

Title: 14/15 PSI - Relativity- Lecture 2

Date: Sep 17, 2014 09:00 AM

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Abstract:

$$\nabla_{\mu} g_{\alpha\beta} = 0$$

Curvature of spacetime - Riemann tensor

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S$$

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\rho =$$

Curvature of spacetime - Riemann tensor

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$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_p =$$

Curvature of spacetime - Riemann tensor

$$\begin{aligned} & (\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu) S \\ & - (\partial_\nu \partial_\mu - \Gamma_{\mu\nu}^\lambda \partial_\lambda) S \end{aligned}$$

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\rho =$$

Curvature of spacetime - Riemann tensor

$$\begin{aligned}
 & (\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu) S \\
 & \quad - \underbrace{(\partial_\nu \partial_\mu - \Gamma_{\mu\nu}^\lambda \partial_\lambda)}_{\text{cancels}} S \\
 & \quad (\Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda) \partial_\lambda S \\
 & \quad = 0
 \end{aligned}$$

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

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 & = 0
 \end{aligned}$$

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$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_p =$$

Curvature of spacetime - Riemann tensor

$$\begin{aligned}
 & (\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu) S \\
 & - (\underbrace{\partial_\nu \partial_\mu - \Gamma_{\mu\nu}^\lambda \partial_\lambda}_{\text{cancel}}) S \\
 & (\Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda) \partial_\lambda S \\
 & \quad \quad \quad = 0 \\
 & \text{by symmetry of } \Gamma_{\mu\nu}^\lambda
 \end{aligned}$$

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\rho = -R^\sigma{}_{\rho\mu\nu} S_\sigma$$

Curvature of spacetime - Riemann tensor

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\rho = -R^\sigma{}_{\rho\mu\nu} S_\sigma$$

Quotient theorem:

if $A_{\mu\nu} S^\lambda$ is a tensor

$\forall$ vectors S^λ , then

$A_{\mu\nu}$ is a tensor.

Curvature of spacetime - Riemann tensor

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\rho = - \underbrace{R^\sigma_{\rho\mu\nu}}_{\text{Riemann tensor}} S_\sigma$$

Quotient theorem:

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Curvature of spacetime - Riemann tensor

$$[\partial_\mu, \partial_\nu] = 0 \quad ; \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

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Quotient theorem:

if $A_{\mu\nu} S^\lambda$ is a tensor
for all vectors S^λ , then
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Calculate $\mathcal{L}$

Calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma)$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) \quad - (\mu \leftrightarrow \nu)$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$- \Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma - \Gamma_{\rho\mu}^\alpha \partial_\alpha S_\nu$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu) \\ - \Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma - \Gamma_{\rho\mu}^\alpha \partial_\nu S_\alpha$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$-\Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma + \Gamma_{\rho\mu}^\alpha \partial_\nu S_\alpha$$

Symm under $\mu \leftrightarrow \nu$

~~cancels w/ $-(\mu \leftrightarrow \nu)$ term.~~

calculate lhs

$$\begin{aligned}
 & \partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu) \\
 & \quad - \underbrace{\Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma + \Gamma_{\rho\mu}^\alpha \partial_\nu S_\alpha}_{\text{Symm under } \mu \leftrightarrow \nu} = \left(-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma \right) S_\sigma - (\mu \leftrightarrow \nu) \\
 & \quad \text{cancels w/ } -(\mu \leftrightarrow \nu) \text{ term.}
 \end{aligned}$$

calculate lhs

$$\partial_\mu (\Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$\cancel{S_\sigma + \Gamma_{\rho\mu}^\alpha \partial_\alpha S_\nu}$$

Symm under $\mu \leftrightarrow \nu$

cancels w/ $-(\mu \leftrightarrow \nu)$ term.

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) S_\sigma - (\mu \leftrightarrow \nu)$$

$$\Rightarrow R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma$$

calculate lhs

$$\cancel{S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma} - \Gamma_{\nu\mu}^\alpha (\cancel{\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma}) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$\cancel{\frac{\partial_\mu S_\sigma - \Gamma_{\mu\alpha}^\sigma \partial_\alpha S_\nu}{\text{Symm under } \mu \leftrightarrow \nu}}$$

$\cancel{\text{cancels w/ } -(\mu \leftrightarrow \nu) \text{ term.}}$

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) S_\sigma - (\mu \leftrightarrow \nu)$$

$$\Rightarrow R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$\underbrace{-\Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma + \Gamma_{\rho\mu}^\alpha \partial_\nu S_\alpha}_{\text{Symm under } \mu \leftrightarrow \nu}$$

~~cancels w/ $-(\mu \leftrightarrow \nu)$ term.~~

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) S_\sigma - (\mu \leftrightarrow \nu)$$

$$\Rightarrow \boxed{R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma}$$

calculate lhs

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha S_\nu - \Gamma_{\alpha\nu}^\sigma S_\sigma) - (\mu \leftrightarrow \nu)$$

$$\underbrace{-\Gamma_{\rho\nu}^\sigma \partial_\mu S_\sigma + \Gamma_{\rho\mu}^\alpha \partial_\nu S_\alpha}_{\text{Symm under } \mu \leftrightarrow \nu}$$

~~cancels w/ $-(\mu \leftrightarrow \nu)$ term.~~

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) S_\sigma - (\mu \leftrightarrow \nu)$$

$$\Rightarrow \boxed{R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma}$$

In a locally flat coord system, $\Gamma^M_{\alpha\beta} = 0$.

