

Title: 14/15 PSI - Relativity- Lecture 11

Date: Sep 22, 2014 09:00 AM

URL: <http://pirsa.org/14090025>

Abstract:

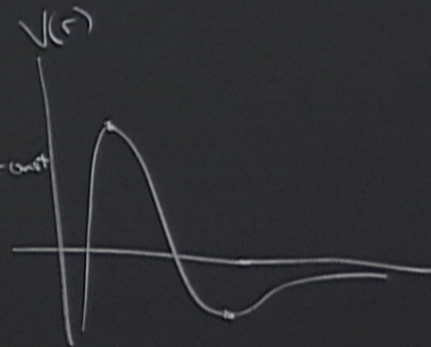
$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$E, J = r^2 \dot{\phi}$$

$$\dot{r} = \frac{dr}{d\tau} \Rightarrow \dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = \text{const.}$$



$$x^2 + cx = \left(x + \frac{c}{2}\right)^2 + \text{const}$$



$$\dot{x}^2 + x^2 = \text{const}$$

$$\Rightarrow r(\phi) \quad \dot{r} = \frac{dr}{d\phi}$$

$$r' = \frac{\dot{r}}{\dot{\phi}} = \frac{\dot{r} r^2}{J}$$

$$\Rightarrow J^2 \frac{r'^2}{r^4} - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = \text{const}$$

$$u \equiv \frac{1}{r} \Rightarrow \boxed{u'^2 - \frac{r_s u}{J^2} + u^2 - r_s u^3 = \text{const}}$$

$$\Delta \text{Newton } u = \frac{r_s}{2J^2} (1 + e \cos \phi) \quad \text{oscillation about minimum}$$

$$\dot{x}^2 + x^2 = \text{const}$$

to x

→ Newton

$$u = \frac{\sqrt{s}}{2J^2} \left(1 + e \cos \varphi \right)$$

min of SHO potn. l

↓ oscillation about minimum

const

min of SHO potn. l

oscillation
about

→ Newton $u = \frac{r_s}{2J^2} (1 + e \cos \varphi)$ minimum

$= u_0$, Newtonian solution.

$$\Rightarrow \frac{J^2 r'^2}{r^4} - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = \text{const}$$

$$u = \frac{1}{r} \Rightarrow \boxed{u'^2 - \frac{r_s u}{J^2} + u^2 - r_s u^3 = \text{const}}$$

SHO potnl
↓ oscillation
about
 $e \cos \varphi$ minimum

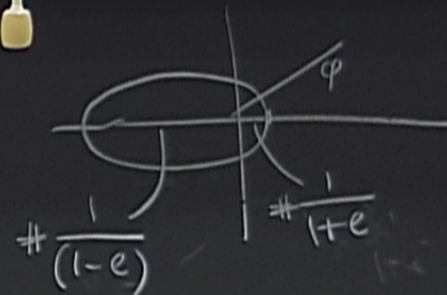
orbits periodic in φ .

Newtonian solution.

↓
perturbation to the potential $V(u)$
which causes orbit to precess.

$V(u)$
process.

$$r = \frac{2J^2}{\Gamma_S(1 + e \cos \varphi)}$$



$u \equiv \frac{1}{r} \Rightarrow \boxed{u'^2 - \frac{r_s}{J^2} u + u^2 - \frac{r_s}{J^2} u^3 = \text{const}} \quad *$

Newton $u = \frac{r_s}{2J^2}(1 + e \cos \varphi)$ minimum
 $= u_0$, Newtonian solution.

