Title: Fault-tolerant logical gates in quantum error-correcting codes

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Abstract: Recently, Bravyi and Koenig have shown that there is a tradeoff between fault-tolerantly implementable logical gates and geometric locality of stabilizer codes. They consider locality-preserving operations which are implemented by a constant depth geometrically local circuit and are thus fault-tolerant by construction. In particular, they shown that, for local stabilizer codes in D spatial dimensions, locality preserving gates are restricted to a set of unitary gates known as the D-th level of the Clifford hierarchy. In this paper, we elaborate this idea and provide several extensions and applications of their characterization in various directions. First, we present a new no-go theorem for self-correcting quantum memory. Namely, we prove that a three-dimensional stabilizer Hamiltonian with a locality-preserving implementation of a non-Clifford gate cannot have a macroscopic energy barrier. Second, we prove that the code distance of a D-dimensional local stabilizer code with non-trivial locality-preserving m-th level Clifford logical gate is upper bounded by L^{D+1-m}. For codes with non-Clifford gates (m>2), this improves the previous best bound by Bravyi and Terhal. Third we prove that a qubit loss threshold of codes with non-trivial transversal m-th level Clifford logical gate is upper bounded by 1/m. As such, no family of fault-tolerant codes with transversal gates in increasing level of the Clifford hierarchy may exist. This result applies to arbitrary stabilizer and subsystem codes, and is not restricted to geometrically-local codes. Fourth we extend the result of Bravyi and Koenig to subsystem codes. A technical difficulty is that, unlike stabilizer codes, the so-called union lemma does not apply to subsystem codes. This problem is avoided by assuming the presence of error threshold in a subsystem code, and the same conclusion as Bravyi-Koenig is recovered. This is a joint work with Fernando Pastawski. arXiv:1408.1720

Fault-tolerant logical gates in quantum error-correcting codes

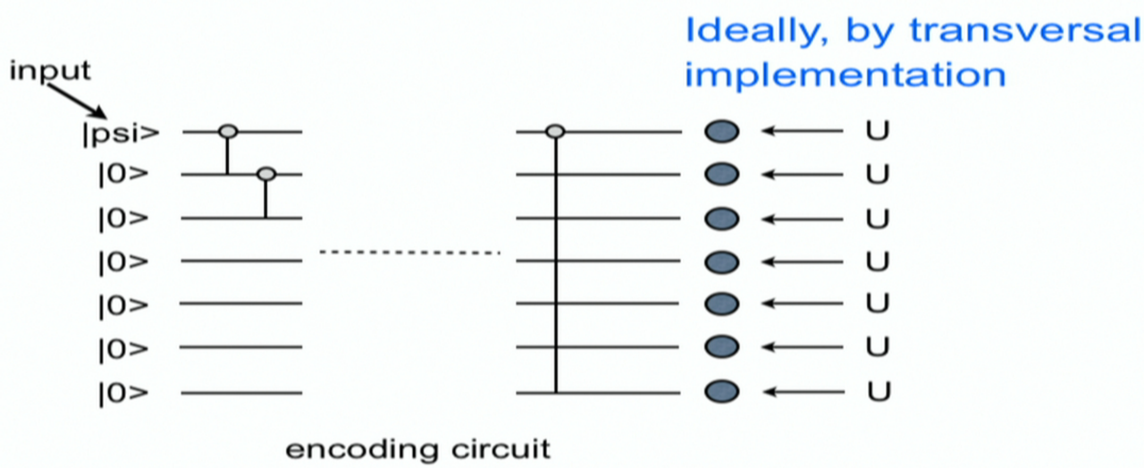
Fernando Pastawski and Beni Yoshida (Caltech)

arXiv:1408.1720

Sep 2014 @ Perimeter Institute (Waterloo, Canada)

Fault-tolerant logical gates

- How do we implement a logical gate fault-tolerantly ?



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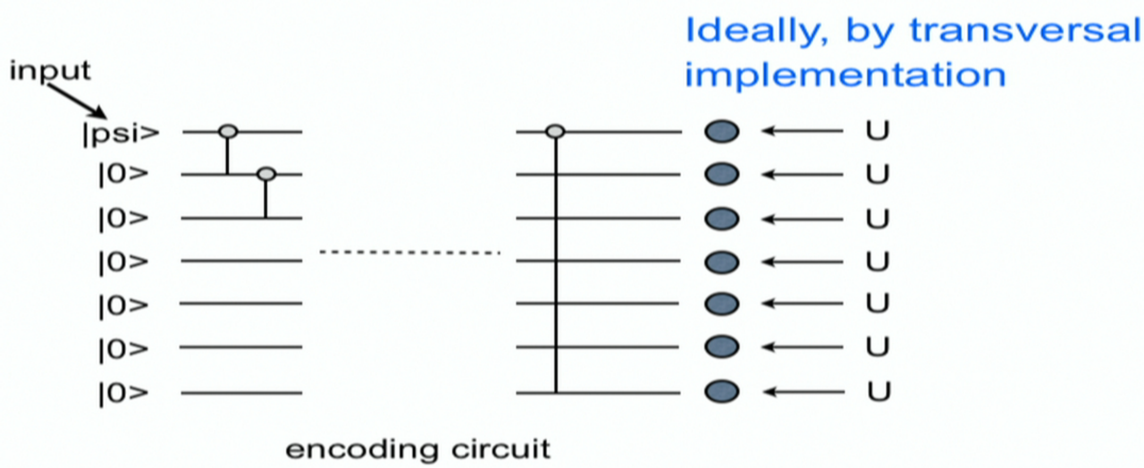


