Title: An invariant of topologically ordered states under local unitary transformations

Date: Sep 10, 2014 04:00 PM

URL: http://pirsa.org/14090030

Abstract: For an anyon model in two spatial dimensions described by a modular tensor category, the topological S-matrix encodes the mutual braiding statistics, the quantum dimensions, and the fusion rules of anyons. It is nontrivial whether one can compute the topological S-matrix from a single ground state wave function. In this talk, I will show that, for a class of Hamiltonians, it is possible to define the S-matrix regardless of the degeneracy of the ground state. The definition manifests invariance of the S-matrix under local unitary transformations (quantum circuits). The defined S-matrix depends only on the ground state, in the sense that it can be computed by any Hamiltonian in the class of which the state is a ground state. This property, together with the local unitary invariance implies that any quantum circuit that connects two ground states of distinct topological S-matrices must have depth that is at least linear in the diameter of the system. A higher dimensional analog is straightforward. [arXiv:1407.2926]

Invariant of topologically ordered states under local unitaries

Jeongwan Haah, MIT

Perimeter Institute,
Sep. 10, 2014

arXiv:1407.2926

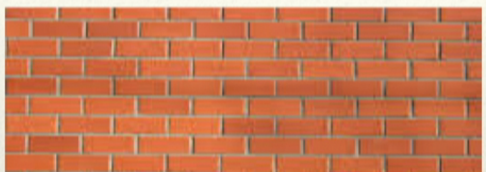
Quantum circuits

~ Local Unitary Transformations

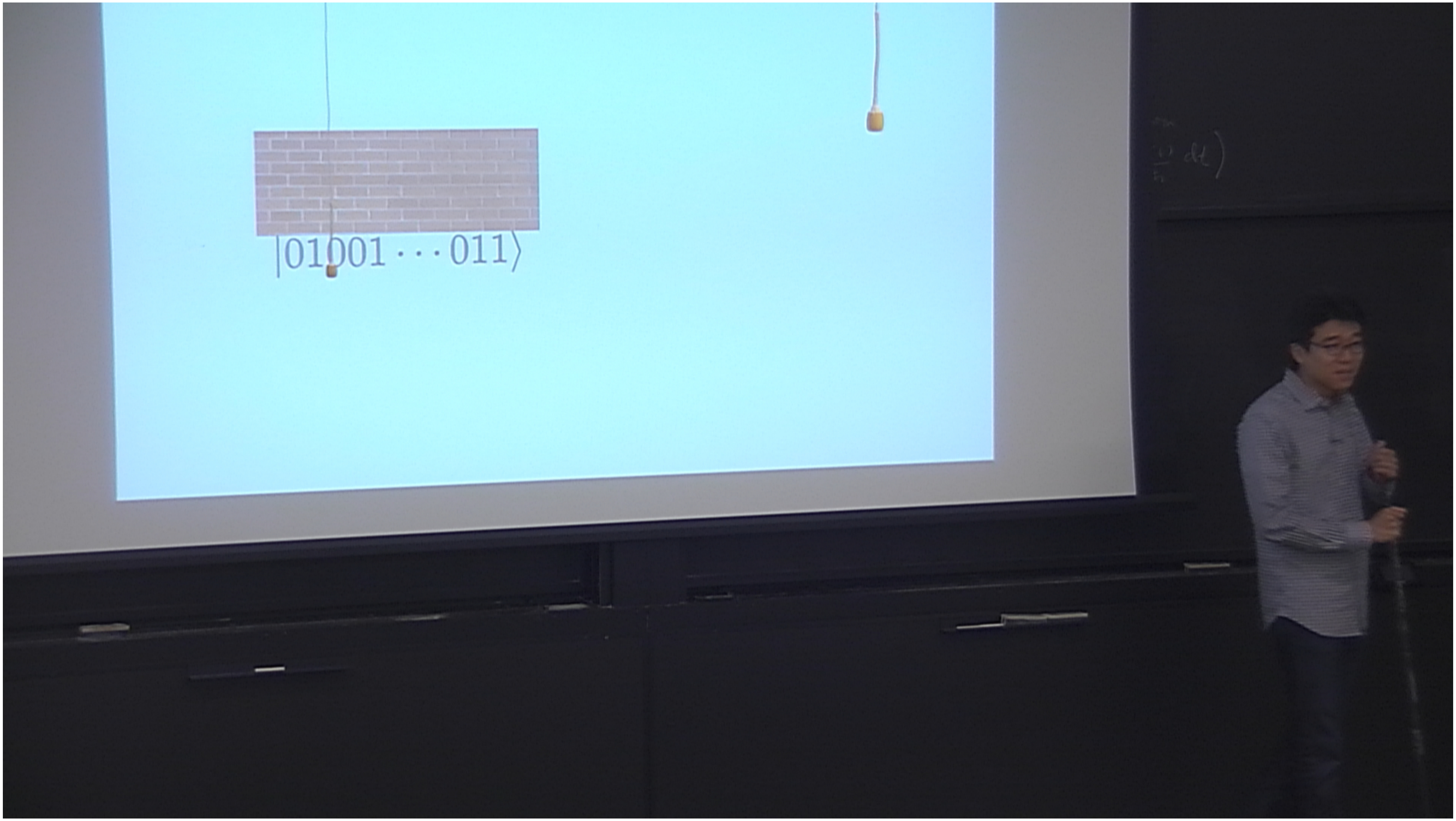
$$|01001 \cdots 011\rangle$$

Quantum circuits

~ Local Unitary Transformations



$$|01001\cdots011\rangle$$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Quantum circuits