↓
 perturbation to the potential $V(u)$
 which causes orbit to precess.

in fact, perturbation in the solution grows with φ

$\frac{d}{d\varphi} (*) \quad u'' = - \left(u - \frac{r_s}{2J^2} \right) + \underbrace{\frac{3r_s}{2} u^2}_{\text{small}}$

set $u = u_0 + \delta u$, solve to linear order in δu

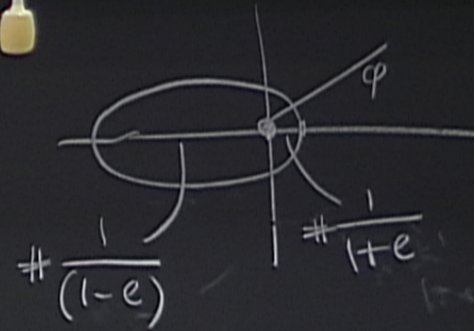
$\delta u'' = -\delta u + \frac{3}{2} \frac{r_s^2}{J^4} u_0^2 \Rightarrow \delta u'' = -\delta u + \frac{3}{8} \frac{r_s^3}{J^4} (1 + e \cos \varphi)^2$

resonance

$(1 + 2e \cos \varphi + e^2 \cos^2 \varphi)$
 $\downarrow$
 $w=3, w=0$

potential $V(u)$
it to precess.

$$r = \frac{2J^2}{r_s(1 + e \cos \varphi)}$$



resonance
↑↑

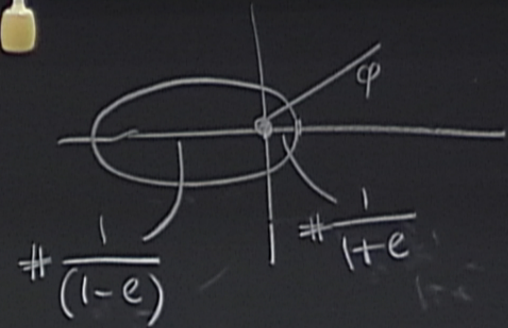
$$\frac{3}{8} \frac{r_s^3}{J^4} (1 + 2e \cos \varphi + e^2 \underbrace{\cos^2 \varphi}_{\substack{\downarrow \\ \ell=2, \ell=0}})$$

↗

$$\Rightarrow \delta u'' = -\delta u + \frac{3}{4} \frac{r_s^3}{J^4} e \cos \varphi$$

$V(u)$
recess.

$$r = \frac{2J^2}{r_s(1 + e \cos \varphi)}$$



resonance

$$\uparrow \uparrow$$

$$(1 + 2e \cos \varphi + e^2 \underbrace{\cos^2 \varphi}_{\substack{\downarrow \\ w=2, w=0}})$$

$\cos \varphi)^2$

$$\delta u'' = -\delta u + \frac{3}{4} \frac{r_s^3}{J^4} e \cos \varphi$$

$$\Rightarrow \text{PI } \delta u = \frac{3 r_s^3}{8 J^4} e \varphi \sin \varphi$$

Set $u = u_0 + \delta u$, solve to linear order in δu small

$$\delta u'' = -\delta u + \frac{3}{2} \frac{r_s^2}{J^2} u_0^2 \Rightarrow \delta u'' = -\delta u + \frac{3}{8} \frac{r_s^3}{J^4} (1 + e \cos \varphi)^2$$

$(1 + 2e \cos \varphi + e^2 \cos^2 \varphi)$
 $\downarrow$
 $w=2, w=0$

$$\Rightarrow u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin \varphi$$

secular growing term.

$$\approx \frac{r_s}{2J^2} (1 + e \cos((1 - \epsilon)\varphi)) \quad , \text{ where } \epsilon = \frac{3r_s^2}{4J^2}$$

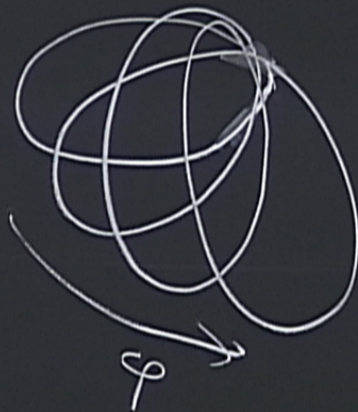
$$\Rightarrow u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin \varphi$$

secular growing term.

