Title: Self-assembling tensor networks and holography in disordered spin chains

Date: Sep 16, 2014 03:30 PM

URL: http://pirsa.org/14090029

Abstract: We show that the numerical strong disorder renormalization group algorithm (SDRG) of Hikihara et.\ al.\ [{Phys. Rev. B} {\bf 60}, 12116 (1999)] for the one-dimensional disordered Heisenberg model naturally describes a tree tensor network (TTN) with an irregular structure defined by the strength of the couplings. Employing the holographic interpretation of the TTN in Hilbert space, we compute expectation values, correlation functions and the entanglement entropy using the geometrical properties of the TTN. We find that the disorder averaged spin-spin correlation scales with the average path length through the tensor network while the entanglement entropy scales with the minimal surface connecting two regions. Furthermore, the entanglement entropy increases with both disorder and system size, resulting in an area-law violation. Our results demonstrate the usefulness of a self-assembling TTN approach to disordered systems and quantitatively validate the connection between holography and quantum many-body systems.

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Self-assembling tensor networks and holography in disordered spin chains

Andrew M. Goldsborough, Rudolf A. Römer

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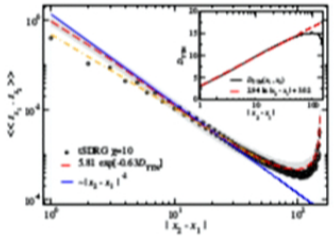
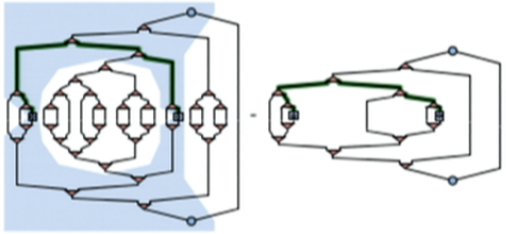
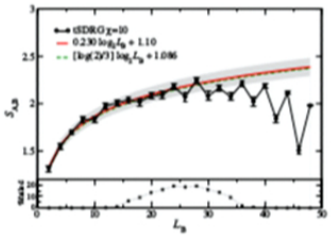
The University of Warwick







A. M. Goldsborough, and R. A. Römer, Phys. Rev. B 89, 214203 (2014)
A. M. Goldsborough, S. A. Rautu, and R. A. Römer, arXiv:1406.4079 (2014)

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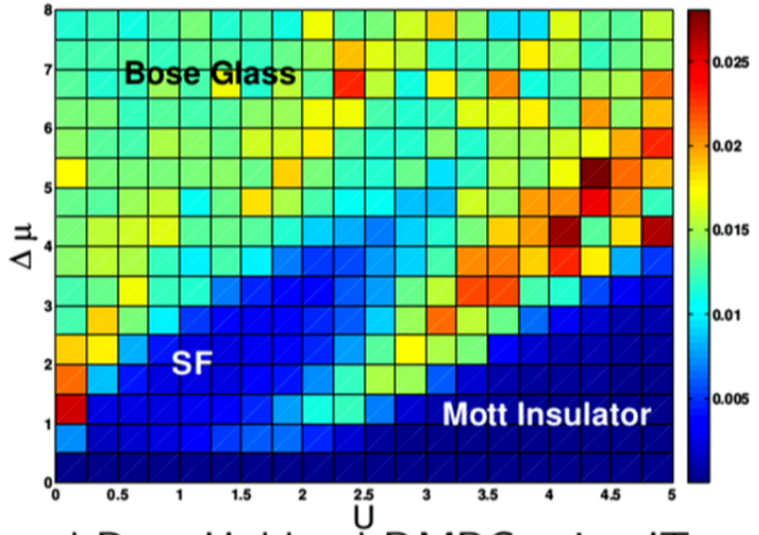
Disorder and Condensed Matter Physics

- Quenched disorder can dramatically effect observed phase
- Smears phase transitions
- New phases (L. Pollet, Comptes Rendus Physique 14, 712 (2013))
- Localization (P. W. Anderson, Phys. Rev. 109, 1492 (1958))
- Not integrable, but a lot gained from renormalization group (F. Iglói, and C. Monthus, Physics Reports 412, 277 (2005))
- Numerically challenging



Disordered Bose-Hubbard DMRG using ITensor

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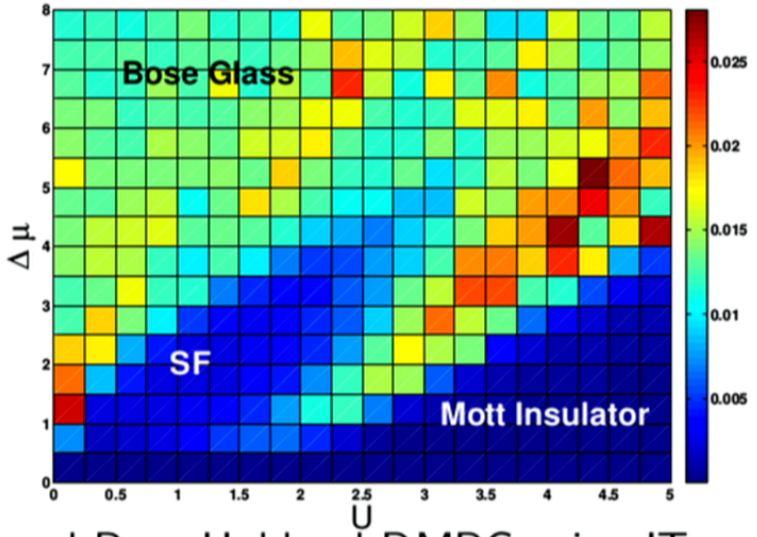

Disordered Bose-Hubbard DMRG using ITensor

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1D Disordered Heisenberg chain

$$H = \sum_{i=1}^{L-1} J_i \vec{s}_i \cdot \vec{s}_{i+1}$$

- Paradigmatic model for quantum interacting system
- Open boundary
- Analytic solution when clean (Bethe Ansatz)
- Coupling constants take a random value $0 < J_i < J_{max}$
- Antiferromagnetic (AFM)
- Use a *box* type distribution

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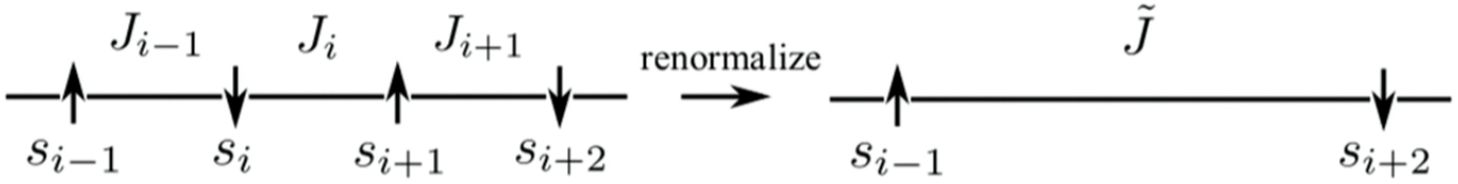
The strong disorder renormalization group (SDRG)

- The Ma, Dasgupta, Hu (MDH) algorithm for the random anti-ferromagnetic (AFM) chain.
- Remove the strongest coupled pair of spins and take the energy of the singlet as the contribution to the total energy.
- Renormalize the coupling between the neighbouring spins.



$$\tilde{J} = \frac{J_{i-1} J_{i+1}}{J_i}$$

S. K. Ma, C. Dasgupta, and C. K. Hu, Phys. Rev. Lett. 43, 1434 (1979)

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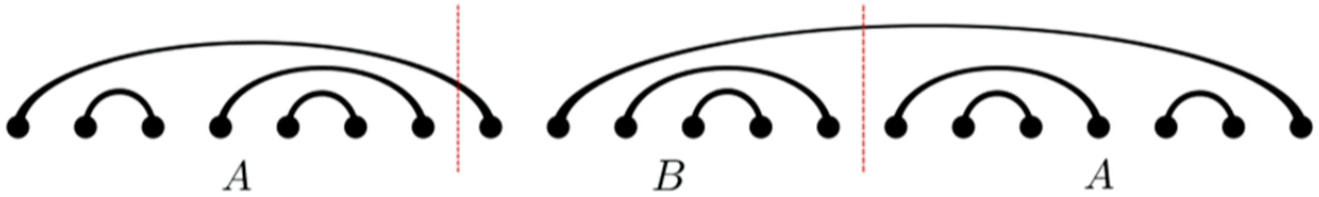
The strong disorder renormalization group (SDRG)