The Eastin-Knill theorem (2008)

- Transversal logical gates are not universal for QC

PRL **102,** 110502 (2009) PHYSICAL REVIEW LETTERS week ending 20 MARCH 2009

Restrictions on Transversal Encoded Quantum Gate Sets

Bryan Eastin* and Emanuel Knill
National Institute of Standards and Technology, Boulder, Colorado 80305, USA
(Received 28 November 2008; published 18 March 2009)

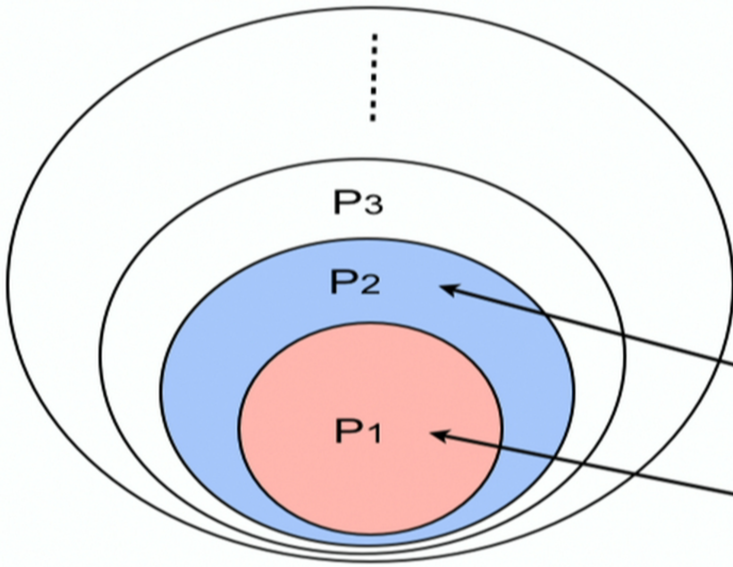
Transversal gates play an important role in the theory of fault-tolerant quantum computation due to their simplicity and robustness to noise. By definition, transversal operators do not couple physical subsystems within the same code block. Consequently, such operators do not spread errors within code blocks and are, therefore, fault tolerant. Nonetheless, other methods of ensuring fault tolerance are required, as it is invariably the case that some encoded gates cannot be implemented transversally. This observation has led to a long-standing conjecture that transversal encoded gate sets cannot be universal. Here we show that the ability of a quantum code to detect an arbitrary error on any single physical subsystem is incompatible with the existence of a universal, transversal encoded gate set for the code.

DOI: 10.1103/PhysRevLett.102.110502 PACS numbers: 03.67.Lx, 03.67.Pp

Don't panic ! Fault-tolerant computation is still possible.

Clifford hierarchy (Gottesman & Chuang)

Sets of unitary transformations on N qubits



$$P_m \text{ Pauli } P_m^\dagger = P_{m-1}$$

$$P_3 \text{ Pauli } P_3^\dagger = P_2$$

$$P_2 \text{ Pauli } P_2^\dagger = P_1$$

Pauli

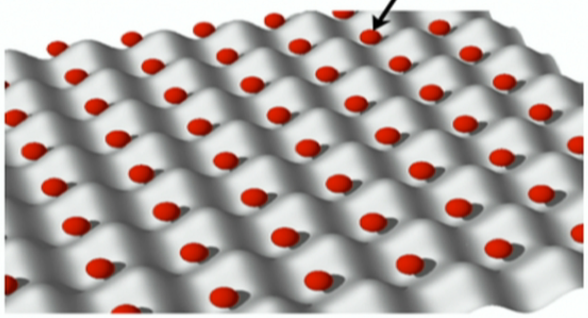
Clifford gates
CNOT, Hadamard, R2

Pauli operators
X,Y,Z

The Bravyi-Koenig theorem (2012)

- Under a more physically realistic setting

Logical gate U : low-depth unitary gate (i.e. Local unitary)



D-dim lattice

Theorem

- For a stabilizer Hamiltonian in D dim, fault-tolerantly implementable gates are restricted to the D-th level of the Clifford hierarchy.

???