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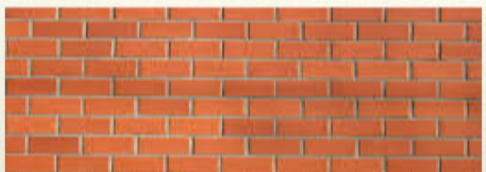

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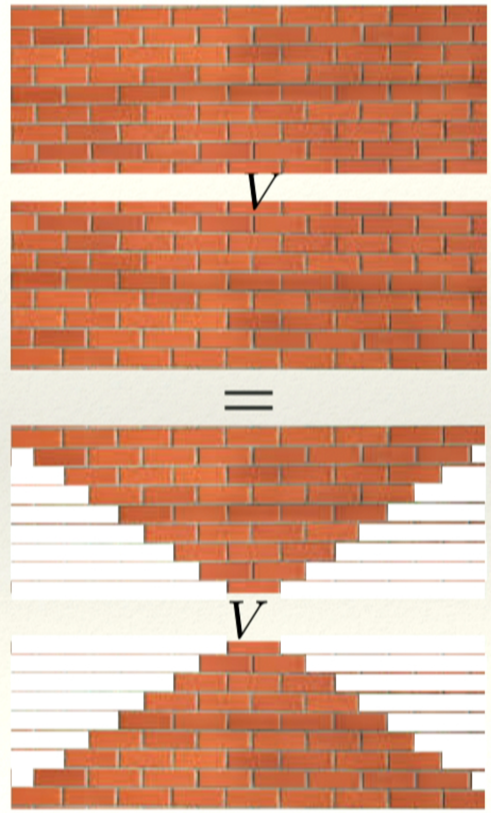


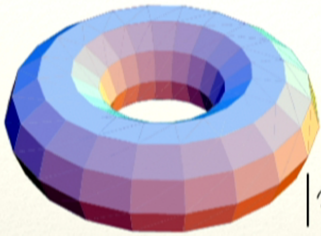
Long-range Entanglement

They all cannot be generated by a constant-depth quantum circuit

Hardness of Generation

[Wolfram MathWorld]

Bravyi, Hastings, Verstraete (2006)