In a locally flat coord system, $\Gamma_{\alpha\beta}^{\mu} = 0$.

$$\partial_{\mu} \Gamma_{\rho\nu}^{\sigma} =$$

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda})$$

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda})$$

$$g_{\alpha\beta,\lambda} = 0$$

$$g^{\alpha\beta}_{,\lambda}$$

$$g^{\alpha\beta} g_{\beta\delta} = \delta^{\alpha}_{\delta}$$

$$\Rightarrow (g^{-1})_{,\alpha} = -g^{-1}_{\lambda\alpha} g^{\lambda\beta} g_{\beta\gamma,\alpha}$$

In a locally flat coord system, $\Gamma_{\alpha\beta}^{\mu} = 0$.

in locally flat coords $\partial_{\mu} \Gamma_{\rho\nu}^{\sigma} = \frac{1}{2} \eta^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$

In a locally flat coord system, $\Gamma_{\alpha\beta}^{\mu} = 0$.

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$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \frac{1}{2} \eta^{\sigma\lambda} (g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$$

In a locally flat coord system, $\Gamma_{\alpha\beta}^M = 0$

in locally flat coords $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} \eta^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$

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$$\Rightarrow R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$$

In a locally flat coord system, $\Gamma_{\alpha\beta}^M = 0$.

In locally flat coords $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} \eta^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$

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$$\Rightarrow R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$$

Symmetries of Riemann tensor

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$$A_{\mu\nu} = A_{\nu\mu}$$

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$$\textcircled{1} \mathcal{R}$$

Symmetries of Riemann tensor

① symmetric under interchange
of first and last pair of indices

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$$

f Riemann tensor

- ① symmetric under interchange of first and last pair of indices

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$$

- ② antisymmetry under exchanging the first two indices or the last two indices.

$$R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\nu\mu} = +R_{\rho\sigma\nu\mu}.$$

① symmetric under interchange
of first and last pair of indices

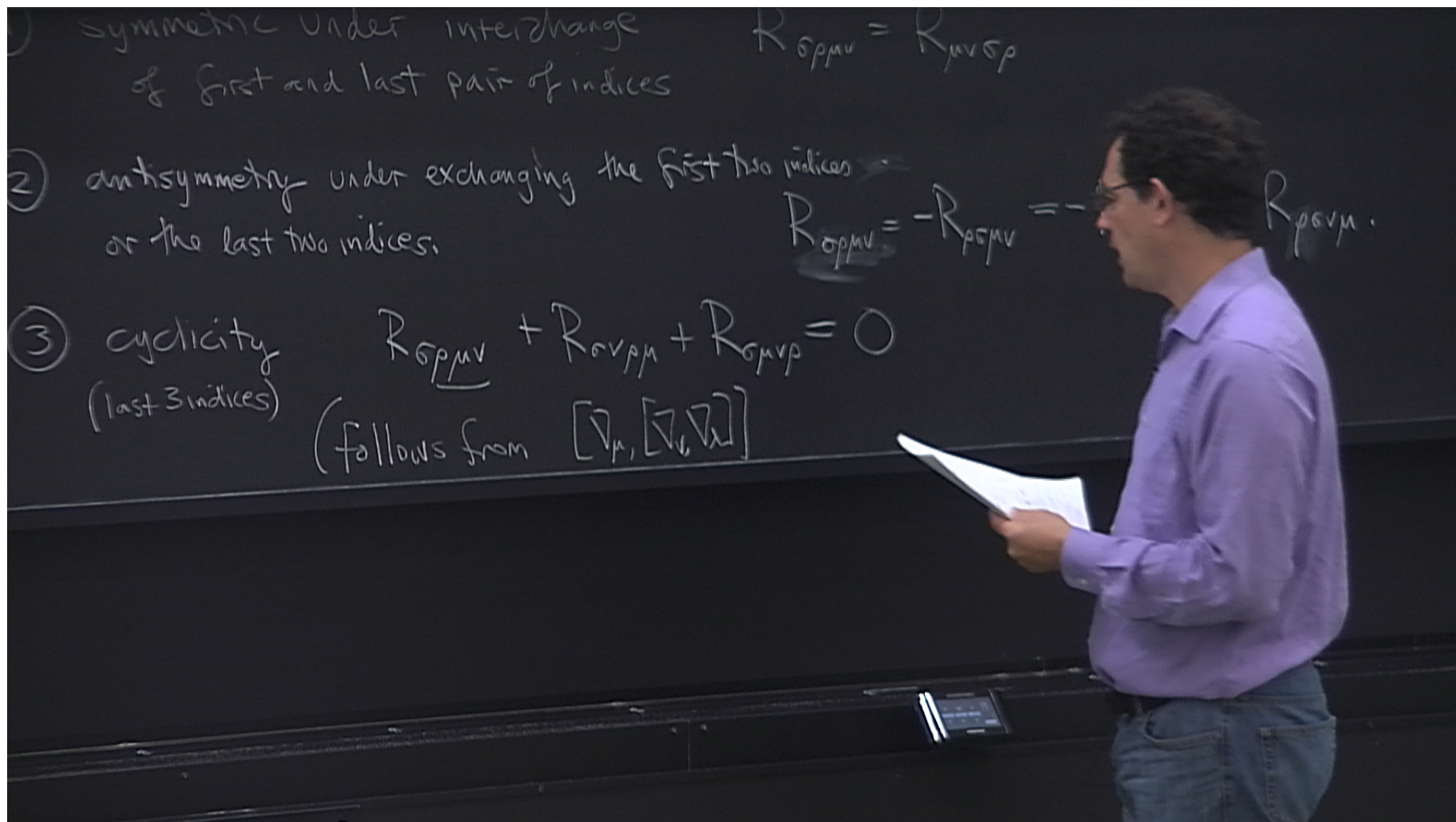
$$R_{\sigma\mu\nu} = R_{\mu\nu\sigma}$$

② antisymmetry under exchanging the first two indices
or the last two indices.