Clifford hierarchy (Gottesman & Chuang)

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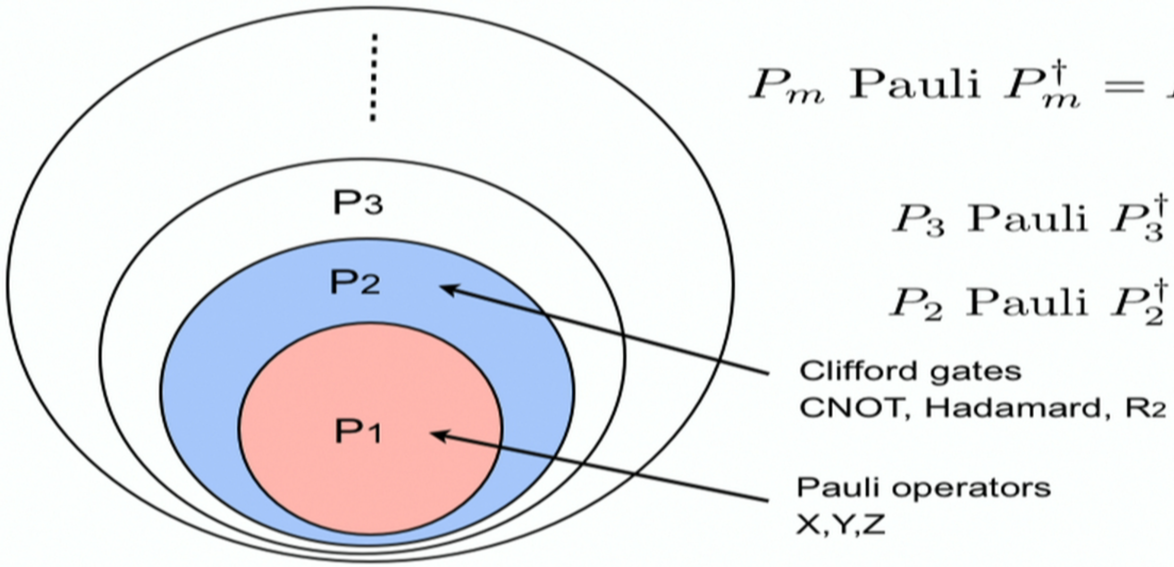

$$P_m \ \text{Pauli} \ P_m^\dagger = P_{m-1}$$

$$P_3 \ \text{Pauli} \ P_3^\dagger = P_2$$

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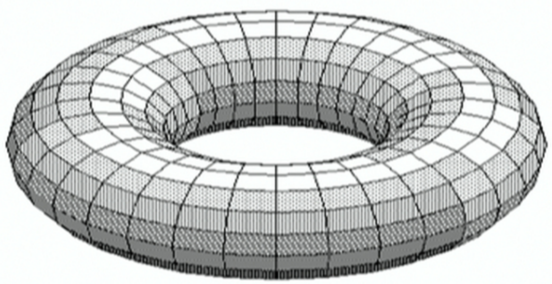
Pauli

Complexity of wavefunctions

- Classification of topological phases via local unitary ?