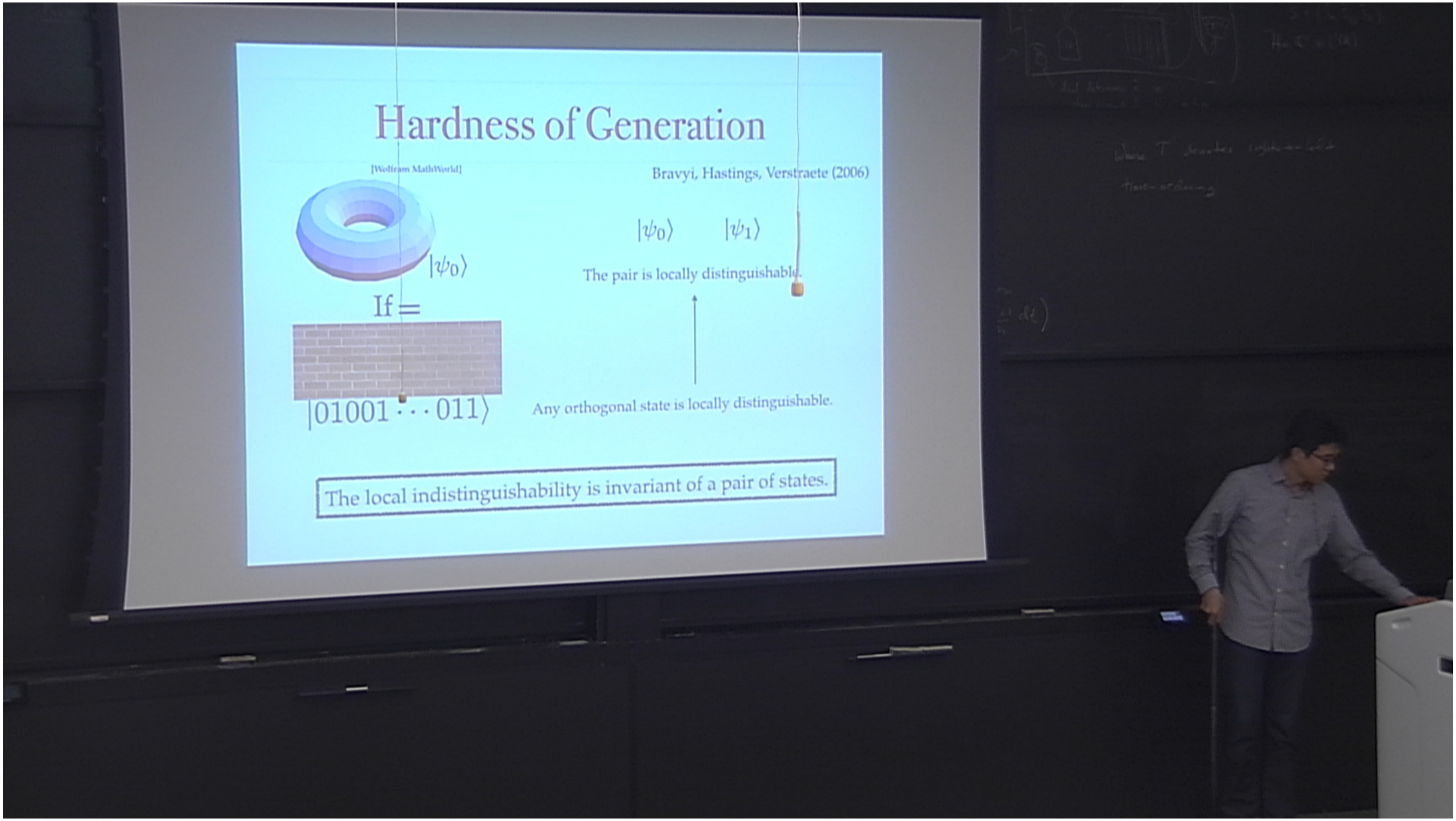


$|\psi_0\rangle$

If $=$

$|01001\cdots011\rangle$

Any orthogonal state is locally distinguishable.


Hardness of Generation
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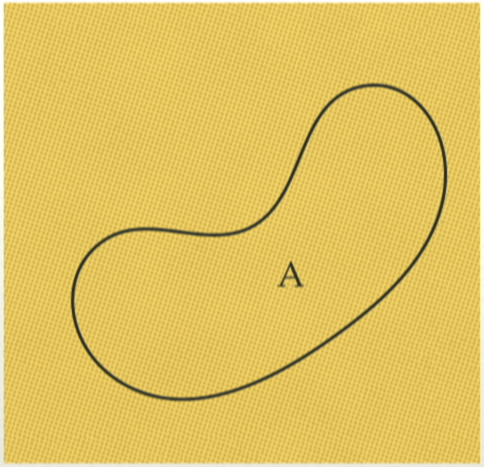
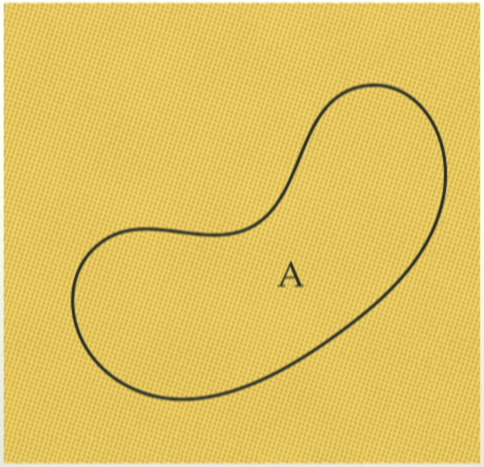
$|\psi_0\rangle$
If =
$|01001\cdots011\rangle$
$|\psi_0\rangle$ $|\psi_1\rangle$
The pair is locally distinguishable.
Any orthogonal state is locally distinguishable.
The local indistinguishability is invariant of a pair of states.

Topological Entanglement Entropy

Kitaev, Preskill; Levin, Wen (2006)



Topological Entanglement Entropy



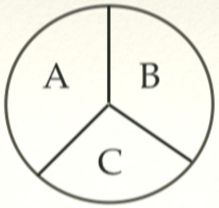
Kitaev, Preskill; Levin, Wen (2006)

$$S_A = \alpha L - \gamma$$

$$\gamma = \log \sqrt{\sum_a d_a^2}$$

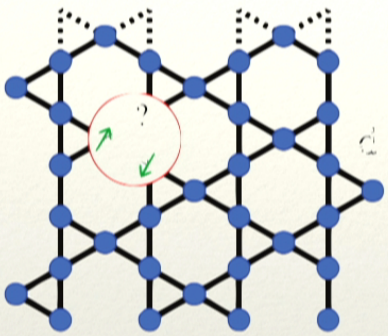
total quantum dimension

Kitaev-Preskill Argument



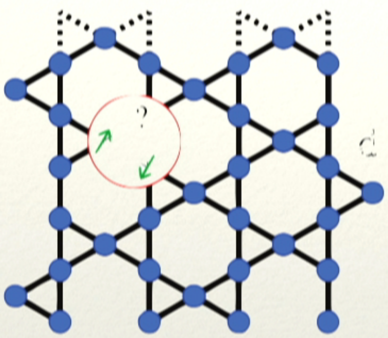
$$S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$