$$R_{\sigma\mu\nu} = -R_{\mu\sigma\nu} = -R_{\sigma\nu\mu} = +R_{\nu\sigma\mu}.$$

③ cyclicity
(last 3 indices)

$$R_{\sigma\mu\nu} + R_{\sigma\nu\mu} + R_{\mu\nu\sigma} = 0$$



① symmetric under interchange of first and last pair of indices

$$R_{\sigma\mu\nu} = R_{\mu\nu\sigma}$$

② antisymmetry under exchanging the first two indices or the last two indices.

$$R_{\sigma\mu\nu} = -R_{\mu\sigma\nu} = -R_{\sigma\nu\mu} = +R_{\nu\sigma\mu}.$$

③ cyclicity (last 3 indices)

$$R_{\sigma\mu\nu} + R_{\sigma\nu\mu} + R_{\sigma\mu\nu} = 0$$

(follows from $[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\mu, \nabla_\lambda]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0$)

Jacobi identity

How many independent components are there in the Riemann tensor?

(follows from $[V_\mu, [V_\nu, V_\lambda]] + [V_\lambda, [V_\mu, V_\nu]] + [V_\nu, [V_\lambda, V_\mu]] = 0$)

How many independent components are there in the Riemann tensor?

first, take ① and ② into account : think of $R_{\mu\nu\rho\sigma}$ as a symmetric matrix
each index takes $\frac{D(D-1)}{2}$ values

(follows from $[V_\mu, [V_\nu, V_\lambda]] + [V_\lambda, [V_\mu, V_\nu]] + [V_\nu, [V_\lambda, V_\mu]] = 0$)

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$$\Rightarrow \frac{D(D-1)}{2} \left(\frac{D(D-1)}{2} + 1 \right)$$

(follows from $[V_\mu, [V_\nu, V_\lambda]] + [V_\lambda, [V_\mu, V_\nu]] + [V_\nu, [V_\lambda, V_\mu]] = 0$)

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each index takes $\frac{D(D-1)}{2}$ values

$$\begin{array}{c} A_{ab} \\ N \\ \frac{N(N+1)}{2} \end{array}$$

$$\Rightarrow \frac{1}{2} \left(\frac{D(D-1)}{2} \right) \left(\frac{D(D-1)}{2} + 1 \right)$$

(follows from $[V_\mu, [V_\nu, V_\lambda]] + [V_\lambda, [V_\mu, V_\nu]] + [V_\nu, [V_\lambda, V_\mu]] = 0$)

How many independent components are there in the Riemann tensor?

first, take ① and ② into account : think of $R_{\mu\nu\rho\sigma}$ as a symmetric matrix
each index takes $\frac{D(D-1)}{2}$ values

$$\begin{array}{c} A_{ab} \\ N \\ \frac{N(N+1)}{2} \end{array}$$

$$\Rightarrow \frac{1}{2} \left(\frac{D(D-1)}{2} \right) \left(\frac{D(D-1)}{2} + 1 \right) = \frac{1}{8} D(D-1) (D^2 - D + 2)$$

Symm under $\mu \leftrightarrow \nu$

/cancels w/ $-(\mu \leftrightarrow \nu)$
term.

$$\Rightarrow R^\sigma_{\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} - \Gamma^\alpha_{\mu\rho} \Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\nu\rho} \Gamma^\sigma_{\alpha\mu}$$

cyclicity: an equation which is totally antisymmetric under exchange of any two indices.

Symmetries of Riemann tensor

$$A_{\mu\nu} = A_{\nu\mu}$$

- ① symmetric under interchange of first and last pair of indices

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$$

- ② antisymmetry under exchanging the first two indices or the last two indices.

$$R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$$

- ③ cyclicity (last 3 indices)

$$R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

(follows from $[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\mu, \nabla_\lambda]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0$)

tem.

$$R_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu = \alpha_\mu \partial_\nu + \alpha_\nu \partial_\mu$$

uation which is totally antisymmetric under exchange of any two indices.

ie. it is $\frac{D(D-1)(D-2)(D-3)}{4!}$ equations.

$$\Rightarrow R_{\mu\nu} = \partial_\mu \Gamma_{\nu}^{\rho} - \partial_\nu \Gamma_{\mu}^{\rho} - \Gamma_{\mu\rho}^{\sigma} \Gamma_{\sigma\nu}^{\rho} + \Gamma_{\nu\rho}^{\sigma} \Gamma_{\sigma\mu}^{\rho}$$

cyclicality: an equation which is totally antisymmetric under exchange of any

ie. it is $\frac{D(D-1)(D-2)(D-3)}{4!}$ equations.

$\Rightarrow$ # of independent components of Riemann taking all these symmetries into account

$$= \frac{1}{8} D(D-1)(D^2-D+2) - \frac{D(D-1)(D-2)(D-3)}{4!} = \frac{D^2}{12} (D^2-1) !$$

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu} - \partial_\nu \Gamma_{\mu} - \Gamma_{\mu} \Gamma_{\nu} + \Gamma_{\nu} \Gamma_{\mu}$$

is antisymmetric under exchange of any two indices.

(D-3) equations.

taking all these symmetries into account

$$\frac{(D-1)(D-2)(D-3)}{4!} = \frac{D^2}{12}(D-1) \quad \Big|_6$$

ie it magically equals the number of second derivatives of the metric tensor which cannot be set zero at a point by a coordinate transformation.