- Ground state is the *random singlet phase*
- Mean spin-spin correlation function decays as a power-law[†]

$$\overline{\langle \vec{s}_i \cdot \vec{s}_j \rangle} \propto \frac{(-1)^{|i-j|}}{|i-j|^2}$$

- Mean entanglement entropy ($S_{A,B} = -\mathrm{Tr}\rho_A \log_2 \rho_A$) scales logarithmically with subsystem size[‡]

$$S_{A,B} \sim \frac{\log 2}{3} \log_2 L_B \simeq 0.231 \log_2 L_B$$



[†] *D. S. Fisher, Phys. Rev. B 50, 3799 (1994).*
[‡] *G. Refael and J. E. Moore, Phys. Rev. Lett. 93, 260602 (2004).*

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Some Applications of SDRG

- Random AF spin-1/2 Heisenberg chain (S. Ma, C. Dasgupta, C. Hu. PRL 43, 1434 (1979))
- Random transverse field Ising model (D. Fisher PRL. 69, 534 (1992))
- Quantum rotor model (E. Altman, Y. Kafri, A. Polkovnikov, G. Refael. Physical Review B 81, 174528 (2010))
- Random AF spin-1 Heisenberg chain with bilinear and biquadratic interactions (V. Quinto, J. Hoyos, E. Miranda. ArXiv:1404.1924 (2014))
- 2 dimensional systems (O. Motrunich, S.-C. Mau, D. A. Huse, and D. S. Fisher, Phys. Rev. B 61, 1160 (2000))
- Recent reviews: G. Refael, and E. Altman, Comptes Rendus Physique 14, 725 (2013). F. Iglói, and C. Monthus, Physics Reports 412 277-431 (2005).

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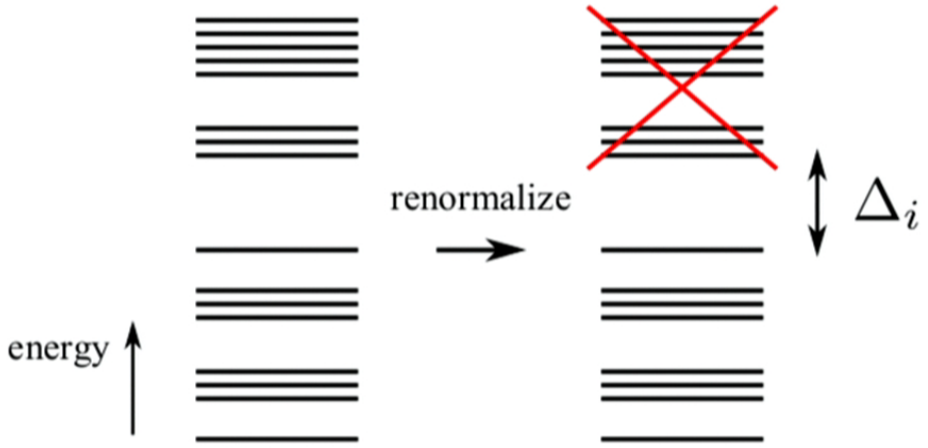
The numerical renormalization group approach

- Numerical renormalization group (NRG) scheme for the FM/AFM chain.
- Keep multiple eigenvectors at each step to increase accuracy.
- Only keep full SU(2) blocks to retain *spin-like* energy spectrum.
- Gap defined by the energy difference between the highest block retained and lowest discarded.



T. Hikihara, A. Furusaki, and M. Sigrist, Phys. Rev. B 60,12116 (1999).

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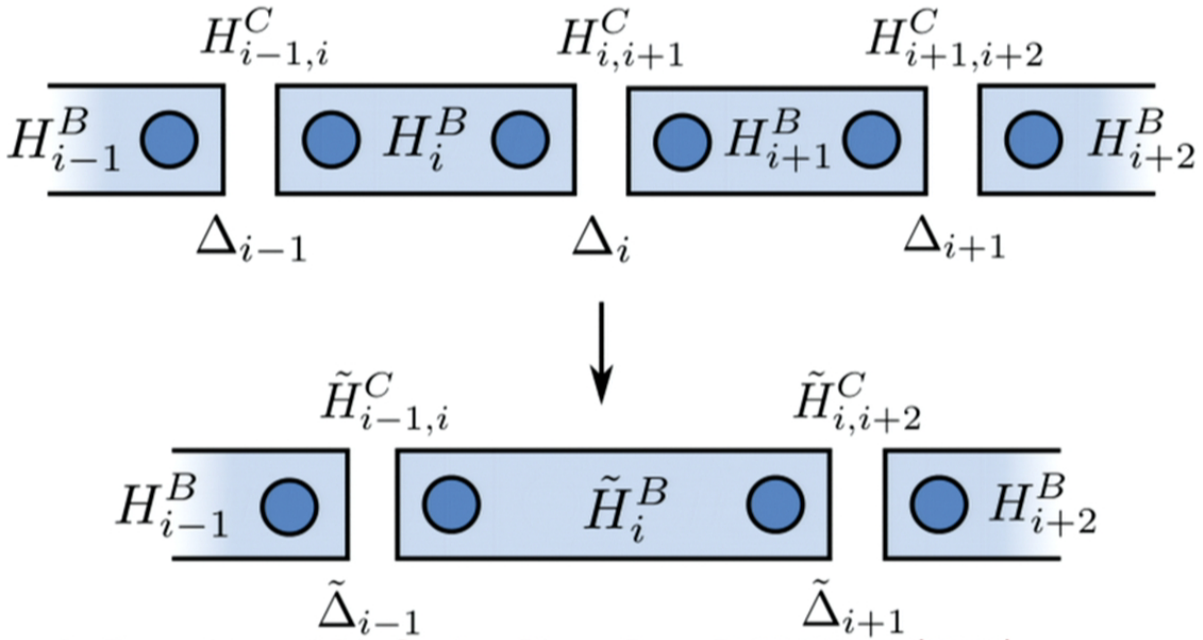
The numerical renormalization group approach

- Combine the blocks with the largest gap into a new block.
- Diagonalize the block keeping multiple eigenvectors.
- Express coupling Hamiltonians in terms of new blocks.
- Diagonalize the couplings to update the distribution of gaps.



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Motivation

- Want to model quantum many-body physics.
- The Hilbert space grows exponentially with system size.

$$\mathcal{H} \sim d^L$$

- Want efficient simulation (polynomial in memory and time)
- Want to be able to model various Hamiltonians (e.g. Bosons and Fermions)
- Want to avoid the *sign problem* of quantum Monte Carlo (M. Troyer, U. Wiese, Phys. Rev. Lett. 94, 170201 (2005))
- Want to measure physical properties and observables

R. Orús, Ann. Phys. 349, 117 (2014)

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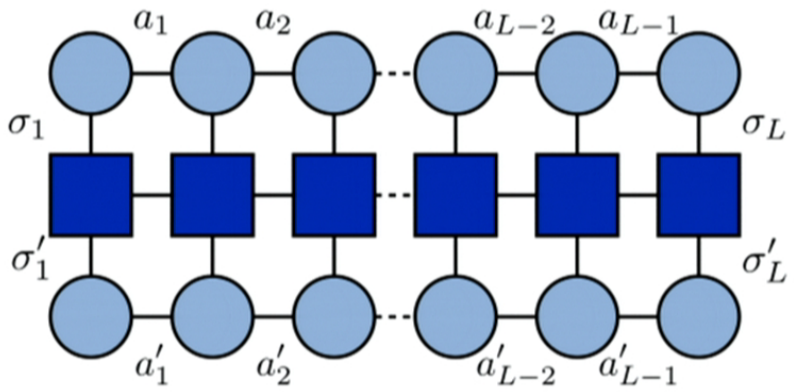
Notation