AFH on Kagome



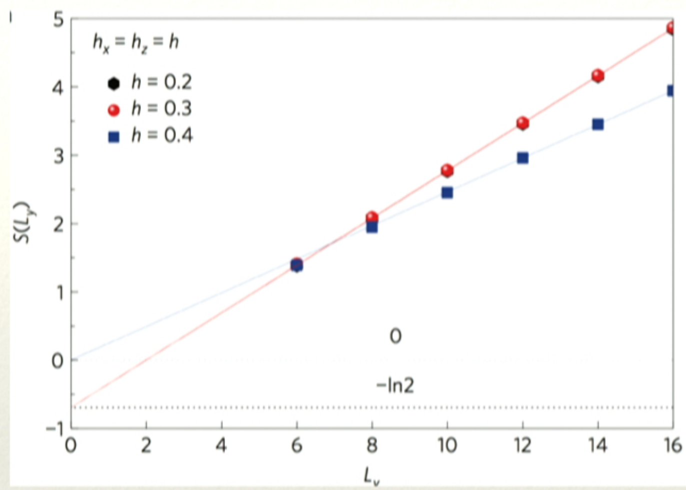
Yan, Huse, White (2011)

AFH on Kagome



Yan, Huse, White (2011)

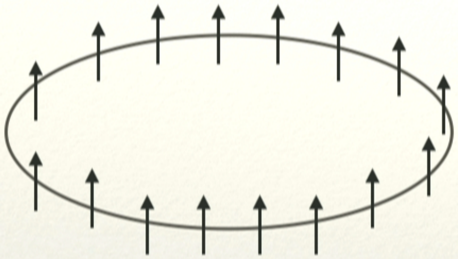
Found no ordering under perturbations



Jiang, Wang, Balents (2012)

Bravyi's Counterexample

From his talk in 2008



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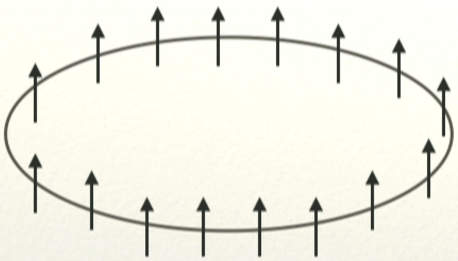

$$H = -\sum_i X_i$$

QC of depth 2

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

Bravyi's Counterexample

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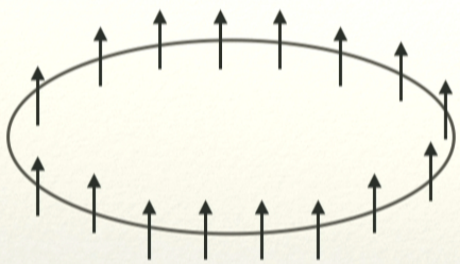
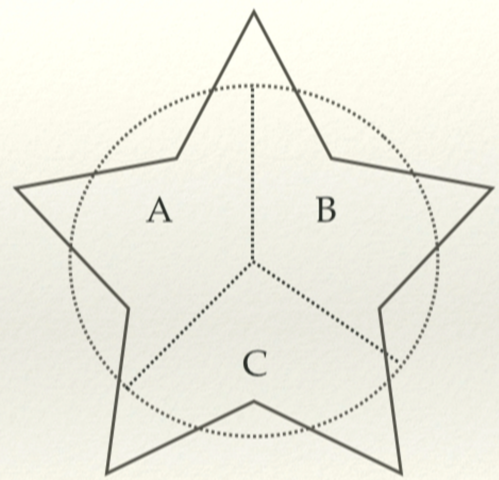