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu}$

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

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$\Rightarrow$ only possible

from 'Riemann, we' can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

$\Rightarrow$ only possible action
(second order in derivatives)

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Ricci tensor $R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}_{\sigma}$

$\Rightarrow$ only possible action
(second order in derivatives)

$$\underbrace{\int d^4x \sqrt{-g}}_{\text{coordinate invariant}}$$

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x$$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

$\Rightarrow$ only possible action
(second order in derivatives)

$$\int d^4x \sqrt{-g}$$

coordinate invariant

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x$$

$$g'_{\mu\nu} = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} g$$

$$\Rightarrow \det g' = \left| \frac{\partial x}{\partial x'} \right|^2 \det g \Rightarrow \text{only possible action}$$

(second order in derivatives)

$$\int d^4x \sqrt{-g}$$

coordinate invariant

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

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(second order in derivatives)

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coordinate invariant

from Riemann, we can form

Ricci tensor $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

Ricci scalar $R = g^{\sigma\nu} R_{\sigma\nu} = R^{\sigma}{}_{\sigma}$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x$$

$$g'_{\mu\nu} = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} g$$

$$\Rightarrow \det g' = \left| \frac{\partial x}{\partial x'} \right|^2 \det g \Rightarrow \text{only possible action}$$

(second order in derivatives)

$$\underbrace{\int d^4x \sqrt{-g}}_{\text{coordinate invariant}} R$$

Scalar

ten.

$$R_{\rho\mu\nu} = \partial_\mu \Gamma_{\rho\nu} - \partial_\nu \Gamma_{\rho\mu} - \Gamma_{\rho\mu} \Gamma_{\nu\sigma} + \Gamma_{\rho\nu} \Gamma_{\mu\sigma}$$

Weyl tensor : $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma})$
 $+ \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$

ten.

$$\Rightarrow R_{\rho\mu\nu} = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}) \\ + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

- has the property that it is traceless on all pairs of indices i.e. $C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

ten.

$$\Rightarrow R_{\mu\nu} = \partial_\mu \Gamma_{\nu}^{\sigma} - \partial_\nu \Gamma_{\mu}^{\sigma} - \Gamma_{\mu\sigma}^{\alpha} \Gamma_{\alpha\nu}^{\sigma} + \Gamma_{\nu\sigma}^{\alpha} \Gamma_{\alpha\mu}^{\sigma}$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\rho} - g_{\mu\rho} R_{\nu\sigma}) \\ + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

- has the property that it is traceless on all pairs of indices i.e. $C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

of indep components $\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} \overset{\text{Ricci tensor}}{=} \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\rho} - g_{\mu\rho} R_{\nu\sigma}$$

$$2) (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is traceless on all pairs of indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{2} \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl
1	0	1	$\frac{D(D+1)}{2}$
	1		
	6		
	20		
	$\rho_{\sigma} = 0$		etc.

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$2) (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is traceless on all pairs of indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
1	0	0	0
2	1	1	
3	6		
4	20		

$C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

$$R_{1212} = 1$$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$2) (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is traceless on all parts of

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} \text{ Ricci tensor } (D^2 - 1)$$

$$\frac{9.8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
1	0	0	0
2	1	1	
3	6		
4	20		

$C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

$$g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$2) (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is traceless on all pairs of indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
1	0	0	0
2	1	1	(trace on all indices identically zero)
3	6		
4	20		

$C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

$$R_{1212} = \dots$$

$$g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$2) (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is tracers on all parts of indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9.8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
1	0	0	0
2	1	1	(trace on all indices identically zero)
3	6	6	0
4	20		

$C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

$$g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma})$$

s traceless on all indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D-1)}{12} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

components of

D	Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
1	0	0	0
2	1	1	(trace on all indices identically zero)
3	6	6	identically zero
4	20	10	identically zero

$C_{\mu}^{\rho}{}_{\rho\sigma} = 0$ etc.

$R_{1212} =$

$g^{\rho\sigma} R_{\rho\mu\sigma\nu}$

$$g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma}$$

$$(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

is traceless on all pairs of indices i.e.

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9.8}{12} = 6$$

D	# components of Riemann	Ricci (tensor)	Weyl $\frac{D(D+1)}{2}$
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4	20	10	10

$$C_{\mu}^{\rho}{}_{\rho\sigma} = 0 \text{ etc.}$$

$$R_{1212}$$

$$g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

ten.

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu} - \partial_\nu \Gamma_{\mu} - \Gamma_{\mu\rho} \Gamma_{\nu}^{\rho} - \Gamma_{\nu\rho} \Gamma_{\mu}^{\rho}$$

Weyl tensor : $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\rho} - g_{\mu\rho} R_{\nu\sigma})$
 $+ \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$

$h_{\mu\nu}$

- has the property that it is traceless on all pairs of indices

of indep components $\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} \right)$

Weyl: 1) use full in describing gravitational radiation

$\frac{9 \cdot 8}{12} = 6$

2) $g'_{\mu\nu}(x) = \int \tilde{L}(x) g_{\mu\nu}(x)$

ten.

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu} - \partial_\nu \Gamma_{\mu} - \Gamma_{\mu\rho} \Gamma_{\nu}^{\rho} - \Gamma_{\nu\rho} \Gamma_{\mu}^{\rho}$$

Weyl tensor : $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\rho} - g_{\mu\rho} R_{\nu\sigma})$
 $+ \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$

$h_{\mu\nu}$

- has the property that it is traceless on all pairs of indices

Weyl: 1) use full in describing gravitational radiation

of indep components

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

2) $g'_{\mu\nu}(x) = \int \mathcal{L}(x) g_{\mu\nu}(x)$

Weyl transform

ten.

