

Title: 14/15 PSI - Relativity- Lecture 2

Date: Sep 19, 2014 09:00 AM

URL: <http://pirsa.org/14090022>

Abstract:

last time: showed that most general spherically symmetric solution

$$G_{\mu\nu} = 0, \text{ is}$$

$$-dt^2\left(1 + \frac{k}{r}\right) + \frac{dr^2}{\left(1 + \frac{k}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

spherically symmetric solution of the source-free Einstein equations,

$$(\sin^2\theta d\varphi^2)$$

last time: showed that most general spherically symmetric solution

$$G_{\mu\nu} = 0, \text{ is}$$

$$-dt^2 \left(1 + \frac{k}{r}\right) + \frac{dr^2}{\left(1 + \frac{k}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$\text{look at large } r \Rightarrow g_{\mu\nu} \approx \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\sim \frac{k}{r}} \quad \text{recall that } h_{00} = -\frac{2\phi}{c^2}$$

spherically symmetric solution of the source-free Einstein equations,

$$(\sin^2 \theta d\varphi^2)$$

$g_{\phi\phi} = -\frac{2\phi}{c^2}$, recall that $\phi = -\frac{GM}{r}$ is the Newtonian potential outside a spherical mass M .

$$\rightarrow h_{00} \approx -\frac{k}{r} = -\frac{2GM}{rc^2}$$

$\Rightarrow$ Schwarzschild metric represents the exterior of a spherical mass,

$$-dt^2 \left(1 - \frac{2GM}{rc^2}\right) + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

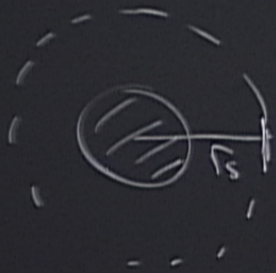
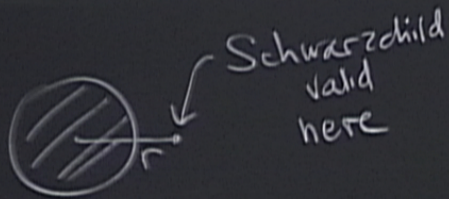
$$\rightarrow h_{00} \approx -\frac{k}{r} = -\frac{2GM}{rc^2}$$

$\Rightarrow$ Schwarzschild metric represents the exterior of a spherical mass,

$$-dt^2 \left(1 - \frac{2GM}{rc^2}\right) + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(in fact this leads to the definition of the mass of a body in G.R.)

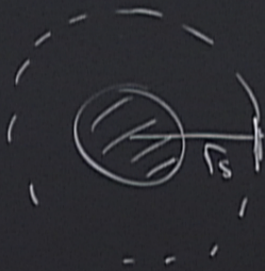
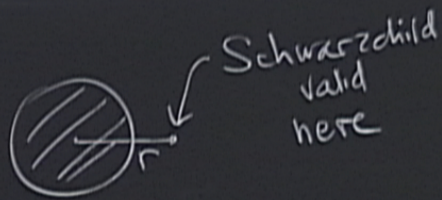
always inside r , it looks like metric becomes singular at $r = \frac{2GM}{c^2}$
 $= r_s$
Schwarzschild
radius



provided the mass M is always inside r , it
 we shall see that this is a coordinate singularity, r
 is not singular at $r=r_s$.

always inside r , it looks like metric becomes singular at $r = \frac{2GM}{c^2}$
coordinate singularity, $\exists$ coordinates in which $g_{\mu\nu}$

$= r_s$
Schwarzschild
radius



: provided the mass M is always inside

we shall see that this is a coordinate
is not singular at $r=r_s$.

There is a huge physical effect at $r=r_s$:

always inside r , it looks like metric becomes singular at $r = \frac{2GM}{c^2}$

coordinate singularity, $\exists$ coordinates in which $g_{\mu\nu}$
gravitational time dilation

Recall.
$$\underline{\underline{\Delta\tau}} = \frac{\Delta t}{\left(-g_{00}\right)^{\frac{1}{2}}}$$

$$\Rightarrow \Delta t = \frac{\Delta\tau}{\left(-g_{00}\right)^{\frac{1}{2}}} = \frac{\Delta\tau}{\sqrt{1 - \frac{2GM}{rc^2}}} \Rightarrow \infty$$

as r decreases
to r_s .

$$= r_s$$

Schwarzschild
radius

black hole
Milky way S2

$$r_s = \frac{2GM}{c^2} \quad ; \quad \text{for Sun, } \frac{2 \times 6.7 \times 10^{-11} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} = 3 \text{ km.}$$

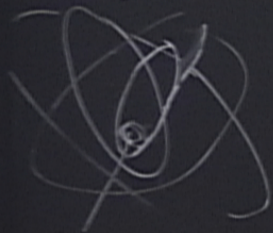
obviously, $r_{\text{sun}} > r_{\text{schw}}$. Sun is not a black hole.