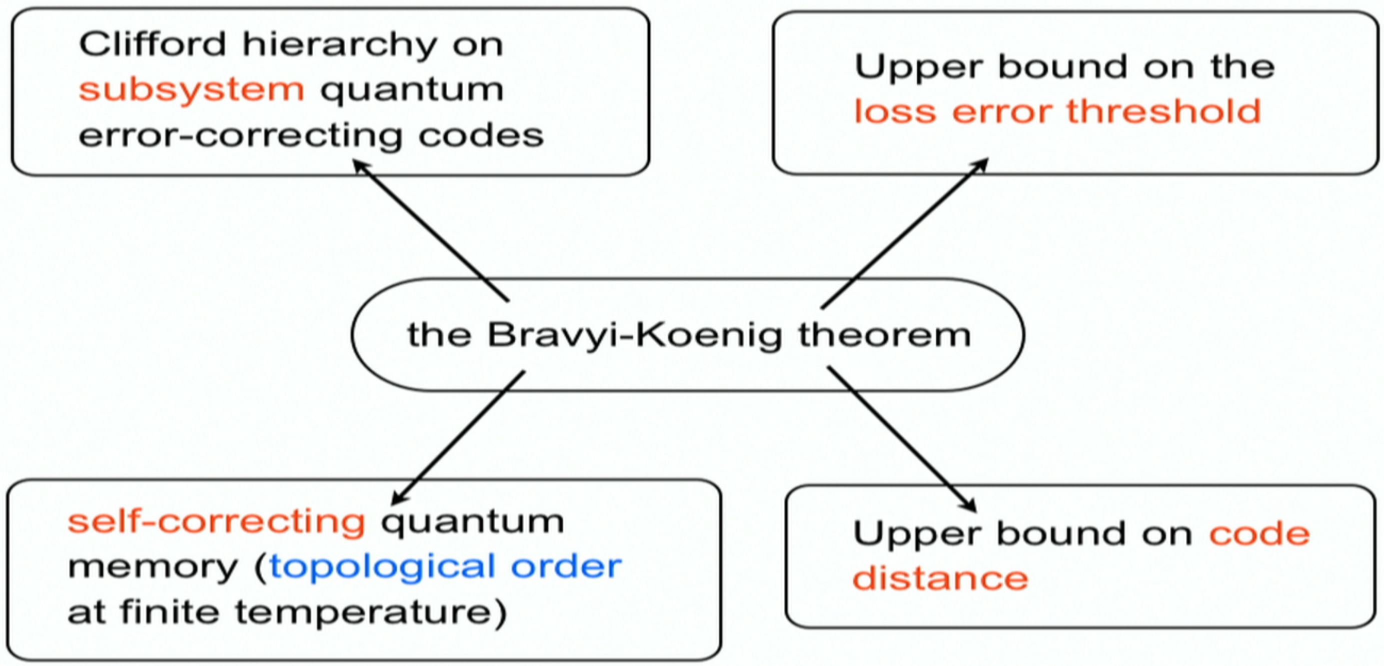
toric code

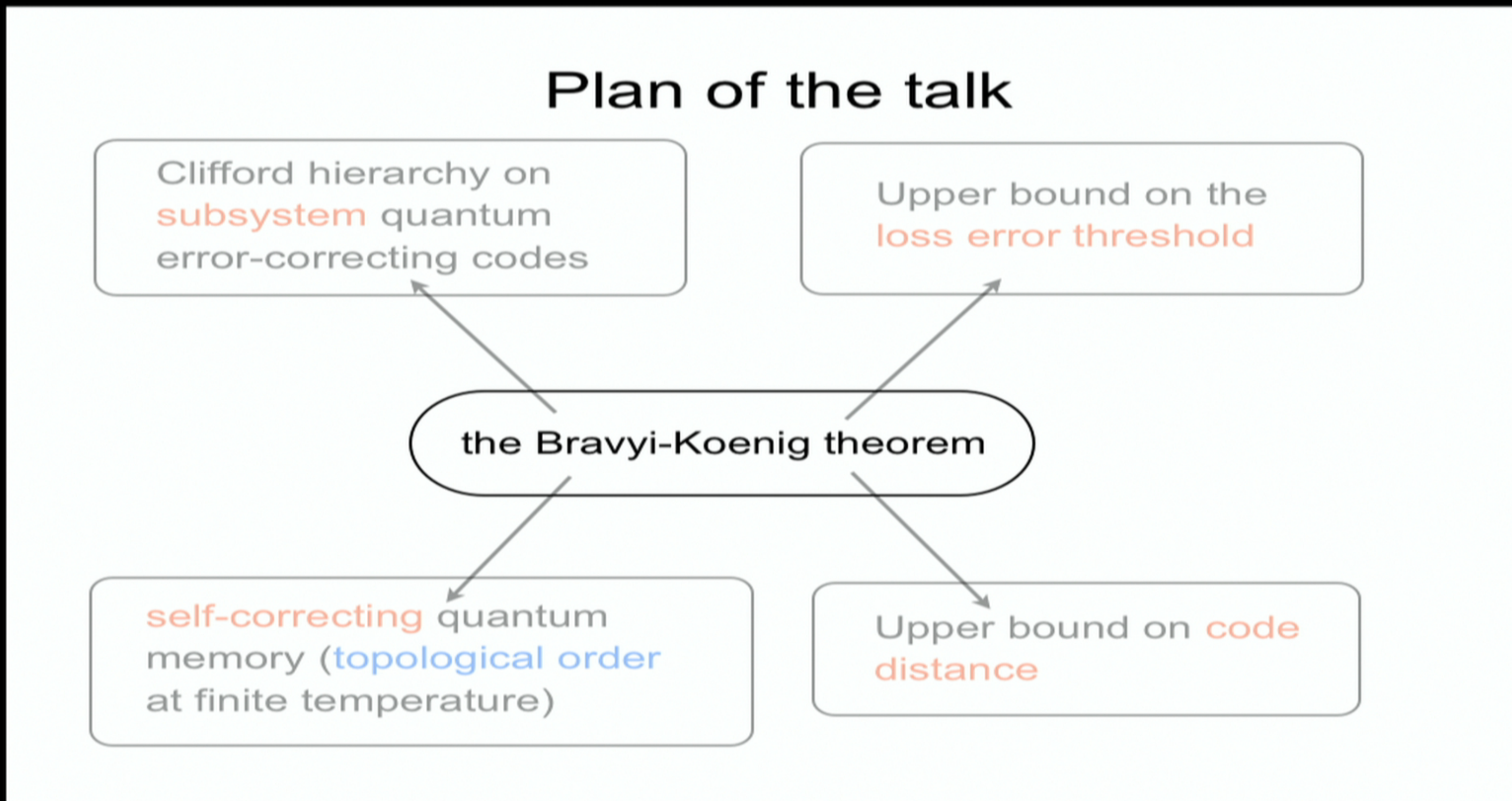
(Z_2 spin liquid)