$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{\sigma'_1,\ldots,\sigma'_L} \sum_{a_1,\ldots,a_L} \sum_{a'_1,\ldots,a'_L} \sum_{b_1,\ldots,b_L}$$

$$A^{*\sigma_1}_{1,a_1} W^{\sigma_1,\sigma'_1}_{1,b_1} A^{\sigma'_1}_{1,a'_1} A^{*\sigma_2}_{a_1,a_2} W^{\sigma_2,\sigma'_2}_{b_1,b_2} A^{\sigma'_2}_{a'_1,a'_2} \cdots$$

$$\cdots A^{*\sigma_{L-1}}_{a_{L-2},a_{L-1}} W^{\sigma_{L-1},\sigma'_{L-1}}_{b_{L-2},b_{L-1}} A^{\sigma'_{L-1}}_{a'_{L-2},a'_{L-1}} A^{*\sigma_L}_{a_{L-1},1} W^{\sigma_L,\sigma'_L}_{b_{L-1},1} A^{\sigma'_L}_{a'_{L-1},1}$$



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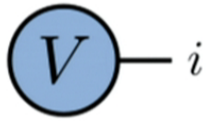
Notation

Tensor networks are often represented using a graphical notation.

Vector:

$$V_i = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

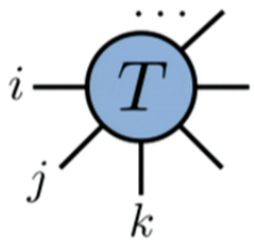


Matrix:

$$M_{ij} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Tensor:

$$T_{ijk\ldots}$$



S. Singh, R. N. C. Pfeifer, and G. Vidal, Phys. Rev. B 83, 115125 (2011)

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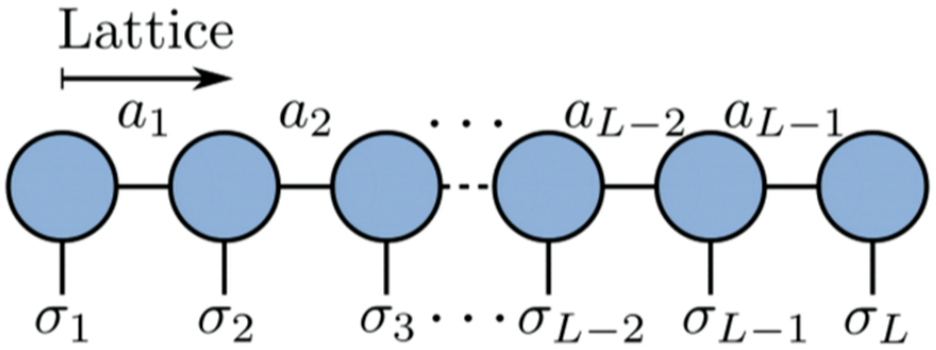
Matrix product states (MPS)

The matrix product state takes the form:

$$|\Psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} \sum_{a_1,\ldots,a_{L-1}} M^{\sigma_1}_{a_1} M^{\sigma_2}_{a_1 a_2} \ldots M^{\sigma_{L-1}}_{a_{L-2} a_{L-1}} M^{\sigma_L}_{a_{L-1}} |\sigma_1\rangle \otimes \cdots \otimes |\sigma_L\rangle$$



Basis of modern density-matrix renormalization group (DMRG) algroithms.

U. Schollwöck, Ann. Phys. 326, 96 (2011).

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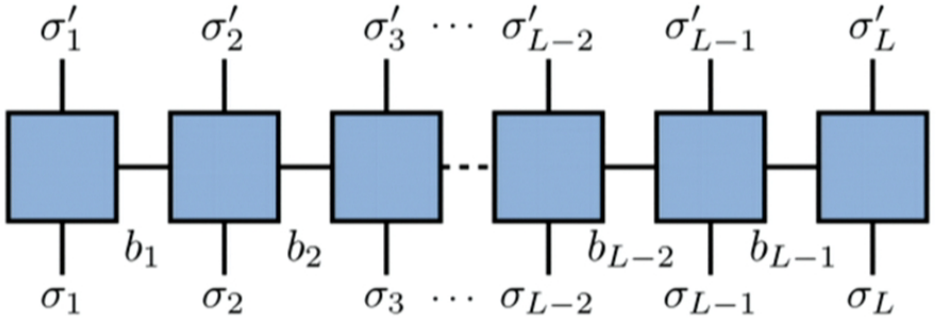
The matrix product operator (MPO)

Operators can also be represented as the product of tensors:

$$\mathcal{O} = \sum_{\substack{\sigma_1,\ldots\sigma_L \\ \sigma'_1,\ldots,\sigma'_L \\ b_1,\ldots,b_{L-1}}} W^{\sigma_1,\sigma'_1}_{b_1} W^{\sigma_2,\sigma'_2}_{b_1,b_2} \ldots W^{\sigma_{L-1},\sigma'_{L-1}}_{b_{L-2},b_{L-1}} W^{\sigma_L,\sigma'_L}_{b_{L-1}} \left|\sigma_1 \ldots \sigma_L\right\rangle \left\langle\sigma'_1 \ldots \sigma'_L\right|$$



I. P. McCulloch, J. Stat. Mech. P10014 (2007)

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Nearest neighbour interactions

The most simple one dimensional nearest-neighbour Hamiltonian with OBCs takes the form:

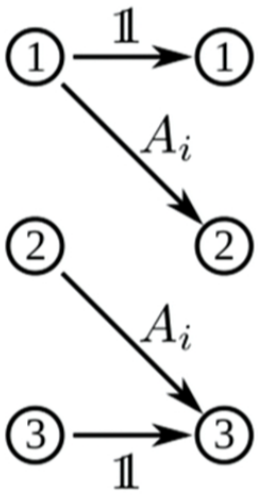
$$H = \sum_{i=1}^{L-1} A_i A_{i+1}$$

which can be written as a MPO of the form:

$$W^{[1]}_{1,b_1} = \begin{pmatrix} \mathbb{1} & A_1 & 0 \end{pmatrix},$$

$$W^{[i]}_{b_{i-1},b_i} = \begin{pmatrix} \mathbb{1} & A_i & 0 \\ 0 & 0 & A_i \\ 0 & 0 & \mathbb{1} \end{pmatrix},$$

$$W^{[L]}_{b_{L-1},1} = \begin{pmatrix} 0 \\ A_L \\ \mathbb{1} \end{pmatrix}.$$



G. M. Crosswhite and D. Bacon, Phys. Rev. A 78, 012356 (2008).

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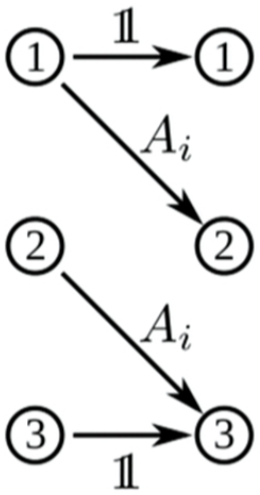

G. M. Crosswhite and D. Bacon, Phys. Rev. A 78, 012356 (2008).

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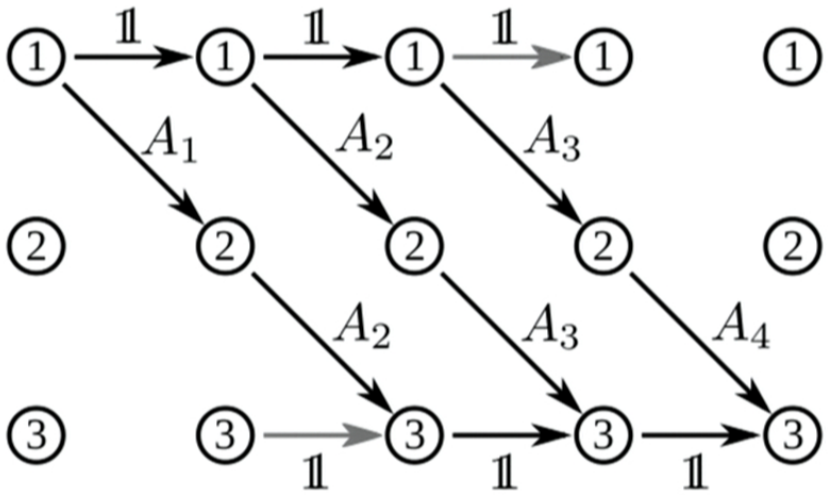
Contracting the MPO

Contracting the MPO along the virtual indices produces the desired Hamiltonian.



$$H = A_1A_2 + A_2A_3 + A_3A_4$$

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On-site terms

On-site terms encode on-site potentials and magnetic fields. E.g.

