

Title: 14/15 PSI - Relativity- Lecture 2

Date: Sep 16, 2014 09:00 AM

URL: <http://pirsa.org/14090017>

Abstract:

last time:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

last time:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

, massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

last time:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

, massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

$$\text{recall: } \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha})$$

$$\frac{dx^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \quad , \text{ massive particle, } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

$$\text{recall: } \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha})$$

$$\text{Consider a static metric } \frac{\partial}{\partial x^0} = 0$$

$$\frac{dx^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \quad , \text{ massive particle, } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

recall: $\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha})$

Consider a static metric $\frac{\partial}{\partial x^0} = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$

$$\frac{dx^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \quad , \text{ massive particle, } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

$$\text{recall: } \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\alpha\lambda,\nu})$$

Consider a static metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.

$$\frac{dx^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \quad , \text{ massive particle, } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

$$\text{recall: } \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha})$$

$$\text{Consider a static metric } \frac{\partial}{\partial x^0} g_{\mu\nu} = 0 \quad , \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1.$$

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

$\eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, weak perturbation of flat spacetime.

massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

x^μ, α)

1, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, weak perturbation of flat spacetime.
 $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

x^μ, α)

1, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, weak perturbation of flat spacetime.

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$$

massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

x^μ, α)

1, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, weak perturbation of flat spacetime.

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + o(h^2) \quad h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu - h^\mu_\nu$$

massive particle, $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

x^μ, α)

1, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, weak perturbation of flat spacetime.

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + o(h^2) \quad h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu - h^\mu_\nu + h^\mu_\nu + o(h^2) = \delta^\mu_\nu + o(h^2)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad \text{weak perturbation of flat}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + o(h^2) \quad h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$$

$$g^{\mu\kappa} g_{\kappa\nu} = \delta^\mu_\nu - h^\mu_\nu + h^\mu_\nu + o(h^2) = \delta^\mu_\nu + o(h^2)$$

and, assume $\left| \frac{dx^i}{dt} \right| \ll c$

and, assume $\left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$

geodesic equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i$

and, assume $\left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$

etc equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau}$

and, assume $\left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$

geodesic equation becomes

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau}$$

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 \quad "$$

$$\Gamma_{\nu\lambda}^{\mu} \approx \frac{1}{2} \eta^{\mu\alpha} (h_{\nu\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\alpha\lambda,\nu}) \quad \text{and, assume } \left| \frac{dx^i}{dt} \right| \ll c \Rightarrow$$

$$+ o(h^2)$$

geodesic equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau}$

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 \quad "$$

$v_{\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\lambda\nu,\alpha}$ and, assume $\left| \frac{dx^i}{d\tau} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$

the equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau}$

$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 = 0$ up to order h^2 , $\left| \frac{dx^i}{d\tau} \right|^2$.

$$v_{\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\lambda\nu,\alpha}) \quad \text{and, assume} \quad \left| \frac{dx^i}{d\tau} \right| \ll c \quad \Rightarrow \quad \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$$

2)

the equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau}$

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 = 0 \quad \text{up to order } h^2, \left| \frac{dx^i}{d\tau} \right|^2.$$

$$\Rightarrow t \propto \tau$$

$$h_{\alpha,\lambda} + h_{\lambda,\alpha} - h_{\lambda,\nu} - h_{\nu,\alpha}) \quad \text{and, assume} \quad \left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$$

2)

the equation becomes $\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dct}{d\tau} \frac{dct}{d\tau} \quad (1)$

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 = 0 \quad (2) \quad \text{up to order } h^2, \left| \frac{dx^i}{d\tau} \right|^2.$$

$$(2) \Rightarrow t \propto \tau$$

$$\therefore (1) \Rightarrow \frac{d^2 x^i}{dt^2} = -\Gamma_{00}^i c^2 = -$$

$$\Gamma_{\nu\lambda}^{\mu} \approx \frac{1}{2} \eta^{\mu\alpha} (h_{\nu\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\lambda\nu,\alpha}) \quad \text{and, assume } \left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll c$$

$$+ o(h^2)$$

geodesic equation becomes

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dt}{d\tau} \frac{dt}{d\tau} \quad (1)$$

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 \quad " = 0 \quad (2) \quad \text{up to } o(h^2)$$

$$(2) \Rightarrow t \propto \tau$$

$$\therefore (1) \Rightarrow \frac{d^2 x^i}{dt^2} = -\Gamma_{00}^i c^2 = +\frac{1}{2} h_{00;i} c^2$$

But this is just Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial \phi}{\partial x^i}$$

$$m \frac{d^2 x_i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(x)$$

where ϕ is the gravitational potential

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

where ϕ is the gravitational potential

But this is just Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i}$$

$$\Rightarrow \text{identify } h_{00}(\underline{x}) = -2 \frac{\phi(\underline{x})}{c^2}$$

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(x)$$

where ϕ is the gravitational potential

$$-\frac{GM}{r} \sim v^2$$

just Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

$$\Rightarrow \text{identify } h_{00}(\underline{x}) = -2 \frac{\phi(\underline{x})}{c^2} \quad (\text{dimensionless})$$

we shall see that Einstein's equations for $g_{\mu\nu}$ set $\nabla^2 \phi = 4\pi G\rho$

gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

where ϕ is the gravitational

$$h_{ij} = -2 \frac{\phi(\underline{x})}{c^2} \quad (\text{dimensionless})$$

$$-\frac{GM}{r} \sim v^2$$

Einstein's equations for $g_{\mu\nu}$

set

$$\nabla^2 \phi = 4\pi G \rho_{\text{mass}}(\underline{x}) \quad (\text{Poisson's equation})$$

↑ density of mass.

gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

where ϕ is the gravitational

$$g_{00} = -2\frac{\phi(\underline{x})}{c^2} \quad (\text{dimensionless})$$

$$\phi \sim -\frac{GM}{r} \sim v^2$$

Einstein's equations for $g_{\mu\nu}$ set $\nabla^2 \phi = 4\pi G \rho_{\text{mass}}(\underline{x})$ (Poisson's equation)

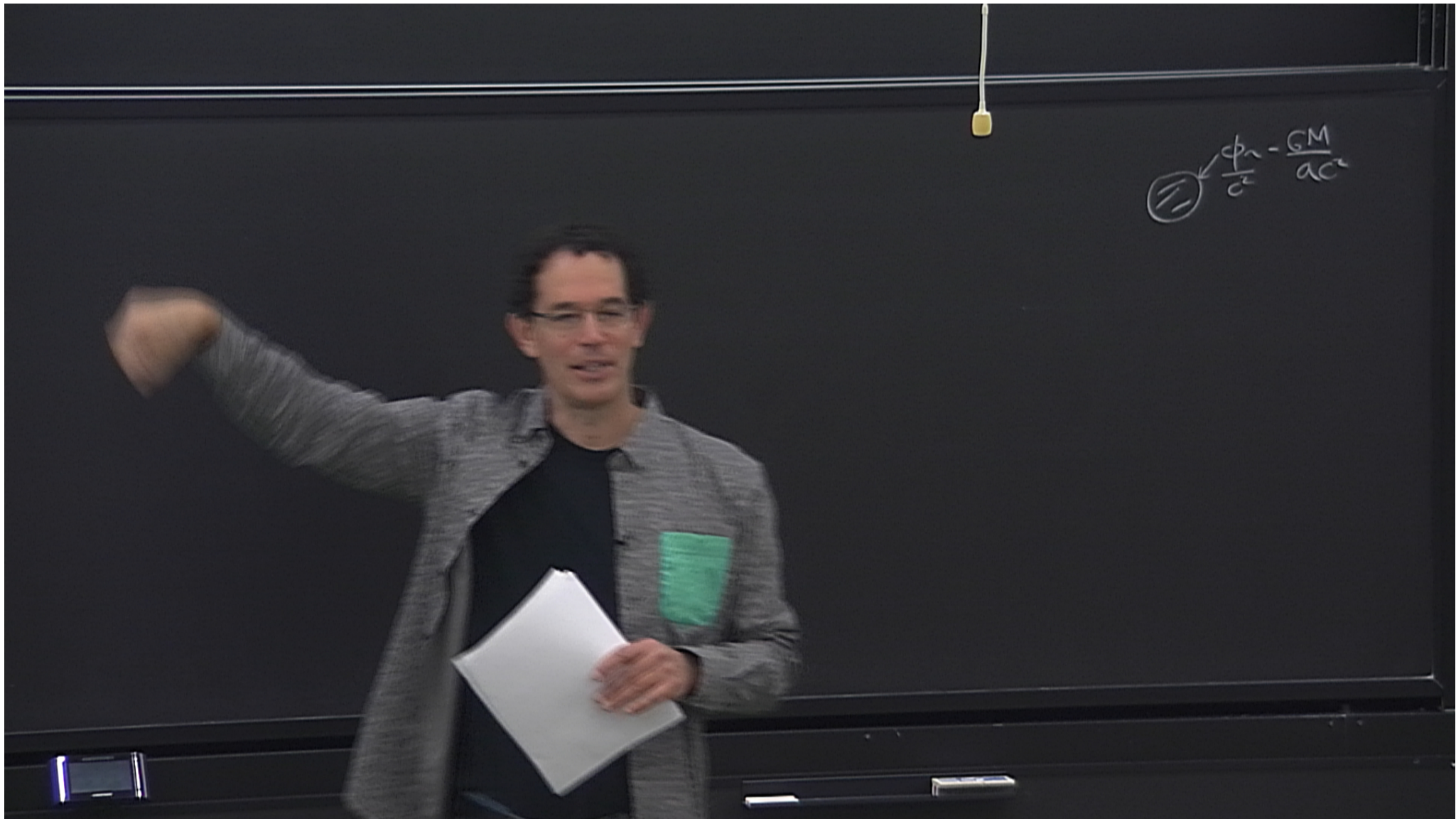
↑ density of mass.