$$\begin{array}{ccc} |0\rangle^{\otimes N} & \xrightarrow{\text{long-range}} & |\psi_0\rangle \\ \Big\downarrow{\scriptstyle \text{long-range}} & & \Big\downarrow{\scriptstyle \text{long-range}} \\ |\psi_1\rangle & \xrightarrow[\text{long-range}]{} & |\psi_0\rangle + e^{i\theta}|\psi_1\rangle \end{array}$$

Even more restrictive for non-Abelian topological order
(2001 Beckman, Gottesman, Kitaev, Preskill)

Plan of the talk

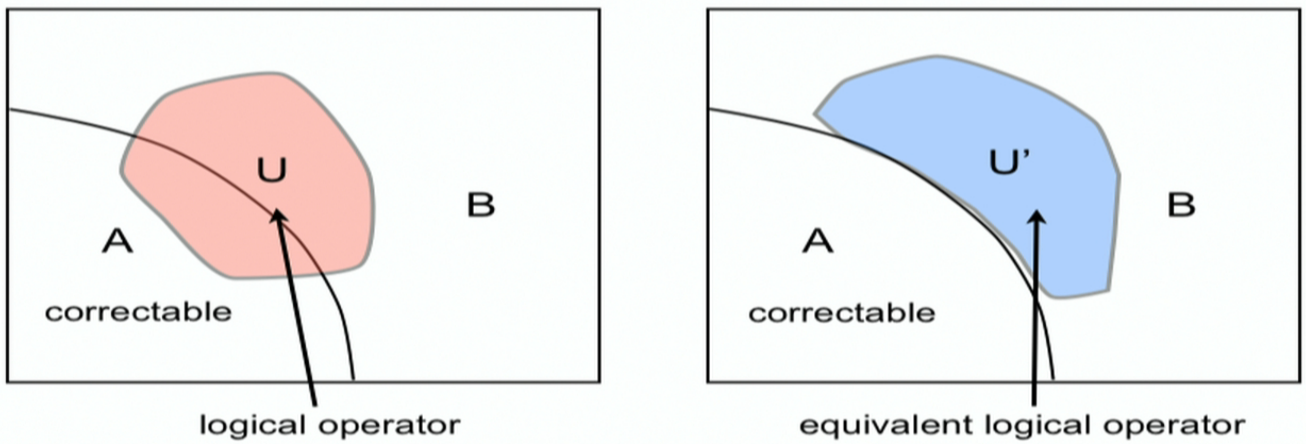



Plan of the talk
Clifford hierarchy on subsystem quantum error-correcting codes
Upper bound on the loss error threshold
the Bravyi-Koenig theorem
self-correcting quantum memory (topological order at finite temperature)
Upper bound on code distance

Logical operator cleaning

- A logical operator can be "cleaned" from a correctable region.

A "correctable region" supports no logical operator.



- Consider arbitrary Pauli logical operators V_0, V_1, ... V_m.

	R_0,	R_1,	R_2,	...	R_{m-1},	R_m	Hierarchy
V_0		✓	✓	⋯	✓	✓	
V_1	✓		✓	⋯	✓	✓	Pauli
⋮				⋮			
V_m	✓	✓	✓	⋯	✓		

- Consider arbitrary Pauli logical operators $V_0, V_1, \ldots V_m$.

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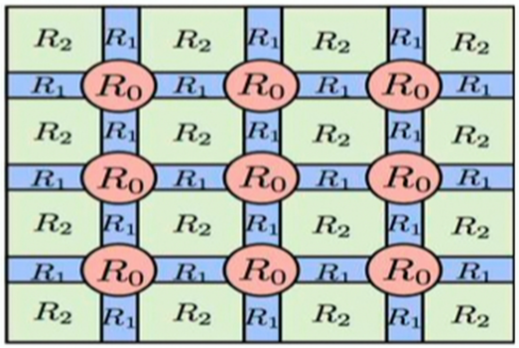
⋮				⋮			
V_m	✓	✓	✓	⋯	✓		goal
U_0	✓	✓	✓	⋯	✓	✓	$\boxed{P_m}$
$U_1=K(U_0,V_0)$		✓	✓	⋯	✓	✓	

commutator : $K(A,B)=ABA^*B^*$

Proof of Bravyi-Koenig theorem

- We can split D-dimensional system into D+1 correctable regions.