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu\alpha} - \partial_\nu \Gamma_{\mu\alpha} - \Gamma_{\mu\alpha} \Gamma_{\nu\beta} + \Gamma_{\nu\alpha} \Gamma_{\mu\beta}$$

tensor :
$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\rho\mu} - g_{\mu\rho} R_{\nu\sigma})$$

$$+ \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

- has the property that it is traceless on all pairs of indices

Weyl: 1) use ful in describing gravitational radiation

of indept components

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2}$$

$$\frac{9.8}{12} = 6$$

2)
$$g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$$
 Weyl transform

$$C'_{\mu\nu\rho\sigma} = \Omega^2 C_{\mu\nu\rho\sigma} \quad (\pm \text{think!})$$

ten.

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu\rho} - \partial_\nu \Gamma_{\mu\rho} - \Gamma_{\mu\rho} \Gamma_{\nu\sigma} + \Gamma_{\nu\rho} \Gamma_{\mu\sigma}$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)} (g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\nu\sigma} R_{\mu\rho} - g_{\mu\rho} R_{\nu\sigma}) \\ + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) R$$

D	# comp
1	0
2	0
3	0
4	0

- has the property that it is traceless on all pairs of indices i.e.

$$C_{\mu}^{\mu}{}_{\rho\sigma} = 0$$

of indep components
ful in describing
gravitational radiation

$$\frac{D^2}{12} (D^2 - 1) - \frac{D(D+1)}{2} = \frac{D(D+1)}{2} \left(\frac{D(D-1)}{6} - 1 \right)$$

$$\frac{9 \cdot 8}{12} = 6$$

$$g'_{\mu\nu}(x) = \int \mathcal{L}(x) g_{\mu\nu}(x) \quad \text{Weyl transform.}$$

$$C'_{\mu\nu\rho\sigma} = \int \mathcal{L}^2 C_{\mu\nu\rho\sigma} \quad (\pm \text{think!}) \text{ transforms covariantly.}$$

Einstein equations

Einstein equations

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

Einstein equations

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

Space-time energy

Einstein equations

Einstein
wanted to
understand how
energy bends
spacetime.

$R^{\rho}_{\sigma\mu\nu}$
spacetime

$\sim T_{\mu\nu}$
energy

Einstein equations

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$$R^{\rho}_{\sigma\mu\nu}$$

Space-time



$$R_{\mu\nu}$$

$$\sim T_{\mu\nu}$$

energy

Einstein equations

Einstein
wanted to
understand how
energy bends
spacetime.

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

Space-time

↓

$$R_{\mu\nu}, g_{\mu\nu} R$$

energy

$$R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

$$R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

key property of $T_{\mu\nu}$ is that it is conserved $\nabla_\mu T^{\mu\nu} = 0$

$$R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

key property of $T_{\mu\nu}$ is that it is conserved

$$\nabla_\mu T^{\mu\nu} = 0$$

It turns out: $\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$ identically.

(follows from Bianchi identity)

$$R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

property of $T_{\mu\nu}$ is that it is conserved $\nabla_\mu(T^{\mu\nu}) = 0$

turns out: $\nabla_\mu \left(\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}_{G^{\mu\nu}} \right) = 0$ identically (follows from Bianchi identity)

$G^{\mu\nu}$, the Einstein tensor.

So, we must have $G_{\mu\nu} = BT_{\mu\nu}$, with B a constant

So, we must have $G_{\mu\nu} = B T_{\mu\nu}$, with B a

work out the constant in
the weak field (Newtonian)
limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
$$|h_{\mu\nu}| \ll 1$$

So, we must have

$$G_{\mu\nu} = B T_{\mu\nu}, \text{ with}$$

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So, we must have

$$G_{\mu\nu} = B T_{\mu\nu}, \text{ with } B$$

work out the constant in
the weak field (Newtonian)
limit.

$$R_{\rho\mu\kappa} =$$

$$\Gamma \sim h$$

$G_{\mu\nu} = B T_{\mu\nu}$, with B a constant

$$R_{\sigma\rho\mu\kappa} = \frac{1}{2} (h_{\nu\sigma,\rho\mu} - h_{\rho\nu,\sigma\mu} - h_{\mu\sigma,\rho\nu} + h_{\rho\mu,\sigma\nu}) + o(h^2)$$

$G_{\mu\nu} = B T_{\mu\nu}$, with B a constant

$$R_{\sigma\rho\mu\kappa} = \frac{1}{2} (h_{\nu\sigma,\rho\mu} - h_{\rho\nu,\sigma\mu} - h_{\mu\sigma,\rho\nu} + h_{\rho\mu,\sigma\nu}) + o(h^2)$$