$$\approx \frac{r_s}{2J^2} \left(1 + e \underbrace{\cos((1-\epsilon)\varphi)}_{\cos \varphi + \epsilon \varphi \sin \varphi} \right), \text{ where } \epsilon = \frac{3r_s^2}{4J^2}$$

angular frequency has changed: for the orbit to come back to the

$$(1-\epsilon)(1+\epsilon)2\pi \approx 2\pi \text{ up to order } \epsilon^2$$



$$\Delta\phi / \text{orbit} = 2\pi\epsilon.$$

$$\Rightarrow u = \frac{r_s}{2J^2} (1 + e \cos \phi)$$

$$\approx \frac{r_s}{2J^2} (1 + e \underbrace{\cos}_{\cos})$$

$\Rightarrow$ angular frequency has changed

$$(1-\epsilon)(1+\epsilon) \cdot 2\pi \approx 2\pi \text{ up to } \epsilon^2$$

$$u + \frac{3\gamma_s}{8J^4} (1 + e \cos \varphi)$$

$$\frac{3\gamma_s^3}{8J^4} e \varphi \sin \varphi$$

secular growing term.

$$e \varphi) \Big), \text{ where } \epsilon = \frac{3\gamma_s^2}{4J^2}$$

$$e \varphi \sin \varphi$$

for the orbit to come back to the same value of u , φ must advance by $2\pi(1+\epsilon)$

For Mercury,

$$\epsilon^2: \text{ For Mercury, } \Delta\varphi = 5 \times 10^{-7} \text{ radians per orbit}$$

415 orbits per year

perihelion advances by $43''$ per century.

$$\frac{15}{J^4} (1 + e \cos \varphi)$$

$p \sin \varphi$
secular growing term.

, where $\epsilon = \frac{3\Gamma_3^2}{4J^2}$

the orbit to come back to the same value of u , φ must advance by $2\pi(1+\epsilon)$

For Mercury,

For Mercury, $\Delta\varphi = 5 \times 10^{-7}$ radians per orbit

415 orbits per century.

perihelion advances by $43''$ per century.

The Action for GR

Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

↑ only scalar involving two derivatives of metric, representing the curvature of spacetime.

$$\delta S = S[g + \delta g] - S[g]$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\delta(\sqrt{-g} R) = \delta(\sqrt{-g}) R + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}$$

$$\begin{array}{c}
 R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} \\
 \chi^3 \quad f(R) \quad \underbrace{\text{const} + R}_{\text{}} + \underbrace{\alpha R^2}_{\substack{\downarrow \\ R_{\mu\nu} R^{\mu\nu}}} + \beta R^3
 \end{array}$$

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

only scalar involving two derivatives of metric,
representing the curvature of spacetime.

$$\delta S = S[g + \delta g] - S[g]$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\delta(\sqrt{-g} R) = \underbrace{\delta(\sqrt{-g}) R + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu}}_{\text{easy, leave to later}} + \underbrace{\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}}_{\text{complicated.}}$$

easy, leave to later

complicated.

$\delta R_{\mu\nu}$:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu}$$

$$\delta R_{\mu\nu} = \delta \Gamma^\lambda_{\mu\lambda,\nu} - \delta \Gamma^\lambda_{\mu\lambda,\nu} + \delta \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\lambda} + \Gamma^\alpha_{\mu\nu} \delta \Gamma^\lambda_{\alpha\lambda} \\ - \delta \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu} - \Gamma^\alpha_{\mu\lambda} \delta \Gamma^\lambda_{\alpha\nu}$$

easy, leave to later

complicated.

$\delta R_{\mu\nu}$:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu}$$

$$\delta R_{\mu\nu} = \delta \Gamma^\lambda_{\mu\nu,\lambda} - \delta \Gamma^\lambda_{\mu\lambda,\nu} + \delta \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\lambda} + \Gamma^\alpha_{\mu\nu} \delta \Gamma^\lambda_{\alpha\lambda} - \delta \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu} - \Gamma^\alpha_{\mu\lambda} \delta \Gamma^\lambda_{\alpha\nu}$$

$$= (\delta \Gamma^\alpha_{\mu\nu})_{;\alpha} - (\delta \Gamma^\lambda_{\mu\lambda})_{;\nu} \quad \text{Palatini identity}$$

easy, leave to later

complicated

$\delta R_{\mu\nu}$:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \Gamma_{\mu\nu,\lambda}^\lambda - \Gamma_{\mu\lambda,\nu}^\lambda + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\lambda}^\lambda - \Gamma_{\mu\lambda}^\alpha \Gamma_{\alpha\nu}^\lambda$$

