Title: 14/15 PSI - Green Functions

Date: Sep 02, 2014 09:00 AM

URL: http://pirsa.org/14090033

Abstract:

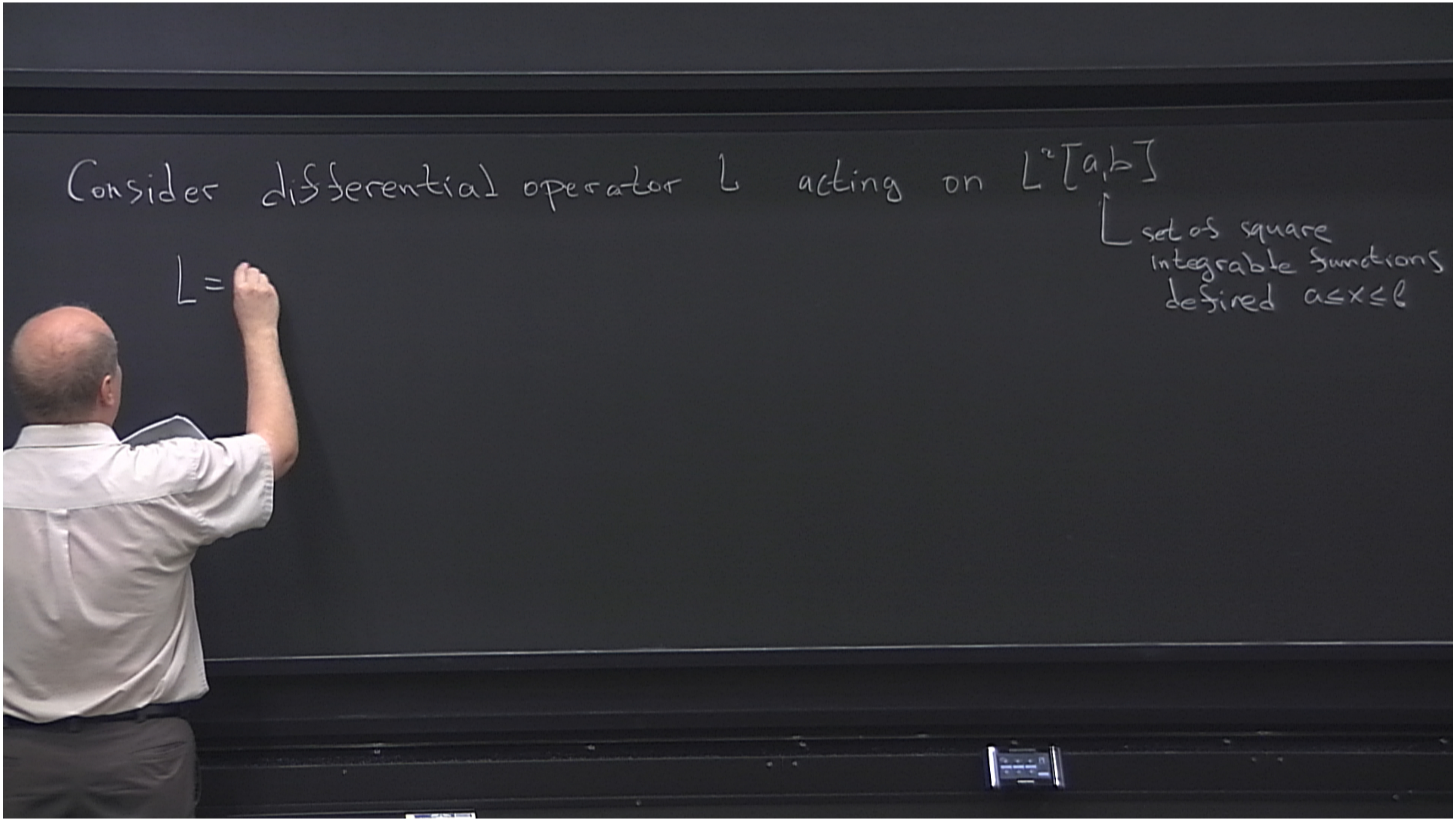
Plan of the course "Green's function"

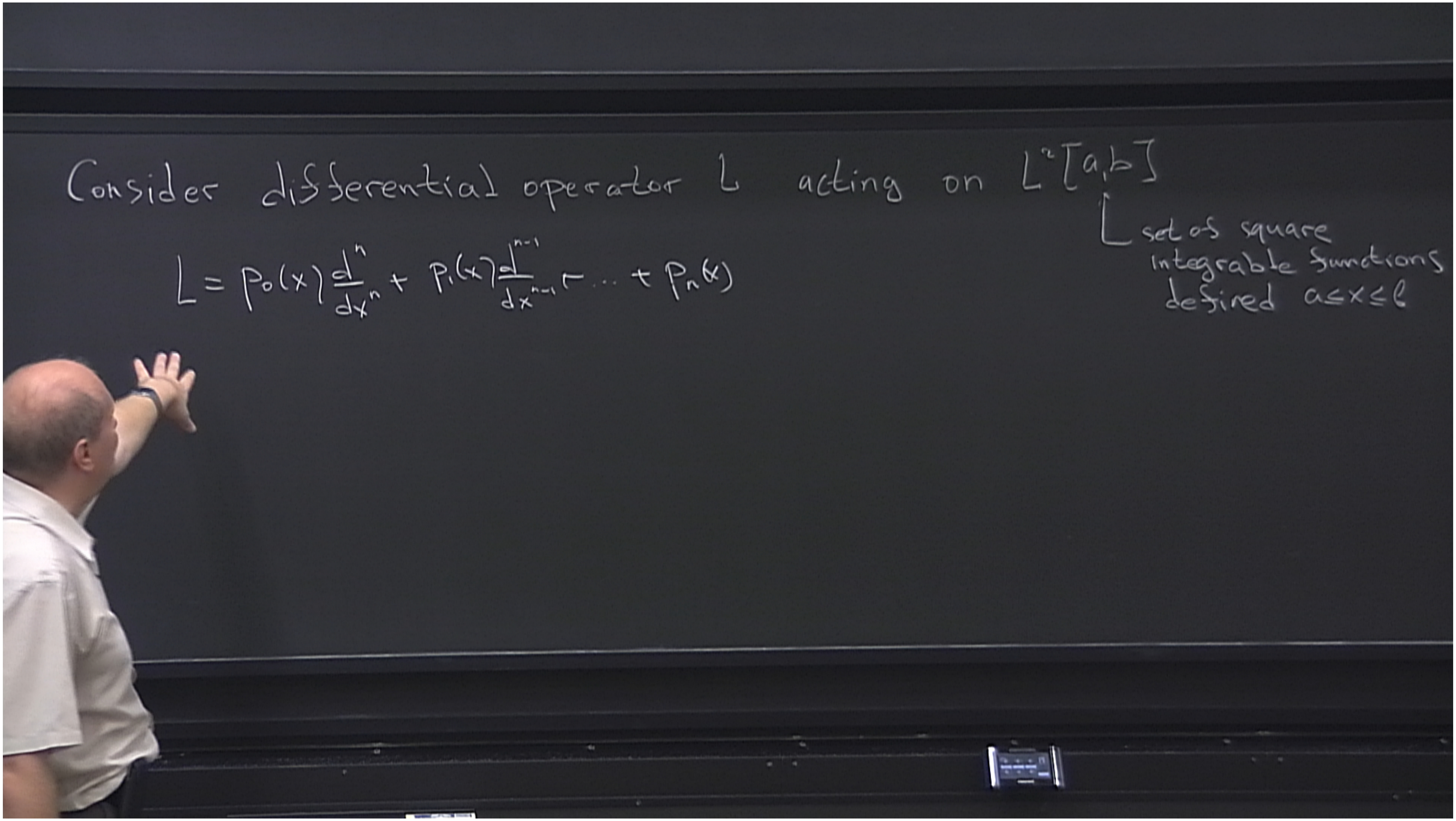
* GF for differential equations; self-adjoint differential operators
* Sturm-Liouville problem and boundary conditions; eigenfunction expansion
* Fourier transform and its properties
* Causal GF

fferential operators

s ; eigenfunction expansion

* GF for partial differential equations
* Diffusion equation and heat kernel
* Laplace and Poisson equations
* Maxwell's equations, retarded potentials
* Propagators in quantum field theory


Consider differential operator L acting on L²[a,b]
set of square integrable functions defined a≤x≤b
L =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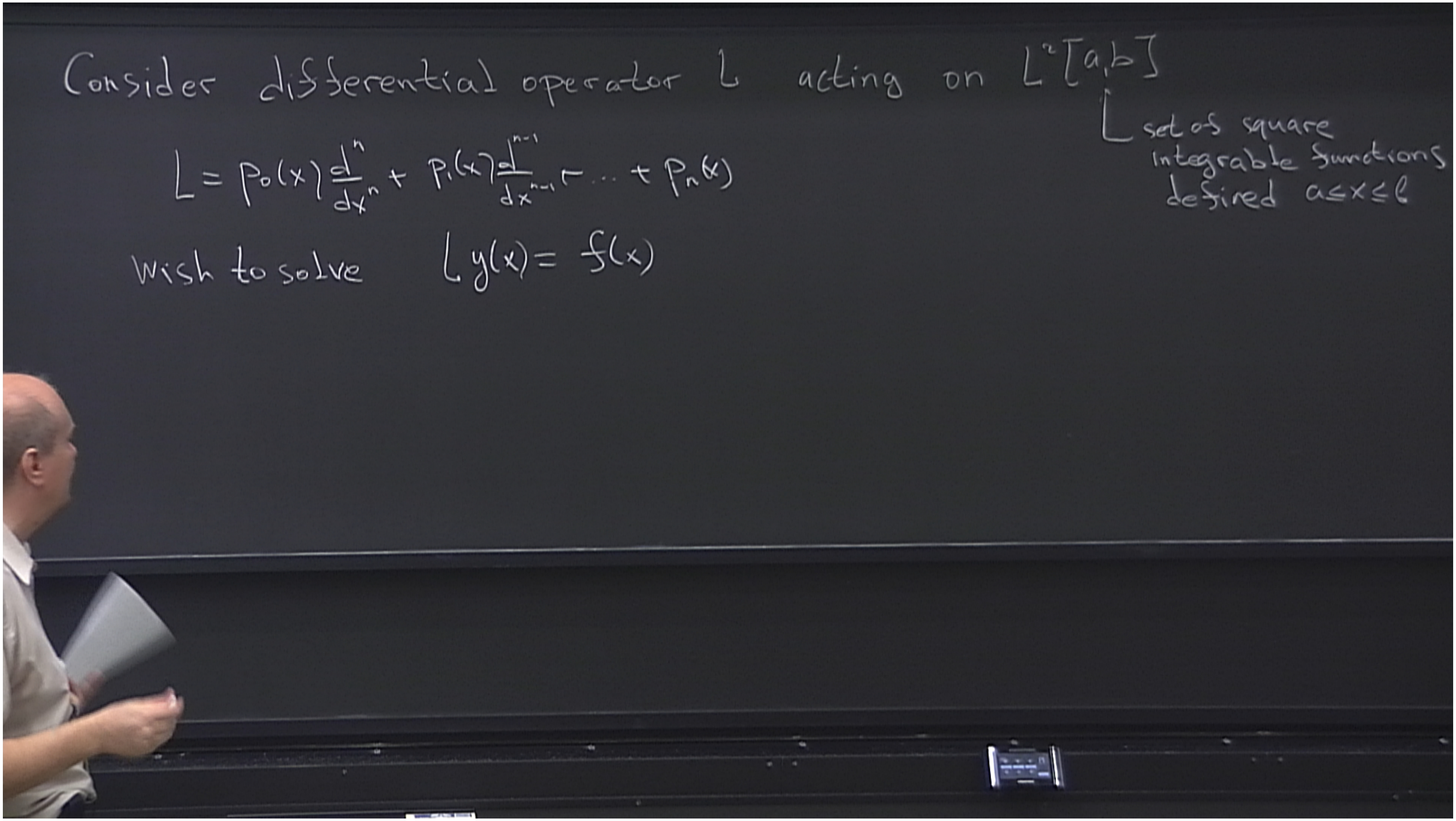
set of square integrable functions defined $a \le x \le b$
$L = p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + p_n(x)$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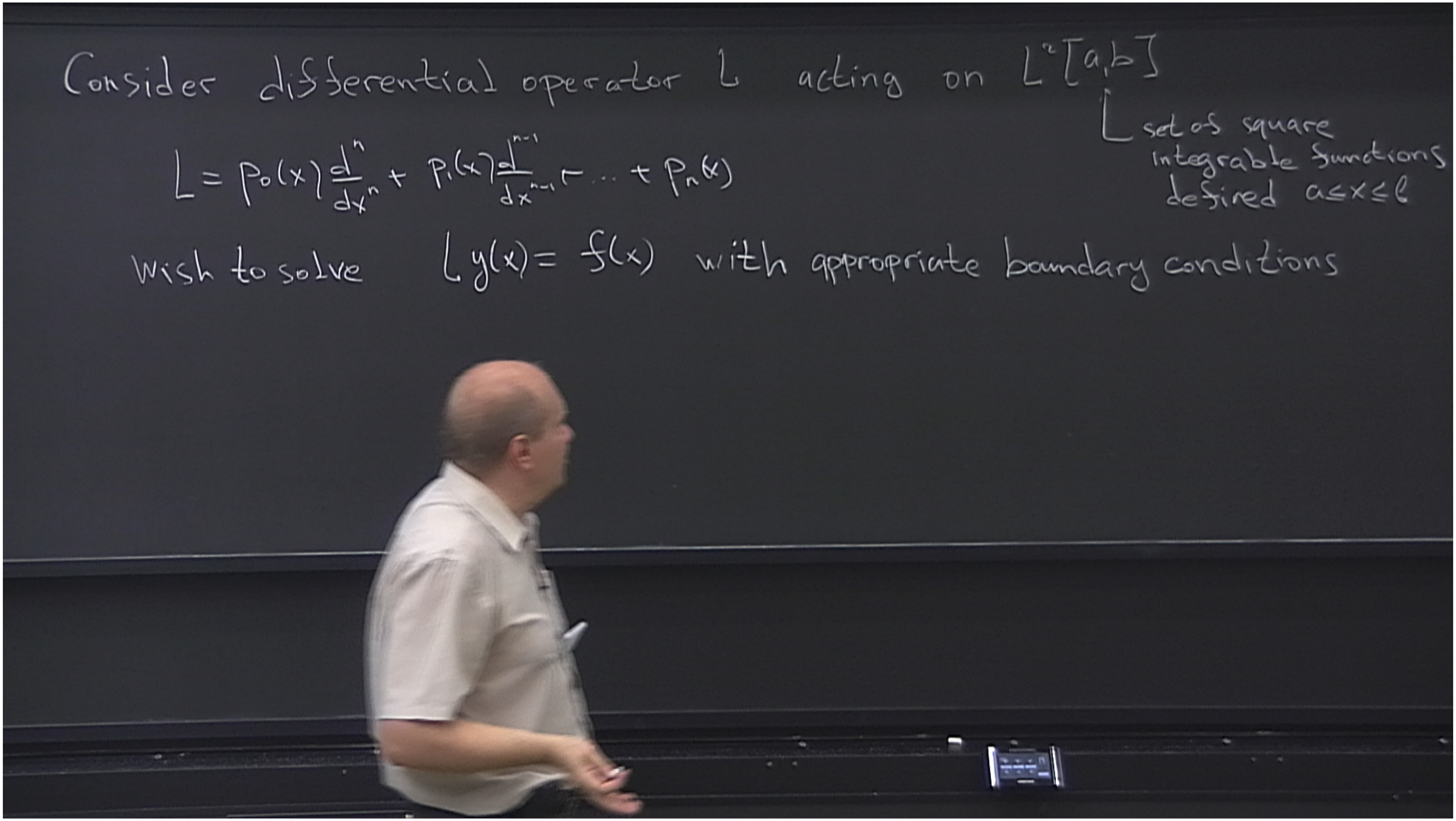
Wish to solve $Ly(x) = f(x)$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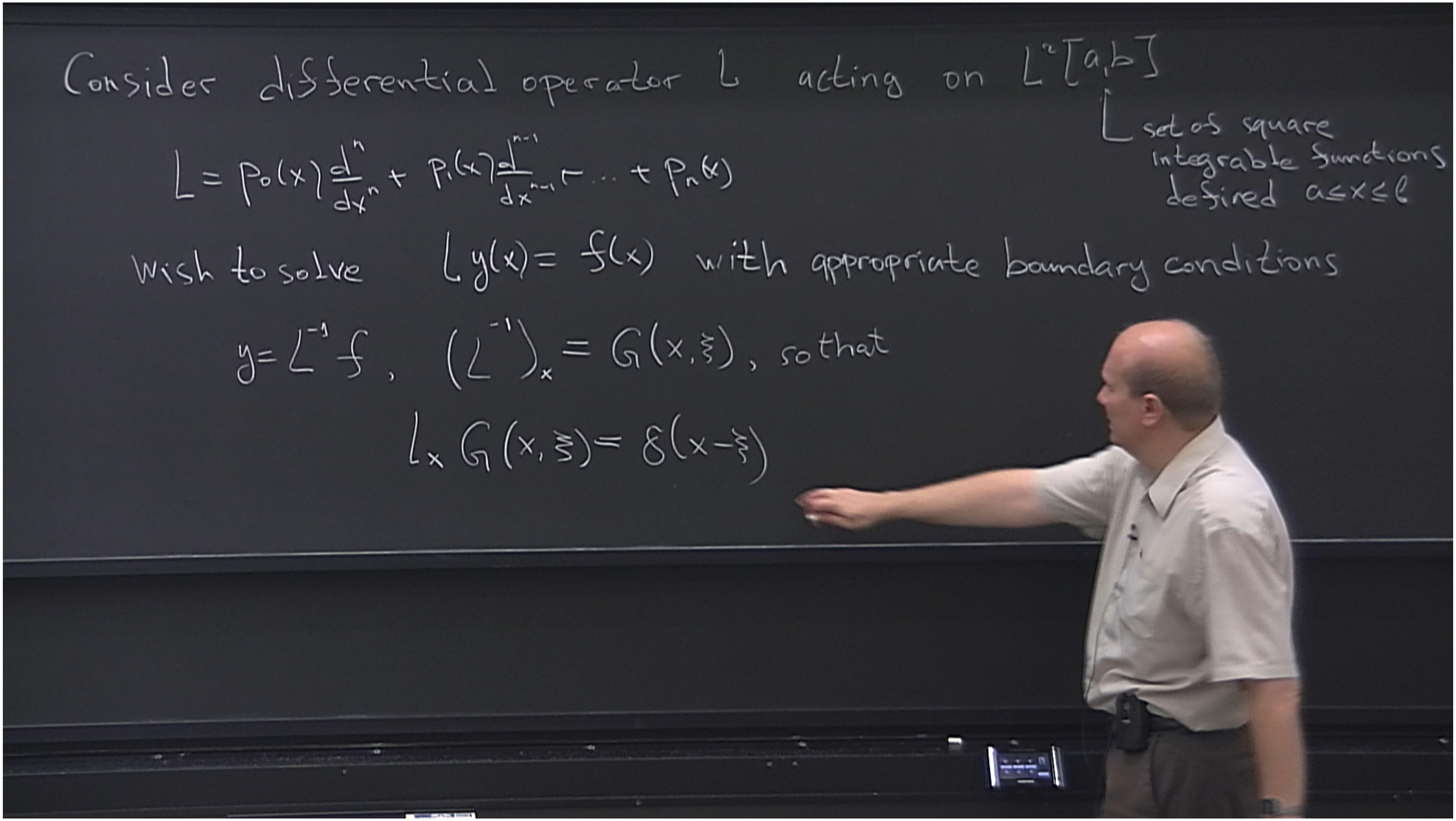
Wish to solve $Ly(x) = f(x)$ with appropriate boundary conditions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Wish to solve $Ly(x) = f(x)$ with appropriate boundary conditions
$y = L^{-1}f$, $(L^{-1})_x = G(x,\xi)$, so that
$L_x G(x,\xi) = \delta(x-\xi)$