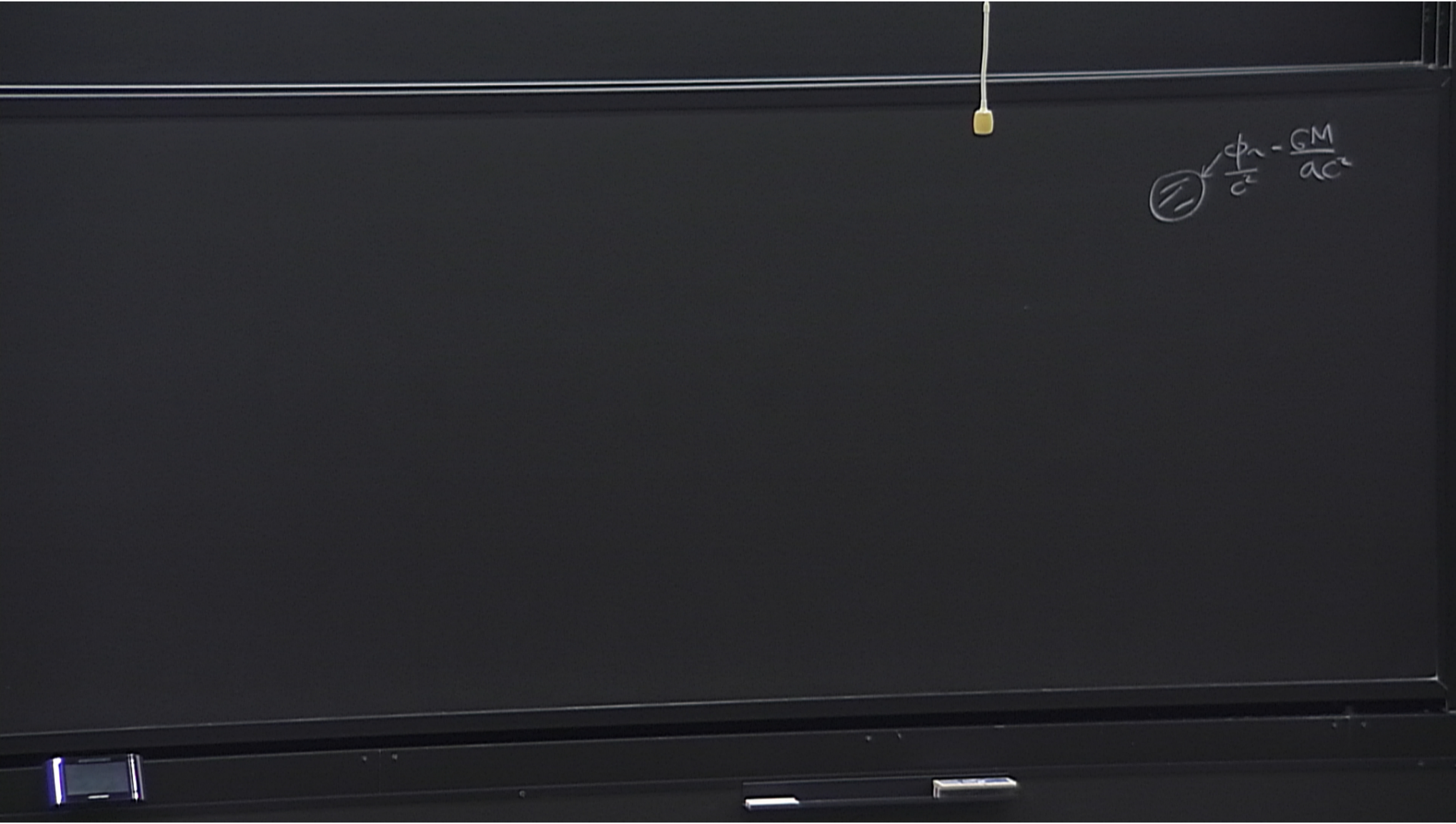
But this is just Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i$$

$\Rightarrow$ identify $\boxed{h_{00}(x) = -2 \frac{\phi(x)}{c^2}}$ (dimensionless)

we shall see that Einstein's equations for $g_{\mu\nu}$




$$\frac{\phi_n}{c^2} - \frac{GM}{ac^2}$$

But this is just Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F$$

$\Rightarrow$ identify $h_{00}(x) = -2 \frac{\phi(x)}{c^2}$ (dimensionless)



we shall see that Einstein's equations for $g_{\mu\nu}$

for a mass M , $\phi = -\frac{GM}{r}$ (outside the mass)

$$\Rightarrow g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$$

Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

where ϕ

$\Rightarrow$ identify $h_{00}(\underline{x}) = -\frac{2\phi(\underline{x})}{c^2}$ (dimensionless)

$$\phi \sim -\frac{GM}{r}$$

we shall see that Einstein's equations for $g_{\mu\nu}$ set

$$\nabla^2 \phi = 4\pi G \rho(\underline{x}) \quad (\text{Poisson})$$

$\uparrow$ density of mass
 10^{-9} at surface of earth

$$= -\frac{GM}{r} \quad (\text{outside the mass})$$

$$\Rightarrow g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$$

, deviation from -1 is

Newton's law for gravity

$$m \frac{d^2 x^i}{dt^2} = F_g^i = -m \frac{\partial}{\partial x^i} \phi(\underline{x})$$

where ϕ

if $h_{00}(\underline{x}) = -2 \frac{\phi(\underline{x})}{c^2}$ (dimensionless)

$$\phi \sim -\frac{GM}{r}$$

we see that Einstein's equations for $g_{\mu\nu}$ set

$$\nabla^2 \phi = 4\pi G \rho(\underline{x}) \quad (\text{Poisson})$$

↑ density of mass

10^{-9}	at surface of earth		
10^{-6}	"	"	Sun
10^{-4}	"	"	white dwarf
10^{-1}	"	"	neutron star

(outside mass)

$$g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$$

, deviation from -1 is

we shall see that Einstein's equations for $g_{\mu\nu}$ set $\nabla\phi = 4\pi G \rho(x)$ (force density of mass)

outside the mass)

$\Rightarrow \approx -\left(1 - \frac{2GM}{rc^2}\right)$, deviation from -1 is

10^{-9}	at surface of earth
10^{-6}	" " " Sun
10^{-4}	" " " white dwarf
10^{-1}	" " " neutron star
1	" " " black hole

