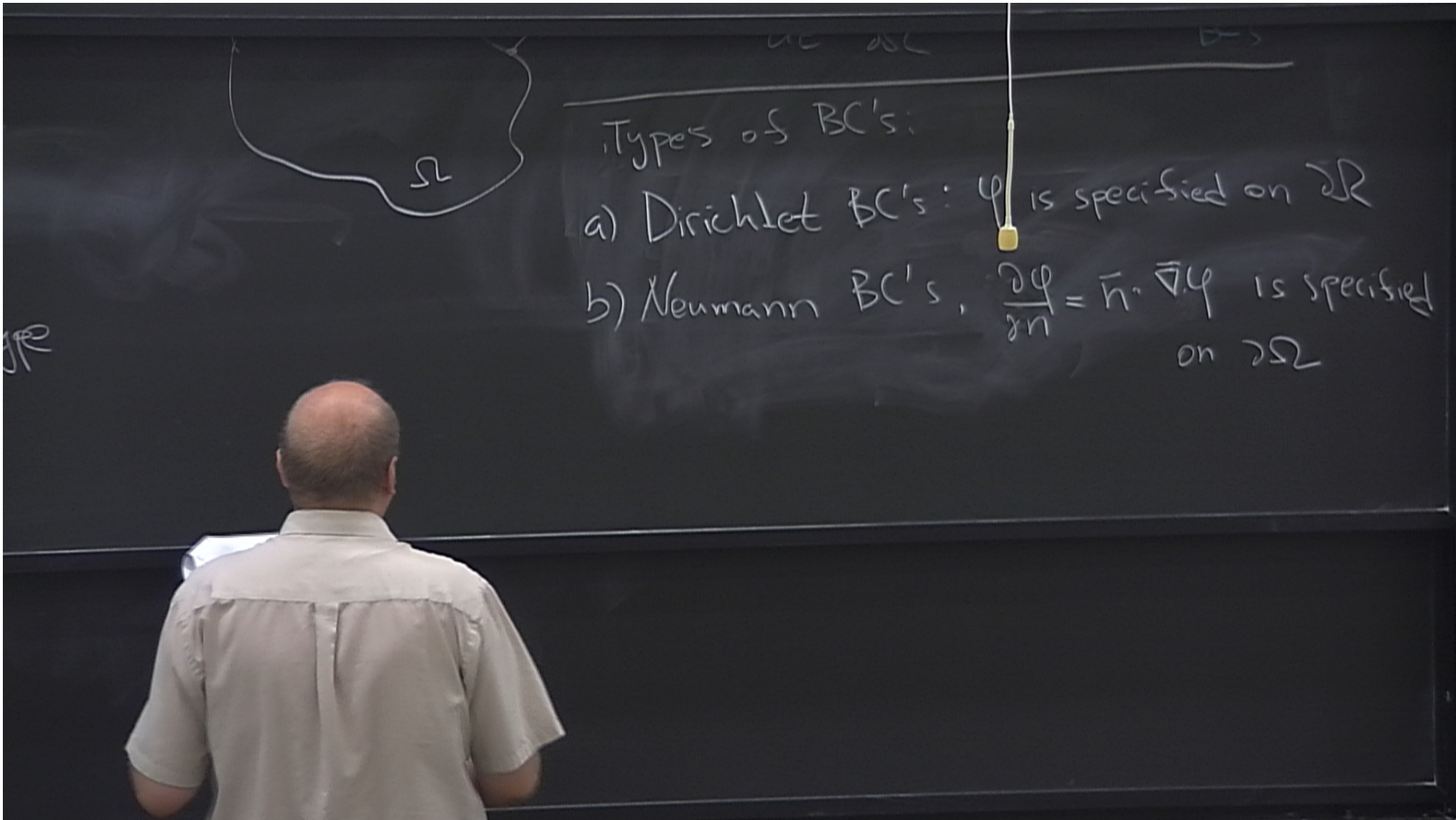
(ii) Poisson (or Laplace if  $f(x, y) = 0$ ) equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y) \quad \leftarrow \text{Elliptic type}$$

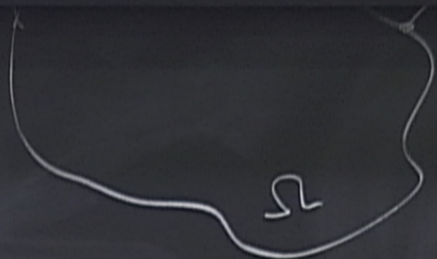
(iii) Wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = f(x, t) \quad \leftarrow \text{Hyperbolic type}$$









Types of BC's:

a) Dirichlet BC's:  $\varphi$  is specified on  $\partial\Omega$

b) Neumann BC's,  $\frac{\partial\varphi}{\partial n} = \vec{n} \cdot \vec{\nabla}\varphi$  is specified on  $\partial\Omega$

c) Cauchy BC's:

Both  $\varphi$  and  $\frac{\partial\varphi}{\partial n}$  are specified