$$H = -\sum_i X_i$$

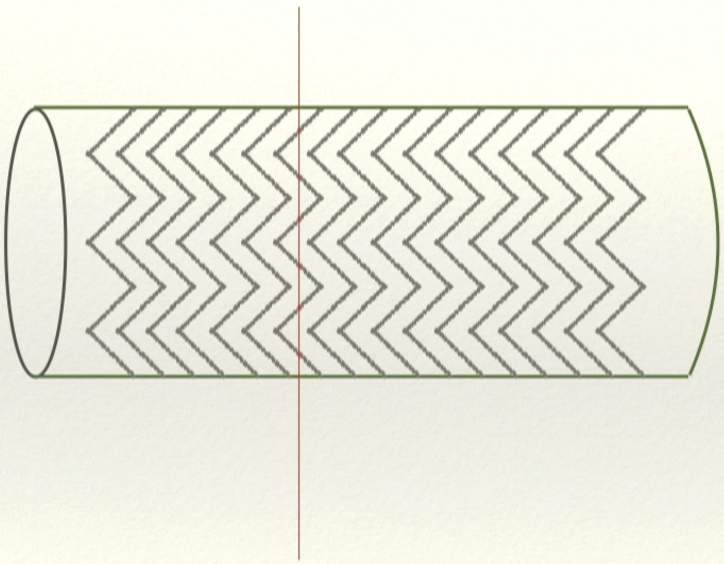
QC of depth 2

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

$$S_{\text{even}} = L/2 - 1$$



Cluster state embedded



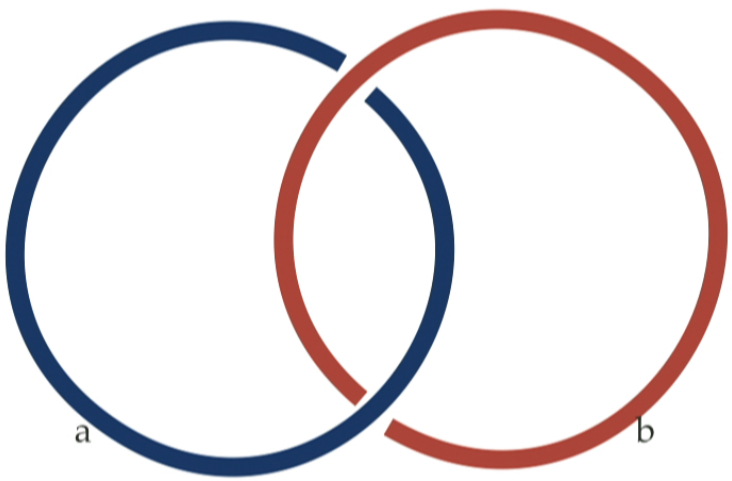
Can we say that TEE is an evidence for topological order?

Topological S-matrix

Quantum amplitude of braiding process

Topological S-matrix

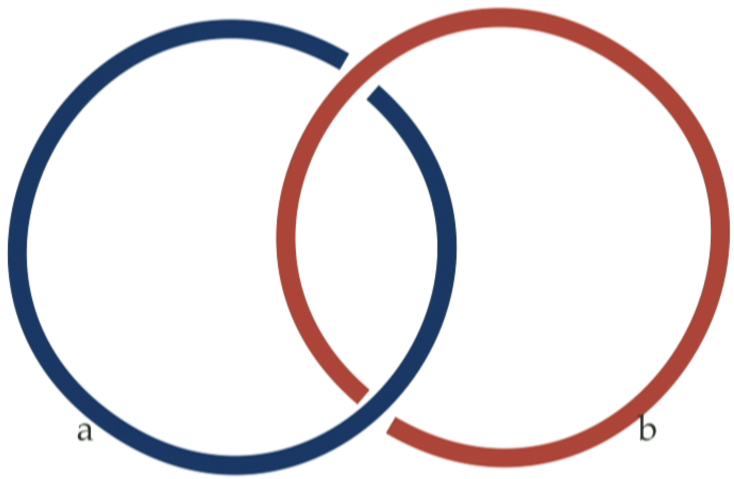
Quantum amplitude of braiding process



Topological S-matrix

Quantum amplitude of braiding process

$$S_{ab} = \frac{1}{\mathcal{D}}$$



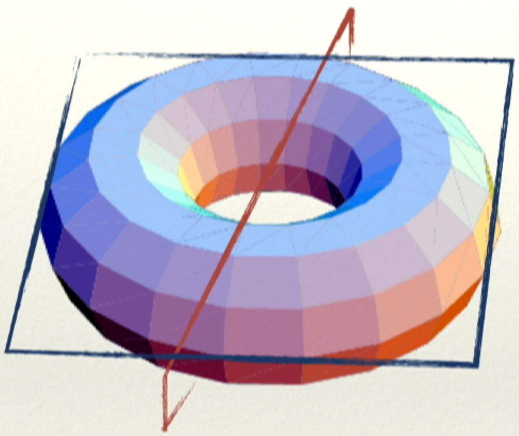
$$\mathcal{D}^2 = \sum_a d_a^2$$

$$\langle\psi| \bigcirc_a |\psi\rangle = d_a$$

Invariant of Hamiltonian or state?

Minimally Entangled States

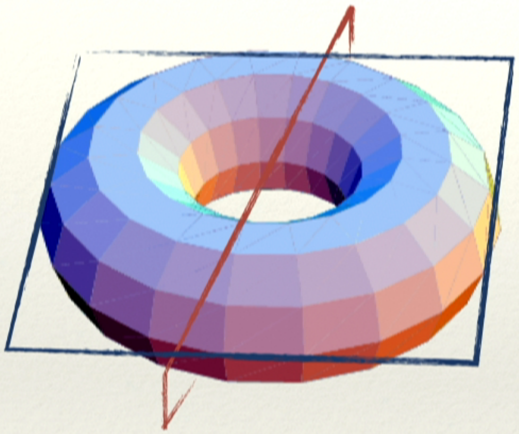
Zhang, Grover, Oshikawa, Vishwanath (2012)



- Start with full ground space.

Minimally Entangled States

Zhang, Grover, Oshikawa, Vishwanath (2012)



- Start with full ground space.
- Compute minimal ent. states.
- Compute overlap.

$$S_{ab} = \langle \psi_a^H | \psi_b^V \rangle$$

?

- Phase ambiguity on overlap computation = Trivial particle
- Particle label matching?

Toric code vs Double semion

$$\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

Back to Complexity Question

- S-matrix is defined with excitations or degenerate g.s.
- Is this invariant under small perturbations?

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- S-matrix is defined with excitations or degenerate g.s.
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- "It takes a linear depth Q. circuit to have a nontrivial topological S-matrix starting from a product state."