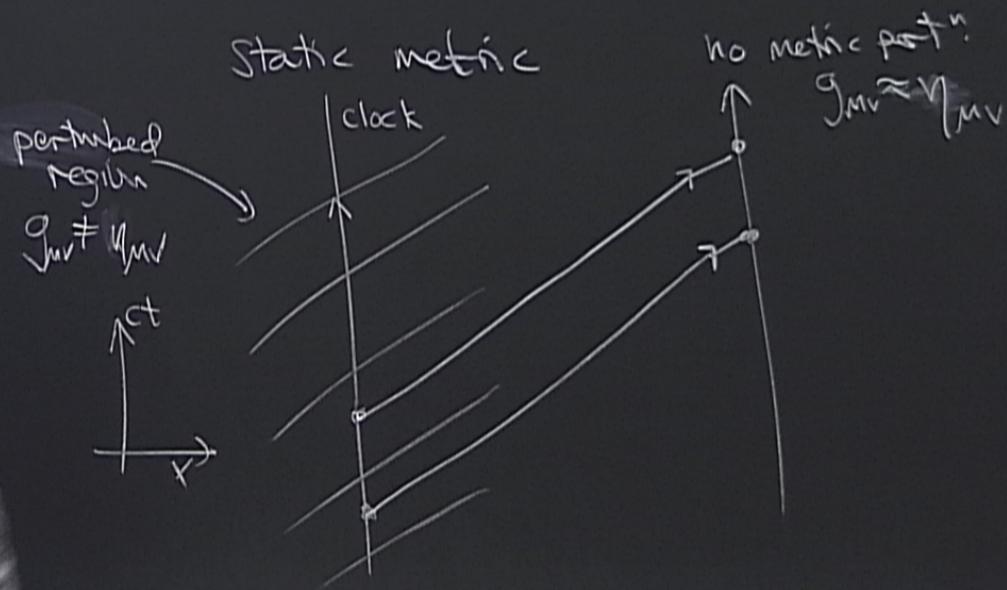
Gravitational time dilation

Gravitational time dilation

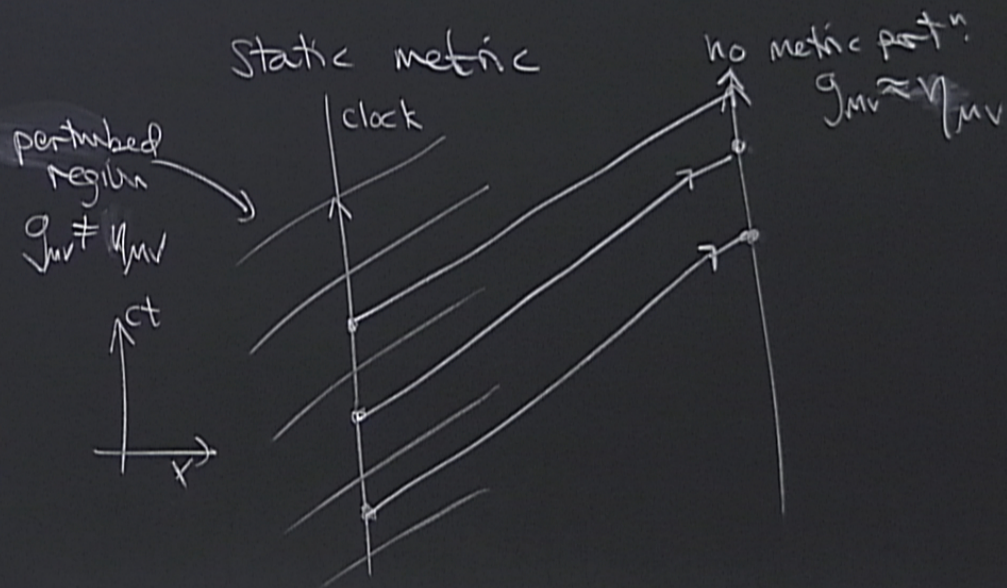
Static metric



Gravitational time dilation



Gravitational time dilation



clock : ticks are separated by $\Delta\tau = (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is stationary

$$\Delta\tau = \Delta t.$$

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is static

$$\Delta\tau = \Delta t.$$

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances.

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock
 $\Delta\tau = \Delta t$.

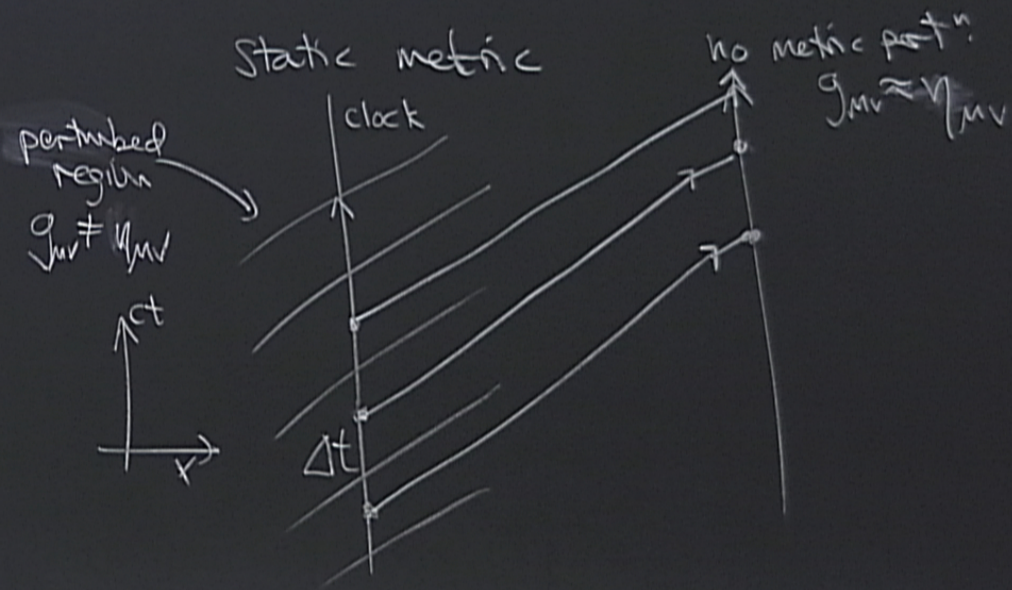
clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances. The time in this coordinate system is t .

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock
 $\Delta\tau = \Delta t$.

Gravitational time dilation



clock :

r.h.s. is coordinate
it in a coordinate
static, and
The time

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances. The time in this coordinate system is t .

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is static
clock ticks are $\Delta\tau$ apart in these
coords.

Gravitational time dilation

Static metric

clock

$g_{00} \neq 1$
 $\uparrow ct$
 $\rightarrow r$

Δt

no metric perturbation

$$g_{\mu\nu} \approx \eta_{\mu\nu}$$

Δt

clock :

r.h.s. is coordinate

it in a coordinate

static, and

The time

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2} *$

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances. The time in this coordinate system is t .

So from $*$, $\Delta\tau$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is
clock ticks are $\Delta\tau$ apart in coord

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$ *

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances. The time in this coordinate system is t .

So from *,
$$\Delta\tau = \frac{1}{c}(-g_{00}\Delta t^2)$$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is
clock ticks are $\Delta\tau$ apart in coord

clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$ *

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances.

The time in this coordinate system is t .

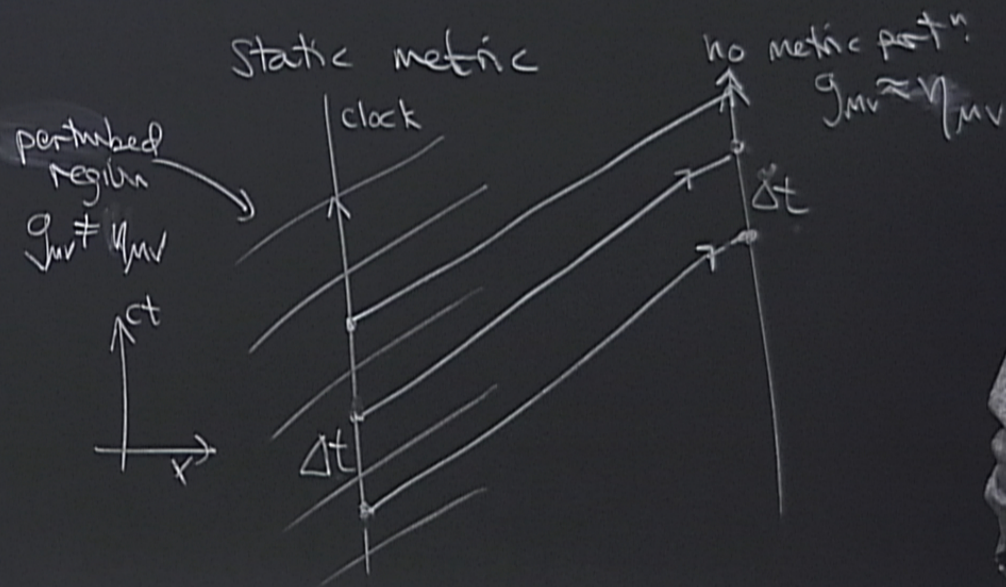
So from *, $\Delta\tau = (-g_{00}\Delta t^2)^{1/2} \Rightarrow$

$$\Delta t = \frac{\Delta\tau}{\sqrt{-g_{00}(x_{\text{clock}})}}$$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is
clock ticks are $\Delta\tau$ apart in coord

Gravitational time dilation



ion
 $\eta_{\mu\nu}$
 clock : ticks are separated by $\Delta\tau = \frac{1}{c}(-g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}} \quad *$

r.h.s. is coordinate invariant - let us evaluate it in a coordinate system in which $g_{\mu\nu}$ is static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$ at large distances.

The time in this coordinate system is t .