○ However, there is a black hole in the centre of our galaxy.

for at large r $\Rightarrow$ $\frac{2GM}{c^2} \approx \frac{2 \times 10^{30}}{(3 \times 10^8)^2} \approx 2.2 \times 10^{-6} \text{ m}$

$$r_s = \frac{2GM}{c^2} \quad ; \quad \text{for Sun, } \frac{2 \times 6.7 \times 10^{-11} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} = 3 \text{ km.}$$

black hole
Milky way S2



Candidate b. hole

obviously, $r_{\text{sun}} > r_{\text{schw}}$. Sun is not a black hole.

However, there is a black hole in the centre of our galaxy. $M \approx 4 \times 10^6 M_{\odot}$

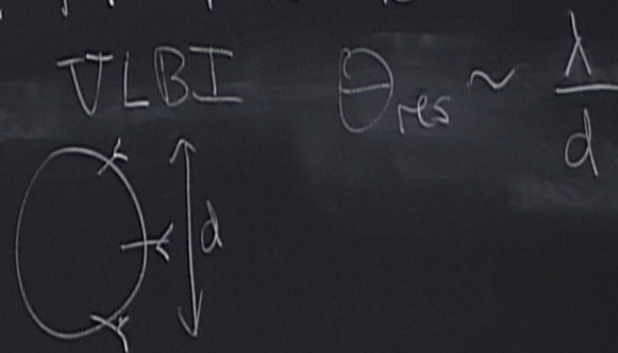
at larger $\Rightarrow r_{\text{schw}} \approx 10^{10} \text{ m}$ $\left. \begin{array}{l} \text{radius} \\ \text{distance} \approx 2 \times 10^{20} \text{ m.} \end{array} \right\} \theta \sim 10^{-10} \text{ radians.}$

$$\text{Sun, } \frac{2 \times 6.7 \times 10^{-11} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} = 3 \text{ km.}$$

obviously, $r_{\text{sun}} > r_{\text{schw}}$. Sun is not a black hole.

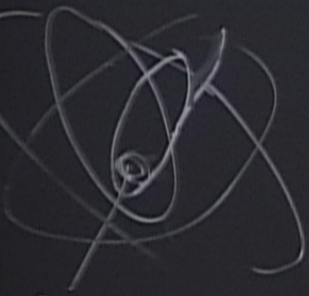
black hole in the centre of our galaxy. : $M \approx 4 \times 10^6 M_{\odot}$

$$\Rightarrow \left. \begin{array}{l} r_{\text{schw}} \approx 10^{10} \text{ m} \\ \text{distance} \approx 2 \times 10^{20} \text{ m.} \end{array} \right\} \theta \sim 10^{-10} \text{ radians.}$$



$$r_s = \frac{2GM}{c^2} \quad ; \quad \text{for Sun, } \frac{2 \times 6.7 \times 10^{-11} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} = 3 \text{ km.}$$

black hole
milky way S2



Candidate b.hole

obviously, $r_{\text{sun}} > r_{\text{schw}}$. Sun is not a b

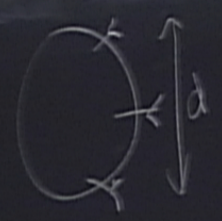
However, there is a black hole in the centre of our galaxy.

$$\left. \begin{array}{l} r_s \sim 695,000 \text{ km} \\ \quad \quad \quad \underline{10^9 \text{ m}} \end{array} \right\} \Rightarrow \left. \begin{array}{l} r_{\text{schw}} \approx 10^{10} \text{ m} \\ \text{distance} \approx 2 \times 10^{20} \text{ m.} \end{array} \right\} \theta \sim 10^{-10} \text{ radians.}$$

~~solution of the source-free Einstein equation~~

a black hole.