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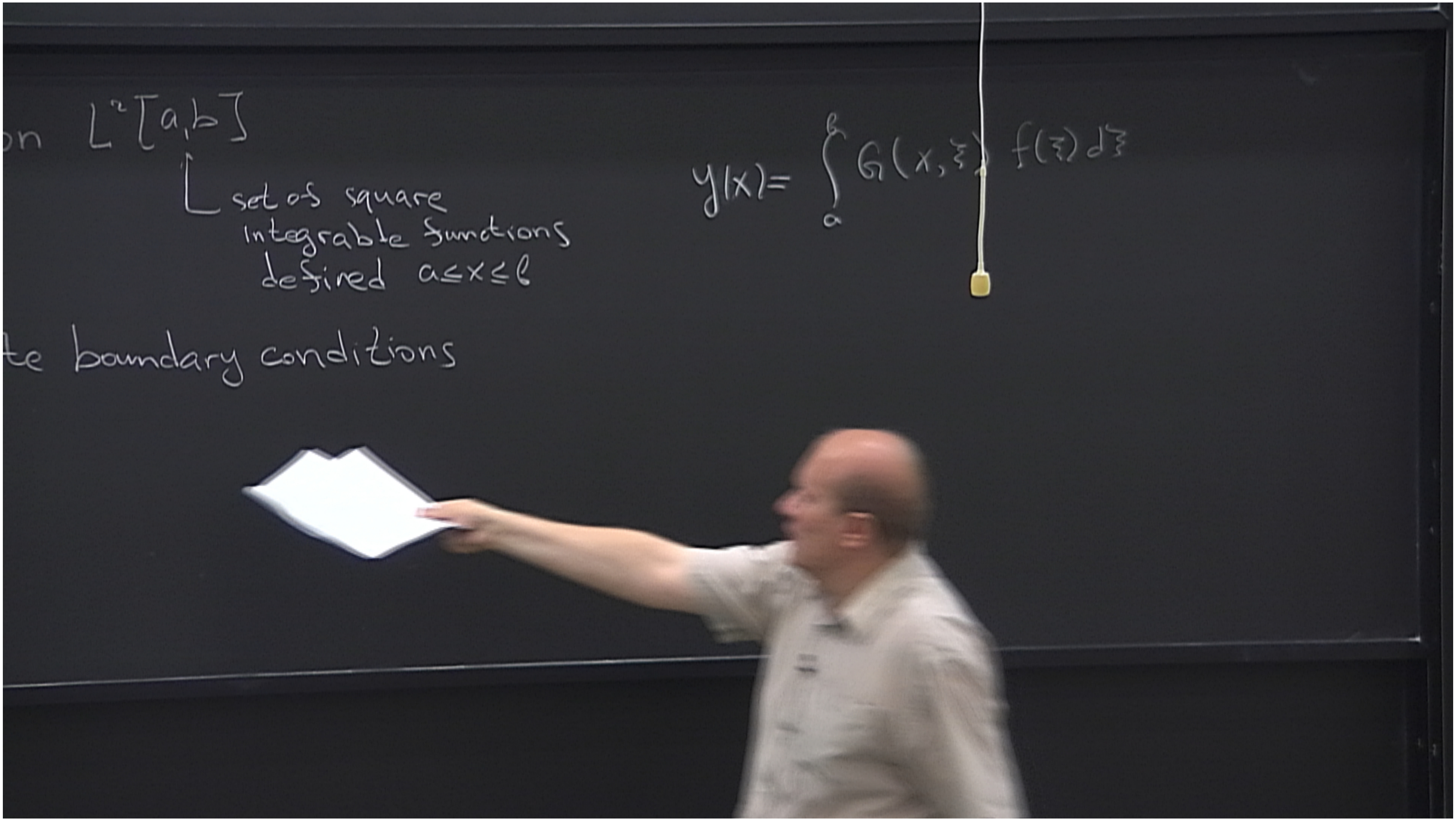
on $L^2[a,b]$
set of square
integrable functions
defined $a \le x \le b$
$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi$
te boundary conditions

on $L^2[a,b]$

set of square integrable functions defined $a \le x \le b$

te boundary conditions

$$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi$$

$$Ly = \int_a^b \delta(x-\xi)\, f(\xi)\, d\xi = f(x)$$

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Question. How to find $G(x,\xi)$

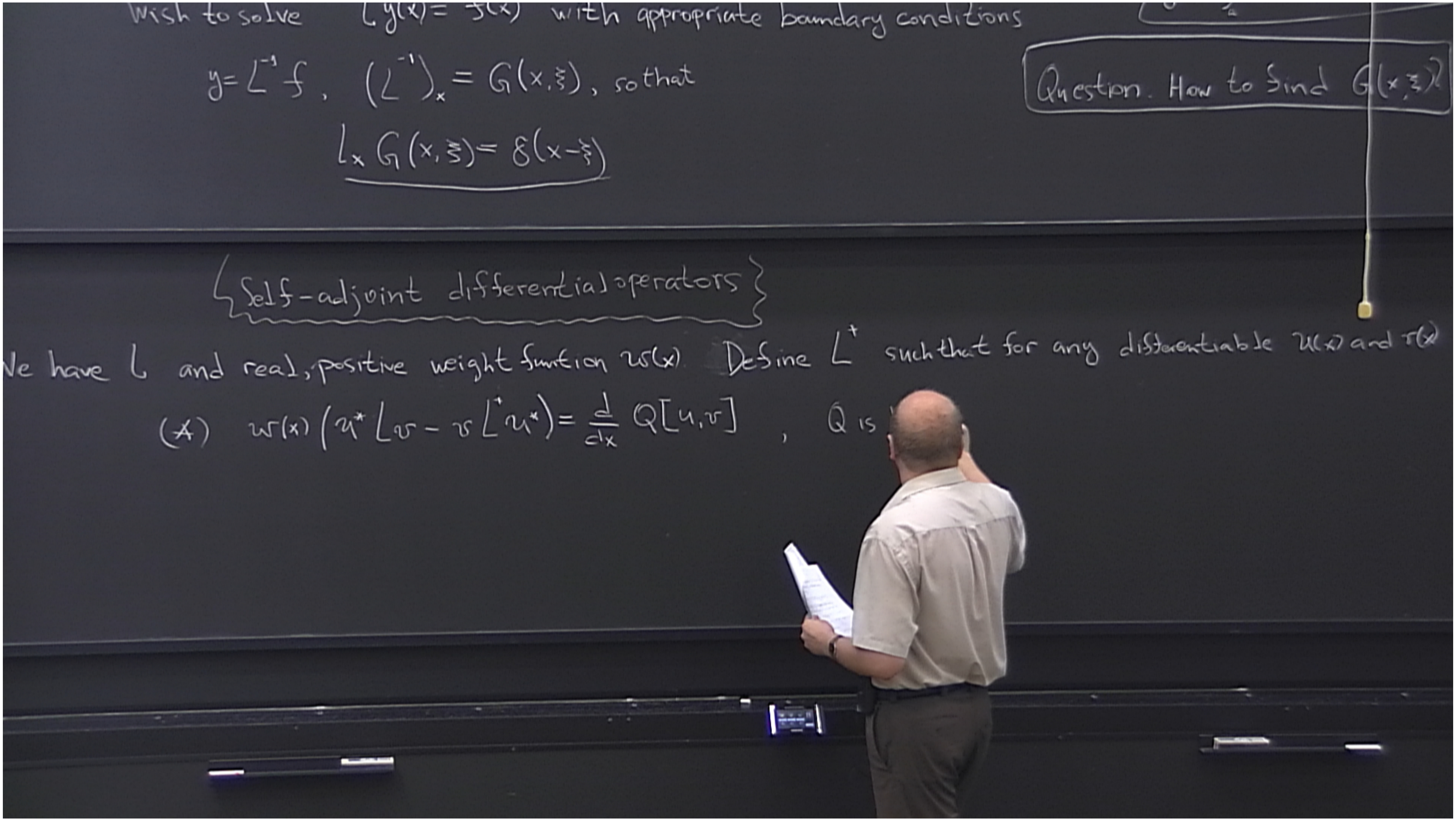
set of square integrable functions defined $a \le x \le b$

$g(x) = \int_a$

$$Ly = \int_a^b \delta(x-\xi) f(\xi) d\xi = f(x)$$

...e boundary conditions

Question. How to find $G(x,\xi)$?


Wish to solve $L y(x) = f(x)$ with appropriate boundary conditions
$y = L^{-1} f$, $(L^{-1})_x = G(x,\xi)$, so that
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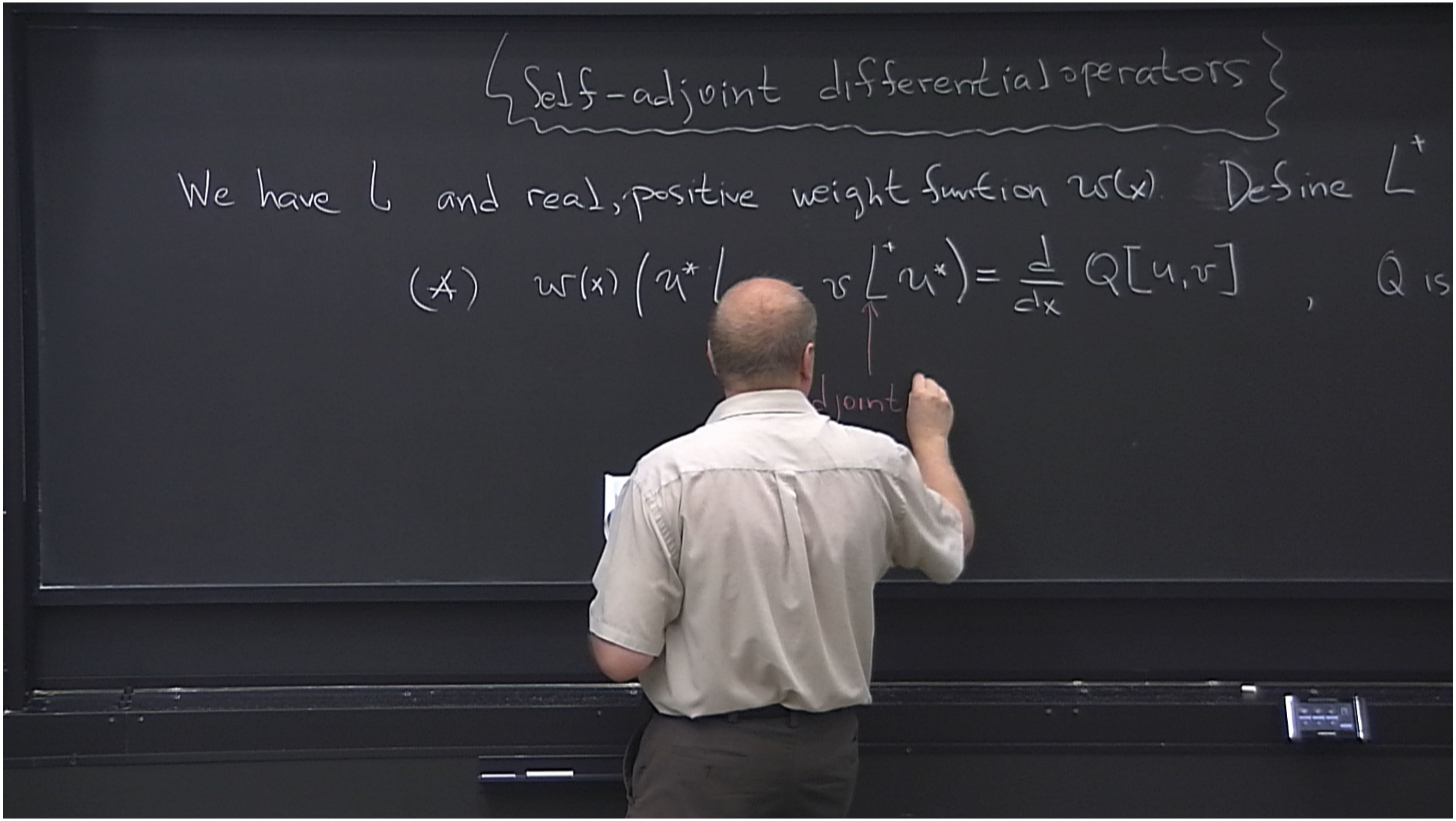
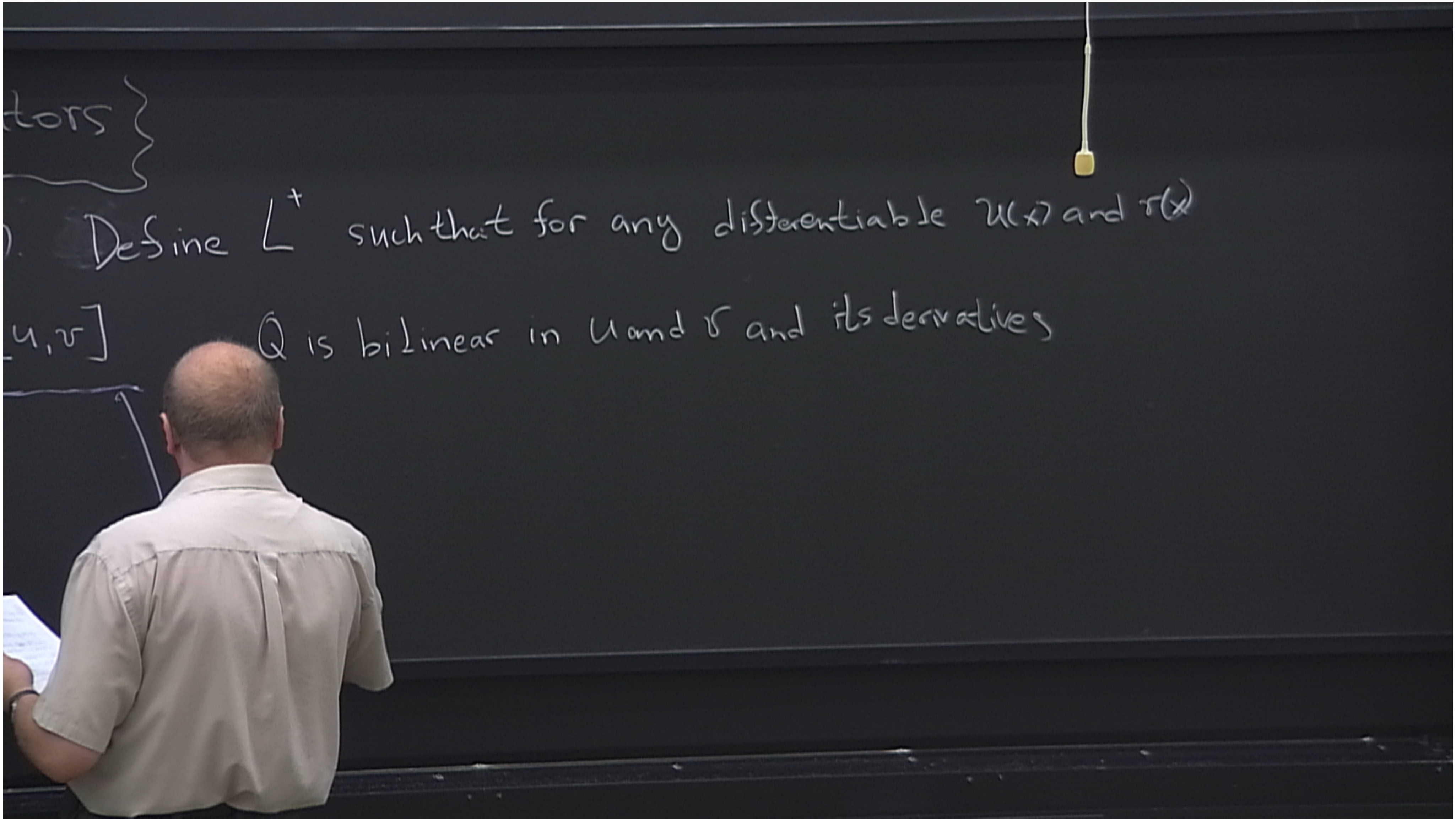
Question. How to find $G(x,\xi)$?
{Self-adjoint differential operators}
We have L and real, positive weight function $w(x)$. Define $L^\dagger$ such that for any differentiable $u(x)$ and $v(x)$
(*) $w(x)\left(u^* L v - v L^\dagger u^*\right) = \frac{d}{dx} Q[u,v]$, Q is


Self-adjoint differential operators
We have L and real, positive weight function w(x). Define L^+
(*) w(x) (u* L v - v L^+ u*) = d/dx Q[u,v] , Q is
djoint


tors
Define L† such that for any differentiable u(x) and v(x)
[u,v]
Q is bilinear in u and v and its derivatives