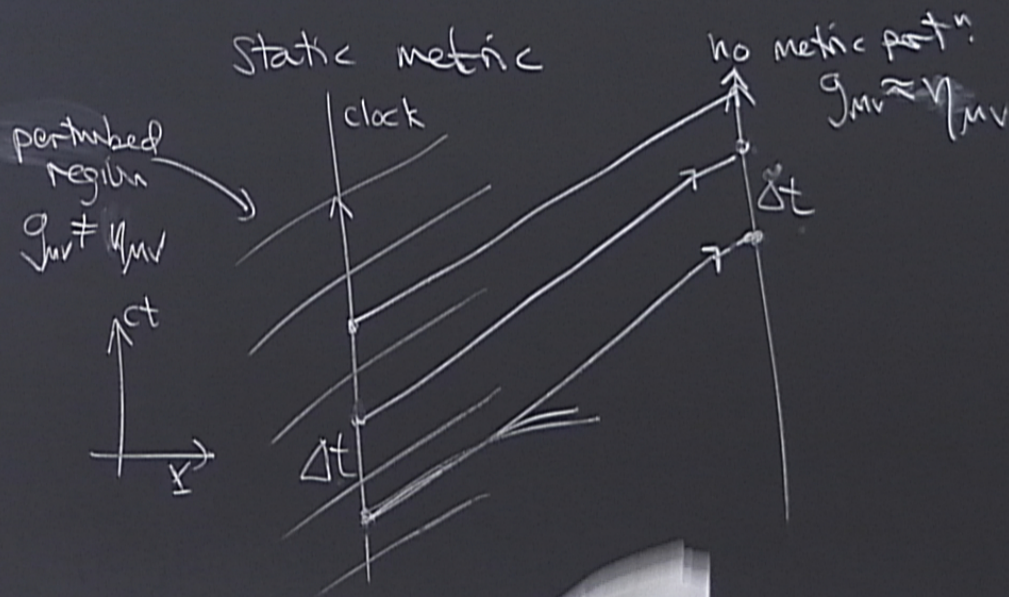
So from $*$, $\Delta\tau = (-g_{00}\Delta t^2)^{\frac{1}{2}} \Rightarrow$

$$\Delta t = \frac{\Delta\tau}{\sqrt{-g_{00}(x_{\text{clock}})}} > \Delta\tau$$

1) $g_{\mu\nu} = \eta_{\mu\nu}$

2) Lorentz frame in which clock is
 clock ticks are $\Delta\tau$ apart in coord

Gravitational time dilation



clock : ticks are

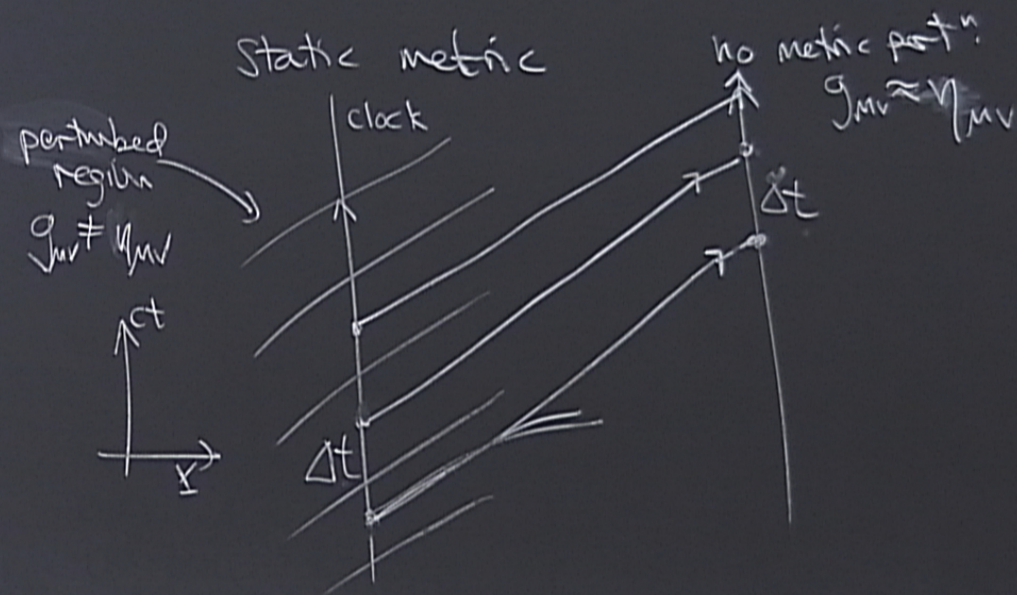
r.h.s. is coordinate invariant in a coordinate system

static, and $g_{\mu\nu} \approx \eta_{\mu\nu}$

The time in this coordinate system

So from *, $\Delta\tau =$

Gravitational time dilation



clock :

r.h.s. is coordinate

it in a coordinate

static, and

The time

So from *,

$\Rightarrow$ frequency shift

$$\text{So from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \Delta t = \frac{\Delta\tau}{\sqrt{-g_{00}(x_{\text{clock}})}} > \Delta\tau$$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface of sun.

ebka

So from *, $\Delta\tau = \sqrt{-g_{00}\Delta t} \Rightarrow \Delta\tau = \sqrt{-g_{00}}\Delta t$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms

expts: Pound-Rebka

$$\text{So from } *, \Delta\tau = \sqrt{-g_{00}\Delta t} \Rightarrow \Delta\tau = \sqrt{-g_{00}}\Delta t$$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms

expts: Pound-Rebka confirmed $\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few per cent
 Vessot et al masers on satellite/earth, confirmed GR to $\sim 2 \times 10^{-4}$.

The time in this coordinate system is t .
So from *, $\Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \Delta t = \sqrt{-\frac{\Delta\tau^2}{g_{00}}}$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for at

expts: Pound-Rebka confirmed $\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few per cent
Vessot et al masers on satellite/earth, confirmed GR to $\sim 2 \times 10^{-4}$.

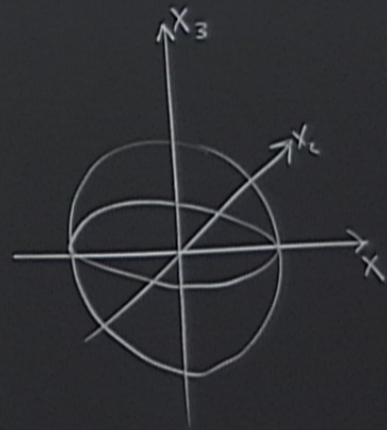
into : tensor

Math interlude : tensors and covariant derivatives

Math interlude : tensors and covariant derivatives

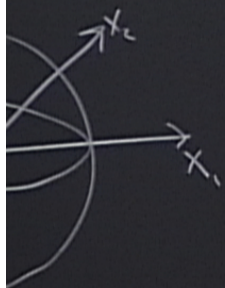
$g_{\mu\nu}$ defines the line element on spacetime

Simplest example of a curved space : a sphere S^2
 $x_1^2 + x_2^2 + x_3^2 = 1$



derivatives

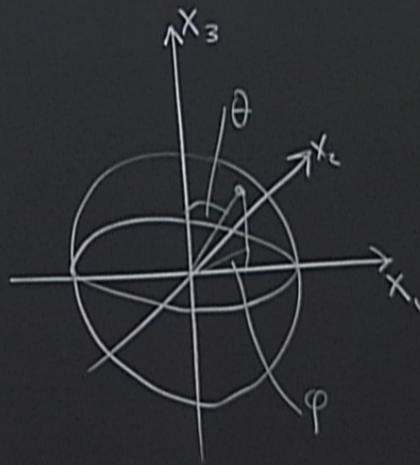
coordinates on S^2 : spherical
 e.g. polars $x_1 = \sin \theta \cos \varphi$
 $x_2 = \sin \theta \sin \varphi$
 $x_3 = \cos \theta$



tensors and covariant derivatives

at on spacetime

ce : a sphere S^2
 $x_3^2 = 1$



coordinates on S^2 : spherical
e.g. polars

$$\begin{aligned}x_1 &= \sin\theta \cos\phi \\x_2 &= \sin\theta \sin\phi \\x_3 &= \cos\theta\end{aligned}$$

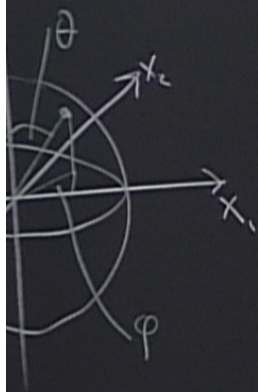
ant derivatives

coordinates on S^2 : spherical
e.g. polars

$$x_1 = \sin \theta \cos \varphi$$

$$x_2 = \sin \theta \sin \varphi$$

$$x_3 = \cos \theta$$



$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \quad | \quad x_1^2 + x_2^2 + x_3^2 = 1$$

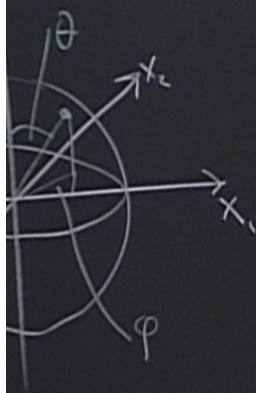
ant derivatives

coordinates on S^2 : spherical e.g. polars

$$x_1 = \sin\theta \cos\varphi$$

$$x_2 = \sin\theta \sin\varphi$$

$$x_3 = \cos\theta$$



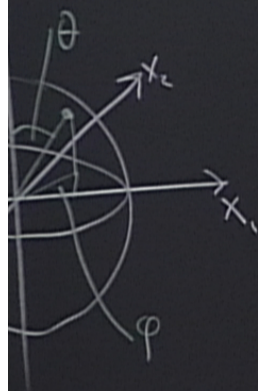
$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$= d\theta^2 + \sin^2\theta d\varphi^2$$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

ant derivatives

coordinates on S^2 : spherical e.g. polars

$$\begin{aligned}x_1 &= \sin\theta \cos\varphi \\x_2 &= \sin\theta \sin\varphi \\x_3 &= \cos\theta\end{aligned}$$


$$\begin{aligned}ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\&= d\theta^2 + \sin^2\theta d\varphi^2\end{aligned}$$