Back to Complexity Question

- S-matrix is defined with excitations or degenerate g.s.
- Is this invariant under small perturbations?
- "It takes a linear depth Q. circuit to have a nontrivial topological S-matrix starting from a product state."
- Proof?

$$\begin{tikzcd} H_0 \arrow[d] & & UH_0U^\dagger \neq H_1 \arrow[d] \\ |\psi_0\rangle \arrow[rr, dotted] & & U|\psi_0\rangle = |\psi_1\rangle \end{tikzcd}$$

Goal

- Find a quantity such that
  - It is defined by a state.
  - It is independent of boundary conditions.
  - It is invariant under local unitary transformations.
  - (It can be computed given a wave function.)

What is anyon?

- "Superselection sector"
  - A set of states related by local operators, not necessarily unitary.
  - No symmetry constraint.

Recall: Total spin

- Allowed operators,

$$[J_x, J_y] = iJ_z$$

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)$$

To define particle types

- Allowed operators,

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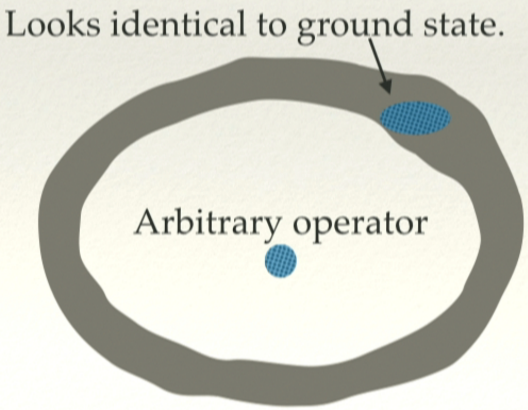
$$Mat(D) \otimes \mathcal{A}$$

- Allowed operators,



To define particle types

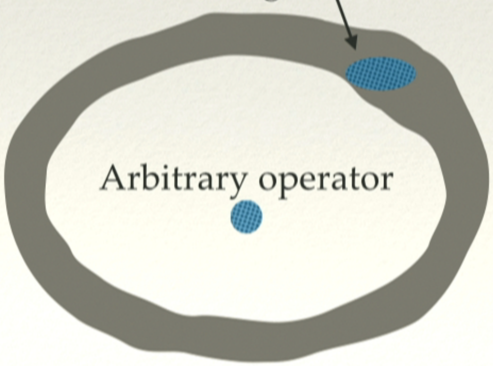
$$Mat(D) \otimes \mathcal{A}$$

Any local term of H should commute

Structure theorem
of C*-algebra

- Allowed operators,
- Find an operator in the center of the operator algebra.

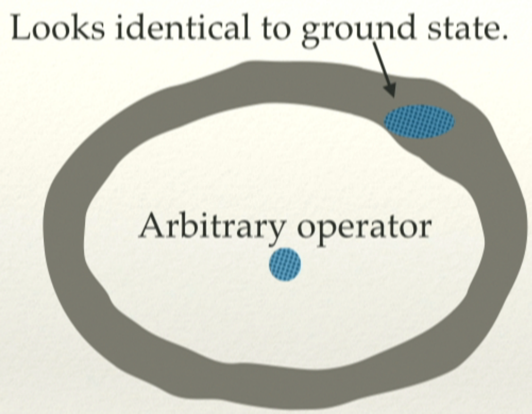
Looks identical to ground state.



C^*-algebra

- Algebra over complex numbers (finite dimensional)
- Enough to think of matrix algebra closed under dagger.
- Completely decomposes into (a direct sum of) full matrix algebras
- Projections onto components generate the center.

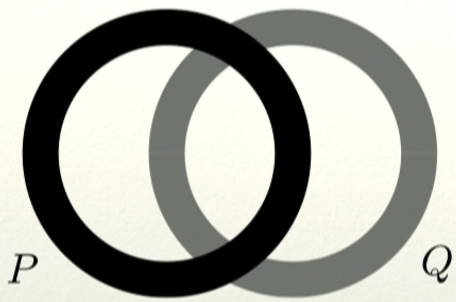
Null operators



Twist product

Ordinary product PQ

Ordinary product QP

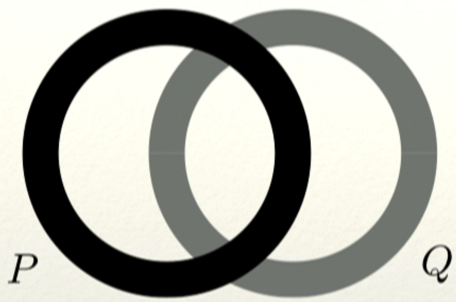


Twist Product

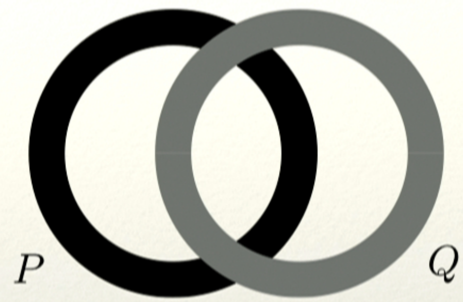
Well-defined as long as intersection is separated.