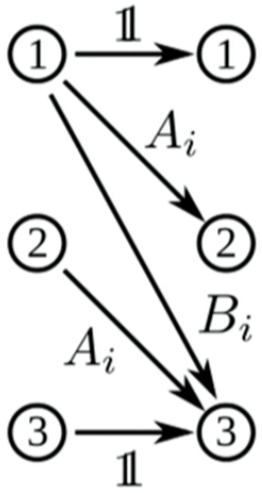
$$H = \sum_{i=1}^{L-1} A_i A_{i+1} + \sum_{i=1}^{L} B_i$$

where B_i are the on site terms.

$$W^{[1]}_{1,b_1} = \begin{pmatrix} \mathbb{1} & A_1 & B_1 \end{pmatrix}$$

$$W^{[i]}_{b_{i-1},b_i} = \begin{pmatrix} \mathbb{1} & A_i & B_i \\ 0 & 0 & A_i \\ 0 & 0 & \mathbb{1} \end{pmatrix}$$

$$W^{[L]}_{b_{L-1},1} = \begin{pmatrix} B_L \\ A_L \\ \mathbb{1} \end{pmatrix}$$



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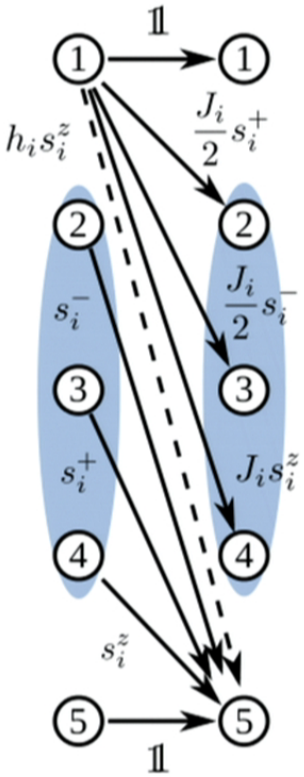
Heisenberg MPO Hamiltonian

- Encode the Heisenberg Hamiltonian into an MPO:

$$H = \sum_{i=1}^{L-1} J_i \vec{s}_i \cdot \vec{s}_{i+1}$$

$$W_{b_{i-1},b_i} = \begin{pmatrix} \mathbb{1} & \frac{J_i}{2} s_i^+ & \frac{J_i}{2} s_i^- & J_i s_i^z & 0 \\ 0 & 0 & 0 & 0 & s_i^- \\ 0 & 0 & 0 & 0 & s_i^+ \\ 0 & 0 & 0 & 0 & s_i^z \\ 0 & 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$



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- e.g. $W_{b_2,b_3}W_{b_3,b_4}$

$$W_{b_2,b_3}W_{b_3,b_4} = \begin{pmatrix} \mathbb{1} & \frac{J_4}{2}s_4^+ & \frac{J_4}{2}s_4^- & J_4 s_4^z & \frac{J_3}{2}s_3^+s_4^- + \frac{J_3}{2}s_3^-s_4^+ + J_3 s_3^z s_4^z \\ 0 & 0 & 0 & 0 & s_3^- \\ 0 & 0 & 0 & 0 & s_3^+ \\ 0 & 0 & 0 & 0 & s_3^z \\ 0 & 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

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Heisenberg MPO Hamiltonian

- Encode the Hamiltonian into an MPO:

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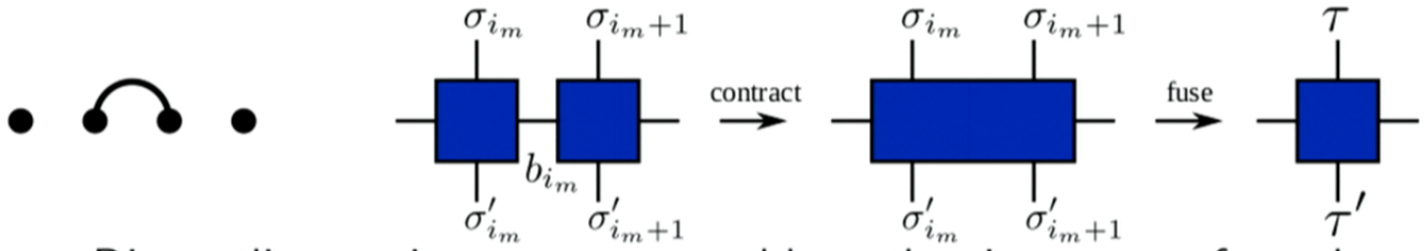
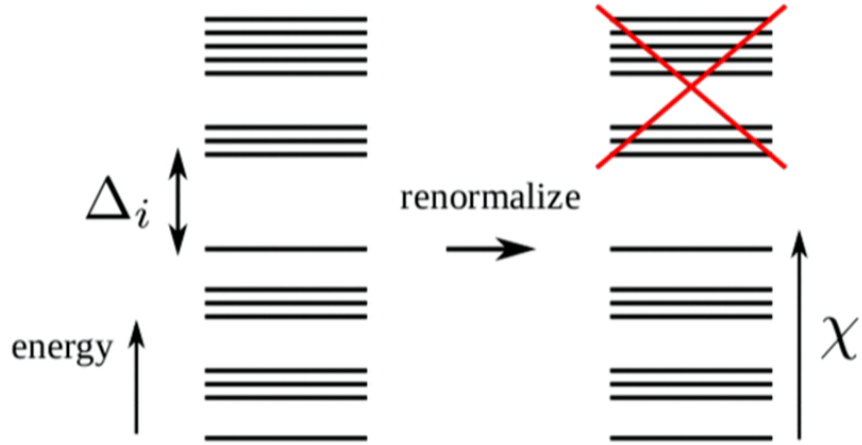
SDRG on matrix product operators

- Contract MPO tensors with largest gap



- Diagonalize on site component and keep the eigenvectors from the lowest χ eigenvalues but keeping only full SU(2) blocks.



A. M. Goldsborough and R. A. Römer, Phys. Rev. B 89, 214203 (2014)

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- e.g. $W_{b_2,b_3}W_{b_3,b_4}$

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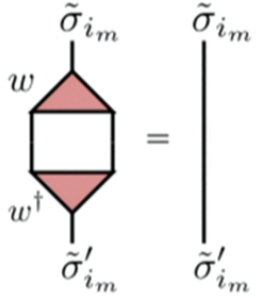
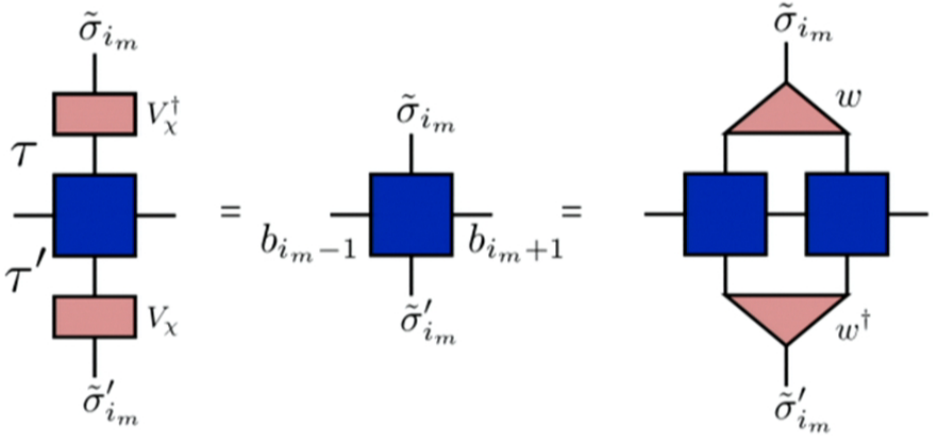
SDRG on matrix product operators