Γ not a tensor

$$\delta R_{\mu\nu} = \delta \Gamma_{\mu\nu,\lambda}^\lambda - \delta \Gamma_{\mu\lambda,\nu}^\lambda + \delta \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\lambda}^\lambda + \Gamma_{\mu\nu}^\alpha \delta \Gamma_{\alpha\lambda}^\lambda - \delta \Gamma_{\mu\lambda}^\alpha \Gamma_{\alpha\nu}^\lambda - \Gamma_{\mu\lambda}^\alpha \delta \Gamma_{\alpha\nu}^\lambda$$

$\delta \Gamma$

$$= (\delta \Gamma_{\mu\nu}^\alpha)_{;\alpha} - (\delta \Gamma_{\mu\lambda}^\lambda)_{;\nu} \quad \text{Palatini identity}$$

: both terms of form.

$$\Rightarrow \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \sqrt{-g} (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha)_{;\alpha} - \sqrt{-g} (g^{\mu\nu} \delta \Gamma_{\mu\lambda}^\lambda)_{;\nu}$$

$$\begin{aligned}
 & R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} \\
 & f(R) \quad \underbrace{\text{const} + R + \alpha R^2 + \beta R^3}_{R_{\mu\nu} R^{\mu\nu}} \cdot \boxed{\Gamma^\alpha_{\beta\alpha} = \frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g})}
 \end{aligned}$$

Now use $V^\alpha_{;\alpha} = V^\alpha_{,\alpha} + \Gamma^\alpha_{\beta\alpha} V^\beta$

both terms of form $\sqrt{-g} (V^\alpha)_{;\alpha}$

$$R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}$$

$$f(R)$$

$$\underbrace{\text{const} + R + \alpha R^2 + \beta R^3}_{R_{\mu\nu} R^{\mu\nu}} - \boxed{\Gamma^\alpha_{\beta\alpha} = \frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g})}$$

Now use $V^\alpha_{;\alpha} = V^\alpha_{;\alpha} + \Gamma^\alpha_{\beta\alpha} V^\beta = V^\alpha_{;\alpha} + \frac{\partial_\beta \sqrt{-g}}{\sqrt{-g}} V^\beta$

$$= \frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g} V^\beta)$$

both terms

$$\int d^4x \partial_\mu V^\mu$$

$$= (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^\lambda)_{,\lambda} - (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\lambda}^\lambda)_{,\nu}$$

$\uparrow$
 ordinary
 derivative

i.e. it is a total divergence, in ordinary sense!

$$\int_{V_4} d^4x \partial_\mu V^\mu$$

// divergence theorem

$$\int_{S_3} V^\mu dS_\mu$$

$$= (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^\alpha)_{,\alpha} - (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\lambda}^\lambda)_{,\nu}$$

↑
ordinary derivative

i.e. it is a total divergence, in ordinary sense!

$\Rightarrow$ get only surface terms.

$$= (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^{\lambda})_{,\lambda} - (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\lambda}^{\lambda})_{,\nu}$$

↑
ordinary derivative

$$\int_{\mathbb{R}^4} d^4x \partial_\mu V^\mu$$

// divergence theorem

$$\int_{S_3} V^\mu dS_\mu$$

i.e. it is a total divergence, in ordinary sense!

$\Rightarrow$ get only surface terms, can be neglected

$$\Rightarrow \delta S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \underbrace{\left(\frac{1}{2} g^{\mu\nu} R - g^{\mu\nu} R \right)}_{=-G^{\mu\nu}} \delta g_{\mu\nu}$$

$$\delta \sqrt{-g} = \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu}$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha} \delta g_{\alpha\beta} g^{\beta\nu}$$