Twist product

Ordinary product PQ



Ordinary product QP



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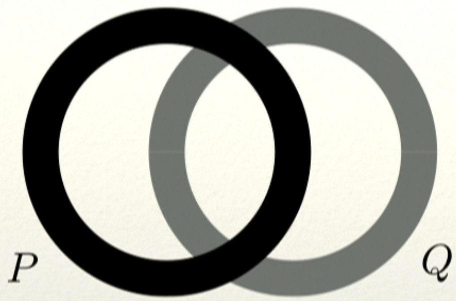

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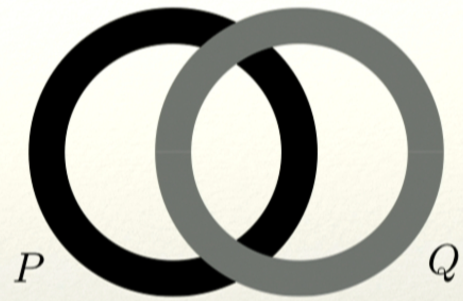

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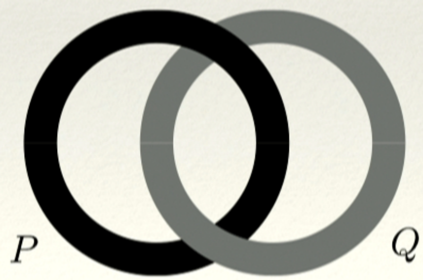


Well-defined as long as intersection is separated.

My S-matrix

$$\tilde{S}_{PQ} = \langle \psi | \; \cdots \; | \psi \rangle$$

Particle type projectors

- Input: (commuting) Hamiltonian (ground state)

My S-matrix

$\tilde{S}_{PQ} = \langle \psi |$  $| \psi \rangle$

Particle type projectors

- Input: (commuting) Hamiltonian (ground state)
- No special boundary; just some large disk.
- No phase ambiguity.
- The trivial particle ("1") projector is distinguished.

Relation to ord. S-matrix

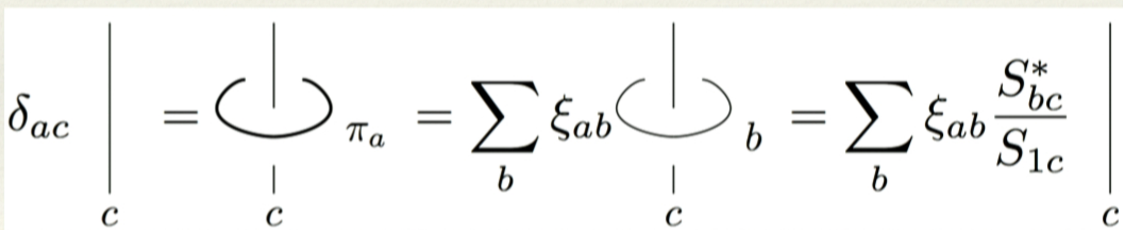
$$\tilde{S}_{ab} = \frac{d_a d_b}{D} S_{ab}$$

Relation to ord. S-matrix

$$\tilde{S}_{ab} = \frac{d_a d_b}{D} S_{ab}$$

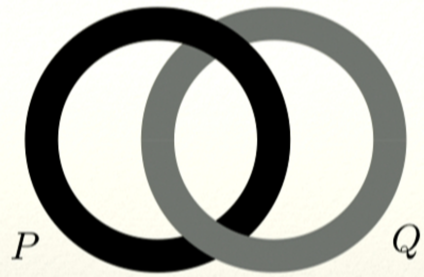
It contains the same data!

Proof:



Invariance under local unitaries

$$\langle\psi|\, W^{\dagger}W$$



$$W^{\dagger}W\,|\psi\rangle$$

Particle type projectors

Invariance under local unitaries

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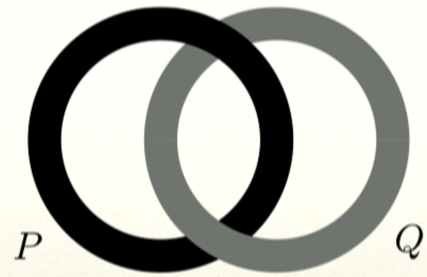

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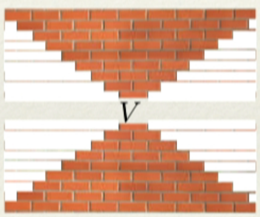


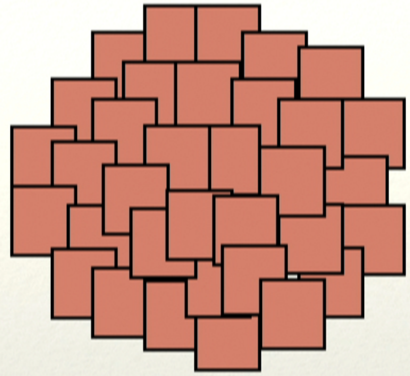
$$W(P\infty Q)W^\dagger = (WPW^\dagger)\infty(WQW^\dagger)$$

as long as the depth of W is smaller than the separation of the intersection.