- Eigenvectors from diagonalization create an isometric tensor



- Contract to perform renormalization on couplings



- Diagonalize couplings to update distribution of gaps

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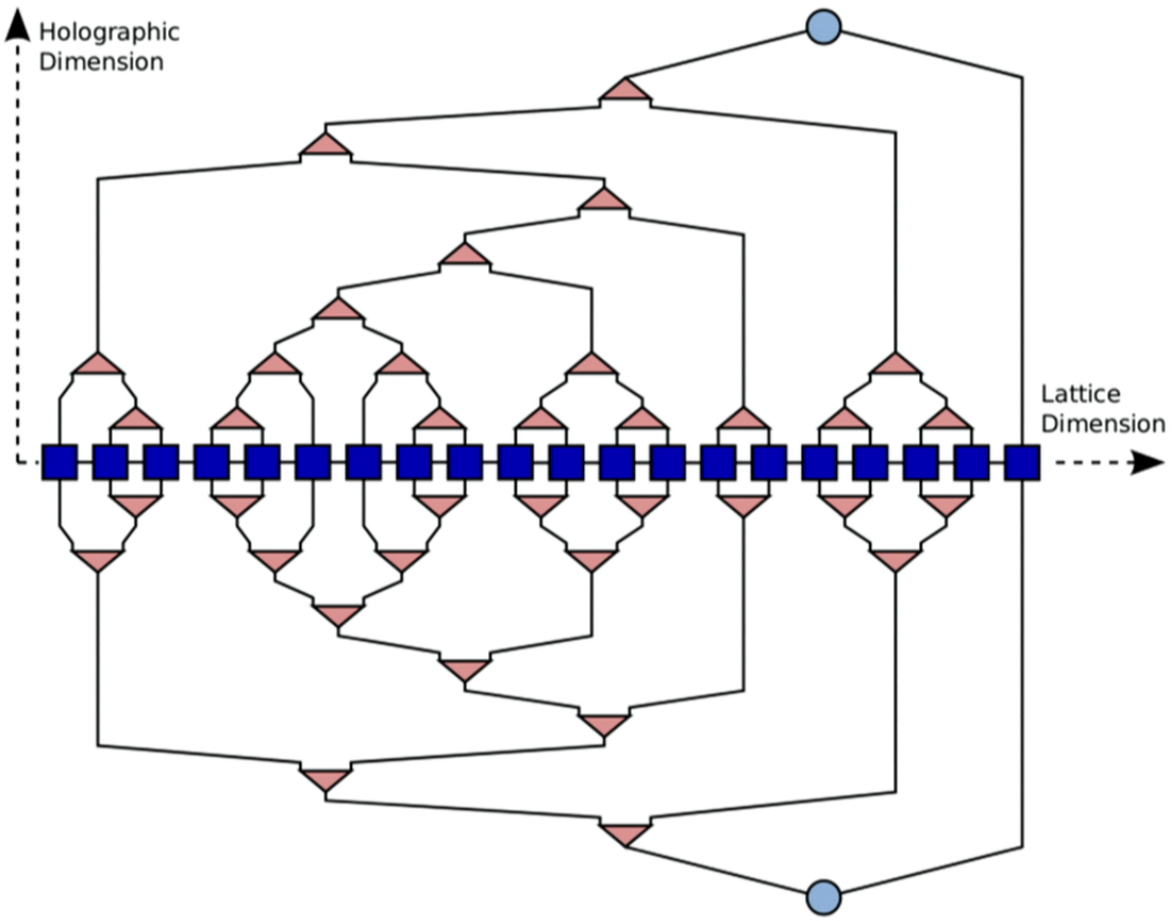
SDRG as a tree tensor network



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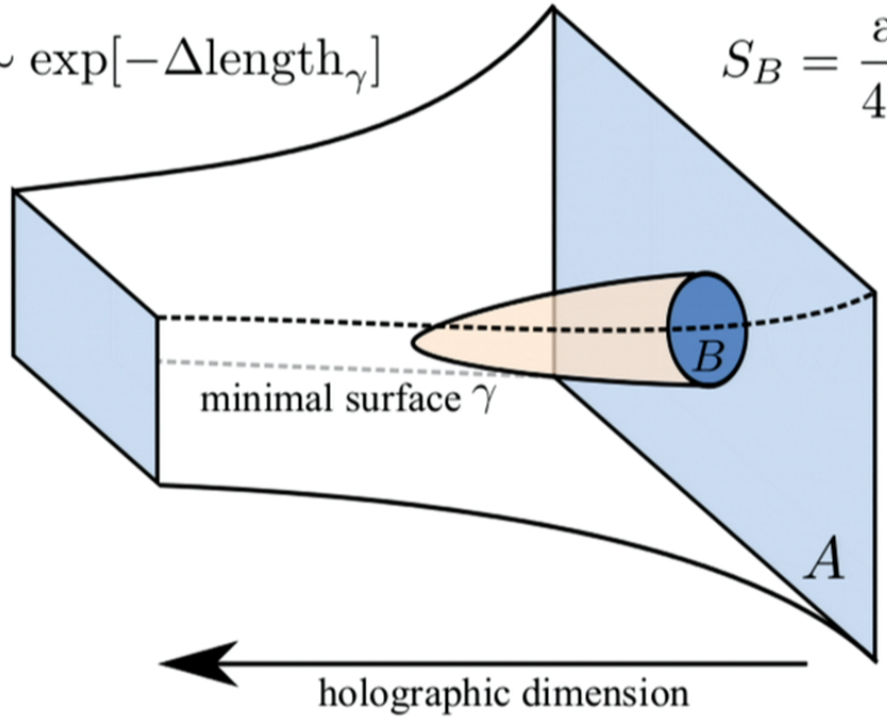
Holography and the AdS/CFT correspondence

$$\langle \mathcal{O}(x_i)\mathcal{O}(x_j)\rangle \sim \exp[-\Delta \text{length}_\gamma]$$

$$S_B = \frac{\text{area}_\gamma}{4G_N^{d+2}}$$



S. Ryu, and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006)
B. Swingle, Phys. Rev. D 86, 065007 (2012)
G. Evenbly, and G. Vidal, J. Stat. Phys. 145, 891 (2011)

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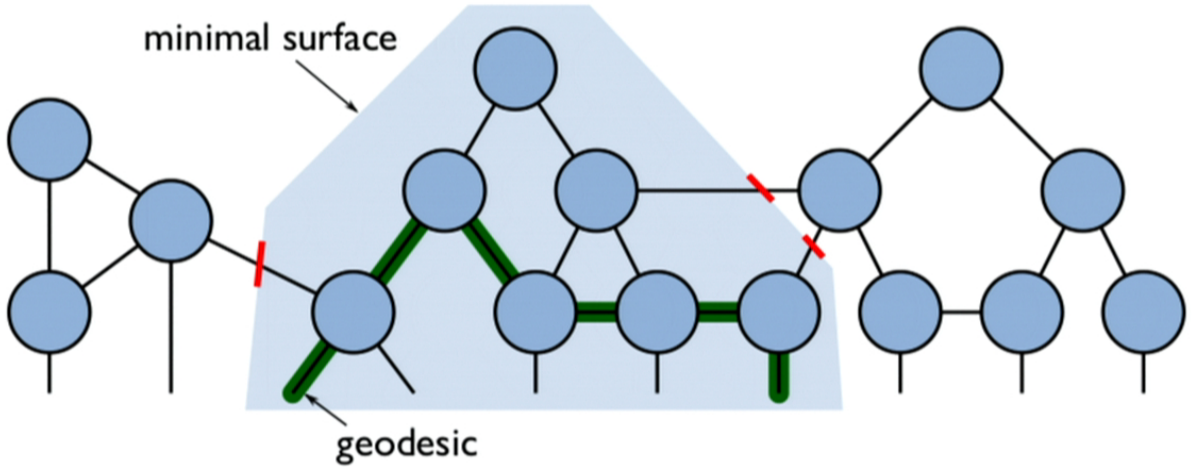
Geometry and tensor networks