ea R_{00}

$$G_{\mu\nu} = BT_{\mu\nu}, \text{ with } B \text{ a constant}$$

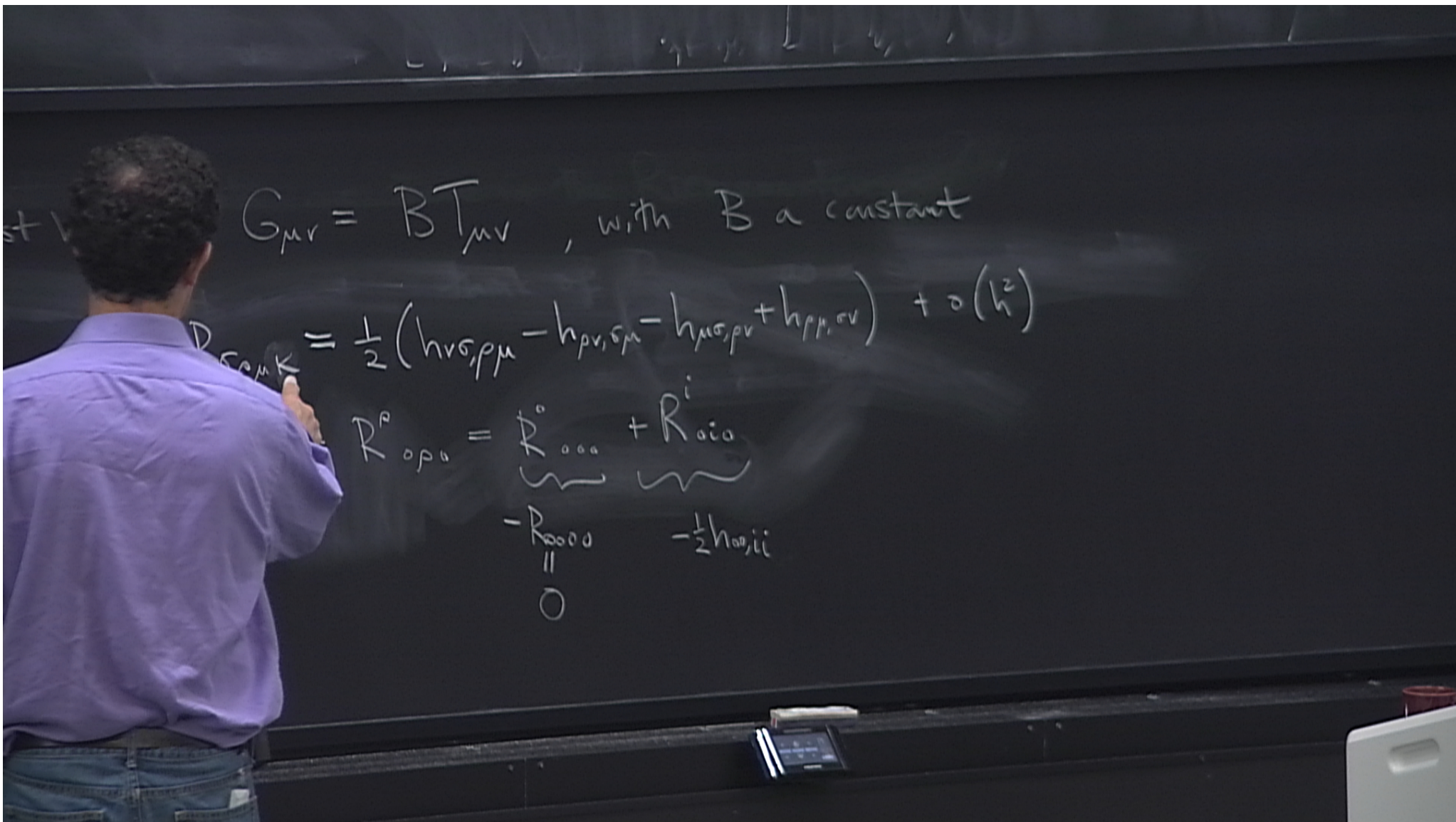
$$R_{\sigma\rho\mu\kappa} = \frac{1}{2} (h_{\sigma\rho,\mu\kappa} - h_{\rho\kappa,\sigma\mu} - h_{\mu\kappa,\sigma\rho} + h_{\sigma\mu,\rho\kappa}) + o(h^2)$$

$$\text{e.g. } R_{00} = R^{\rho}{}_{0\rho 0} = R^{\rho}{}_{00\rho} + R^i{}_{0i0}$$

st have $G_{\mu\nu} = BT_{\mu\nu}$, with B a constant

$$R_{\sigma\mu\kappa} = \frac{1}{2} (h_{\sigma,\rho\mu} - h_{\rho,\sigma\mu} - h_{\mu,\sigma\rho} + h_{\rho\mu,\sigma\nu}) + o(h^2)$$

need $R_{00} = R^{\rho}{}_{0\rho 0} = \underbrace{R^{\rho}{}_{00\rho}}_{=0} + R^i{}_{0i0}$



$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$|h_{\mu\nu}| \ll 1$$

So, we must have

$$G_{\mu\nu} = B T_{\mu\nu}, \text{ with } B \text{ a}$$

work out the constant in
the weak field (Newtonian)

limit, assuming $\frac{\partial}{\partial x^0} h_{\mu\nu} = 0$

$\Gamma \sim h$

$$R_{\sigma\rho\mu\kappa} = \frac{1}{2} (h_{\sigma\rho,\mu\kappa} - h_{\rho\sigma,\mu\kappa} - h_{\mu\sigma,\rho\kappa} + h_{\mu\rho,\sigma\kappa})$$

$$\text{need } R_{00} = R^{\rho}{}_{0\rho 0} = \underbrace{R^{\alpha}{}_{0\alpha 0}}_{=0} + \underbrace{R^i{}_{0i0}}_{=-\frac{1}{2}h_{00,ii}}$$

o, we must have $G_{\mu\nu} = B T_{\mu\nu}$, with B a constant

constant in
(Newtonian)

eg: $\frac{\partial}{\partial x^0} h_{\mu\nu} = 0$

$$R_{\sigma\rho\mu\nu} = \frac{1}{2} (h_{\nu\sigma,\rho\mu} - h_{\rho\nu,\sigma\mu} - h_{\mu\sigma,\rho\nu} + h_{\rho\mu,\sigma\nu}) + o(h^2)$$

need $R_{00} = R^{\rho}{}_{0\rho 0} = \underbrace{R^0{}_{000}}_{=0} + \underbrace{R^i{}_{0i0}}_{= -\frac{1}{2} h_{00,ii}}$