Assumptions

1. Local topological order
   - Local ground state matches the global one
2. Stable logical algebra
   - logical algebra does not depend on the size of the support

Local Topological Order



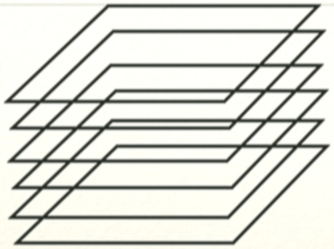
Stable Logical Algebra

Isomorphic

$\mathcal{A}/\mathcal{N}$

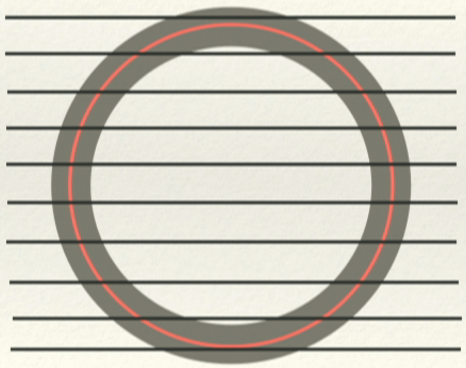
Regardless of the thickness

Finiteness of particle types



Infinite stack of 2D layers

A particle is separated by a sphere with thick wall.



Side View

Complexity of transformation

- Any transformation between states with distinct S-matrices requires a deep (linear in diameter) circuit.

$$
\begin{tikzcd}
H_0 \arrow[d] & & U H_0 U^\dagger \neq H_1 \arrow[d] \\
|\psi_0\rangle \arrow[rr, dotted] & & U|\psi_0\rangle = |\psi_1\rangle
\end{tikzcd}
$$

Complexity of transformation

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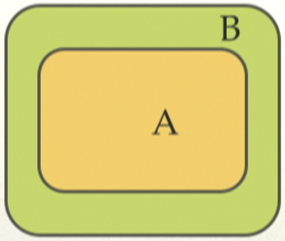
$$\begin{array}{ccc} H_0 & & UH_0U^\dagger \neq H_1 \\ \downarrow & & \qquad\quad\;\downarrow \\ |\psi_0\rangle & \cdots\cdots\cdots\rightarrow & U|\psi_0\rangle = |\psi_1\rangle \end{array}$$

- In view of quasi-adiabatic evolution,
  the energy gap must close at some point in any path between Hamiltonians with distinct S-matrices.

Crucial notion: Local Invisibility



$$\mathrm{Tr}_{B^c}|\phi\rangle\langle\phi| = \mathrm{Tr}_{B^c}|\psi\rangle\langle\psi|$$

$$\Downarrow$$

$$\mathrm{Tr}_{A^c}\left(O|\phi\rangle\langle\phi|O^\dagger\right) \propto \mathrm{Tr}_{B^c}|\psi\rangle\langle\psi|$$

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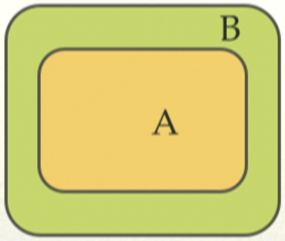

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$$\Downarrow$$

$$\mathrm{Tr}_{A^c}\left(O|\phi\rangle\langle\phi|O^\dagger\right) \propto \mathrm{Tr}_{B^c}|\psi\rangle\langle\psi|$$

- Usual string operators satisfy this.
- No Hamiltonian involved.

Sketch of independence proof

$$\mathcal{A}_t/\mathcal{N}_t \to \mathcal{I}_t/\mathcal{M}_t \to \mathcal{A}_{t+w}/\mathcal{N}_{t+w}$$

- Logical algebra to locally invisible operators
  - They are naturally invisible thanks to local topological order condition.
- Locally invisible operators to logical algebra
  - "Symmetrize" so locally invisible operators is dressed to commute with the Hamiltonian

$$\mathcal{A}_t^{H_1}/\mathcal{N}_t^{H_1} \to \mathcal{I}_t/\mathcal{M}_t \to \mathcal{A}_{t+w}^{H_2}/\mathcal{N}_{t+w}^{H_2}$$

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Toric code state

$\mathcal{A}/\mathcal{N}$ is diagonal matrix algebra of dimension d^2

$$\tilde{S}^{(d)}_{(a_x a_z),(a'_x a'_z)} = \frac{1}{d^2}\omega_d^{a_z a'_x + a_x a'_z}.$$

- Two assumptions are satisfied, as verified by direct computation.
- Rows and columns unsorted except for the distinguished "1".
- Verlinde formula recovers the fusion (group) rules.

Single ground state knows S-matrix

$$\tilde{S}_{PQ} = \langle \psi |$$

$$| \psi \rangle$$

- We have given a class of ground states, for which S-matrix can be defined.
- Only a patch of a ground state is needed; insensitive to boundary.
- Indeed invariant under perturbations.
- 2D is not particularly used.
- Any heuristic algorithm would be interesting.
- Perhaps, in 2D stable logical algebra assumption is redundant.