Entanglement scales with minimal surface separating regions A and B:

$$S_{A,B} \propto \partial A.$$

Correlations scale with the length of the geodesic connecting the sites:

$$C(x_1, x_2) \propto e^{-\alpha D(x_1, x_2)}$$

G. Evenbly and G. Vidal, J. Stat. Phys., 145, 891, 2011.

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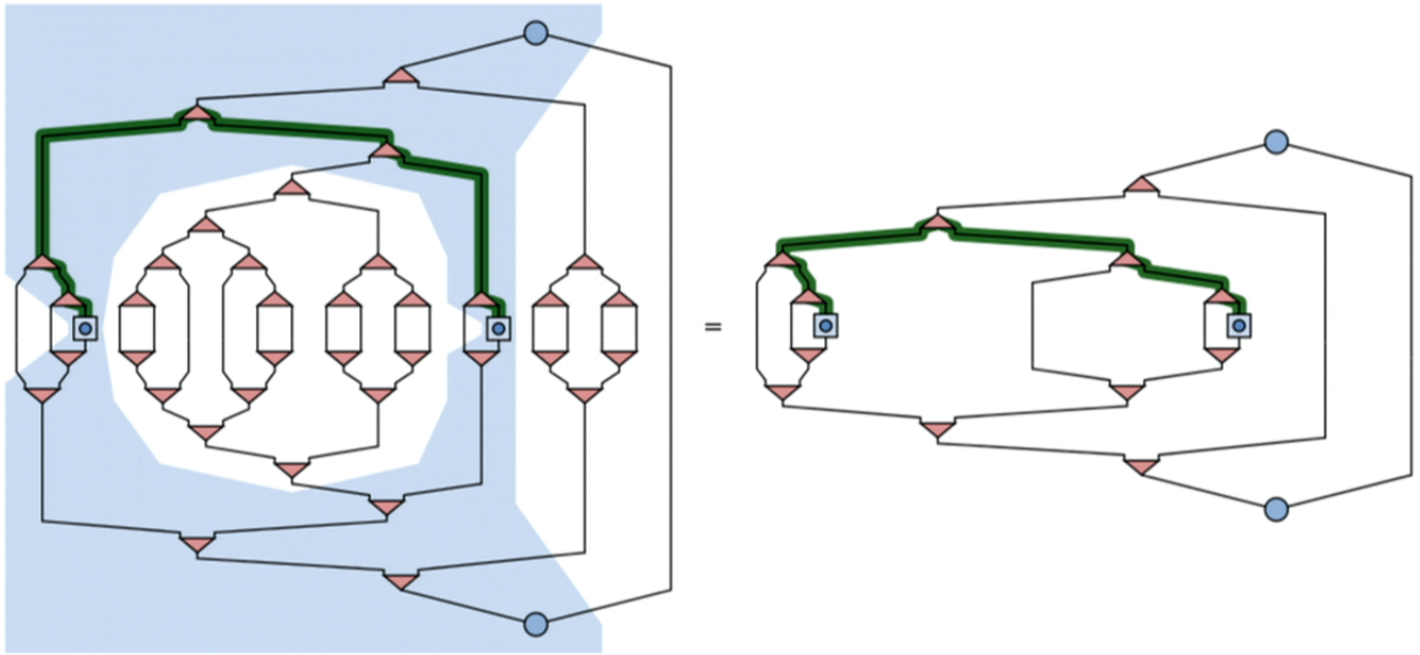
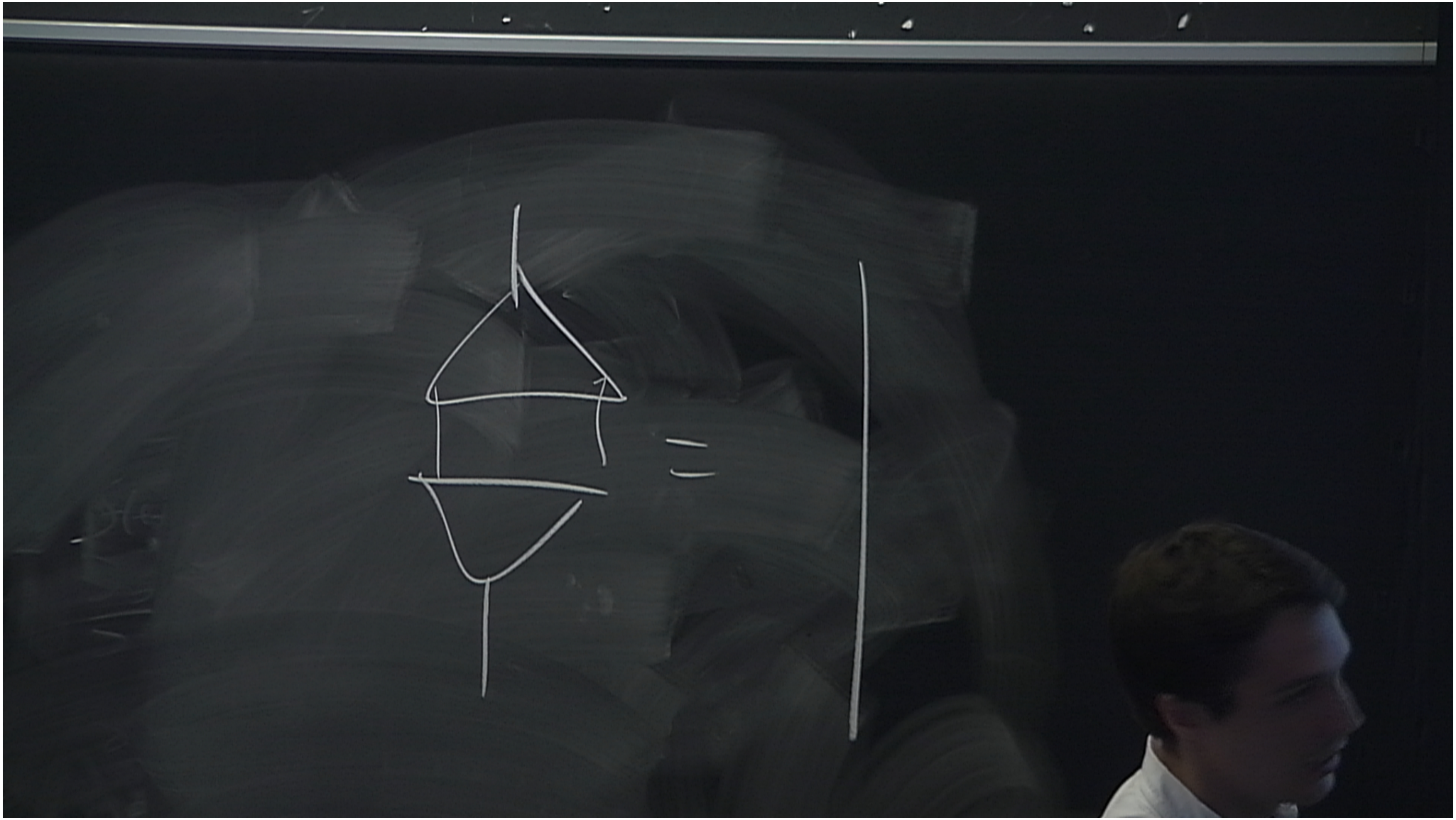
Correlation Functions