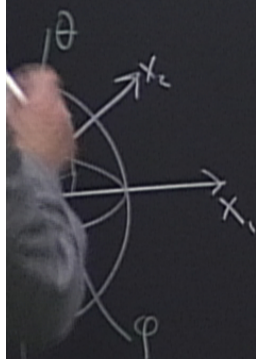
$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

These coordinates

ant derivatives

coordinates on S^2 : spherical e.g. polars

$$\begin{aligned}x_1 &= \sin\theta \cos\varphi \\x_2 &= \sin\theta \sin\varphi \\x_3 &= \cos\theta\end{aligned}$$


$$\begin{aligned}ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\&= d\theta^2 + \sin^2\theta d\varphi^2\end{aligned} \quad \left| \begin{aligned}x_1^2 + x_2^2 + x_3^2 &= 1\end{aligned} \right.$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

These coordinates are singular at poles.

ant derivatives

coordinates on S^2 :

spherical
e.g. polars

$$\begin{aligned}x_1 &= \sin\theta \cos\varphi \\x_2 &= \sin\theta \sin\varphi \\x_3 &= \cos\theta\end{aligned}$$

$$\begin{array}{cc}NP & SP \\0 \leq \theta \leq \pi & \\0 < \varphi \leq 2\pi & \end{array}$$

identified

$$\begin{aligned}ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\&= d\theta^2 + \sin^2\theta d\varphi^2\end{aligned}$$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

These coordinates are singular at poles.

ant derivatives

coordinates on S^2 :

spherical
e.g. polars

$$\begin{aligned}x_1 &= \sin\theta \cos\varphi \\x_2 &= \sin\theta \sin\varphi \\x_3 &= \cos\theta\end{aligned}$$

$$\begin{array}{cc}NP & SP \\0 \leq \theta \leq \pi & \\0 < \varphi \leq 2\pi & \end{array}$$

↑
identified

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

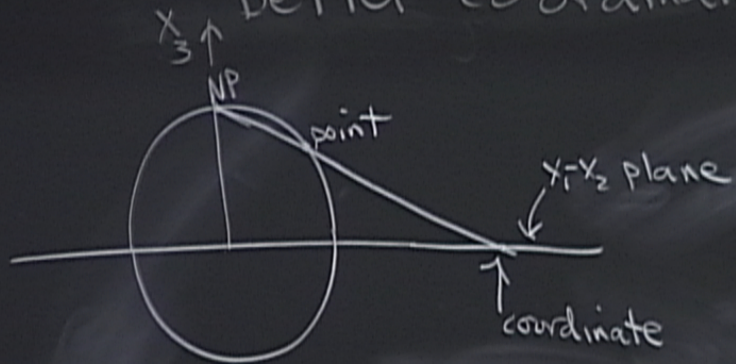
$$= d\theta^2 + \sin^2\theta d\varphi^2$$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

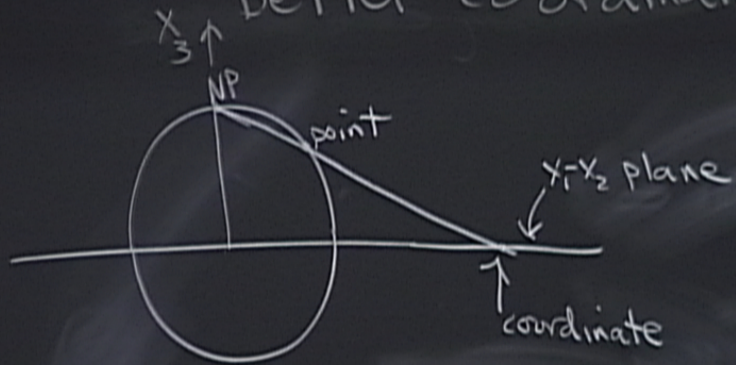
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

These coordinates are singular at poles.

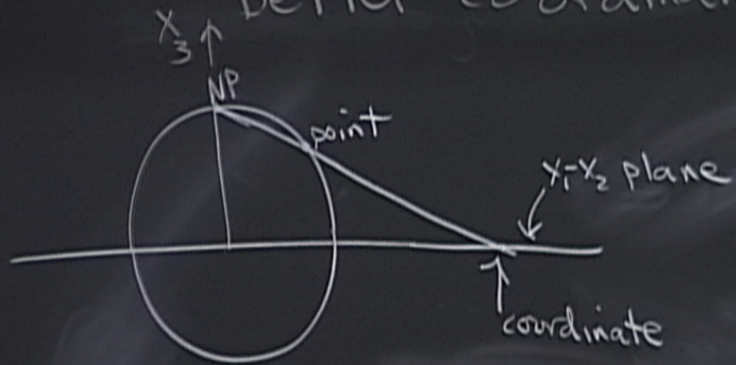
better coordinates



better coordinates in projective coordinates

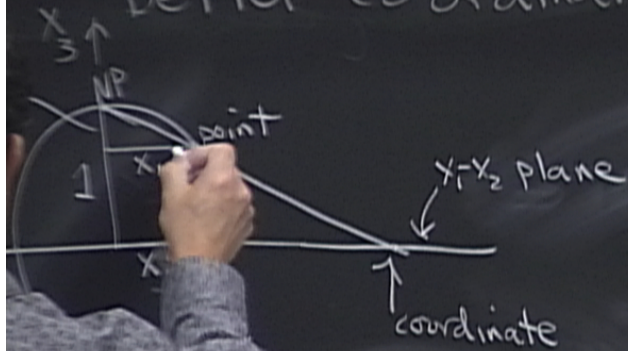


better coordinates in projective coordinates



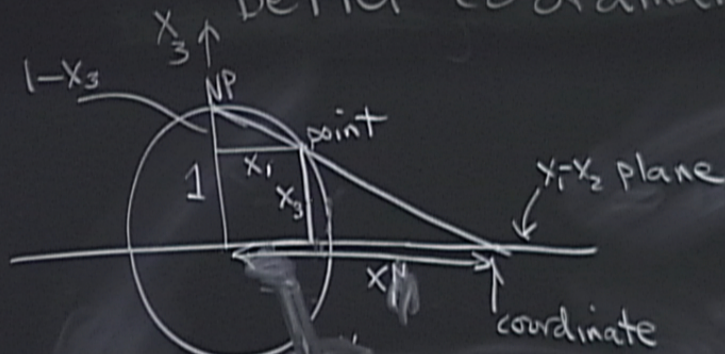
N.P. proj coords: (x_1^N, x_2^N)

better coordinates : projective coordinates



N.P. proj coords: $(x_1^N, x_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$

better coordinates in projective coordinates



N.P. proj coords: $(x_1^N, x_2^N) =$

$$\frac{x_1}{1-x_3} = \frac{x_1^N}{1}$$

projective coordinates

N.P. proj coords: $(x_1^N, x_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$

S.P. proj coords: $(x_1^S, x_2^S) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)$

projective coordinates

N.P. proj coords: $(X_1^N, X_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$

S.P. proj coords: $(X_1^S, X_2^S) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)$

$$-\infty < X_1^{N,S}, X_2^{N,S} < \infty.$$

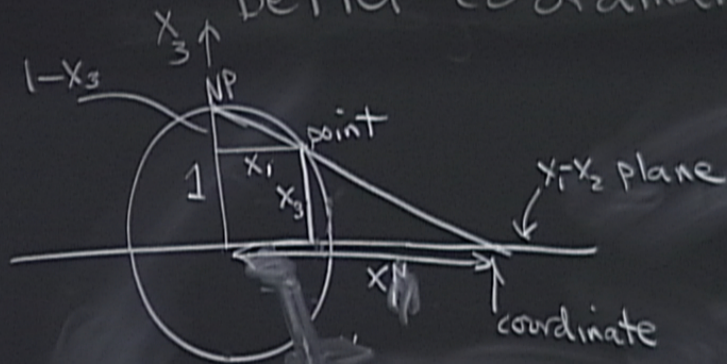
projective coordinates

N.P. proj coords: $(X_1^N, X_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$

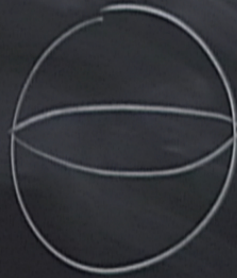
S.P. proj coords: $(X_1^S, X_2^S) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)$

$$-\infty < X_1^{NS}, X_2^{NS} < \infty.$$

better coordinates in projective coordinates



$$\frac{x_1}{1-x_3} = \frac{x_1^N}{1}$$



N.P. proj coords:

S.P. proj coords:

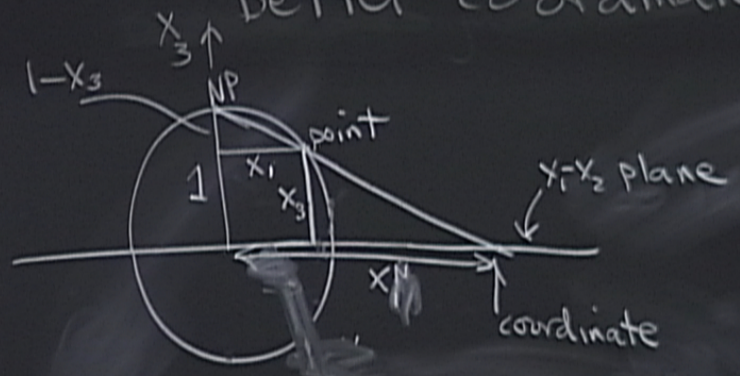
$$(X_1^N, X_2^N) = \left(\frac{X_1}{1-X_3}, \frac{X_2}{1-X_3} \right) \quad \text{Singular at N.P.,}$$

$$(X_1^S, X_2^S) = \left(\frac{X_1}{1+X_3}, \frac{X_2}{1+X_3} \right) \quad \text{Singular at SP,}$$

$-\infty < X_1^{N,S}, X_2^{N,S} < \infty.$

but regular and 1-1 everywhere else

better coordinates : projective coordinates



$$\frac{x_1}{1-x_3} = \frac{x_1^N}{1}$$



N.P. $(x_1^N, x_2^N) =$
S.P. (x_1^S, x_2^S)

projective coordinates

N.P. proj coords: $(X_1^N, X_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)$ - singular at N.P. $\left. \begin{array}{l} -\infty < X_1^{N,S}, X_2^{N,S} < \infty \\ \text{but regular and 1-1} \end{array} \right\}$

S.P. proj coords: $(X_1^S, X_2^S) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)$ - singular

Transformation between them is good everywhere except

$$X_1^N = \frac{X_1^S}{X_1^{S2} + X_2^{S2}}, \quad X_2^N = \frac{X_2^S}{X_1^{S2} + X_2^{S2}}$$

projective coordinates

N.P. proj coords: $(X_1^N, X_2^N) = \left(\frac{X_2}{1-X_3} \right)$ - singular at N.P. $\left. \begin{array}{l} -\infty < X_1^{N,S}, X_2^{N,S} < \infty \\ \text{but regular and 1-1} \end{array} \right\}$

S.P. proj coords: $(X_1^S, X_2^S) = \left(\frac{X_2}{1+X_3} \right)$ - singular at S.P.

Transformation between them is good

except the poles
("transition functions")

projective coordinates

$$\text{N.P. proj coords: } (X_1^N, X_2^N) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right) \quad -\infty < X_1^{N,S}, X_2^{N,S} < \infty$$

$$\text{S.P. proj coords: } (X_1^S, X_2^S) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right) \quad \left. \begin{array}{l} \text{Singular at N.P.} \\ \text{Singular at S.P.} \end{array} \right\} \text{ but regular and 1-1 e}$$

Transformation between them is good everywhere except the poles

$$X_1^N = \frac{X_1^S}{X_1^{S2} + X_2^{S2}}, \quad X_2^N = \frac{X_2^S}{X_1^{N2} + X_2^{N2}} \quad (\text{"transition functions"})$$

and, assume $\left| \frac{dx^i}{dt} \right| \ll c \Rightarrow \left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dt}{d\tau} \right|$

$$\therefore (1) \Rightarrow \frac{d^2 x}{dt^2} = -1_{00} C = +\frac{1}{2} h_{00,i} C$$

line elem

$$ds^2 = \frac{4(dx_1^{N^2} + dx_2^{N^2})}{(1 + x_1^{N^2} + x_2^{N^2})^2}$$

and similarly for x^S coords.

(check x^S coord)

$$\therefore (1) \Rightarrow \frac{d^2 x}{dt^2} = -1_{00} c = +\frac{1}{2} h_{00,i} c$$

line element $ds^2 = \frac{4(dx_1^{N^2} + dx_2^{N^2})}{(1 + x_1^{N^2} + x_2^{N^2})^2}$

and similarly for x^S coords.

(check via $x_1^N = \cot \frac{\theta}{2} \cos \varphi$
 $x_2^N = \cot \frac{\theta}{2} \sin \varphi$)

line element $ds^2 = \frac{4(dx_1^{N^2} + dx_2^{N^2})}{(1 + x_1^{N^2} + x_2^{N^2})^2}$

in general, have an ∞ number
of coord systems, have to
check that everything we do
is equivalently described by any of them

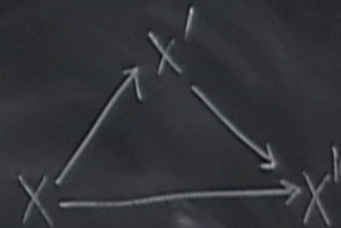
(check via $x_1^N = \cot \theta/2 \cos \varphi$
 $x_2^N = \cot \theta/2 \sin \varphi$)



line element $ds^2 = \frac{4(dx_1^{N^2} + dx_2^{N^2})}{(1 + x_1^{N^2} + x_2^{N^2})^2}$

in general, have an ∞ number
of coord systems, have to
check that everything we do
is equivalently described by any of them

(check via $x_1^N = \cot \frac{\theta}{2} \cos \varphi$
 $x_2^N = \cot \frac{\theta}{2} \sin \varphi$)



and similarly for x^S coords.

ie, need $X''(x) = X''(X'(x))$

Tensors

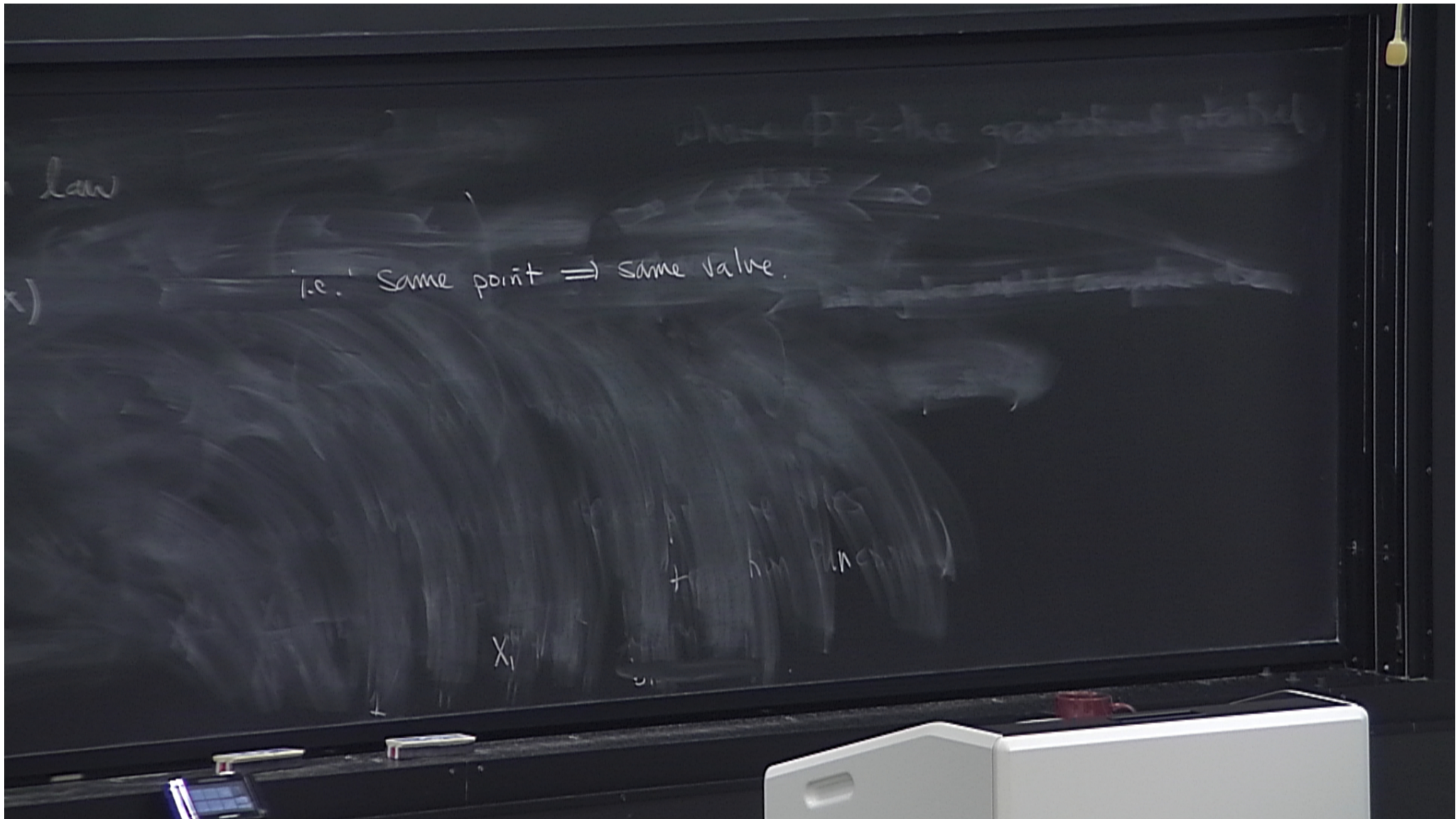
Simplest - Scalars

Tensors

Transformation law

Simplest - Scalars

$$\phi'(x') = \phi(x)$$



Tensors

Transformation law

Simplest - Scalars

$$\phi'(x') = \phi(x)$$

vectors

Contravariant
(trees)

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x)$$

Covariant
(valleys)

$$V'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x)$$

Transformation law

Scalars

$$\phi'(x') = \phi(x)$$

'i.e.' same point $\Rightarrow$ same value.