$L_x G(x,\xi) = \delta(x-\xi)$

{Self-adjoint differential operators}

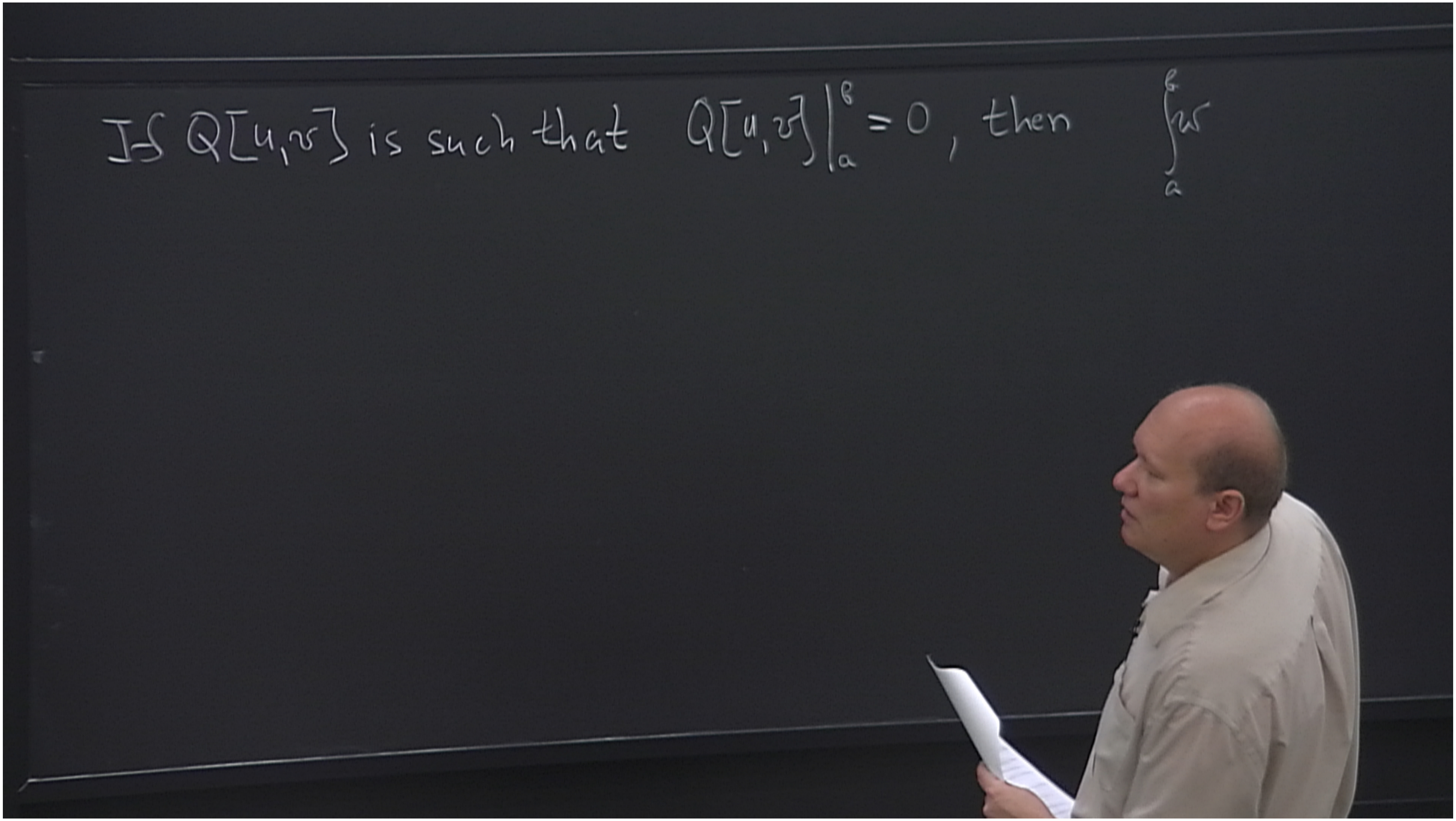
e L and real, positive weight function $w(x)$. Define L^{+} such that for any different

$$(*) \quad w(x)\left(u^* L v - v L^{+} u^*\right) = \frac{d}{dx} Q[u,v]$$

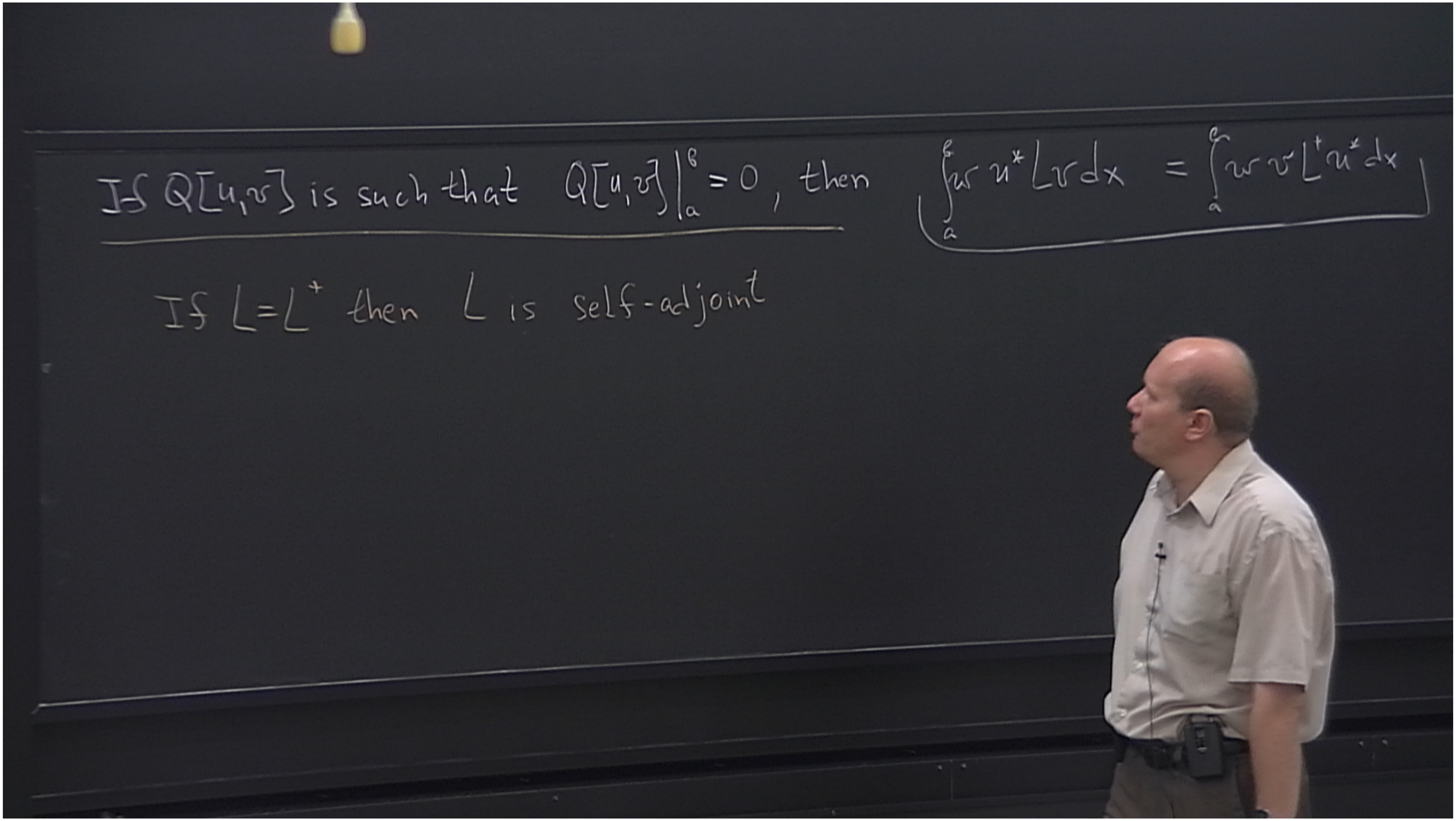
, Q is bilinear in u and v and i

adjoint to L

Lagrange's identity


If $Q[u,v]$ is such that $Q[u,v]\Big|_a^b = 0$, then $\int_a^b v$

If $Q[u,v]$ is such that $Q[u,v]\Big|_a^b = 0$, then $\boxed{\int_a^b w\, u^* L v\, dx = \int_a^b w\, v\, L^\dagger u^*\, dx}$


If $Q[u,v]$ is such that $Q[u,v]\Big|_a^b = 0$, then $\int_a^b w\, u^* L v\, dx = \int_a^b w\, v\, L^\dagger u^*\, dx$
If $L = L^\dagger$ then L is self-adjoint

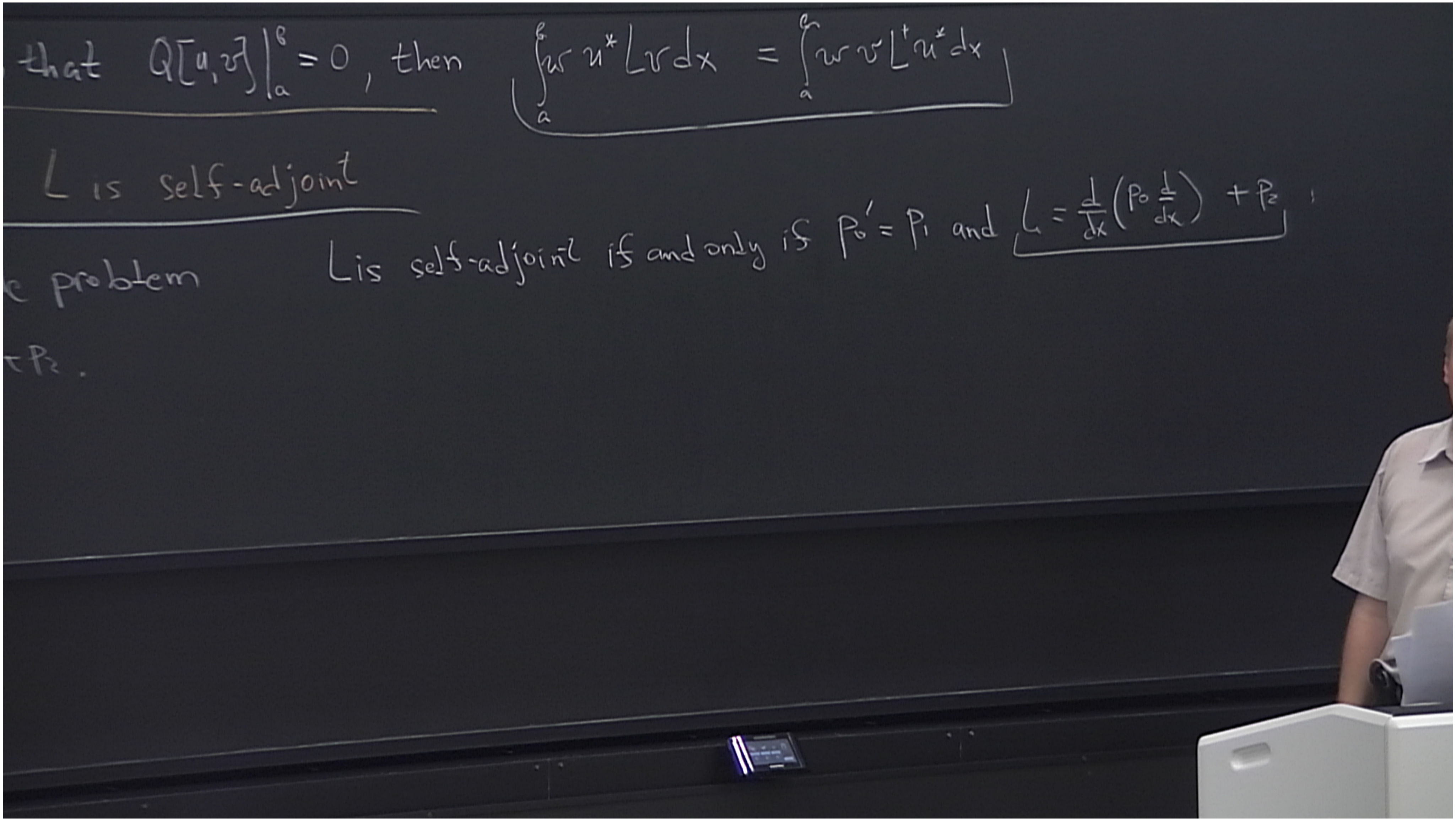
If $Q[u,v]$ is such that $Q[u,v]\Big|_a^b = 0$, then $\int_a^b w\, u^* L v\, dx = \int_a^b w\, v\, L^\dagger u^*\, dx$

If $L = L^+$ then L is self-adjoint

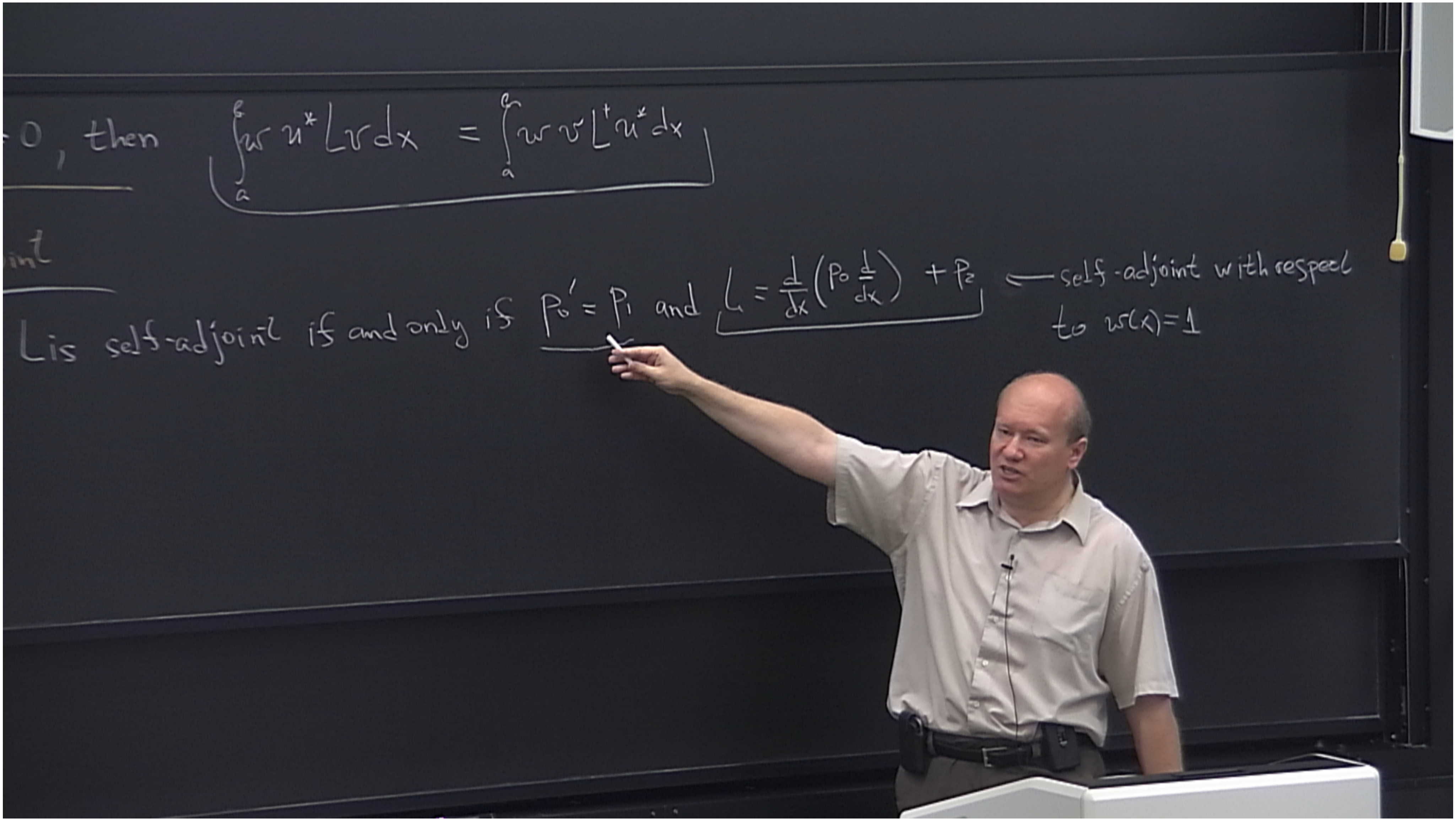
Sturm-Liouville problem

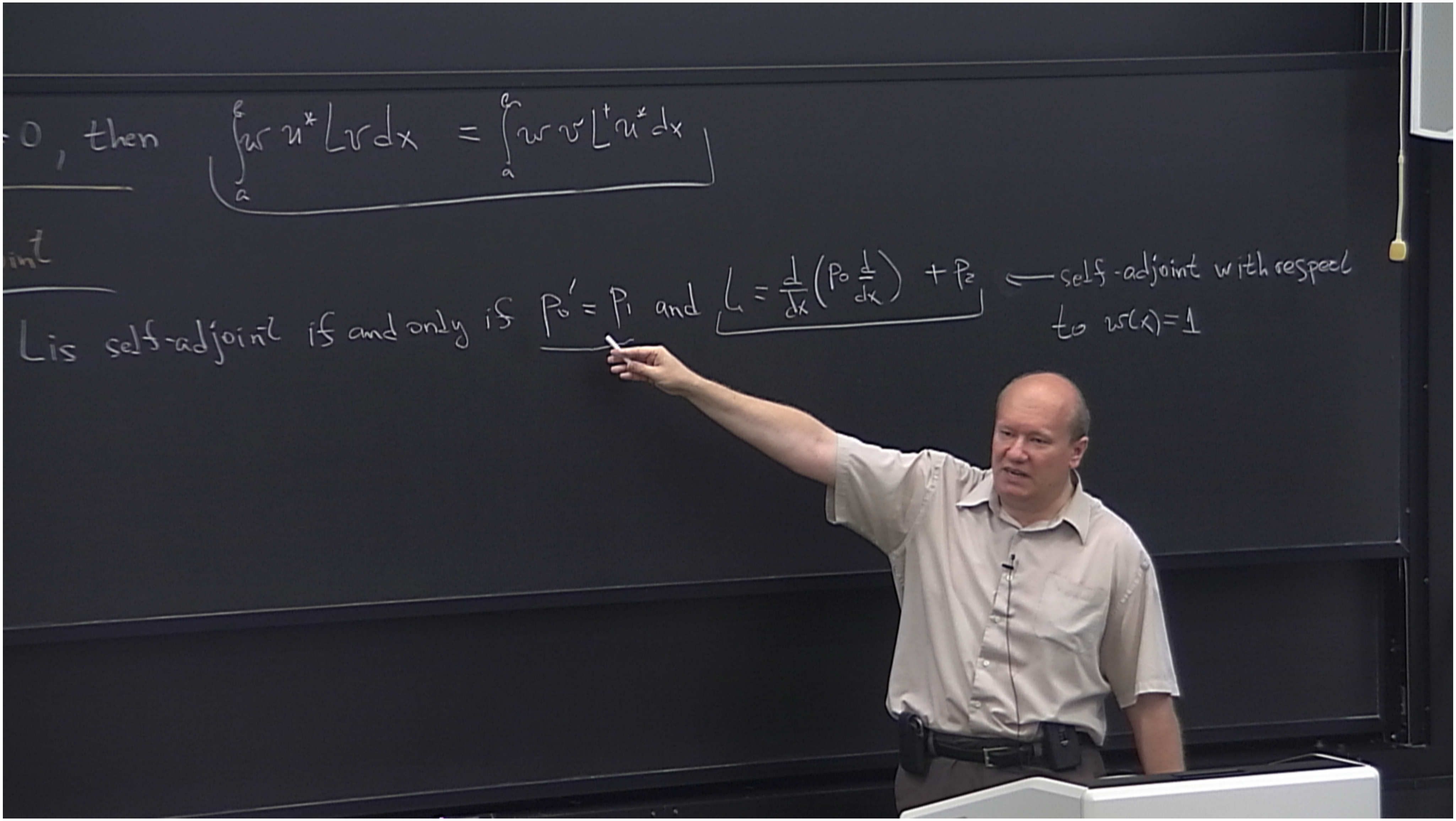
$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$.