- Only need to contract the tensors in the *causal cone*





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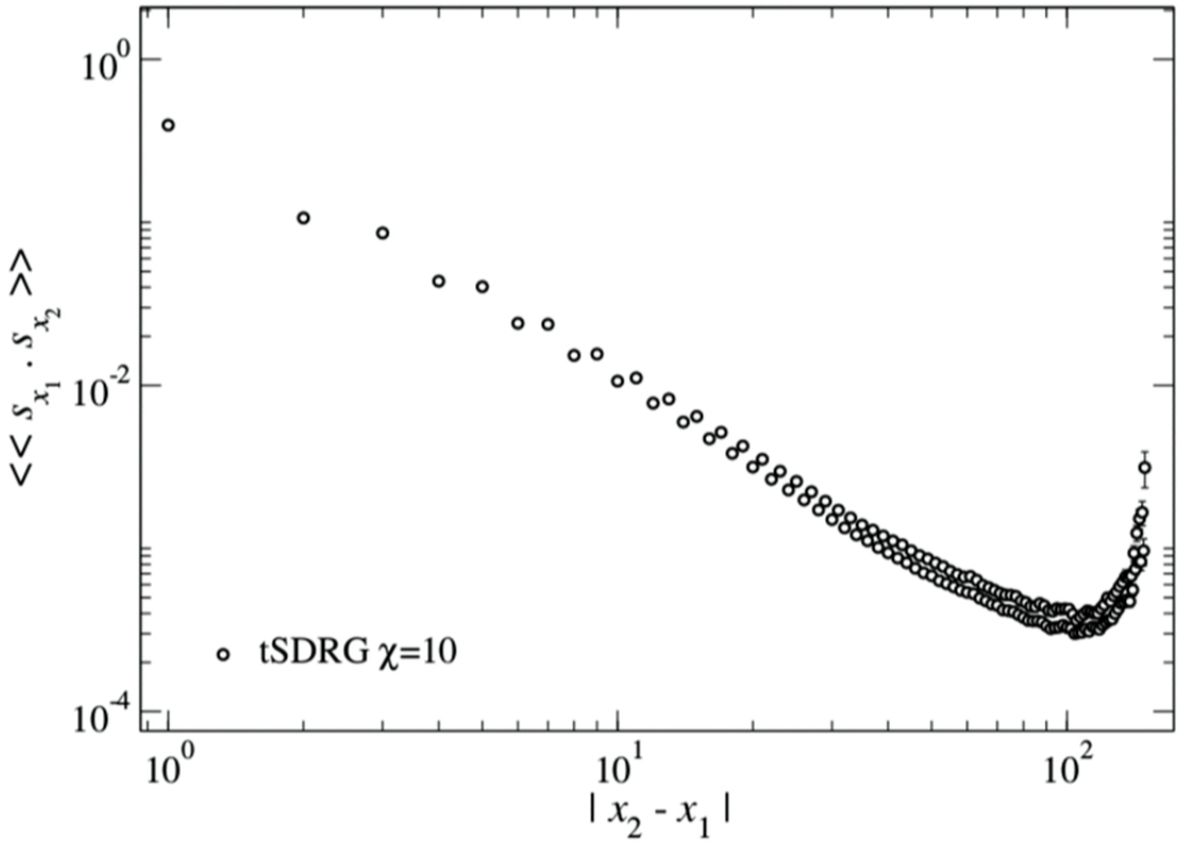
Correlation functions

L=150, AFM $0 < J_i < 2$, 2000 disorder realizations, $\chi = 10$.



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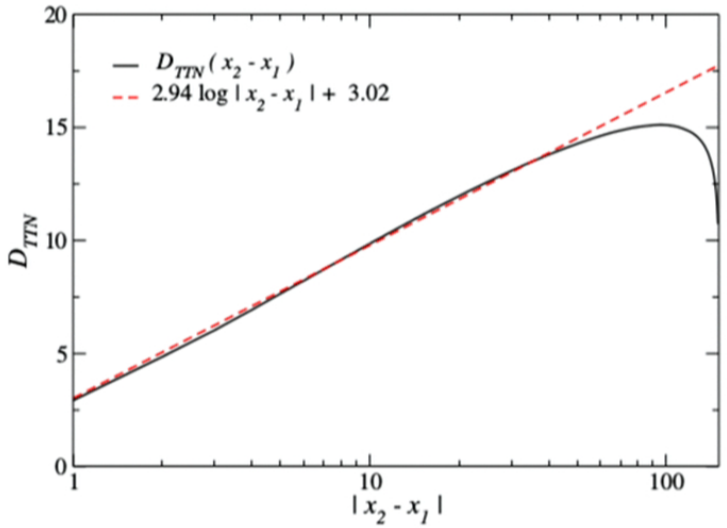
Correlation Functions

- SDRG prediction† of even-odd difference by a factor of 1.25.
- Two-point correlation function is related to the average path length joining the two points.
- Proposal:

$$\langle\langle \vec{s}_i \cdot \vec{s}_j \rangle\rangle \propto \langle e^{-\alpha D(x_1,x_2)} \rangle \sim e^{-\alpha \langle D(x_1,x_2) \rangle} \propto e^{-a \log|x_2 - x_1|} \propto |x_2 - x_1|^a$$



† *J. Hoyos, A. Vieira, N. Laflorencie, E. Miranda, Phys. Rev. B 76, 174425 (2007).*

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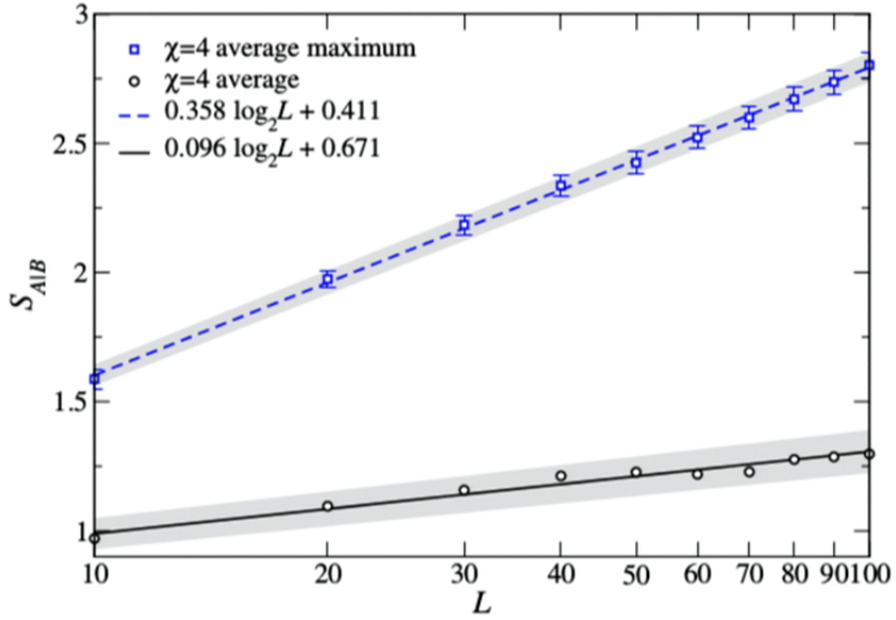
Entanglement entropy for bipartitions ($S_{A|B}$)

- Average and average maximum entanglement entropy scale logarithmically with system size
- Need to increase bond dimension with length for DMRG to capture the entanglement



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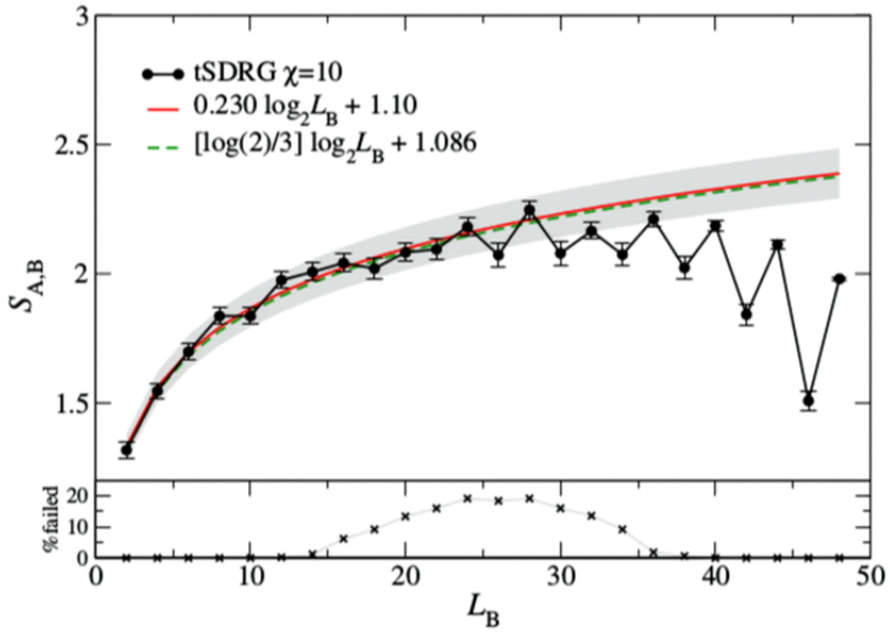
Entanglement entropy for blocks ($S_{A,B}$)