Galaxy. : $M \approx 4 \times 10^6 M_\odot$
VLBI

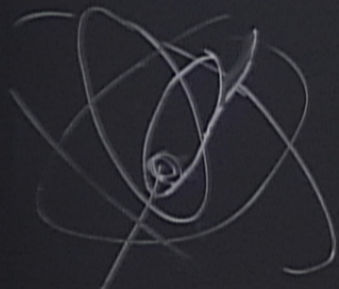


$$\Theta_{\text{res}} \sim \frac{\lambda}{d}$$

100GHz radio waves
 $\downarrow$
 $\frac{10^{-3}}{10^7} \sim 10^{-10}$
 $\uparrow$
 diameter of Earth

close to
being
observed

black hole
Milky way S2



Candidate b.hole

$$r_s = \frac{2GM}{c^2} \quad ; \quad \text{for Sun, } \frac{2 \times 6.7 \times 10^{-11} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} = 3 \text{ km.}$$

obviously, $r_{\text{sun}} > r_{\text{schw}}$. Sun is not a

However, there is a black hole in the centre of our galaxy

$$\left. \begin{array}{l} r_s \sim 695,000 \text{ km} \\ \underline{10^9 \text{ m}} \end{array} \Rightarrow r_{\text{schw}} \approx 10^{10} \text{ m.} \right\} \theta \sim 10^{-10} \text{ radians.}$$

$$\text{distance} \approx 2 \times 10^{20} \text{ m.}$$

$$\frac{10^{10}}{3 \times 10^8} = 30$$

proper surface area of
black hole

$$\begin{aligned} &= 4\pi \cdot r_s^2 = 4\pi \cdot \left(\frac{2GM}{c^2}\right)^2 \\ &= 16\pi \frac{(GM)^2}{c^4} \end{aligned}$$

- completely coordinate independent.

Tests of GR

2 different objects falling in the Earth's gravitational field experience the same acceleration a , up to

① Equivalence principle: $\frac{\Delta a}{a} \lesssim 10^{-13}$ (Adelberger, U. Washington)

proposed STEP satellite 1401.4784 Overduin et al

$$\Rightarrow \frac{\Delta a}{a} \lesssim 10^{-18}$$

② Gravitational redshift: 2 Masers on earth and in orbit



$$= \frac{\Delta V - \Delta V_{th}}{\Delta V_{th}} < 2 \times 10^{-4}$$

③ Deflection of light

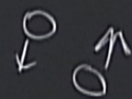
$$\frac{\Delta\phi - \Delta\phi_{\text{th}}}{\Delta\phi_{\text{th}}} \lesssim 3 \times 10^{-3}$$

④ Shapiro time delay



$$\frac{\Delta t - \Delta t_{\text{th}}}{\Delta t_{\text{th}}} \lesssim 2 \times 10^{-5}$$

⑤ Pulsars



binary pulsar (Hulse + Taylor) $\Rightarrow$ confirmation of GR
prediction for gravitational waves

$$3 \times 10^{-3}$$

planet

$$\frac{\Delta t - \Delta t_{\text{GR}}}{\Delta t_{\text{GR}}} \lesssim 2 \times 10^{-5}$$

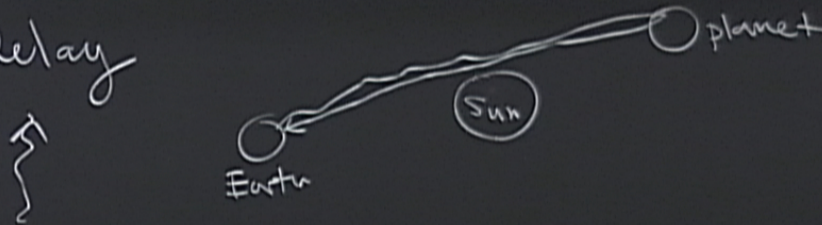
on of GR

for gravitational waves. agrees with GR to $\sim 2 \times 10^{-3}$.

③ Deflection of light

$$\frac{\Delta\phi - \Delta\phi_{GR}}{\Delta\phi_{GR}} \lesssim 3 \times 10^{-3}$$

④ Shapiro time delay



$$\frac{\Delta t - \Delta t_{GR}}{\Delta t_{GR}} \lesssim 2 \times 10^{-5}$$

⑤ Pulsars



binary pulsar
(Hulse + Taylor)

$\Rightarrow$ confirmation of GR
prediction for gravitational waves

agrees w

⑥ Direct Detection of
Gravitational Waves

LIGO

⑦ Perihelion shift of Mercury (Einstein, ~1917)

Orbits of a massive particle in Schwarzschild metric

$$r_s = \frac{2GM}{c^2}$$
$$c = 1$$

$$-\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

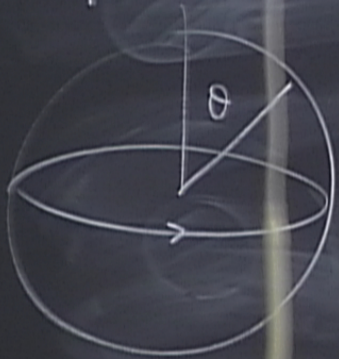
for massive particle, $-\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) = -1$