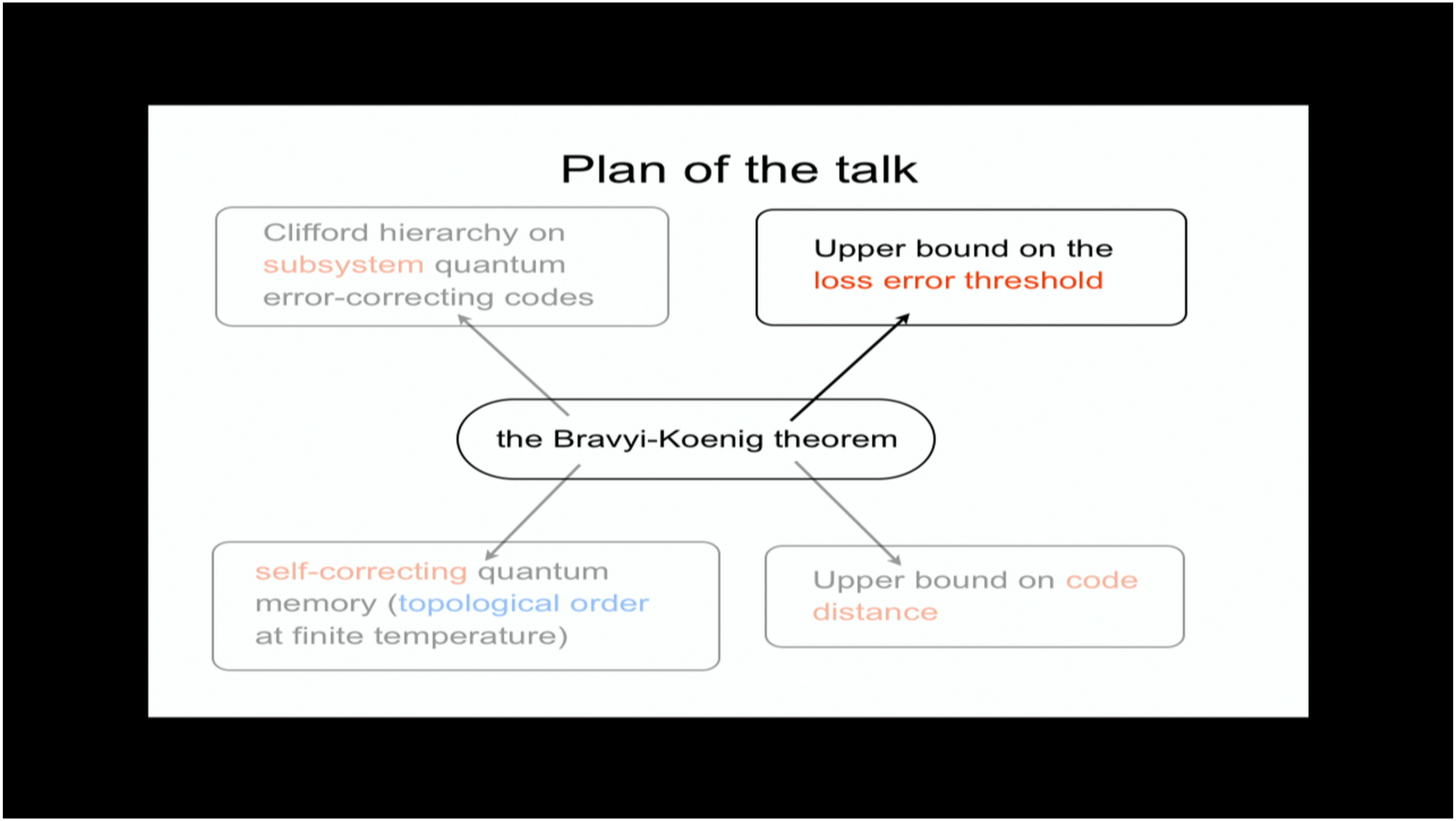
(eg) 2 dim



Fault-tolerant gates are in P2

*Union of spatially disjoint correctable regions = correctable region

*This is not the case for subsystem codes.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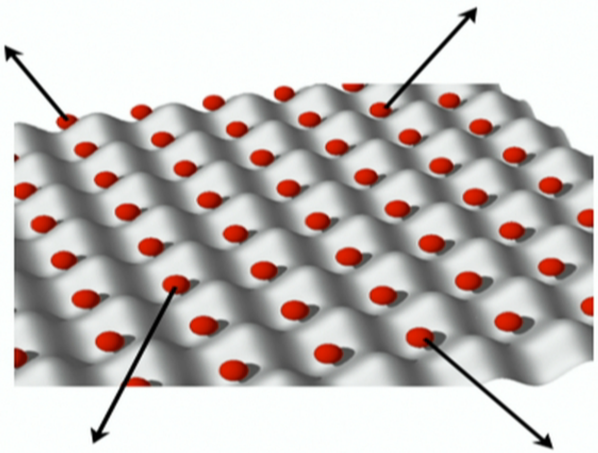
Loss Error Threshold

- Some qubits may be lost (removal errors)...