Contravariant
(trees)

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x)$$

e.g. $\frac{dx^{\mu}}{dx}$

Covariant
(valleys)

$$V'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x)$$

e.g. $\frac{\partial}{\partial x^{\mu}}$

higher valence tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

higher valance tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

general rules for getting tensors from tensors : add $A^{\mu}_{\nu} + B^{\mu}_{\nu}$
multiply $A^{\mu}_{\nu} B^{\alpha}_{\beta}$
contract indices

e tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

getting tensors from tensors : add $A^{\mu}_{\nu} + B^{\mu}_{\nu}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

multiply $A^{\mu}_{\nu} B^{\alpha}_{\beta}$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

contract indices $A^{\mu}_{\nu} B^{\alpha}_{\mu}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

e tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

getting tensors from tensors : add $A^{\mu}_{\nu} + B^{\mu}_{\nu}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

multiply $A^{\mu}_{\nu} B^{\alpha}_{\beta}$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

contract indices $A^{\mu}_{\nu} B^{\alpha}_{\mu}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

etc.

as valence $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

add $A^M_v + B^M_v$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

multiply $A^M_v B^\alpha_\beta$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

contract $A^M_v B^\alpha_\mu$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

etc.

$$T'^{\mu\nu\lambda}_{\alpha\beta}(x') = \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial x} \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} T$$

higher valence tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence

general rules for getting tensors from tensors : add A^μ_ν

$$T^{\mu\alpha\nu\lambda}_{\sigma\beta} = g^{\alpha\gamma} T^{\mu\gamma\nu\lambda}_{\sigma\beta}$$

multiply
contract
indices

higher valence tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence

general rules for getting tensors from tensors : add A^{μ}_{ν}

$$A^{\mu} = (\phi, A^i)$$

$$T^{\mu\alpha\nu\lambda}_{\alpha\beta} = g^{\alpha\gamma} T^{\mu\gamma\nu\lambda}_{\alpha\beta}$$

multiply
contract
indices

higher valence tensors e.g. $T^{\mu\nu\lambda}_{\alpha\beta}$ has valence

general rules for getting tensors from tensors : add A^{μ}_{ν}

$$A^{\mu} = (\phi, A^i)$$

$$T^{\mu\alpha\nu\lambda}_{\alpha\beta} = g^{\alpha\gamma} T^{\mu\gamma\nu\lambda}_{\alpha\beta}$$

multiply
contract
indices

$$g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x} \frac{\partial x^{\nu}}{\partial x}$$

covariant derivative

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\mu\lambda} A^{\lambda\dots}_{\beta\dots} - \Gamma^\lambda_{\mu\beta} A^{\alpha\dots}_{\lambda\dots}$$

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\mu\lambda} A^{\lambda\dots}_{\beta\dots} - \Gamma^\lambda_{\mu\beta} A^{\alpha\dots}_{\lambda\dots}$$

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\gamma\mu} A^{\gamma\dots}_{\beta\dots}$$

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\gamma\mu} A^{\gamma\dots}_{\beta\dots} -$$

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^{\alpha}_{\gamma\mu} A^{\gamma\dots}_{\beta\dots} -$$

$$\frac{\partial x}{\partial x'} \frac{\partial x'}{\partial x} = \delta$$

$$- \square_{\beta\mu}^{\gamma} A_{\gamma\dots}^{\alpha\dots} \text{ etc.}$$

+ derivative

$$A_{\beta\dots}^{\alpha\dots} \equiv \partial_{\mu} A_{\beta\dots}^{\alpha\dots} + \Gamma_{\gamma\mu}^{\alpha} A_{\beta\dots}^{\gamma\dots} - \Gamma_{\beta\mu}^{\gamma} A_{\gamma\dots}^{\alpha\dots} \text{ etc.}$$

i.e. one Γ for each index, + for upper, - for lower

covariant derivative

$$\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^\alpha_{\gamma\mu} A^{\gamma\dots}_{\beta\dots} - \Gamma^\gamma_{\beta\mu} A^{\alpha\dots}_{\gamma\dots} \text{ etc.}$$

i.e. one Γ for each index, + for upper, -

The covariant derivative of a tensor is a tensor.

covariant derivative

$$\partial_\mu \rightarrow \nabla_\mu A^{\alpha\dots}_{\beta\dots} \equiv \partial_\mu A^{\alpha\dots}_{\beta\dots} + \Gamma^{\alpha}_{\delta\mu} A^{\delta\dots}_{\beta\dots} - \Gamma^{\gamma}_{\beta\mu} A^{\alpha\dots}_{\gamma\dots} \text{ etc.}$$

$\frac{dx'}{dx} = \delta$
i.e. one Γ for each index, + for upper, - for lower

The covariant derivative of a tensor is a tensor.

If a tensor is zero in ^{one} coordinate system, it is zero in all coordinate systems.
(everywhere)

Tensors

Transformation law

$$\phi'(x') = \phi(x)$$

e.g. $\nabla_\mu g_{\alpha\beta}$

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha}^r g_{r\nu} - \Gamma_{\nu\alpha}^r g_{\mu r} = g_{\mu\nu,\alpha}$$

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha}^r g_{r\nu} - \Gamma_{\nu\alpha}^r g_{\mu r} = g_{\mu\nu,\alpha}$$

||
 $\frac{\partial}{\partial x^\alpha} g_{\mu\nu}$

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha}^{\rho} g_{\rho\nu} - \Gamma_{\nu\alpha}^{\rho} g_{\mu\rho} = \underbrace{g_{\mu\nu,\alpha}}_{\partial_{\overline{x}^\alpha} g_{\mu\nu}}$$

$$\Gamma_{\mu\alpha}^{\rho} = \frac{1}{2} g^{\rho\beta} (g_{\mu\beta,\alpha} + g_{\alpha\beta,\mu} - g_{\mu\alpha,\beta})$$

in locally flat coords

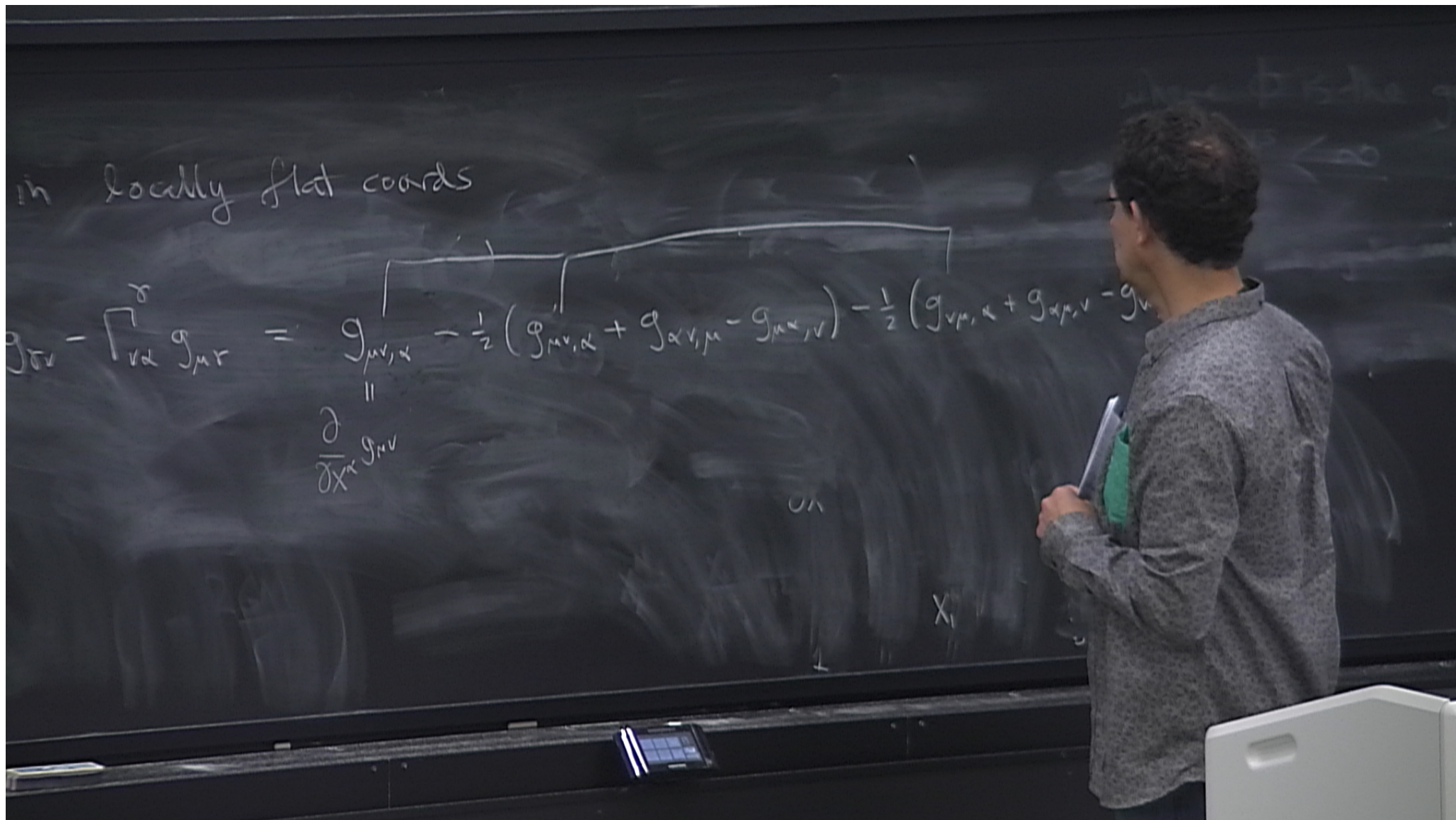
$$g_{\alpha\alpha} - \Gamma_{\alpha\alpha}^{\alpha} g_{\mu\mu} = g_{\mu\nu,\alpha} + \frac{1}{2} (g_{\mu\nu,\alpha} + g_{\alpha\nu,\mu} - g_{\mu\alpha,\nu}) - \frac{1}{2} (g_{\nu\mu,\alpha} + g_{\alpha\mu,\nu} - g_{\alpha\nu,\mu})$$