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1$$

pt of $t, \varphi \Rightarrow$ conserved quantities (Noether) $\mathcal{G}_{t\mu} \dot{X}^\mu, \mathcal{G}_{\varphi\mu} \dot{X}^\mu$

spherical symmetry



$\Rightarrow \theta = \pi/2$ should be a solution,

Conserved quantities: metric is indept of $t, \varphi. \Rightarrow$ con

$$\begin{array}{l} \Rightarrow \text{energy} \quad E = \left(1 - \frac{r_s}{r}\right) \dot{t} \\ \text{angular} \quad J = r^2 \dot{\varphi} \\ \text{Momentum} \end{array} \quad \left. \vphantom{\begin{array}{l} E \\ J \end{array}} \right\} \text{conserved}$$

$\Rightarrow$ 3 eqⁿs for 3 unknowns, $\dot{t}, \dot{r}, \dot{\phi} \Rightarrow$ can solve.

eliminate $\dot{t}, \dot{\phi}$

$$\Rightarrow -\frac{E^2}{(1-\frac{r_s}{r})} + \frac{\dot{r}^2}{(1-\frac{r_s}{r})} + \frac{J^2}{r^2} = -1$$

$$\Rightarrow \dot{r}^2 + V(r) = E^2 - 1 = \text{constant}$$

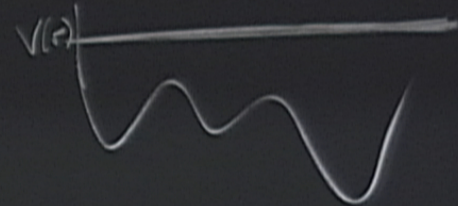
3×10^{-3}

planet

$$\frac{\Delta t - \Delta t_{GR}}{\Delta t_{GR}} \lesssim 2 \times 10^{-5}$$

of GR

for gravitational waves. agrees with GR to $\sim 2 \times 10^{-3}$.



$\Rightarrow 3 \text{ eq}^n\text{'s for 3 unknowns, } \dot{t}, \dot{r}, \dot{\varphi} \Rightarrow \text{can solve.}$

eliminate $\dot{t}, \dot{\varphi}$

$$\Rightarrow -\frac{E^2}{(1-\frac{r_s}{r})} + \frac{\dot{r}^2}{(1-\frac{r_s}{r})} + \frac{J^2}{r^2} = -1$$

$$\Rightarrow \dot{r}^2 + V(r) = E^2 - 1 = \text{constant}$$

$$V(r) = -\frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3}$$

usual Newtonian potential $\swarrow$
usual antipetal force (angular momentum barrier) $\swarrow$

$\Rightarrow$ 3 eqⁿs for 3 unknowns, $\dot{t}, \dot{r}, \dot{\varphi} \Rightarrow$ can solve.

eliminate $\dot{t}, \dot{\varphi}$

$$\Rightarrow -\frac{E^2}{(1-\frac{\sqrt{5}}{r})} + \frac{\dot{r}^2}{(1-\frac{\sqrt{5}}{r})} + \frac{J^2}{r^2} = -1$$

$$\Rightarrow \dot{r}^2 + V(r) = E^2 - 1 = \text{constant}$$

$$V(r) = -\frac{\sqrt{5}}{r} + \frac{J^2}{r^2} - \frac{\sqrt{5}J^2}{r^3}$$

usual Newtonian potential $\swarrow$

usual antipetal force (ang. momentum barrier) $\swarrow$

GR correction $\swarrow$

3 unknowns, $\dot{t}, \dot{r}, \dot{\varphi} \Rightarrow$ can solve.

$$\frac{\ddot{r}^2}{(1-\frac{\sqrt{5}}{r})} + \frac{J^2}{r^2} = -1$$

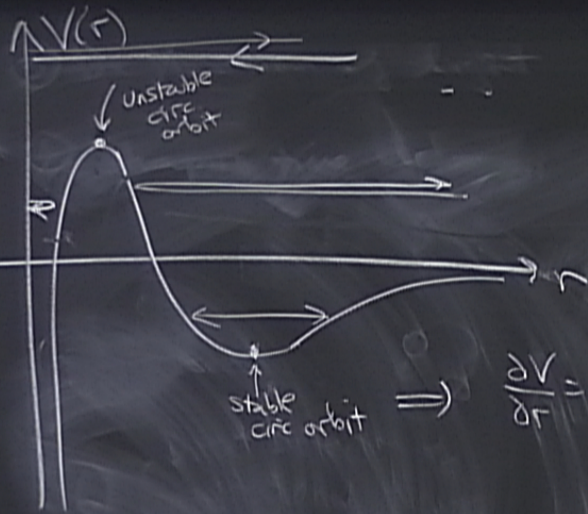
$$= E^2 - 1 = \text{constant}$$

usual Newtonian potential
usual antipetal force (ang. momentum barrier)
GR correction (down by $\frac{1}{c^2}$)