L is self-adjoint if and only if $p_0' = p_1$ and $L = $



that $Q[u,v]\Big|_a^b = 0$, then $\int_a^b w u^* L v \, dx = \int_a^b w v L^\dagger u^* dx$
L is self-adjoint
L is self-adjoint if and only if $p_0' = p_1$ and $L = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$
e problem
$+ p_2$.


0, then ∫ w u* L v dx = ∫ w v L† u* dx
L is self-adjoint if and only if p0' = p1 and L = d/dx (p0 d/dx) + p2
self-adjoint with respect to w(x) = 1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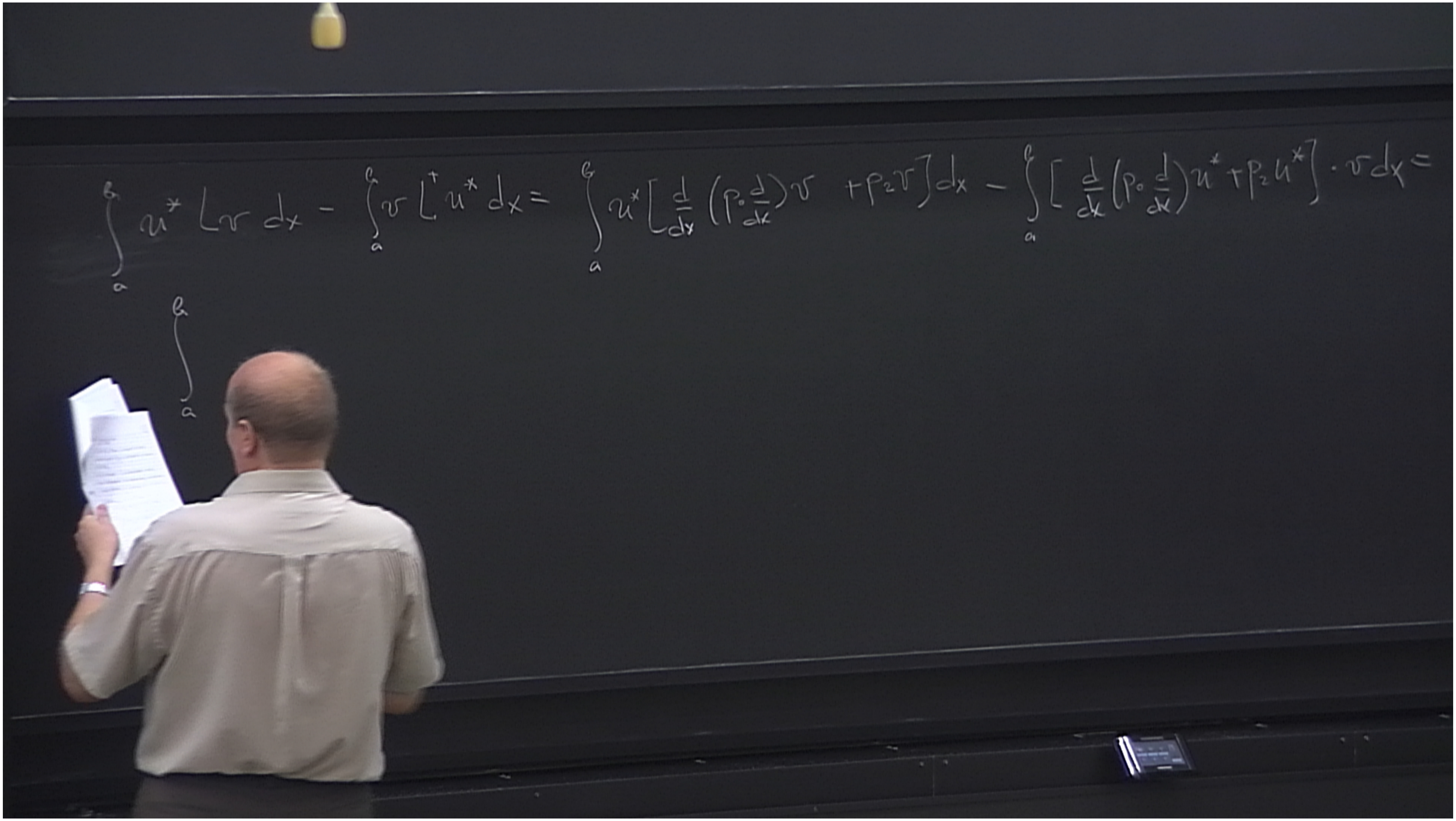
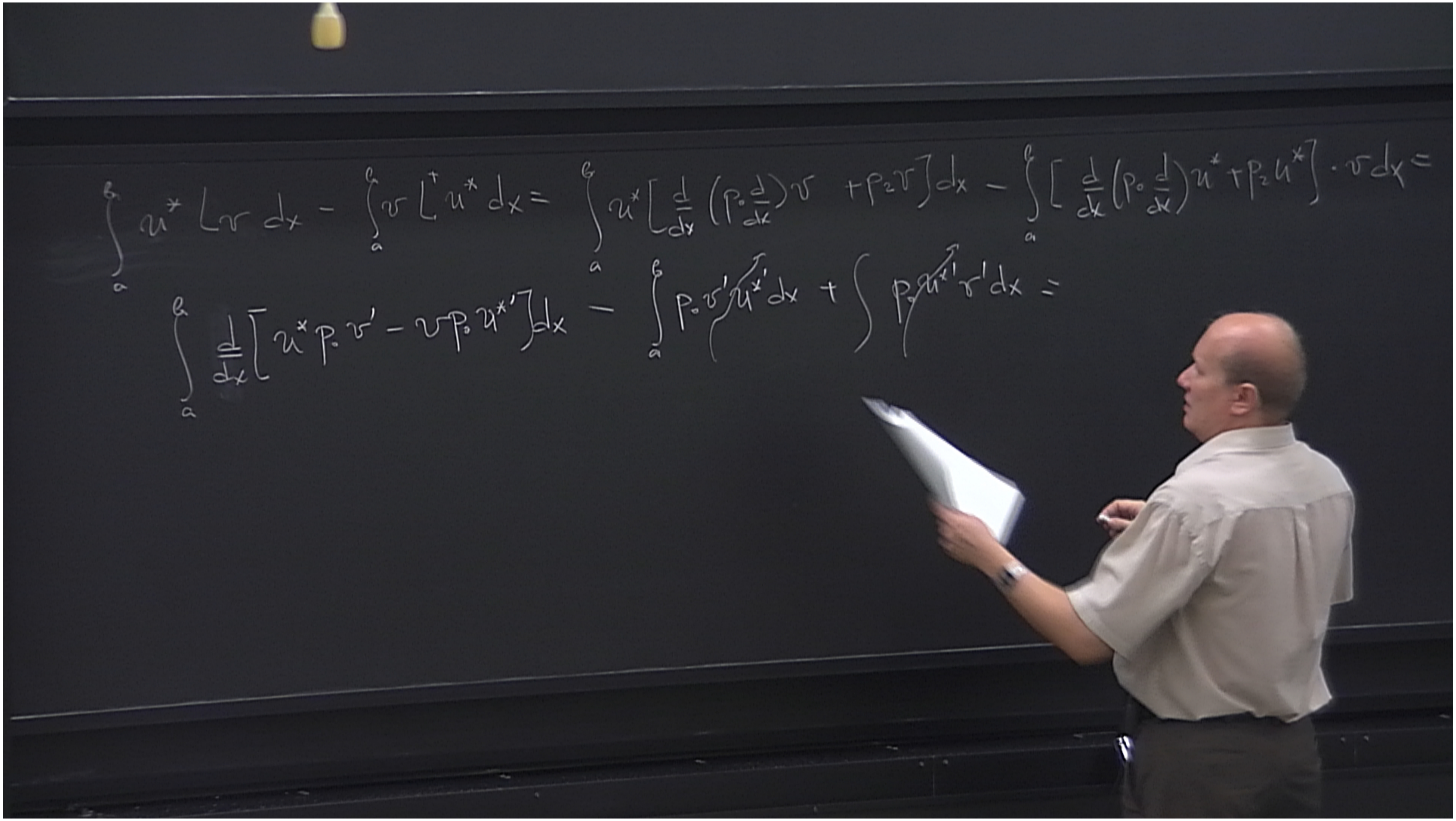
L is self-adjoint if and only if $p_0' = p_1$ and $L = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$
self-adjoint with respect to $w(x) = 1$


$\int_a^b u^* L v \, dx - \int_a^b v L^\dagger u^* \, dx = \int_a^b u^* \left[\frac{d}{dx} \left(p_0 \frac{d}{dx} \right) v + p_2 v \right] dx - \int_a^b \left[\frac{d}{dx} \left(p_0 \frac{d}{dx} \right) u^* + p_2 u^* \right] \cdot v \, dx =$
$\int_a^b$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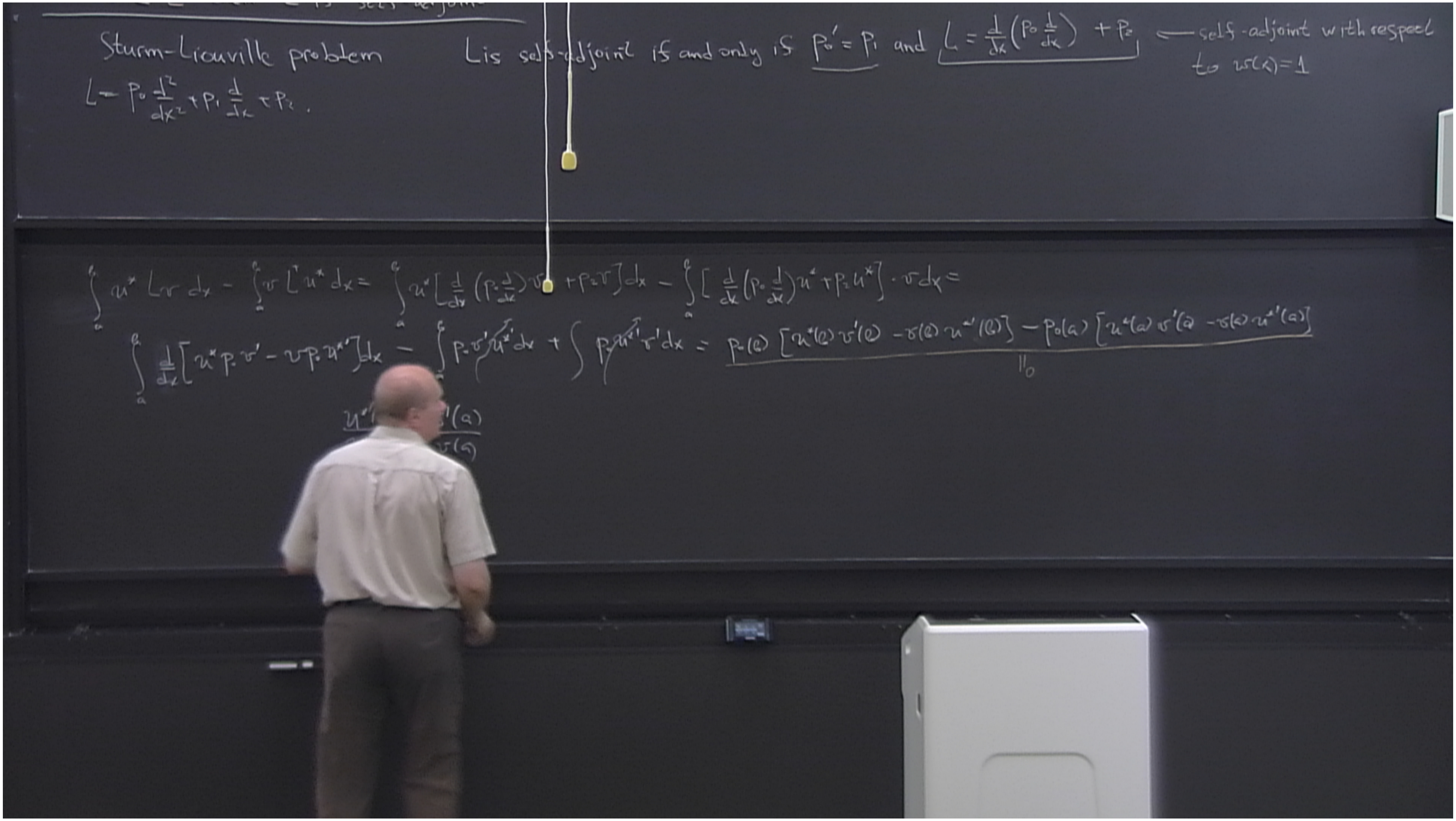
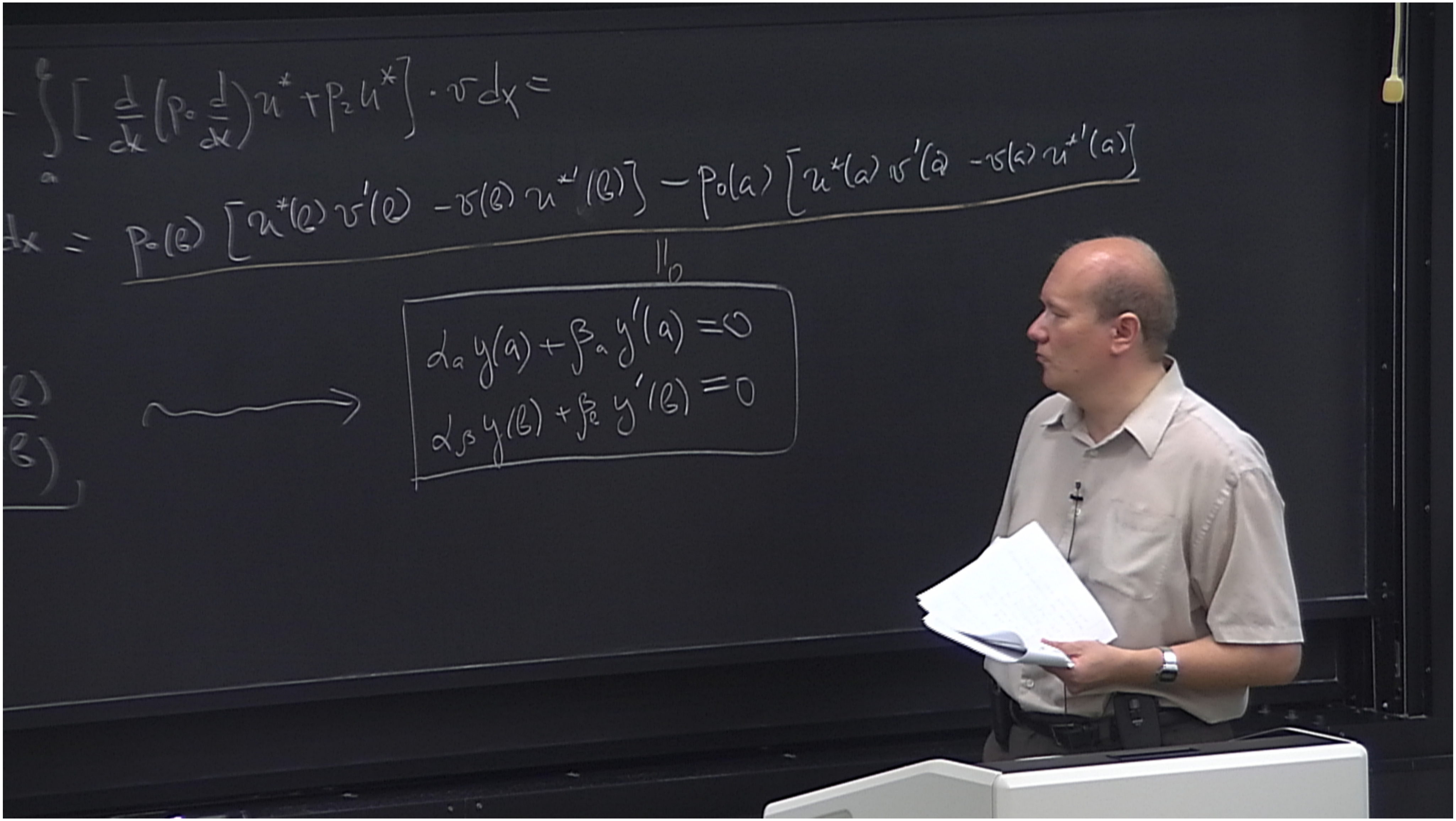
$\int_a^b \frac{d}{dx}\left[u^* p_0 v' - v p_0 u^{*\prime}\right]dx - \int_a^b p_0 v' u^{*\prime} dx + \int p_0 u^{*\prime} v' dx =$