- Rafael/Moore†: effective CFT with central charge $\tilde{c} = 1 \cdot \log 2$ and

$$S_{A,B} \simeq \frac{\log 2}{3} \log_2 L_B \simeq 0.231 \log L_B$$



† *G. Refael and J. E. Moore, Phys. Rev. Lett. 93, 260602 (2004).*

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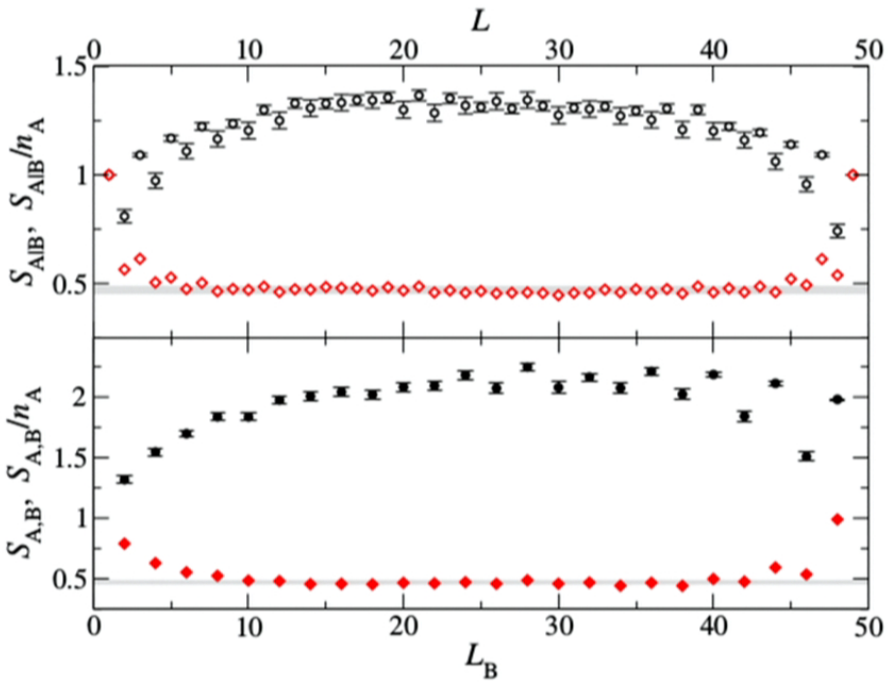
Entanglement per bond

- entanglement per bond the same for both bipartitions and blocks

$$\frac{S}{n_A} = 0.47 \pm 0.02$$



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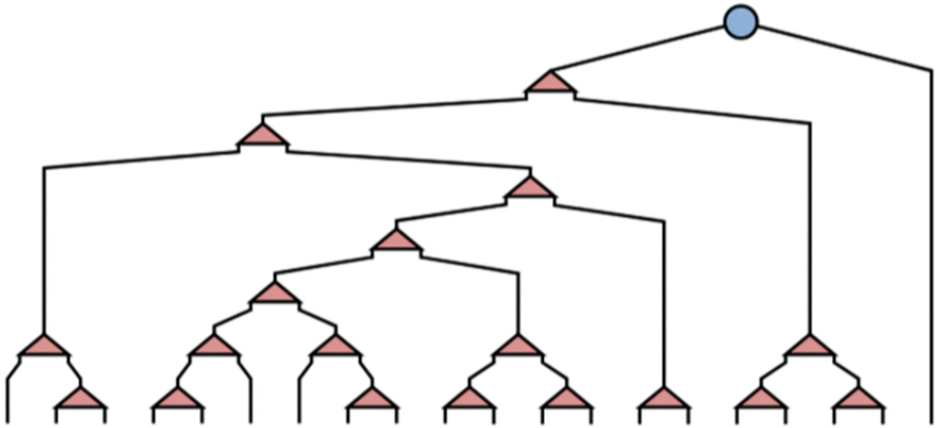
Conclusions

- We have made a self constructing, inhomogeneous tensor network based on SDRG.
- We reproduce the power law mean correlation function and logarithmic entanglement entropy scaling.
- The correlation functions and entanglement entropy support the notion of a holographic tensor network state.
- Outlook: Random spin-1, spin-3/2 Heisenberg model.



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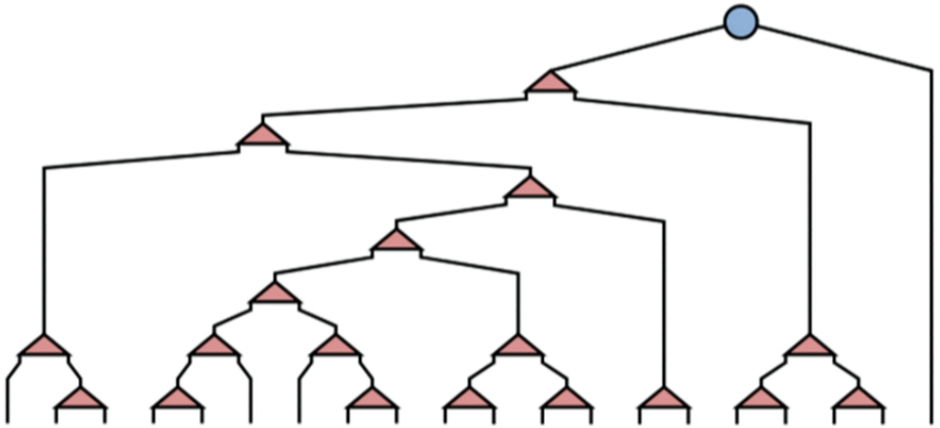

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Ongoing work

- ITensor DMRG for disordered systems (itensor.org)
- Phase diagram for finite disordered Bose-Hubbard model using DMRG
- Path lengths in regular m-ary tree graphs (arXiv:1406.4079)
- Path lengths in Catalan tree graphs (in preparation)
- Random spin-1, spin-3/2 Heisenberg model
- Excited state DMRG for localized states

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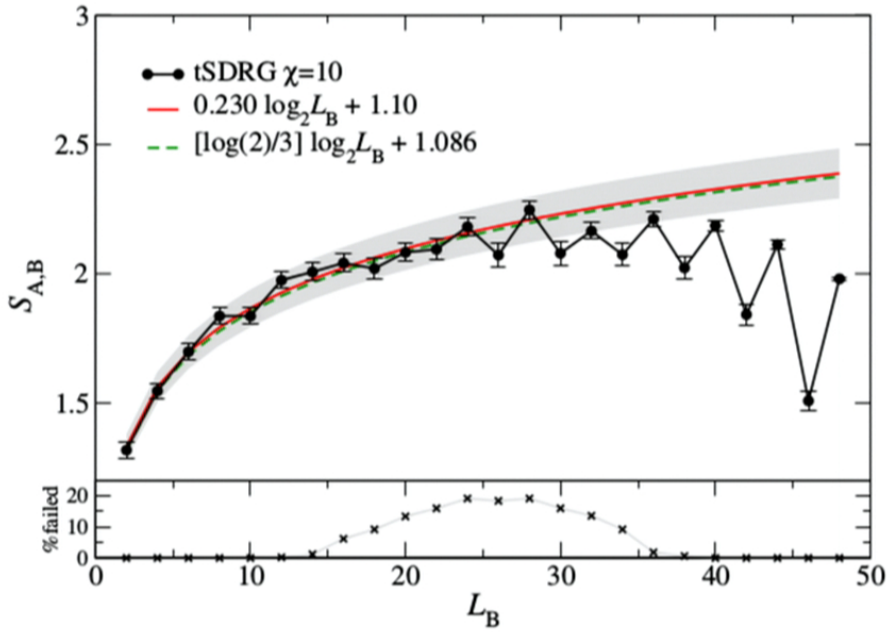

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