$$V(r) = -\frac{\sqrt{5}}{r} + \frac{J^2}{r^2} - \frac{\sqrt{5}J^2}{r^3}$$

② Gravitational redshift: 2 Masers on earth and in orbit

$$= \frac{\Delta V - \Delta \Phi}{\Delta \Phi} < 2 \times 10^{-4} \quad \text{vessel clock.}$$



$$\Rightarrow \frac{\partial V}{\partial r} = 0 = \frac{\sqrt{5}}{r^2} - \frac{2J^2}{r^3} + \frac{3(J^2)^2}{r^4} \Rightarrow r = \frac{J^2}{\sqrt{5}} \left(1 \pm \sqrt{1 - 3\frac{J^2}{J^2}} \right) \quad ; \quad r_{\text{stable}} = \frac{J^2}{\sqrt{5}} \left(1 + \sqrt{1 - 3\frac{J^2}{J^2}} \right)$$

$$+ \frac{3J^2}{r^4} \Rightarrow r = \frac{J^2}{r_s} \left(1 \pm \sqrt{1 - \frac{3r_s^2}{J^2}} \right) ; r_{\text{stable}} = \frac{J^2}{r_s} \left(1 + \sqrt{1 - \frac{3r_s^2}{J^2}} \right)$$

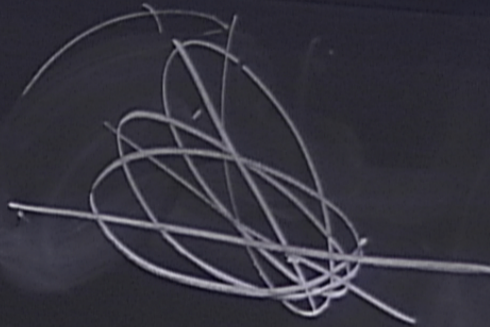
as you decrease J , there is an innermost stable circular orbit : ISCO.

$$+ \frac{3J^2}{r^4} \Rightarrow r = \frac{J^2}{r_s} \left(1 \pm \sqrt{1 - 3r_s^2/J^2} \right) ; r_{\text{stable}} = \frac{J^2}{r_s} \left(1 + \sqrt{1 - 3r_s^2/J^2} \right)$$

as you decrease J , there is an innermost stable circular orbit : ISCO,
 at $J^2 = 3r_s^2$; $r_{\text{stable}} = 3r_s$

ms, $\dot{t}, \dot{r}, \dot{\varphi} \Rightarrow$ can solve.

meast. $(43'' \pm 1'')$ centry.



$= -1$

usual Newtonian potential

usual centripetal force (ang. momentum barrier)

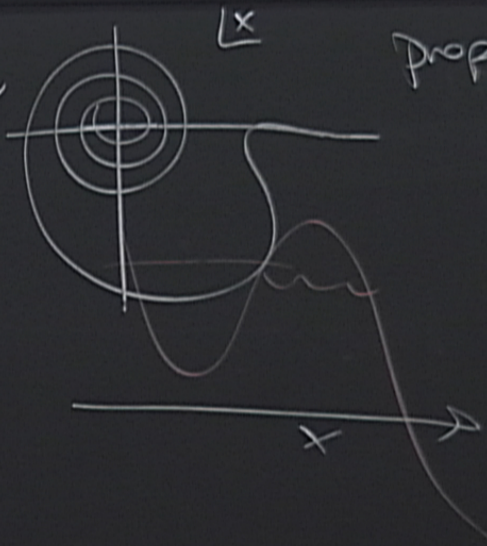
GR correction (down by $\frac{1}{c^2}$)

$$V(r) = -\frac{\sqrt{5}}{r} + \frac{J^2}{r^2} - \frac{\sqrt{5}J^2}{r^3}$$

g in the Earth's gravitational
same acceleration a , up to
(Adelberger, U. Washington)

4784 Overduin et al

$$= \frac{\Delta V - \Delta V_{th}}{\Delta V_{th}} < 2 \times 10^{-4} \quad \text{Vessot et al.}$$



proper surface area of
black hole

$$= 4\pi \cdot r_s^2 = 4\pi \cdot \left(\frac{2GM}{c^2}\right)^2$$

$$= 16\pi \frac{(GM)^2}{c^4}$$

- completely coordinate independent.