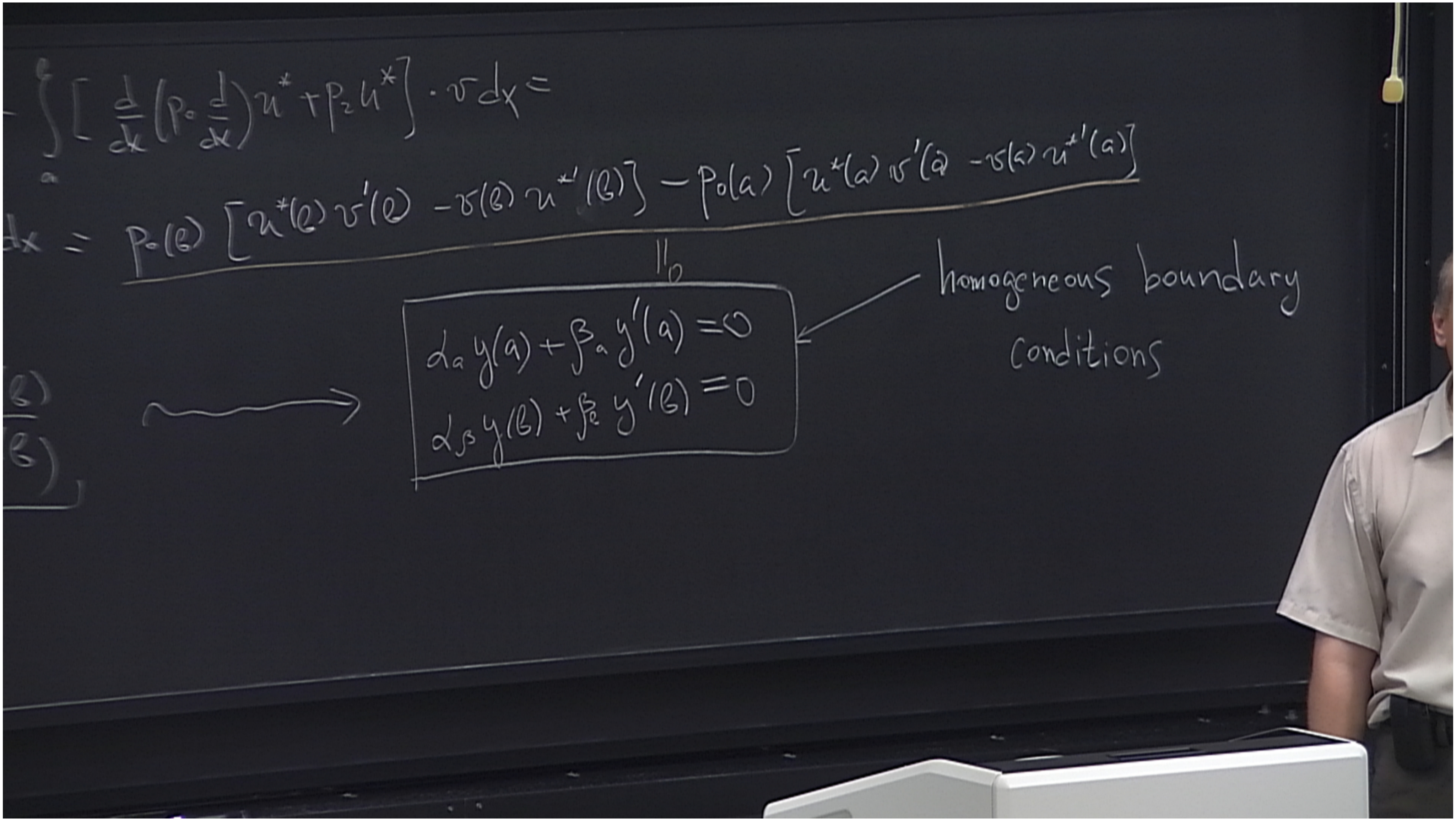

Sturm-Liouville problem
$L = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$.
L is self-adjoint if and only if $p_0' = p_1$ and $L = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$
← self-adjoint with respect to $w(x) = 1$
$\int_a^b u^* L v \, dx - \int_a^b v L u^* \, dx = \int_a^b u^* \left[\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) v + p_2 v\right] dx - \int_a^b \left[\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) u^* + p_2 u^*\right] \cdot v \, dx =$
$\int_a^b \frac{d}{dx}\left[u^* p_0 v' - v p_0 u^{*\prime}\right] dx - \int_a^b p_0 v' u^{*\prime} dx + \int p_0 u^{*\prime} v' dx = p_0(b)\left[u^*(b) v'(b) - v(b) u^{*\prime}(b)\right] - p_0(a)\left[u^*(a) v'(a) - v(a) u^{*\prime}(a)\right]$
$\| 0$

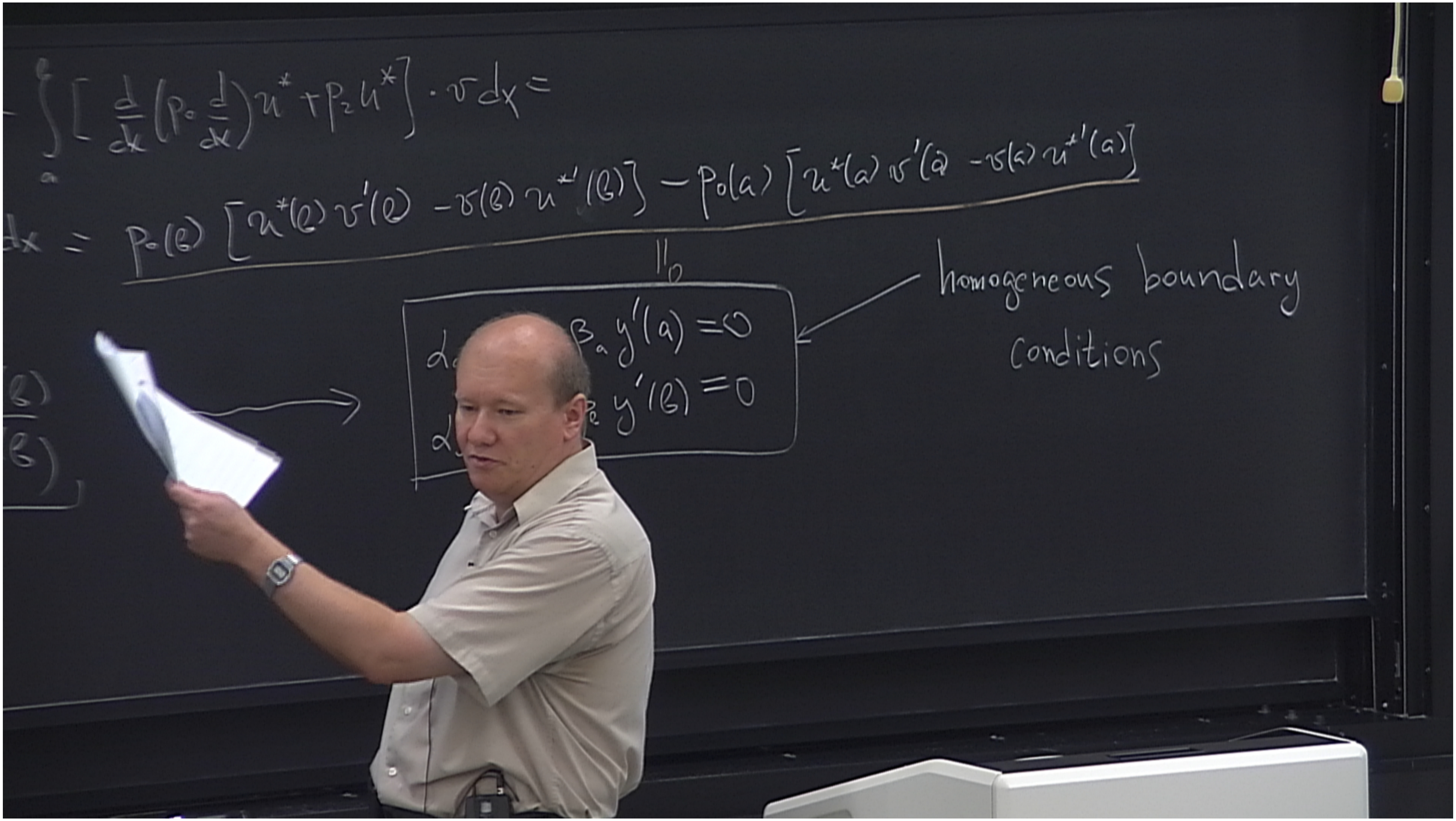
$$\int_a^b u^* L v \, dx - \int_a^b v L^\dagger u^* \, dx = \int_a^b u^* \left[\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) v + p_2 v \right] dx - \int_a^b \left[\frac{d}{dx}\left(p_0 \right.\right.$$

$$\int_a^b \frac{d}{dx}\left[u^* p_0 v' - v p_0 u^{*\prime} \right] dx - \int_a^b p_0 \cancel{v' u^{*\prime}} \, dx + \int p_0 \cancel{u^{*\prime} v'} \, dx = p_0(b)$$

$$\frac{u^{*\prime}(a)}{u^*(a)} = \frac{v'(a)}{v(a)} \quad , \quad \frac{u^{*\prime}(b)}{u^*(b)} = \frac{v'(b)}{v(b)}$$


$\int_a^b [\frac{d}{dx}(p_0 \frac{d}{dx}) u^* + p_2 u^*] \cdot v \, dx =$
$dx = p_0(b)[u^*(b) v'(b) - v(b) u^{*\prime}(b)] - p_0(a)[u^*(a) v'(a) - v(a) u^{*\prime}(a)]$
$= 0$
$\alpha_a y(a) + \beta_a y'(a) = 0$
$\alpha_b y(b) + \beta_b y'(b) = 0$


$\int [\frac{d}{dx}(p_0 \frac{d}{dx}) u^* + p_2 u^*] \cdot v \, dx =$
$p_0(b) [u^*(b) v'(b) - v(b) u^{*\prime}(b)] - p_0(a) [u^*(a) v'(a) - v(a) u^{*\prime}(a)]$
$\alpha_a y(a) + \beta_a y'(a) = 0$
$\alpha_b y(b) + \beta_b y'(b) = 0$
homogeneous boundary conditions


homogeneous boundary
conditions

How to construct $G(x,\xi)$ for a self-adjoint L?

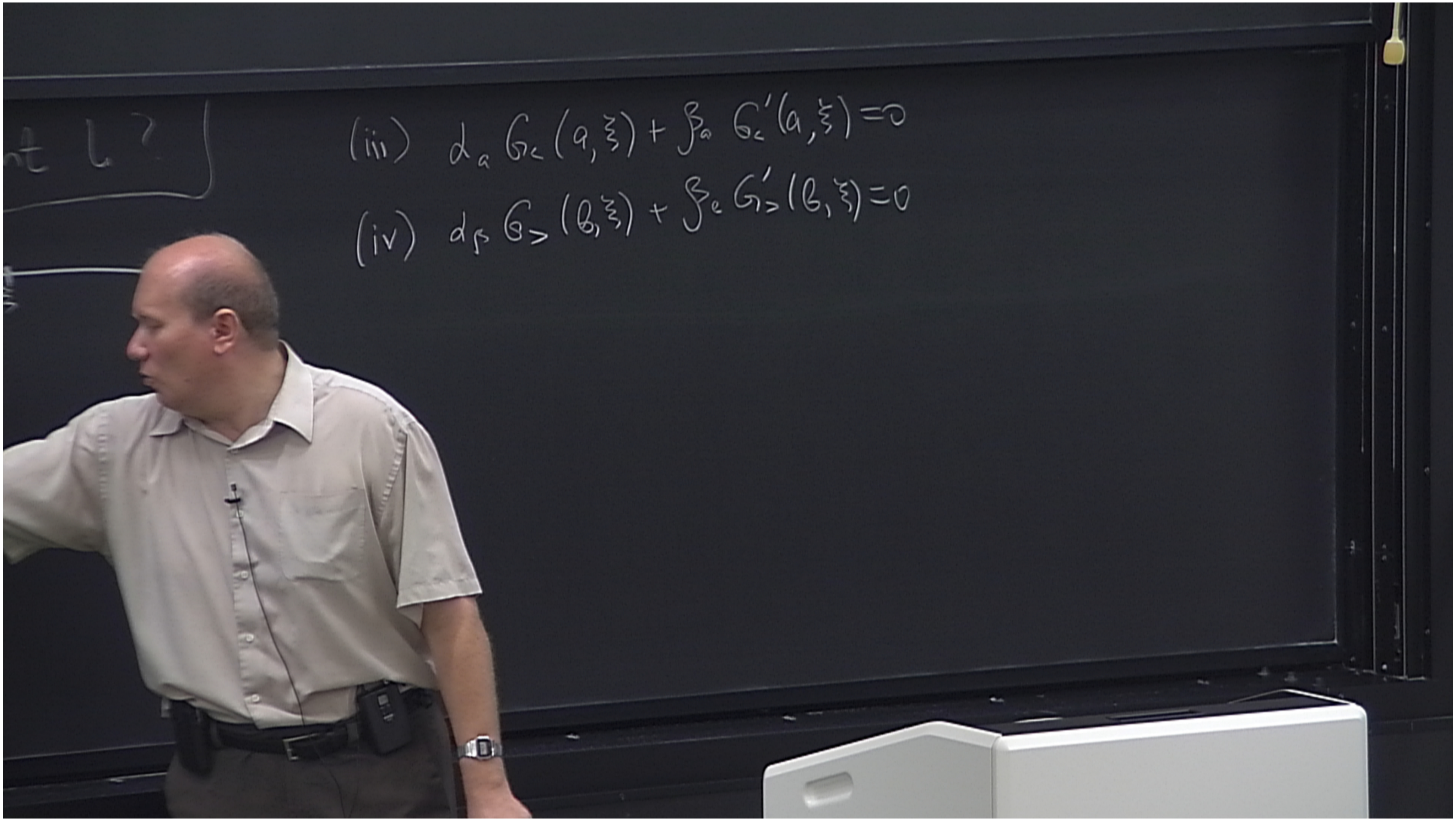
$$L_x\, G(x,\xi) = 0 \quad \text{away from } x = \xi$$

How to construct $G(x,\xi)$ for a self-adjoint L?

$L_x G(x,\xi) = 0$ away from $x = \xi$

(i) For $x < \xi$, $G_<(x,\xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$

(ii) For $x > \xi$, $G_>(x,\xi) = C(\xi)\, Y_1(x) + D(\xi)\, Y_2(x)$


(iii) $\alpha_a G_<(a,\xi) + \beta_a G_<'(a,\xi) = 0$
(iv) $\alpha_b G_>(b,\xi) + \beta_b G_>'(b,\xi) = 0$

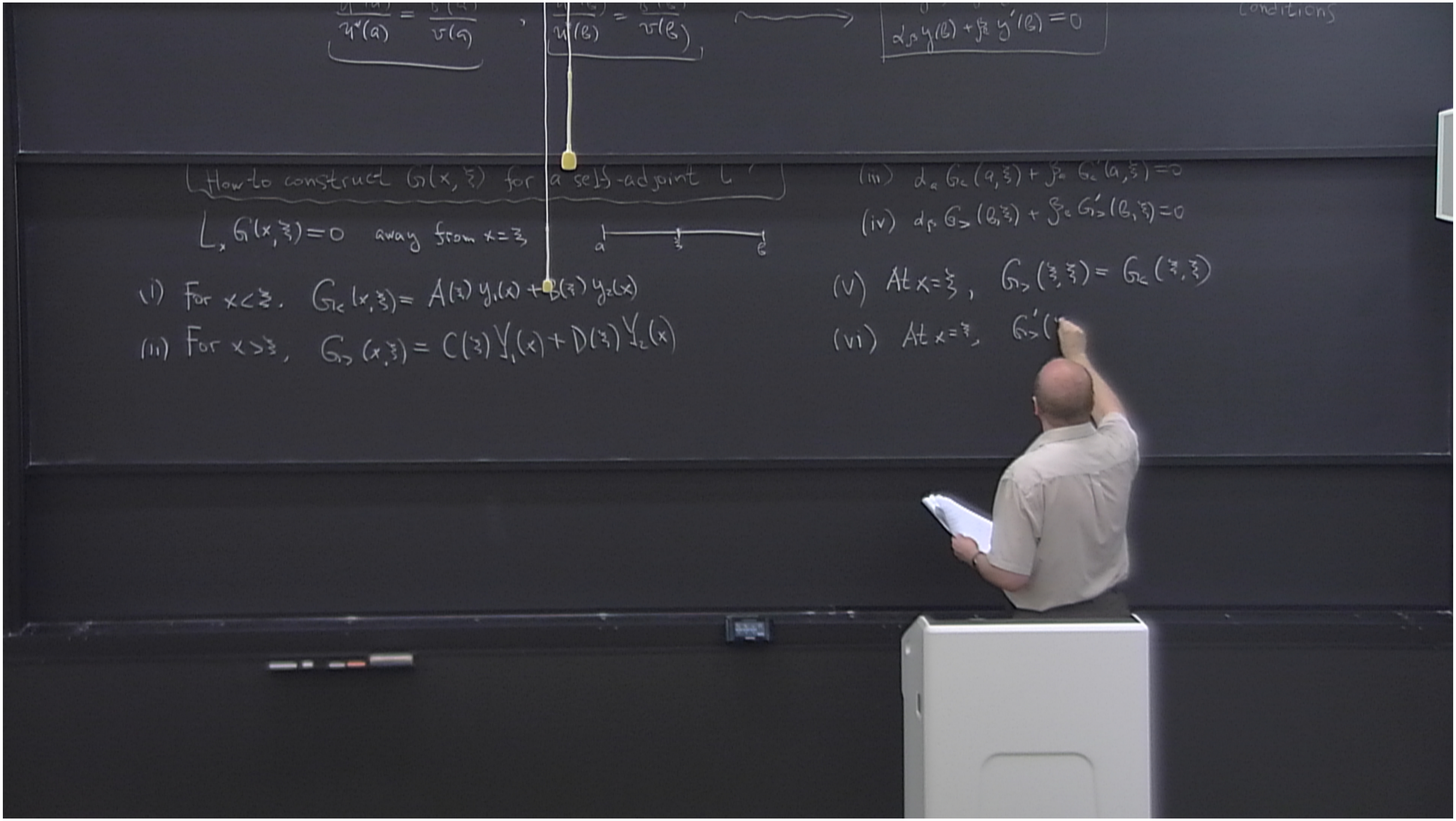
nt L ?