$$G_{\mu\nu} = BT_{\mu\nu}, \text{ with } B \text{ a constant}$$

$$\sigma_{\mu\nu} = \frac{1}{2} (h\nu_{\sigma,\rho\mu} - h\nu_{\rho,\sigma\mu} - h\nu_{\mu\sigma,\rho\nu} + h\nu_{\rho\mu,\sigma\nu}) + o(h^2)$$

$$R_{00} = R^{\rho}{}_{0\rho 0} = \underbrace{R^0{}_{000}} + \underbrace{R^i{}_{0i0}}$$

$$\underbrace{-R_{0000}}_0$$

$$-\frac{1}{2}h_{00,ii}$$

$$= \frac{2\nabla^2\phi}{c^2}$$

$$h_{00} = -\frac{2\phi}{c^2}$$

$\Rightarrow$ Einstein equation reads

$$R = R^{00} + R^{i0}_{00} + R^{0i}_{0i} + R^{ij}_{ij}$$

⇒ Einstein equation reads

$$R = \cancel{R^{00}}_{00} + R^{i0}_{i0} + R^{oi}_{oi} + R^{ij}_{ij}$$

does not depend on h_{00}

$$= -h_{00,ii}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = B T_{\mu\nu}$$

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$\Rightarrow$ Einstein equation reads

$$R = \cancel{R_{00}^{00}} + R_{00}^{00} + R_{0i}^{0i} + R_{ij}^{ij}$$

$\rightarrow$ does not depend on h_{00}

$$= -h_{00,ii}$$

$$R_{00} + \frac{1}{2} R$$

$$= -\nabla^2 h_{00}$$

$$= \frac{2\nabla^2 \phi}{c^2}$$

$$R_{\mu\nu}$$

$$R_{00}$$

$\Rightarrow$ Einstein equation reads

$$R = \cancel{R^{00}}_{00} + R^{i0}_{i0} + R^{0i}_{0i} + R^{ij}_{ij}$$

$\rightarrow$ does not depend on h_{00}

$$= -h_{00,ii}$$

$$\underbrace{R_{00} + \frac{1}{2} R}_{\text{LHS}} = B T_{00}$$

$$- \nabla^2 h_{00} + \frac{2 \nabla^2 \phi}{c^2} = B \rho$$

$R_{\mu\nu}$
 R

⇒ Einstein equation reads

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$$-\nabla^2 h_{00}$$

$$+ \frac{2 \nabla^2 \phi}{c^2}$$

$$= B(\rho_{\text{mass}} c^2)$$

mass density

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = B T_{\mu\nu}$$

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$\Rightarrow$ Einstein equation reads

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$$-\nabla^2 h_{00}$$

$$+ \frac{2 \nabla^2 \phi}{c^2}$$

$$= B T_{00}$$

$$= B (\rho_{\text{mass}} c^2)$$

$\Rightarrow$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = B T_{\mu\nu}$$

$$R + \frac{1}{2} R = B$$

$$\frac{8\pi G}{c^4}$$

R_{ij} does not depend on h_{00}

Poisson

$$\nabla^2 \phi = 4\pi G \rho_{\text{mass}}$$

Einstein equation reads

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$$-\nabla^2 h_{00}$$

$$+ \frac{2 \nabla^2 \phi}{c^2}$$

$$= B T_{00}$$

mass density

$$= B(\rho_{\text{mass}} c^2)$$

$$\Rightarrow B = \frac{8\pi G}{c^4}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Poisson

$$\nabla^2 \phi = 4\pi G \rho_{\text{mass}}$$



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{00} \sim \rho_m c^2 \sim \frac{M}{L^3} c^2$$

$$+ A g_{\mu\nu} R = B T_{\mu\nu}$$

A, B constants.

of $T_{\mu\nu}$ is that it is conserved

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0 \quad \text{identically}$$

$G^{\mu\nu}$, the Einstein tensor

$$\nabla_\mu (T^{\mu\nu}) = 0$$

(Bianchi)

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{00} \sim \rho_m c^2 \sim \frac{M}{L^3} c^2 \quad \cdot \quad \frac{GM}{L} \sim \text{vel}^2.$$

$$R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

$T_{\mu\nu}$ is that it is conserved

$$\nabla_\mu (T^{\mu\nu}) = 0$$

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \quad \text{identically}$$

(follows from Bianchi identity)

$G^{\mu\nu}$, the Einstein tensor.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{00} \sim \rho_m c^2 \sim \frac{M}{L^3} c^2 \quad \cdot \quad \frac{GM}{L} \sim \text{vel}^2.$$

$$\text{rhs} \sim \left(\frac{c^2 \cdot \text{vel}^2}{c^4} \right) \cdot \frac{1}{L^2}$$

$$+ A g_{\mu\nu} R = B T_{\mu\nu} \quad B \text{ constants.}$$

of $T_{\mu\nu}$ is that it is conserved

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

$$\nabla_\mu (T^{\mu\nu}) = 0$$

identically

tensor.

(follows from Bianchi identity)

$$+A g_{\mu\nu} R = B T_{\mu\nu}$$

A, B constants.

of $T_{\mu\nu}$ is that it is conserved

$$\nabla_\mu (T^{\mu\nu}) = 0$$

$$\text{out} - \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \quad \text{identically.}$$

$G^{\mu\nu}$, the Einstein tensor.

$$\text{rhs} \sim \left(\frac{c^2 \text{vel}^2}{c^4} \right) \frac{1}{L^2}$$

$$\text{lhs} \sim \frac{1}{L^2} \quad (2 \text{ derivatives})$$

(follows from Bianchi identity)

Einstein equations

Einstein
wanted to
understand how
energy bends
spacetime.