eg) escape from the trap



$$p < \boxed{p_{\text{loss}}} \quad \Rightarrow \quad \text{Logical qubit is safe}$$

loss error threshold

$$\boxed{p_{\text{error}}} < p_{\text{loss}}$$

against depolarizing error

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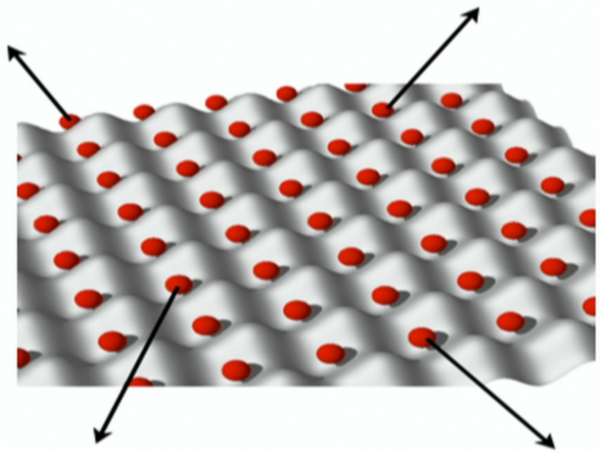

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against depolarizing error

Theorem. [Loss threshold] *Suppose we have a family of subsystem codes with a loss tolerance* $\boxed{p_l > 1/n}$ *for some natural number* n*. Then, any transversally implementable logical gate must belong to* $\boxed{\mathcal{P}_{n-1}}$.

$$\boxed{\mathcal{P}_n \text{ logical gate} \quad \Rightarrow \quad p_\ell \leq 1/n.}$$

Proof sketch

- Assign each qubit to n regions uniformly at random

  R_1, R_2, ... R_n
- All the regions are cleanable since $p_l > 1/n$
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Remarks

- Toric code has p=1/2 threshold (related to percolation).
  It has a transversal P_2 gate (CNOT gate)
- A family of codes with growing n is not fault-tolerant.
- Topological color code in D-dim has P_D gate, so its loss threshold is less than 1/D.

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<u>One additional remark</u> (due to Leonid Pyradko)

Consider a stabilizer code with at most m-body generators.

If the code has transversal P_n logical gate, then

$$\boxed{m > O(n)}$$

- D-dim color code is ~2^D body. Fewer-body code?

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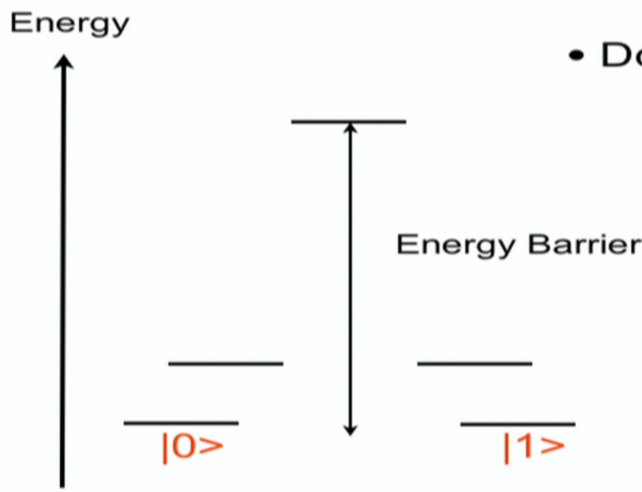
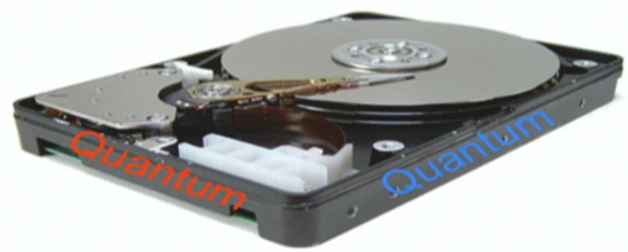
Self-correcting quantum memory

- Can we have self-correcting memory in 3dim?



- Does topological order exist at T>0 ?

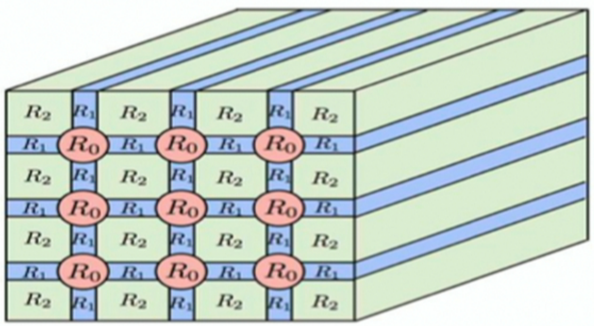


Theorem [Self-correction]

If a stabilizer Hamiltonian in 3 dimensions has fault-tolerantly implementable non-Clifford gates, then the energy barrier is constant.