follows from $g^{\mu\nu} g_{\nu\mu} = \delta^\mu_\mu$

$$\delta S = 0 \Rightarrow G^{\mu\nu} = 0. \quad \text{Einstein eq's}$$

Now include matter in the action: $S = S_G + S_M$

$$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

$$R) \delta g_{\mu\nu}$$

1. Einstein eqⁿs

$$S = S_G + S_M[g, \underbrace{\phi, A, \psi}_{\text{all other fields}}]$$

$$\underbrace{S_G}_{\frac{1}{16\pi G} \int \sqrt{g} R}$$

$$\delta S = 0 \Rightarrow G^{\mu\nu} = 0. \quad \text{Einstein eq's}$$

matter in the action:

$$S_{\text{total}} = S_G + S_M[g, \underbrace{\phi, A, \psi}_{\text{all other fields}}]$$

$$\quad \quad \quad \frac{1}{16\pi G} \int \sqrt{-g} R$$

$$\Rightarrow \delta S_{\text{total}} = \int d^4x \sqrt{-g} \left(-\frac{G^{\mu\nu}}{16\pi G} \delta g_{\mu\nu} + \underbrace{\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}}_{\text{definition of } T^{\mu\nu}} \right) = 0 \Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad \text{Einstein equation with matter}$$

$$\left(-\frac{G^{\mu\nu}}{16\pi G} \delta g_{\mu\nu} + \underbrace{\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}}_{\text{definition of } T^{\mu\nu}} \right) = 0 \Rightarrow G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad \text{Einstein equation with matter.}$$

So Bianchi identities $\nabla_{\mu} G^{\mu\nu} = 0$
covariant
 imply conservation of stress-energy

$$\nabla_{\mu} T^{\mu\nu} = 0.$$

$$\left(-\frac{G^{\mu\nu}}{16\pi G} \delta g_{\mu\nu} + \underbrace{\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}}_{\text{definition of } T^{\mu\nu}} \right) = 0$$

$$\Rightarrow \boxed{G^{\mu\nu} = 8\pi G T^{\mu\nu}}$$

Einstein
equation
with matter

So Bianchi identities $\nabla_{\mu} G^{\mu\nu} = 0$
covariant
 imply conservation of stress-energy

$$\text{or } \nabla_{\mu} T^{\mu\nu} = 0$$

$$\Rightarrow \delta S_{\text{total}} = \int d^4x \sqrt{-g} \left(-\frac{G^{\mu\nu}}{16\pi G} \delta g_{\mu\nu} + \underbrace{\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}}_{\text{definition of } T^{\mu\nu}} \right) = 0 \Rightarrow \boxed{\quad}$$

So Bianchi
implies

i.e.
$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left(\frac{\delta S_M}{\delta g_{\mu\nu}} \right)$$

"functional derivative"

$$\Rightarrow \sqrt{-g} g^{\mu\nu} \partial R_{\mu\nu} = \sqrt{-g} (g^{\mu\nu} \partial_{\mu\nu})_{;\lambda} - \sqrt{-g} (g^{\mu\nu} \partial_{\mu\lambda})_{;\nu}$$

Examples: Higgs field (scalar ϕ)

$$\partial_\mu \phi = \nabla_\mu \phi$$

$$g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$$

"kinetic term"

potential term.

$$S' = \int d^4x \sqrt{-g} \left(-\frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) - V(\phi) \right)$$

Ex: $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right]$

Maxwell $\int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} \right)$; $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

Maxwell

$$\int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) ; F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \Rightarrow T^{\mu\nu} \text{ which we had at start of the course.}$$

$$= \left(\sqrt{-g} g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha \right)_{,\alpha} - \left(\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\lambda}^\lambda \right)_{,\nu}$$

$\uparrow$
 ordinary derivative

$$\partial_\mu V^\mu$$

divergence theorem

$$V^\mu dS_\mu$$

i.e. it is a total divergence, in ordinary sense!

$\Rightarrow$ get only surface terms, can be neglected

$\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \Rightarrow T^{\mu\nu}$ which we had at start of the course.

Point particle
$$S = -m \int d\lambda \sqrt{-g_{\mu\nu}^{(x(\lambda))} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$
$$= -m \int d^4x \int d\lambda \delta^4(x - x(\lambda)) \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} = S[g_{\mu\nu}(x), x^\mu(\lambda)].$$

again, varying wrt $g_{\mu\nu}(x)$, recover the stress energy tensor used earlier in