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

spacetime

↓

$$R_{\mu\nu}, g_{\mu\nu}R \sim T_{\mu\nu}$$

$$R_{\mu\nu\rho\sigma} \quad T_{\mu\nu} g_{\rho\sigma}$$

$$R_{\mu\nu} + \Lambda g_{\mu\nu} R =$$

key property of $T_{\mu\nu}$ is
It turns out $\nabla^\mu T_{\mu\nu} = 0$

Einstein equations

Einstein wanted to understand how energy bends spacetime.

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

spacetime

↓

$$R_{\mu\nu}, g_{\mu\nu}R \sim T_{\mu\nu}$$

$$R_{\mu\nu} + \Lambda g_{\mu\nu} R =$$

key property of $T_{\mu\nu}$ is

It turns out $\nabla_{\mu} T^{\mu\nu} = 0$

$$\nabla_{\mu} T^{\mu\nu} = 0$$

$$R_{\mu\nu\rho\sigma} \quad T_{\mu\nu} g_{\rho\sigma}$$

$$T_{\mu\nu} \sim F_\mu F_\nu + \int \dot{x}_\mu \dot{x}_\nu \delta^4(\dots)$$

$$R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu} \quad A, B \text{ constants.}$$

key property of $T_{\mu\nu}$ is that it is conserved $\nabla_\mu (T^{\mu\nu}) = 0$

It turns out: $\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$ identically. (follows id

$G^{\mu\nu}$, the Einstein tensor.

$$\nabla_\mu T^{\mu\nu} = 0$$

$$T_{\mu\nu} \sim F_{\mu} F_{\nu} + \int \dot{x}_{\mu} \dot{x}_{\nu} \delta^4(\dots) + \int \dot{x}_{\mu} A_{\nu}$$

$$R_{\mu\nu} + A g_{\mu\nu} R = B \delta_{\mu\nu}$$

A, B constants.

key property of $T_{\mu\nu}$ is that it is conserved

$$\nabla_{\mu}(T^{\mu\nu}) = 0$$

It turns out: $\nabla_{\mu} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$ identically

$$\nabla_{\mu} T^{\mu\nu} = 0$$

$G^{\mu\nu}$, the Einstein tensor

Einstein equations

Einstein wanted to understand how energy bends spacetime.

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

spacetime
↓
energy

$$R_{\mu\nu}, g_{\mu\nu}R \sim T_{\mu\nu}$$

$$R_{\mu\nu\rho\sigma}$$

$$T_{\mu\nu} g_{\rho\sigma}$$

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$T^{\mu\nu} \sim A^\mu J^\nu$$

$$T_{\mu\nu} \sim F_\mu F_\nu + \int \dot{x}_\mu \dot{x}_\nu \delta^4(\dots) + \dots$$

$$R_{\mu\nu} + \Lambda g_{\mu\nu} R =$$

key property of $T_{\mu\nu}$ is
It turns out: $\nabla_\mu T^{\mu\nu} = 0$

ly antisymmetric

(D-3) equations.

aking all the

$$\frac{(D-1)(D-2)(D-3)}{4!}$$

exchange of any two indices.

$$\text{Action: } \int \left[(\sqrt{-g} R) + S_M \right] \propto \int \delta g_{\mu\nu} \left(G^{\mu\nu} + T^{\mu\nu} \right)$$

$\equiv$
 δS_M

into account

it magically equals the number of second derivatives of the metric tensor which cannot be set zero at a point by a coordinate transformation.

$\Rightarrow$ Einstein equation reads

$$R = R_{00} + R^{00}_{00} + R^{0i}_{0i} + R^{i0}_{i0} + R^{ij}_{ij}$$

$\rightarrow$ does not depend on h_{00}

$$= -h_{00,ii}$$

Poisson

$$\nabla^2 \phi = 4\pi G \rho_{\text{mass}}$$

$$R_{00} + \frac{1}{2} R = B T_{00}$$

$$-\frac{\nabla^2 h_{00}}{2} + \frac{2\nabla^2 \phi}{c^2}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = B T_{\mu\nu}$$

$$R_{00} + \frac{1}{2} R = B T_{00}$$

mass density

$$= B(\rho_{\text{mass}} c^2) \Rightarrow B = \frac{8\pi G}{c^4}$$

$$\Rightarrow G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{00} \sim \rho_m c^2 \sim \frac{M}{L^3} c^2 \quad \frac{GM}{L} \sim \text{vel}^2$$

Einstein equations

$$T_{\mu\nu} \sim \frac{\int \dot{x}_\mu \dot{x}_\nu \delta^4(x) + \int \dot{x}_\mu A_\nu}{R_{\mu\nu} + A g_{\mu\nu} R = B T_{\mu\nu}}$$

A, B constants

$$\text{rhs} \sim \left(\frac{c^2 \text{vel}^2}{c^4} \right) \frac{1}{L^2}$$

$$\text{lhs} \sim \frac{1}{L^2} \text{ (2 derivatives)}$$

Einstein wanted to understand how energy bends spacetime.

$$R^{\rho}_{\sigma\mu\nu} \sim T_{\mu\nu}$$

spacetime

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$$\nabla_\mu T^{\mu\nu} = 0$$

It turns out: $\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$ identically

(follows from Bianchi identity)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \sim T_{\mu\nu}$$

$G^{\mu\nu}$, the Einstein tensor.

$$R_{\mu\nu\rho\sigma} \quad T_{\mu\nu} g_{\rho\sigma} \quad g^{\mu\nu} F^{\rho\sigma} = J^\nu \quad T^{\mu\nu} \sim A^{\mu} J^\nu$$