Proof sketch

- Consider a partition into R_0, R_1, R_2.



- Suppose that there is no string-like logical operators.
- Then, R_0, R_1, R_2 are cleanable, so the code has P_2 (Clifford gate) at most.
- String-like logical operators imply deconfined particles.

Theorem [Self-correction]

If a stabilizer Hamiltonian in 3 dimensions has fault-tolerantly implementable non-Clifford gates, then the energy barrier is constant.

Remark

- Haah's 3dim cubic code (log(L) barrier) does not have non-Clifford gates.
- MIchnicki's 3dim welded code (poly(L) barrier) does not have non-Clifford gates.
- 6-dim color code ((4,2)-construction) has non-Clifford gate and O(L) barrier.
- 4-dim code with non-Clifford gate and log(L) barrier does exist.

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If a topological stabilizer code in D dimensions has a m-th level logical gate, then its code distance is upper bounded by

$$d \leq O(L^{D+1-m})$$

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- Bravyi-Terhal bound for D-dim stabilizer codes (previous best)

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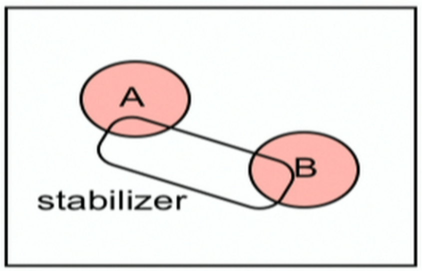
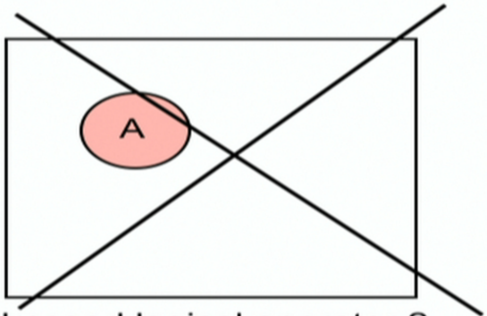
Breakdown of the union lemma

- The union lemma breaks down.



dressed logical operator



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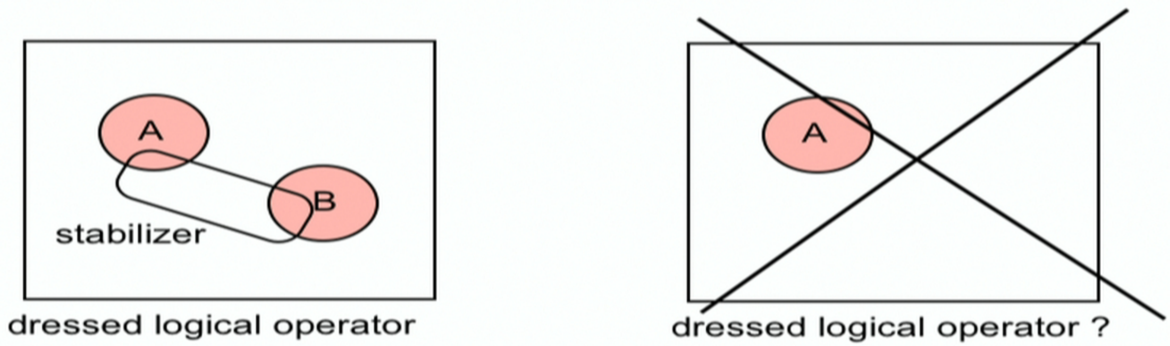

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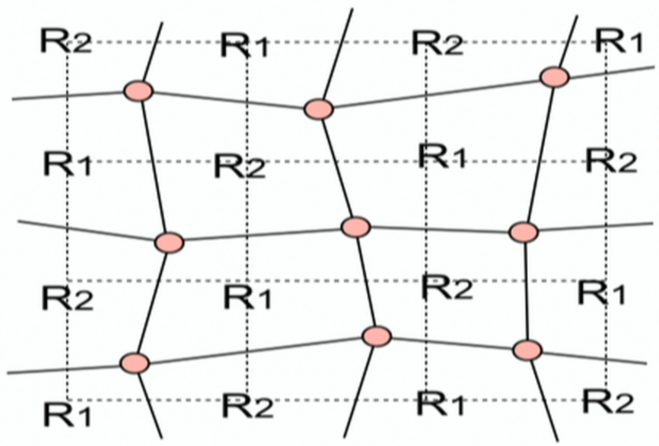
Fault-tolerance of the code

- The code must have a finite error threshold (loss error).



The union of red dots is correctable. (This circumvents the breakdown of the union lemma).

Fault-tolerant logical gates are restricted to P_D.

In D-dimensions, fault-tolerant gates are in P_D.

Open questions

- Fault-tolerant logical gates in TQFT ?
- The number of transversal gates ?

  reducing the overhead of magic state distillations
- Non-local, but finite depth unitary ?

  modular S matrix is implementable

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