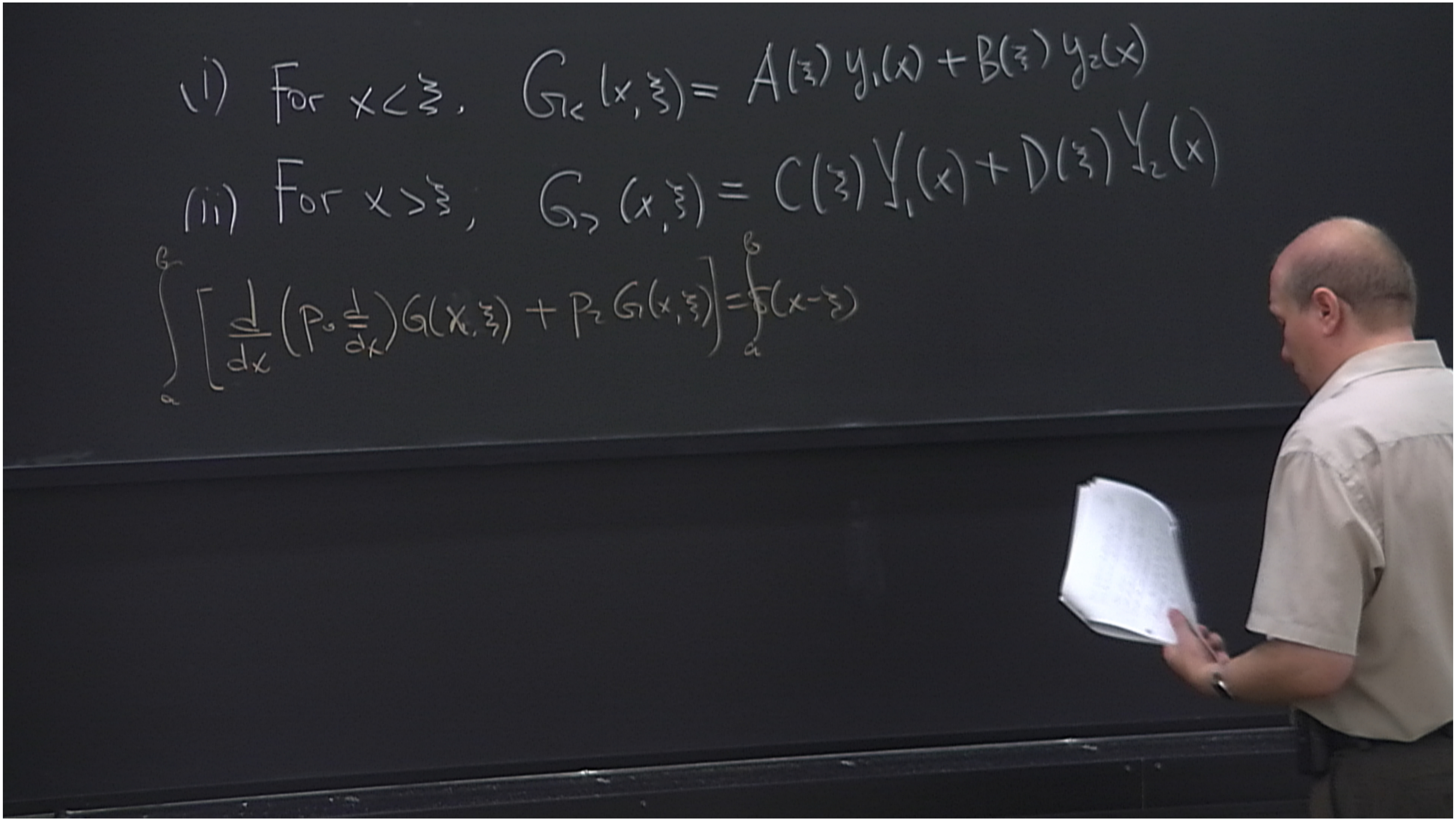
(iii) $\alpha_a G_<(a,\xi) + \beta_a G_<'(a,\xi) = 0$

(iv) $\alpha_b G_>(b,\xi) + \beta_b G_>'(b,\xi) = 0$

b

(v) At $x=\xi$, $G_>(\xi,\xi) = G_<(\xi,\xi)$


conditions
α_b y(b) + β_b y'(b) = 0
How to construct G(x, ξ) for a self-adjoint L
L_x G(x, ξ) = 0 away from x = ξ
a
ξ
b
(i) For x < ξ. G_<(x, ξ) = A(ξ) y_1(x) + B(ξ) y_2(x)
(ii) For x > ξ, G_>(x, ξ) = C(ξ) Y_1(x) + D(ξ) Y_2(x)
(iii) α_a G_<(a, ξ) + β_a G_<'(a, ξ) = 0
(iv) α_b G_>(b, ξ) + β_b G_>'(b, ξ) = 0
(v) At x = ξ, G_>(ξ, ξ) = G_<(ξ, ξ)
(vi) At x = ξ, G_>'(


(i) For $x<\xi$, $G_<(x,\xi) = A(\xi)\,y_1(x) + B(\xi)\,y_2(x)$
(ii) For $x>\xi$; $G_>(x,\xi) = C(\xi)\,Y_1(x) + D(\xi)\,Y_2(x)$
$\int_a^b \left[\frac{d}{dx}\left(p_0\frac{d}{dx}\right)G(x,\xi) + p_2\,G(x,\xi)\right] = \int_a^b \delta(x-\xi)$

(v) At $x=\xi$, $G_>(\xi,\xi) = G_<(\xi,\xi)$

(vi) At $x=\xi$, $G_>'(\xi,\xi) - G_<'(\xi,\xi) = \dfrac{1}{p_0(\xi)}$

(i) For $x < \xi$, $G_<(x, \xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$

(ii) For $x > \xi$, $G_>(x, \xi) = C(\xi)\, Y_1(x) + D(\xi)\, Y_2(x)$

$$\int_{\xi-0}^{\xi+0} \left[\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) G(x, \xi) + p_2\, G(x, \xi) \right] = \int_{\xi-0}^{\xi+0} \delta(x-\xi)$$

How to construct $G(x, \xi)$ for a self-adjoint L?

$L_x G(x, \xi) = 0$ away from $x = \xi$

(i) For $x < \xi$, $G_<(x, \xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$

(ii) For $x > \xi$, $G_>(x, \xi) = C(\xi)\, Y_1(x) + D(\xi)\, Y_2(x)$

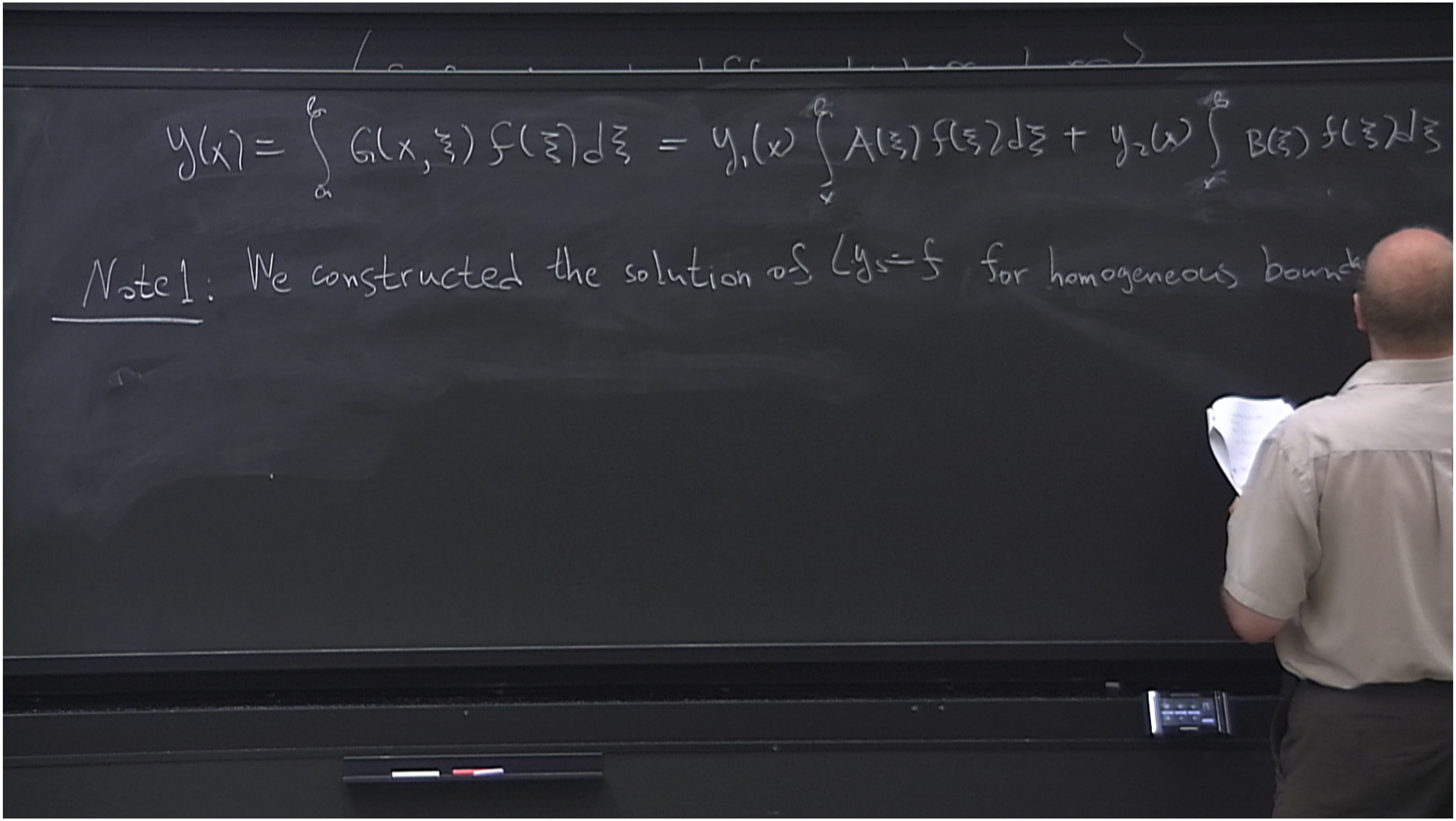
$$\int_{\xi-0}^{\xi+0} \left[\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) G(x, \xi) + p_2\, G(x, \xi) \right] = \int_{\xi-0}^{\xi+0} \delta(x-\xi)$$

$$L_x G(x,\xi) = \delta(x-\xi)$$

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi = y_1(x)\int_x^b A(\xi) f(\xi)\, d\xi + y_2(x)\int_x^b B(\xi) f(\xi)\, d\xi + Y_1(x)\int_a^x C(\xi) f(\xi)\, d\xi + Y_2(x)\int_a^x D(\xi) f(\xi)\, d\xi$$

$$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi = y_1(x) \int_x^a A(\xi)\, f(\xi)\, d\xi + y_2(x) \int_x^b B(\xi)$$

$$y_2(x)\int_x^b B(\xi) f(\xi)\,d\xi + Y_1(x)\int_a^x C(\xi) f(\xi)\,d\xi + Y_2(x)\int_a^x D(\xi) f(\xi)\,d\xi$$

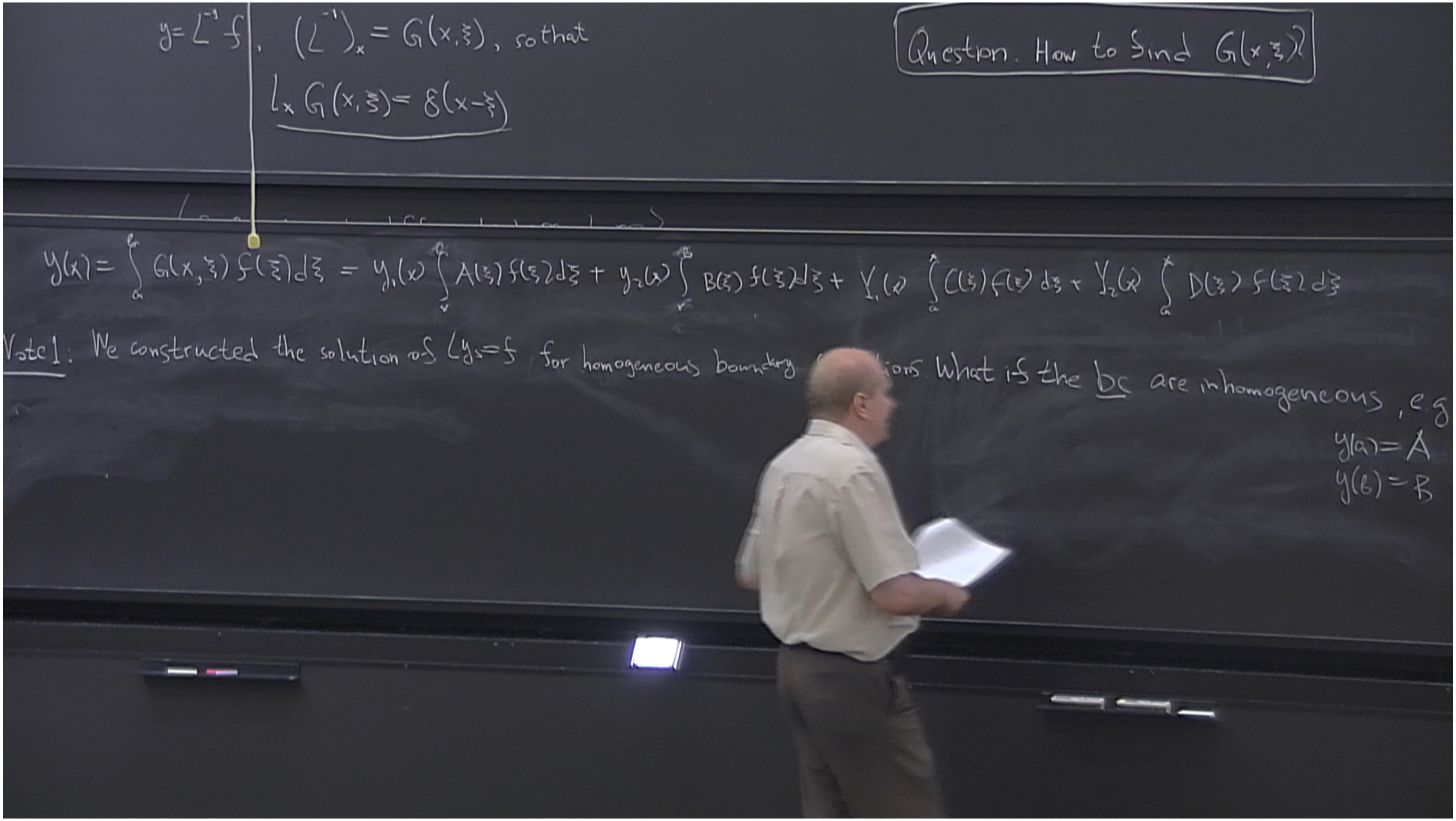


$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi = y_1(x) \int_x^a A(\xi) f(\xi) d\xi + y_2(x) \int_x^b B(\xi) f(\xi) d\xi$
Note 1: We constructed the solution of $Ly_s = f$ for homogeneous bound

$$y_2(x)\int_x^b B(\xi) f(\xi)\, d\xi + Y_1(x)\int_a^x C(\xi) f(\xi)\, d\xi + Y_2(x)\int_a^x D(\xi) f(\xi)\, d\xi$$

ogeneous boundary conditions. What if the bc are inhomogeneous, e.g.

$$y(a) = A$$
$$y(b) = B$$



y = L^-1 f, (L^-1)_x = G(x, ξ), so that
L_x G(x, ξ) = δ(x − ξ)
Question. How to find G(x, ξ)?
y(x) = ∫ G(x, ξ) f(ξ) dξ = y_1(x) ∫ A(ξ) f(ξ) dξ + y_2(x) ∫ B(ξ) f(ξ) dξ + Y_1(x) ∫ C(ξ) f(ξ) dξ + Y_2(x) ∫ D(ξ) f(ξ) dξ
Note 1: We constructed the solution of Ly_s = f for homogeneous boundary
ions. What if the bc are inhomogeneous, e.g.
y(a) = A
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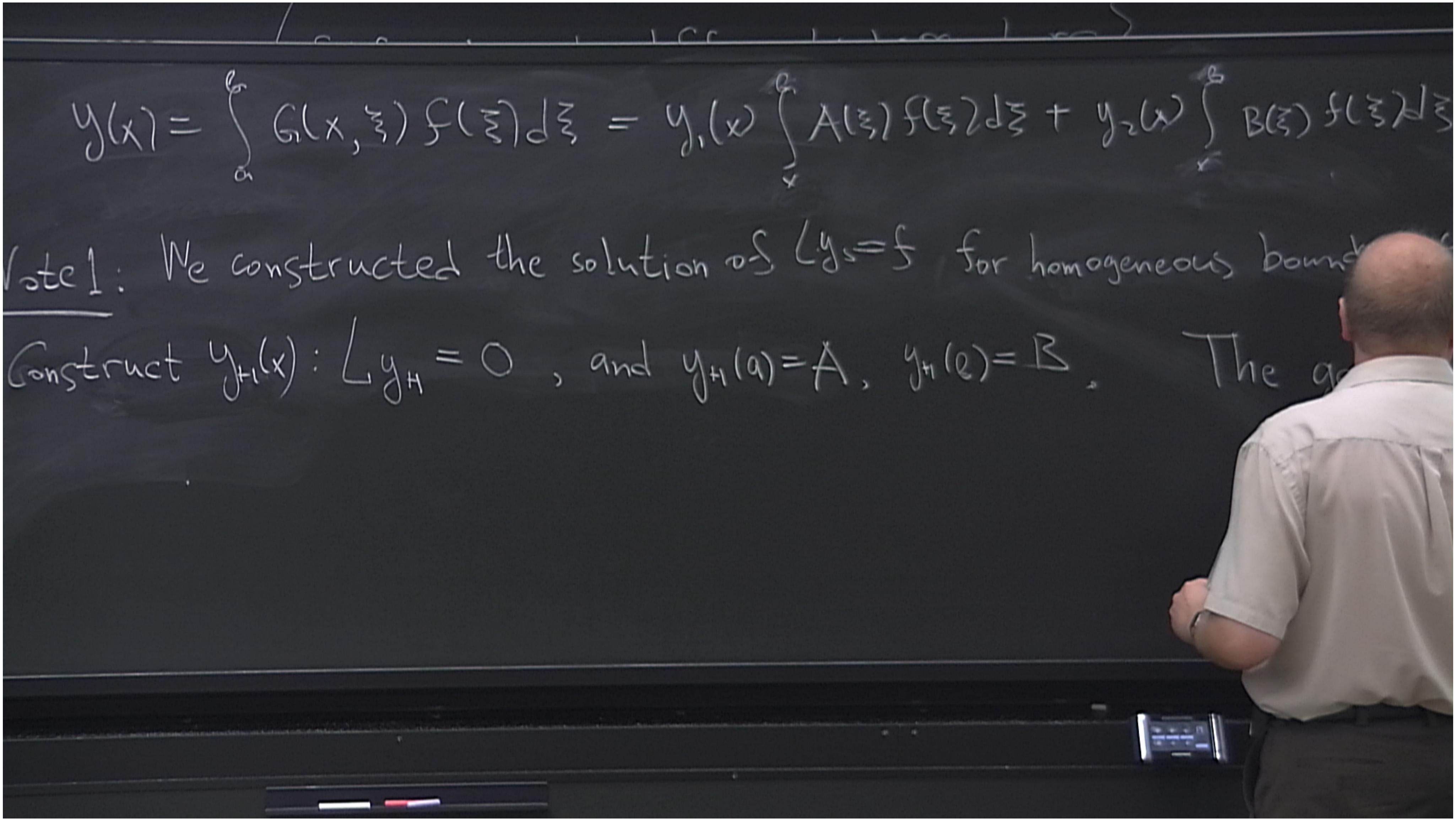

$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi = y_1(x) \int_x^b A(\xi) f(\xi) d\xi + y_2(x) \int_x^b B(\xi) f(\xi) d\xi$
Note 1: We constructed the solution of $Ly_s = f$ for homogeneous bound
Construct $y_H(x)$: $Ly_H = 0$, and $y_H(a) = A$, $y_H(b) = B$. The g

$y_2(x) \int_x^b B(\xi) f(\xi) d\xi + Y_1(x) \int_a^x C(\xi) f(\xi) d\xi + Y_2(x) \int_a^x D(\xi) f(\xi) d\xi$