$\parallel$
 $\frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}$

in locally flat coords

$$g_{\alpha\beta} - \Gamma_{\alpha\beta}^{\gamma} g_{\gamma\delta} = g_{\mu\nu,\alpha} - \frac{1}{2} (g'_{\mu\nu,\alpha} + g_{\alpha\nu,\mu} - g_{\mu\alpha,\nu}) - \frac{1}{2} (g_{\nu\mu,\alpha} + g_{\alpha\mu,\nu})$$

$\parallel$
 $\frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}$



in locally flat coords

$$\partial_{\alpha} - \Gamma_{\alpha}^{\gamma} g_{\mu\gamma} = \underbrace{g_{\mu\nu,\alpha} - \frac{1}{2}(g_{\mu\nu,\alpha} + g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}) - \frac{1}{2}(g_{\nu\mu,\alpha} + g_{\alpha\mu,\nu} - g_{\alpha\nu,\mu})}_{\parallel \frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}} = 0$$

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

$$(\nabla_\mu g_{\alpha\beta})' = \frac{\partial x^\alpha}{\partial x'^\alpha} \frac{\partial x^\beta}{\partial x'^\beta} \frac{\partial x^\gamma}{\partial x'^\gamma} (\nabla_\mu g_{\alpha\beta})$$

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha}^r g_{r\nu} - \Gamma_{\nu\alpha}^r g_{\mu r} = g_{\mu\nu,\alpha} - \frac{1}{2} (g_{\mu\nu,\alpha} + g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu})$$

$$\Gamma_{\mu\alpha}^r = \frac{1}{2} g^{r\beta} (g_{\mu\beta,\alpha} + g_{\alpha\beta,\mu} - g_{\mu\alpha,\beta})$$

$$\frac{\partial}{\partial x^\alpha} g_{\mu\nu}$$

e.g. $\nabla_\mu g_{\alpha\beta} = 0$ in locally flat coords

$$(\nabla_\mu g_{\alpha\beta})' = \frac{\partial x^\alpha}{\partial x'^\alpha} \frac{\partial x^\beta}{\partial x'^\beta} \frac{\partial x^\gamma}{\partial x'^\gamma} (\nabla_\gamma g)$$

Check $\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha}^r g_{r\nu} - \Gamma_{\nu\alpha}^r g_{\mu r} = g_{\mu\nu,\alpha} - \frac{1}{2} (g_{\mu\nu,\alpha} + g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu})$

$\parallel$
 $\frac{\partial}{\partial x^\alpha} g_{\mu\nu}$

$$\Gamma_{\mu\alpha}^r = \frac{1}{2} g^{r\beta} (g_{\mu\beta,\alpha} + g_{\alpha\beta,\mu} - g_{\mu\alpha,\beta})$$

$$\text{So from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \left[\Delta\tau = \sqrt{-g_{00}(x_{\text{clock}})} \right]$$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface

5: Pound-Rebka measured $\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few percent
 Vessot lasers on satellite/earth, confirmed GR to $\sim 2 \times 10^{-4}$.

whereas $[\partial_\mu, \partial_\nu] = 0$, $[\nabla_\mu, \nabla_\nu] \neq 0$

$\Rightarrow$ frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms on sun

Pound-Rebka confirmed $\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few per cent

Vessot et al masers on satellite/earth, confirmed GR 10^{-10} .

$\nabla_\mu T^\alpha_\beta$; whereas $[\partial_\mu, \partial_\nu] = 0$, $[\nabla_\mu, \nabla_\nu]$

$\nabla_\mu \nabla_\nu S_\sigma$ $\xrightarrow{\text{Riemann tensor}}$ $R^\sigma_{\rho\mu\nu} S_\sigma$

So from *, $\Delta\tau = \sqrt{(-g_{00}\Delta t^2)} \Rightarrow \left[\Delta t - \sqrt{-g_{00}(x_{\text{clock}})} \right]$

frequency shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface of sun.

confirmed $\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few per cent

masers on satellite/earth, confirmed GR to $\sim 2 \times 10^{-4}$.

whereas $[\partial_\mu, \partial_\nu] = 0$, $[\nabla_\mu, \nabla_\nu] \neq 0$

T^α_β

$\nearrow$ Riemann tensor

$$\nabla_\mu \nabla_\nu S_p - \nabla_\nu \nabla_\mu S_p = -\overbrace{R^\sigma}_{\text{Riemann tensor}}{}_{\mu\nu} S_\sigma \quad \text{for any } S_p$$

$$\text{from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \Delta t = \sqrt{-g_{00}(x_{\text{clock}})}$$

shift in atomic emission lines $\sim 10^{-6}$ c - atoms on surface of sun.

$$\frac{\Delta\nu}{\nu} \sim 10^{-9} \text{ to a few per cent}$$

in satellite/earth, confirmed GR to ~ 2

whereas $[\partial_\mu, \partial_\nu] = 0$, $[\nabla_\mu, \nabla_\nu] =$ curvature tensor

$$\nabla_\mu \nabla_\nu S_p = \nabla_\nu \nabla_\mu S_p \text{ for any } S_p$$

$$\text{from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \left[\Delta t - \sqrt{-g_{00}(x_{\text{clock}})} \right]$$

shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface of sun.

$\frac{\Delta\nu}{\nu} \sim 10^{-9}$ to a few per cent
in satellite/earth, confirmed GR $\sim 10^{-4}$.

whereas $[\partial_\mu, \partial_\nu] = 0$, $[\nabla_\mu, \nabla_\nu] \neq 0$

$\nabla_\mu \nabla_\nu S_p = -\overbrace{R^\sigma_{p\mu\nu}}^{\text{Riemann tensor}} S_\sigma$ for any S_p

$$\text{from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \Delta\tau = \sqrt{-g_{00}(x_{\text{clock}})}$$

shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface of sun.

$$\frac{\Delta\nu}{\nu} \sim 10^{-9} \text{ to a few per cent}$$

in satellite/earth, confirmed GR to $\sim 2 \times 10^{-4}$

$$\text{whereas } [\partial_\mu, \partial_\nu] = 0, \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$\nabla_\mu \nabla_\nu S_p - \nabla_\nu \nabla_\mu S_p$$

Riemann tensor

$$\nabla_\mu \nabla_\nu S_p - \nabla_\nu \nabla_\mu S_p \text{ for any } S_p$$

$$\frac{\partial f(x(x'))}{\partial x^\alpha} = \frac{\partial x^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$$

$$\frac{\partial}{\partial x^\alpha} \dot{x}^\beta$$

$$\text{from } *, \Delta\tau = \sqrt{-g_{00}\Delta t^2} \Rightarrow \Delta t = \sqrt{-g_{00}(x_{\text{clock}})}$$

shift in atomic emission lines $\sim 10^{-6}$ for atoms on surface of sun.

$$\frac{\Delta\nu}{\nu} \sim 10^{-9} \text{ to a few per cent}$$

in satellite/earth, confirmed GR to $\sim 10^{-5}$

$$\text{whereas } [\partial_\mu, \partial_\nu] = 0, \quad [\nabla_\mu, \nabla_\nu] \neq 0$$

$$\nabla_\mu \nabla_\nu S_p = -$$

$$\frac{\partial f(x(x'))}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

$$\frac{\partial}{\partial x'^\alpha} V^\beta$$

$$\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x'^\beta}{\partial x^\mu} V^\mu \right)$$

→ Riemann tensor

$$\boxed{R^\sigma}_{\rho\mu\nu} S_\sigma \quad \text{for any } S_\rho$$