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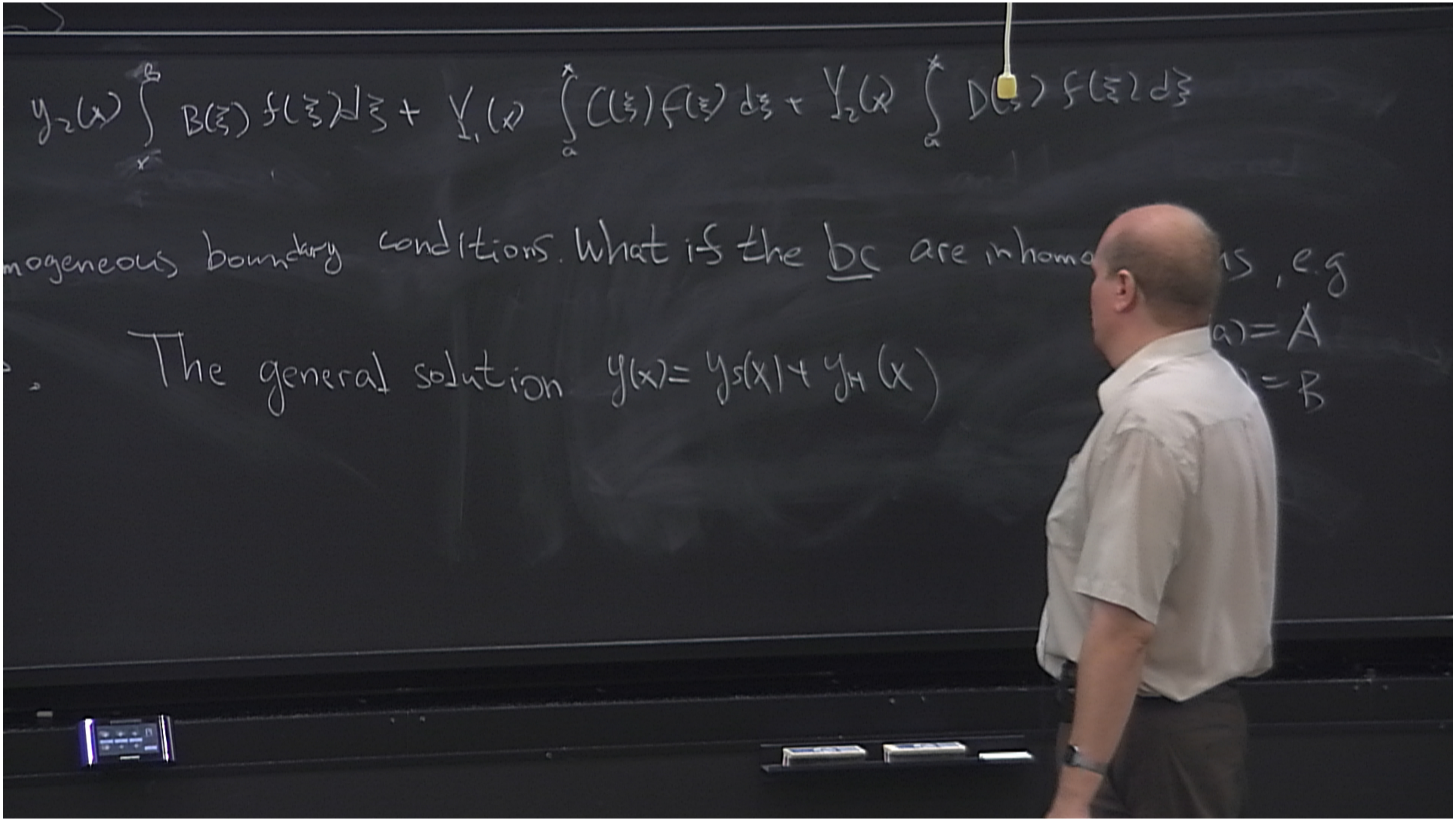
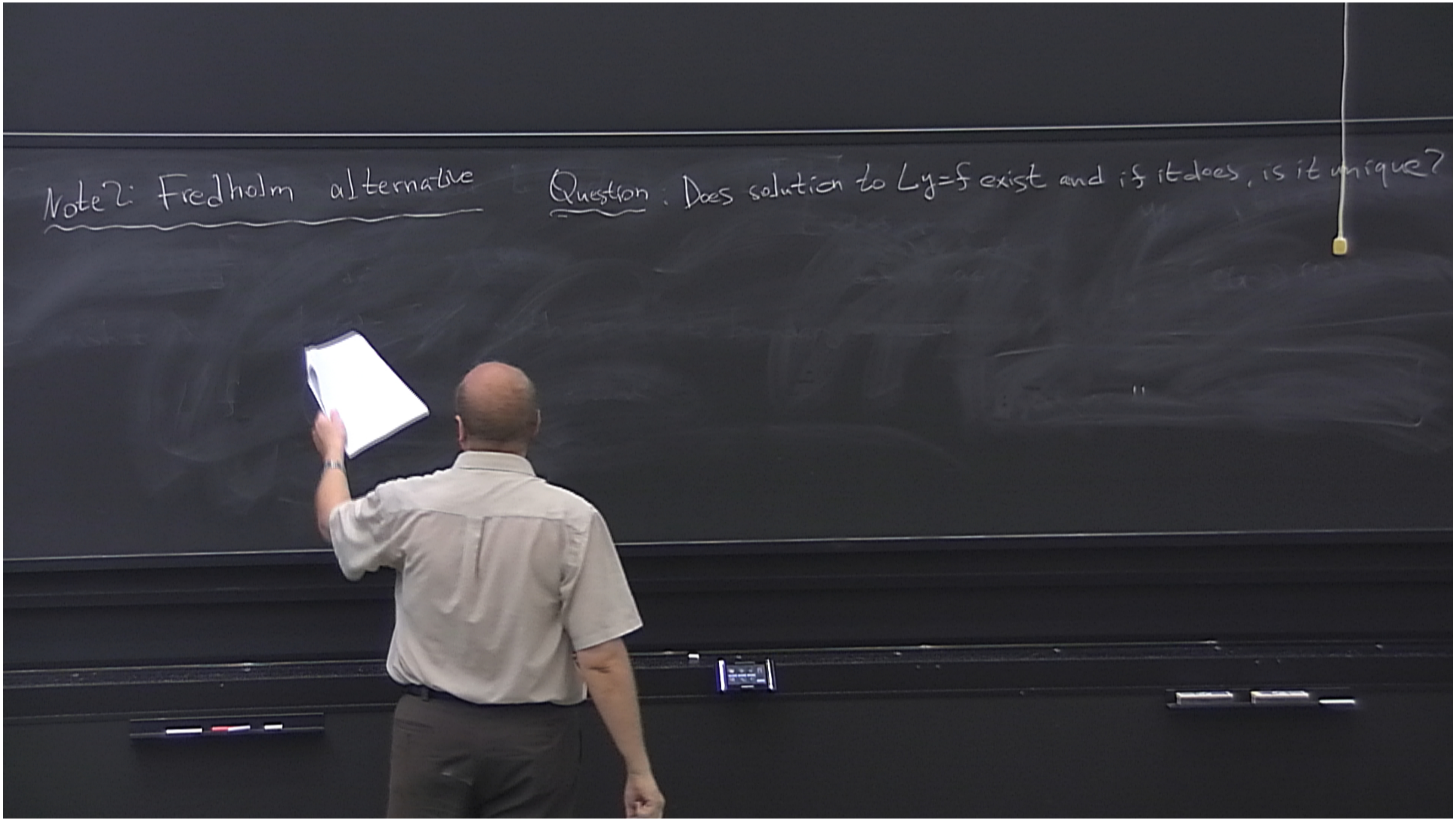
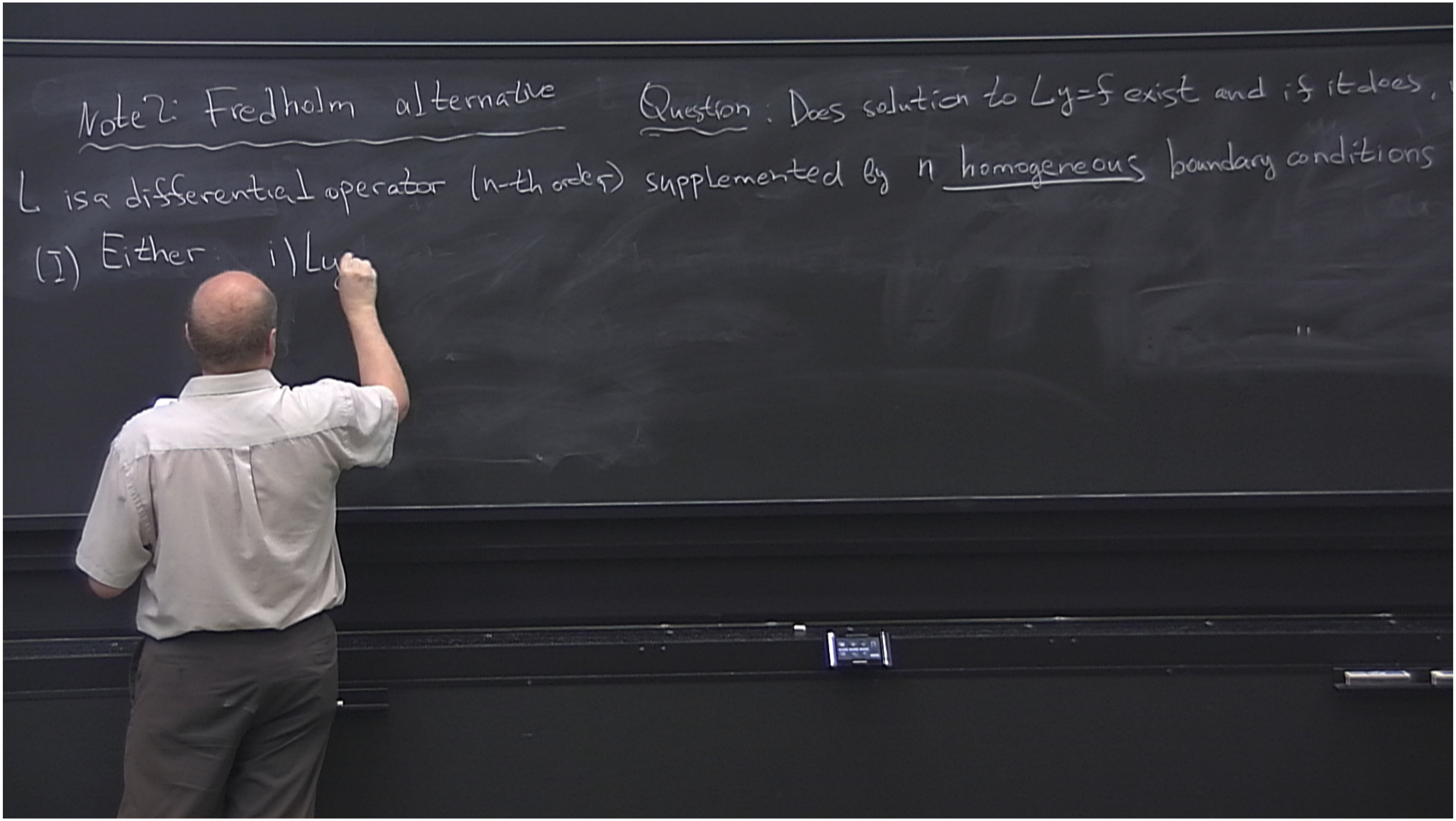
$y(a) = A$

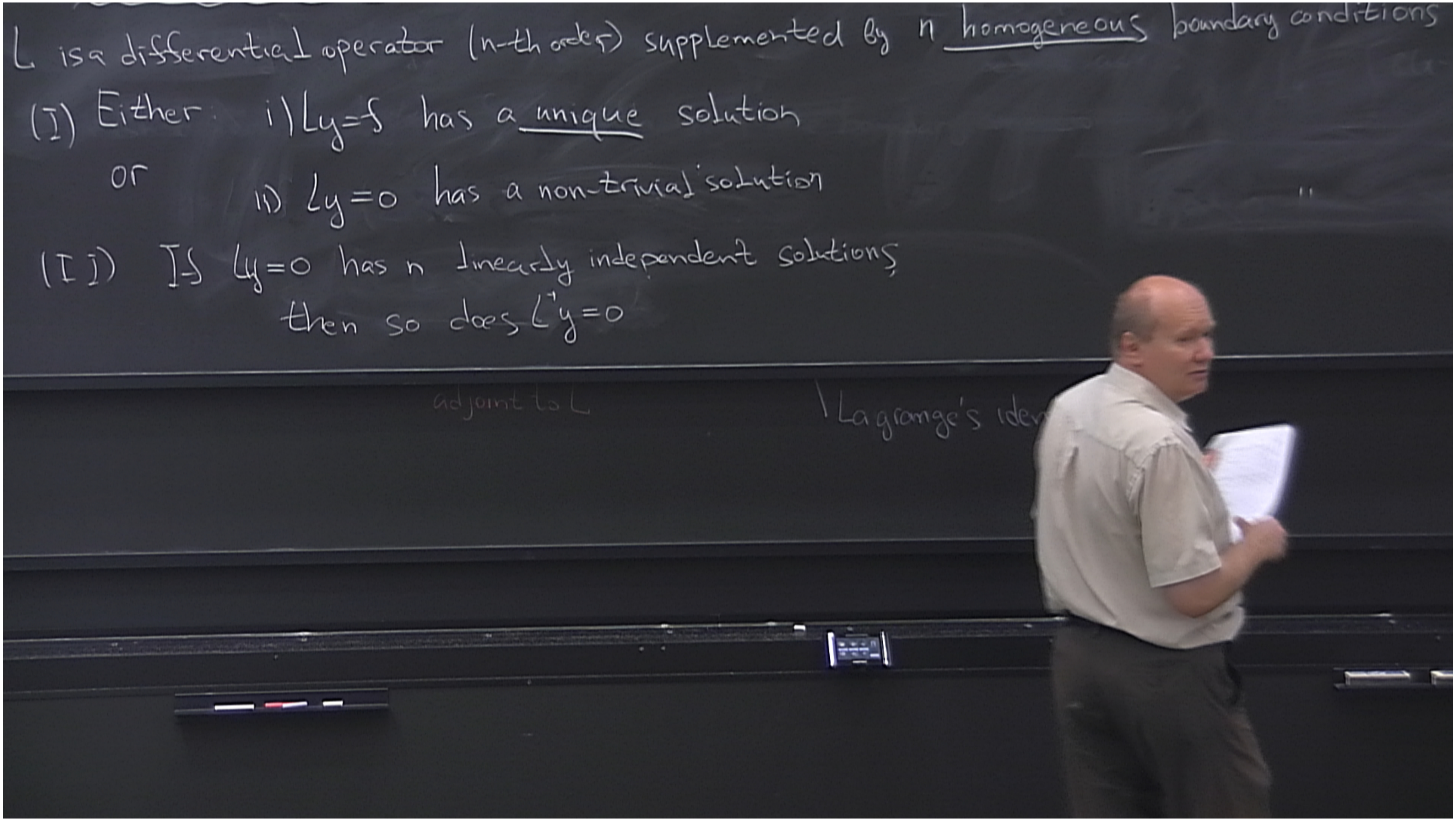
$y(b) = B$

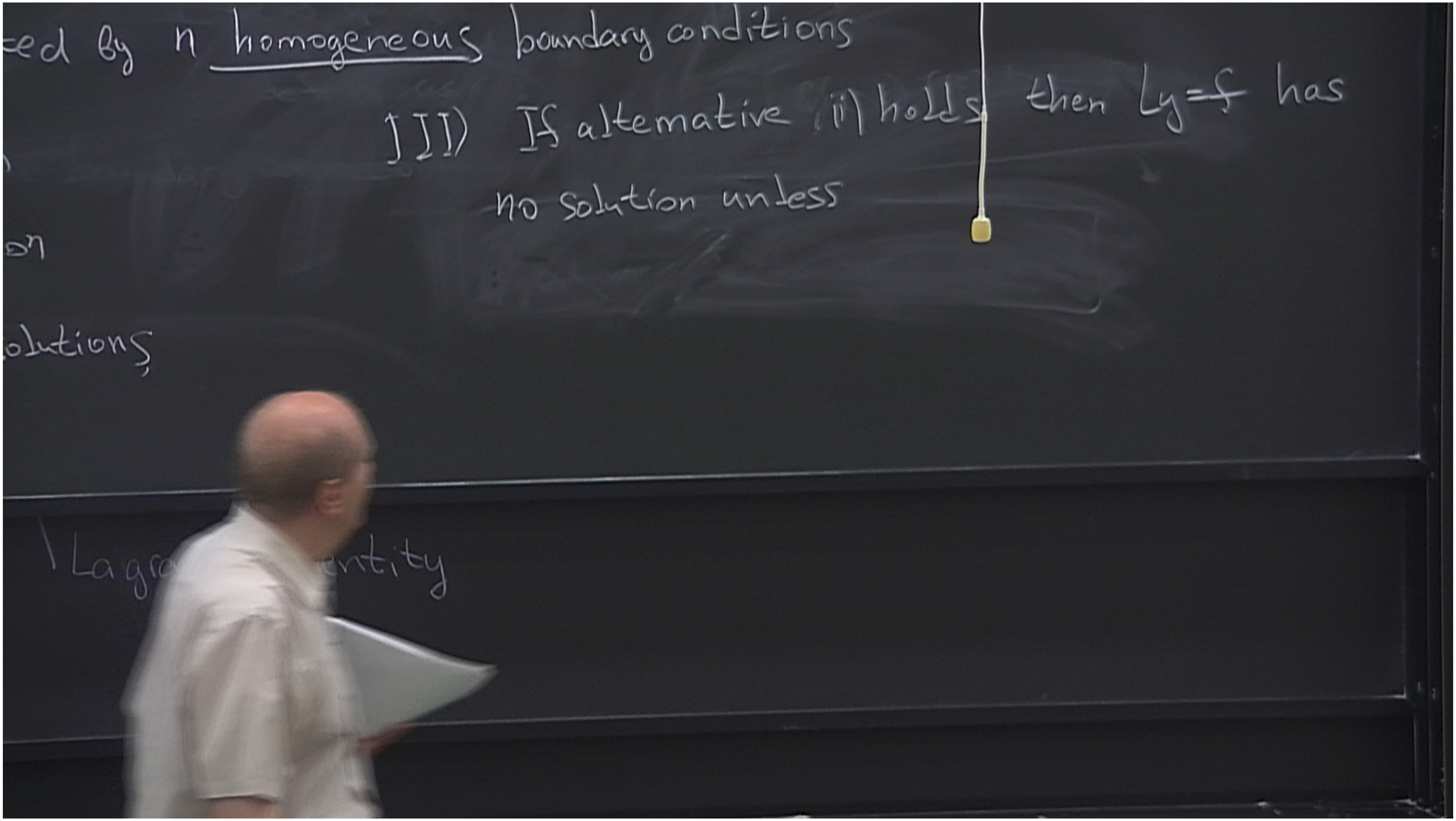
The general solution $y(x) = y_s(x)$


y2(x) ∫x^b B(ξ) f(ξ) dξ + Y1(x) ∫a^x C(ξ) f(ξ) dξ + Y2(x) ∫a^x D(ξ) f(ξ) dξ
mogeneous boundary conditions. What if the bc are inhomo
ns, e.g
a)= A
= B
The general solution y(x)= ys(x) + yH(x)


Note 2: Fredholm alternative
Question: Does solution to Ly=f exist and if it does, is it unique?


Note 2: Fredholm alternative
Question: Does solution to Ly=f exist and if it does,
L is a differential operator (n-th order) supplemented by n homogeneous boundary conditions
(I) Either i) Ly


L is a differential operator (n-th order) supplemented by n homogeneous boundary conditions
(I) Either i) Ly=f has a unique solution
or ii) Ly=0 has a non-trivial solution
(II) If Ly=0 has n linearly independent solutions
then so does L†y=0
adjoint to L
Lagrange's


ed by n homogeneous boundary conditions
III) If alternative (ii) holds then Ly=f has
no solution unless
on
olutions
ntity

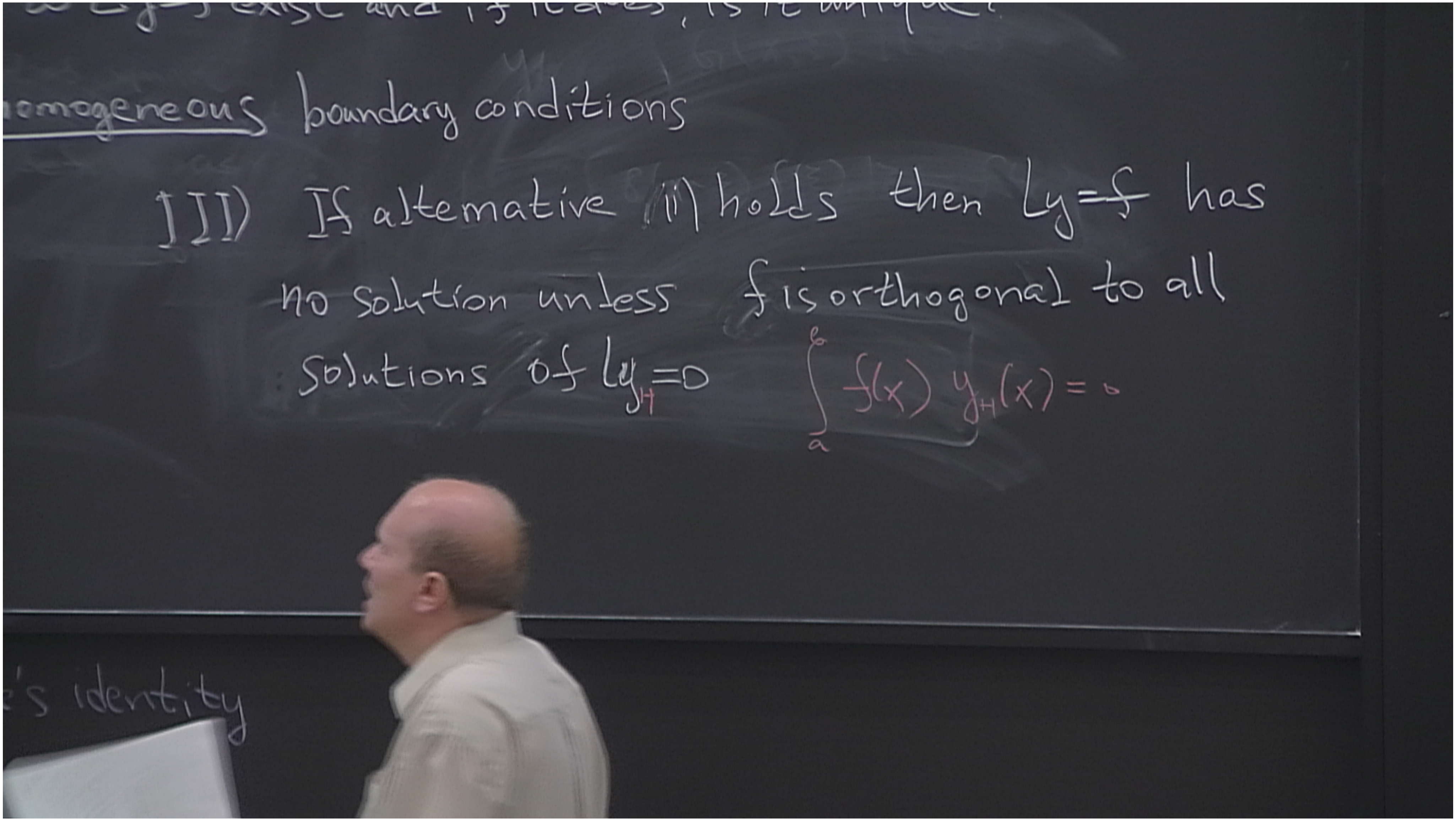
...ed by n <u>homogeneous</u> boundary conditions

on

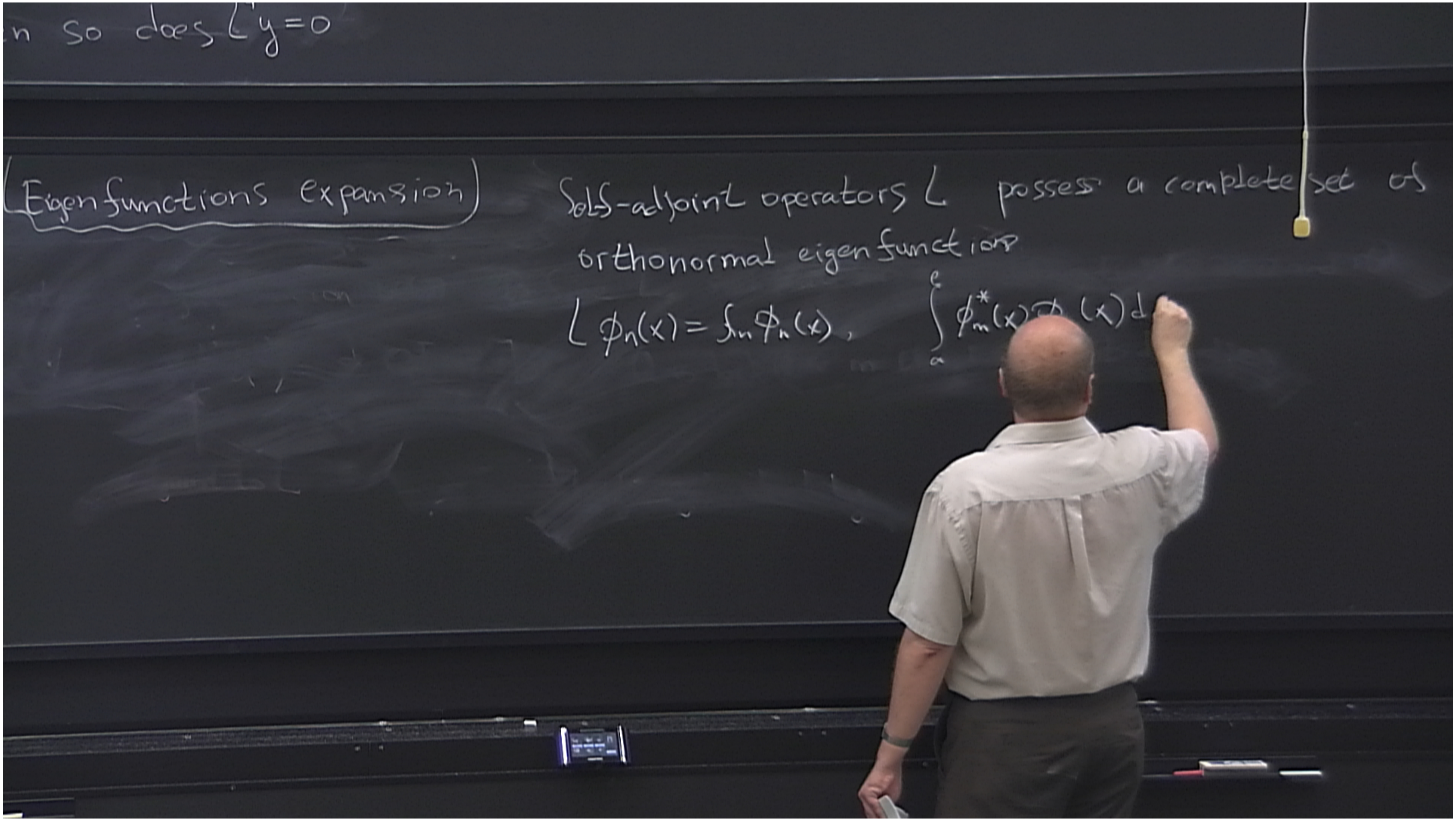
olutions,

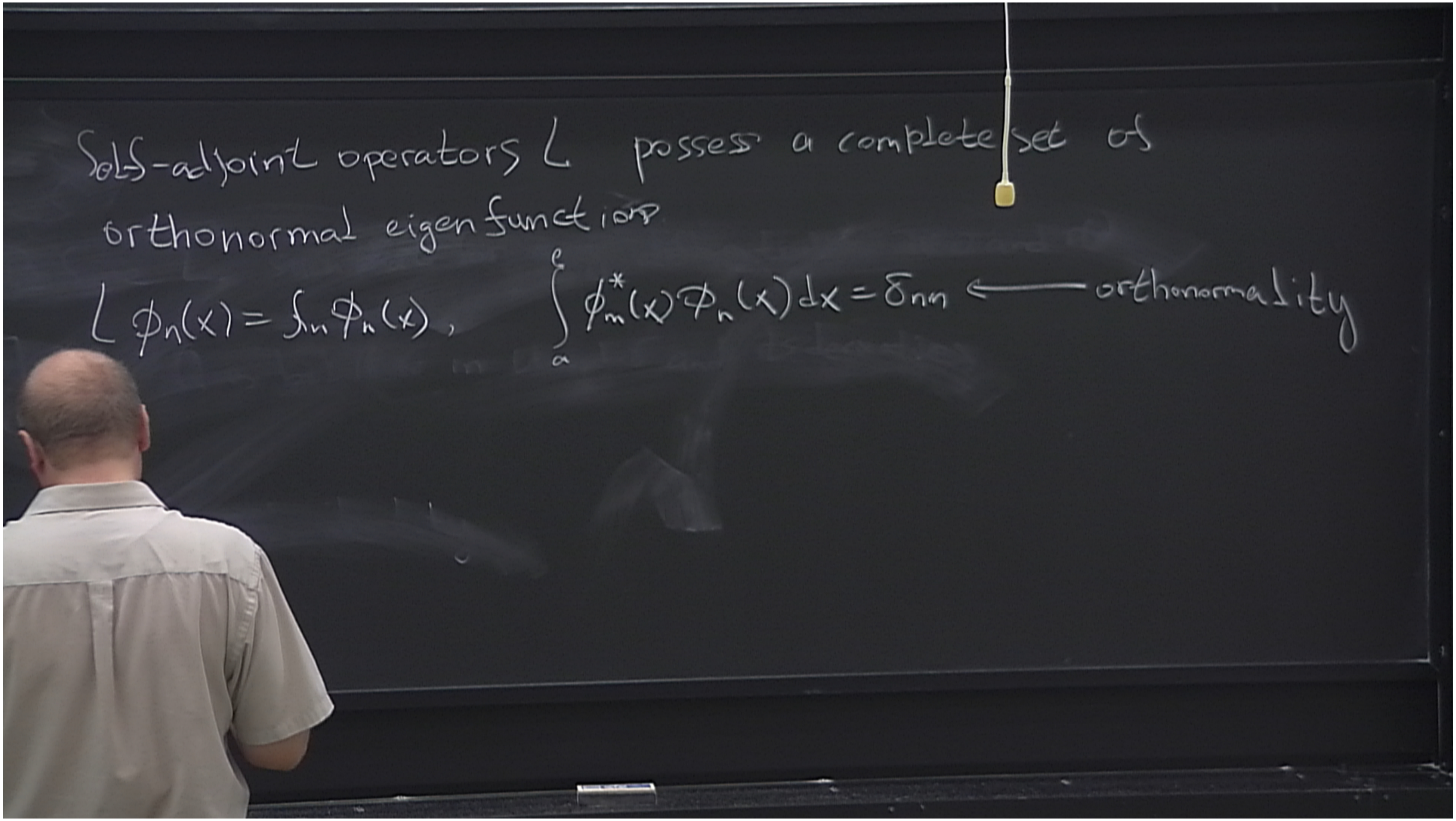
III) If alternative (ii) holds then $Ly=f$ has no solution unless f is orthogonal to all solutions of $Ly=0$

Lagrange's identity



omogeneous boundary conditions
III) If alternative (ii) holds then Ly=f has
no solution unless f is orthogonal to all
solutions of Ly_H=0
∫_a^b f(x) y_H(x) = 0
's identity

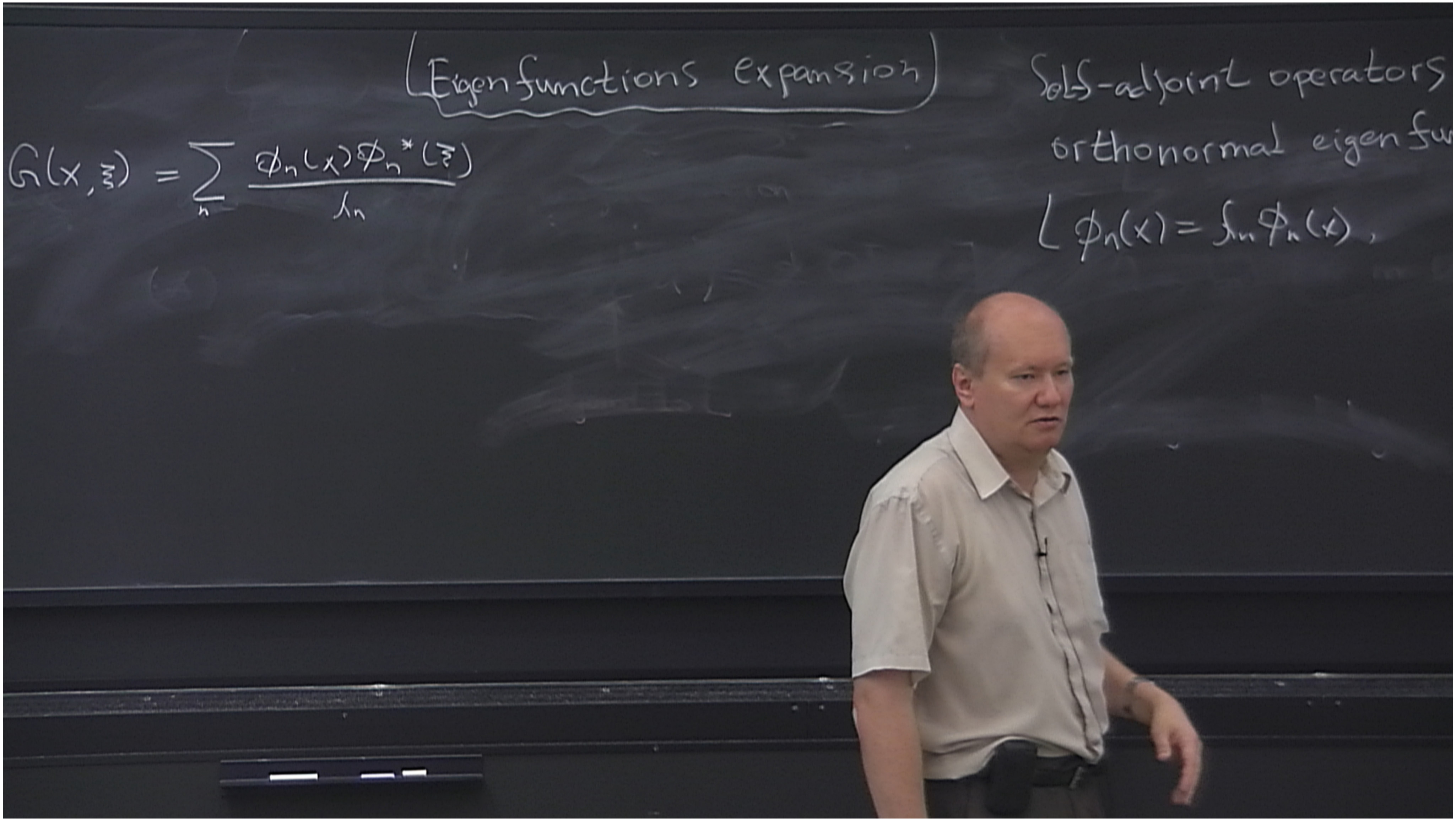

n so does $L^\dagger y=0$
Eigenfunctions expansion
Self-adjoint operators L possess a complete set of orthonormal eigenfunctions
$L\phi_n(x) = \lambda_n \phi_n(x)$, $\int_a^b \phi_m^*(x)\phi_n(x)\,d$


Self-adjoint operators L possess a complete set of
orthonormal eigenfunctions
$L\phi_n(x) = \lambda_n \phi_n(x)$, $\int_a^b \phi_m^*(x)\phi_n(x)\,dx = \delta_{nm}$ ← orthonormality

Self-adjoint operators L possess a complete set of orthonormal eigenfunctions

$$L\,\phi_n(x) = \lambda_n \phi_n(x), \qquad \int_a^b \phi_m^*(x)\,\phi_n(x)\,dx = \delta_{nm} \longleftarrow \text{orthonormality}$$

$$\sum_n \phi_n(x)\,\phi_n^*(\xi) = \delta(x-\xi)$$


Eigenfunctions expansion
$G(x,\xi) = \sum_n \frac{\phi_n(x)\phi_n^*(\xi)}{\lambda_n}$
Self-adjoint operators
orthonormal eigen fu
$L\,\phi_n(x) = \lambda_n \phi_n(x)$,

Eigenfunctions expansion

Self-adjo

orthon

$$L\phi_n(x) =$$

$$G(x,\xi) = \sum_n \frac{\phi_n(x)\phi_n^*(\xi)}{\lambda_n}$$

Proof: $$L_x\left(\sum_n \frac{\phi_n(x)\,\phi_n^*(\xi)}{\lambda_n}\right) = \sum_n \frac{\overbrace{[L_x\phi_n(x)]}^{\lambda_n}\phi_n^*(\xi)}{\lambda_n} =$$

$$= \sum_n \phi_n(x)\phi_n^*(\xi) = \delta(x-\xi)$$