

Title: 14/15 PSI - Relativity- Lecture 2

Date: Sep 15, 2014 09:00 AM

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Abstract:

Volume element on spacetime

recall,

one can always
choose coordinates st.

$$g_{\mu\nu} = \eta_{\mu\nu} \text{ at a point}$$

"locally flat" coordinates
 $\hat{x}^a$

; volume element on flat spacetime

Volume element on spacetime

: volume element on flat spacetime $dV = d\hat{x}^0 d\hat{x}^1 d\hat{x}^2 d\hat{x}^3$ (invariant)

also $dV = \det\left(\frac{\partial \hat{x}^a}{\partial x^\alpha}\right) d^4x$ for arbitrary coords x^μ

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$y(x)$

$$d^4 y = \det\left(\frac{\partial y^\alpha}{\partial x^\mu}\right) d^4 x$$

; volume element on flat spacetime

$$\text{also } dV = \det\left(\frac{\partial \hat{x}^a}{\partial x^\alpha}\right) d^4 x$$

element on spacetime

∴ volume element on flat spacetime $dV = d\hat{x}^0 d\hat{x}^1 d\hat{x}^2 d\hat{x}^3 = d^4\hat{x}$ (invariant)

also $d^4\hat{x} = \det\left(\frac{\partial \hat{x}^a}{\partial x^\alpha}\right) d^4x$ for arbitrary coords x^μ

$$= \epsilon^{\alpha\beta\gamma\delta} \frac{\partial \hat{x}^0}{\partial x^\alpha} \frac{\partial \hat{x}^1}{\partial x^\beta} \frac{\partial \hat{x}^2}{\partial x^\gamma} \frac{\partial \hat{x}^3}{\partial x^\delta} d^4x$$

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also $d^4 \hat{x} = \det\left(\frac{\partial \hat{x}^a}{\partial x^\mu}\right) d^4 x$

$$= \epsilon^{\alpha\beta\gamma\delta} \frac{\partial \hat{x}^0}{\partial x^\alpha} \frac{\partial \hat{x}^1}{\partial x^\beta} \frac{\partial \hat{x}^2}{\partial x^\gamma} \frac{\partial \hat{x}^3}{\partial x^\delta} d^4 x$$

But $g_{\alpha\beta} = \frac{\partial \hat{x}^a}{\partial x^\alpha} \frac{\partial \hat{x}^b}{\partial x^\beta} \eta_{ab} \equiv e^a_\alpha e^b_\beta \eta_{ab}$

coordinates

also $d^4\hat{x} = \det\left(\frac{\partial \hat{x}^a}{\partial x^\alpha}\right) d^4x$ for arbitrary coordinates

$$d^4y = \det\left(\frac{\partial y^\alpha}{\partial x^\mu}\right) d^4x$$

$$= \epsilon^{\alpha\beta\gamma\delta} \frac{\partial \hat{x}^0}{\partial x^\alpha} \frac{\partial \hat{x}^1}{\partial x^\beta} \frac{\partial \hat{x}^2}{\partial x^\gamma} \frac{\partial \hat{x}^3}{\partial x^\delta} d^4x$$

But $g_{\alpha\beta} = \frac{\partial \hat{x}^a}{\partial x^\alpha} \frac{\partial \hat{x}^b}{\partial x^\beta} \eta_{ab} \equiv e^a_\alpha e^b_\beta \eta_{ab}$, $e^a_\alpha = \frac{\partial \hat{x}^a}{\partial x^\alpha}$ is a "local

$$\begin{aligned} d\hat{x} &= \det\left(\frac{\partial \hat{x}^a}{\partial x^\alpha}\right) d^4x \quad \text{for arbitrary coords } x^\mu \\ &= \epsilon^{\alpha\beta\gamma\delta} \frac{\partial \hat{x}^0}{\partial x^\alpha} \frac{\partial \hat{x}^1}{\partial x^\beta} \frac{\partial \hat{x}^2}{\partial x^\gamma} \frac{\partial \hat{x}^3}{\partial x^\delta} d^4x \end{aligned}$$

$$\begin{aligned} d\hat{x}' &= \det\left(\frac{\partial \hat{x}'}{\partial \hat{x}}\right) d^4\hat{x} \\ &= \det \Lambda d^4\hat{x} \\ &= d^4\hat{x} \quad \text{for } \Lambda \end{aligned}$$

$$\equiv e^a_\alpha e^b_\beta \eta_{ab}, \quad e^a_\alpha = \frac{\partial \hat{x}^a}{\partial x^\alpha} \text{ is a "local frame" (or vierbein)}$$

But $g_{\alpha\beta} = \frac{\partial x^a}{\partial x^\alpha} \frac{\partial x^b}{\partial x^\beta} \eta_{ab} \equiv e^a_\alpha e^b_\beta \eta_{ab}$, $e^a_\alpha = \frac{\partial x^a}{\partial x^\alpha}$ is a "local" $\uparrow$ general coordinate index $\nwarrow$ Lorentz index

or $e^T_\alpha \eta e$

$\Rightarrow \det(e^a_\alpha) = \sqrt{-\det g}$

But $g_{\alpha\beta} = \frac{\partial \hat{x}^a}{\partial x^\alpha} \frac{\partial \hat{x}^b}{\partial x^\beta} \eta_{ab} \equiv e^a_\alpha e^b_\beta \eta_{ab}$, $e^a_\alpha = \frac{\partial \hat{x}^a}{\partial x^\alpha}$ is a "local" Lorentz index

↑
general coordinate index

$$\text{or } g = e^T \eta e$$

$$\Rightarrow \det g = -(\det e)^2 \Rightarrow \det(e^a_\alpha) = \sqrt{-\det g}$$

$$\Rightarrow \boxed{dV = \sqrt{-g} d^4x}$$

Geodesics in curved spacetime

Action for a free particle is

$$S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

But $g_{\alpha\beta} = \frac{\partial \hat{x}^a}{\partial x^\alpha} \frac{\partial \hat{x}^b}{\partial x^\beta} \eta_{ab} \equiv e^a_\alpha e^b_\beta \eta_{ab}$, $e^a_\alpha = \frac{\partial \hat{x}^a}{\partial x^\alpha}$ is a "local frame"

$\nwarrow$ Lorentz index
 $\nearrow$ general coordinate index

or $g = e^T \eta e$

$e^a_\alpha \eta_{ab} e^b_\beta$
 $(e^T)_\alpha \eta_{ab} e^b_\beta$

$$\Rightarrow \det g = -(\det e)^2 \Rightarrow \det(e^a_\alpha) = \sqrt{-\det g}$$

$$\Rightarrow \boxed{dV = \sqrt{-g} d^4x}$$

$$d^4\hat{x} = \det e d^4x = \sqrt{-\det g} d^4x$$

invariant under $x^\mu \rightarrow x'^\mu(x)$

$$g_{\mu\nu} \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

$$\int \sqrt{-g_{\mu\nu}(x(\lambda))} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda$$

$$\frac{dx'^\mu}{d\lambda} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda}$$

$$\text{and } \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\mu_\nu$$

Geodesics in curved spacetime

Action for a free particle is

$$S = -mc \int$$

$x^\mu(\lambda)$

$$S = -mc \int \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

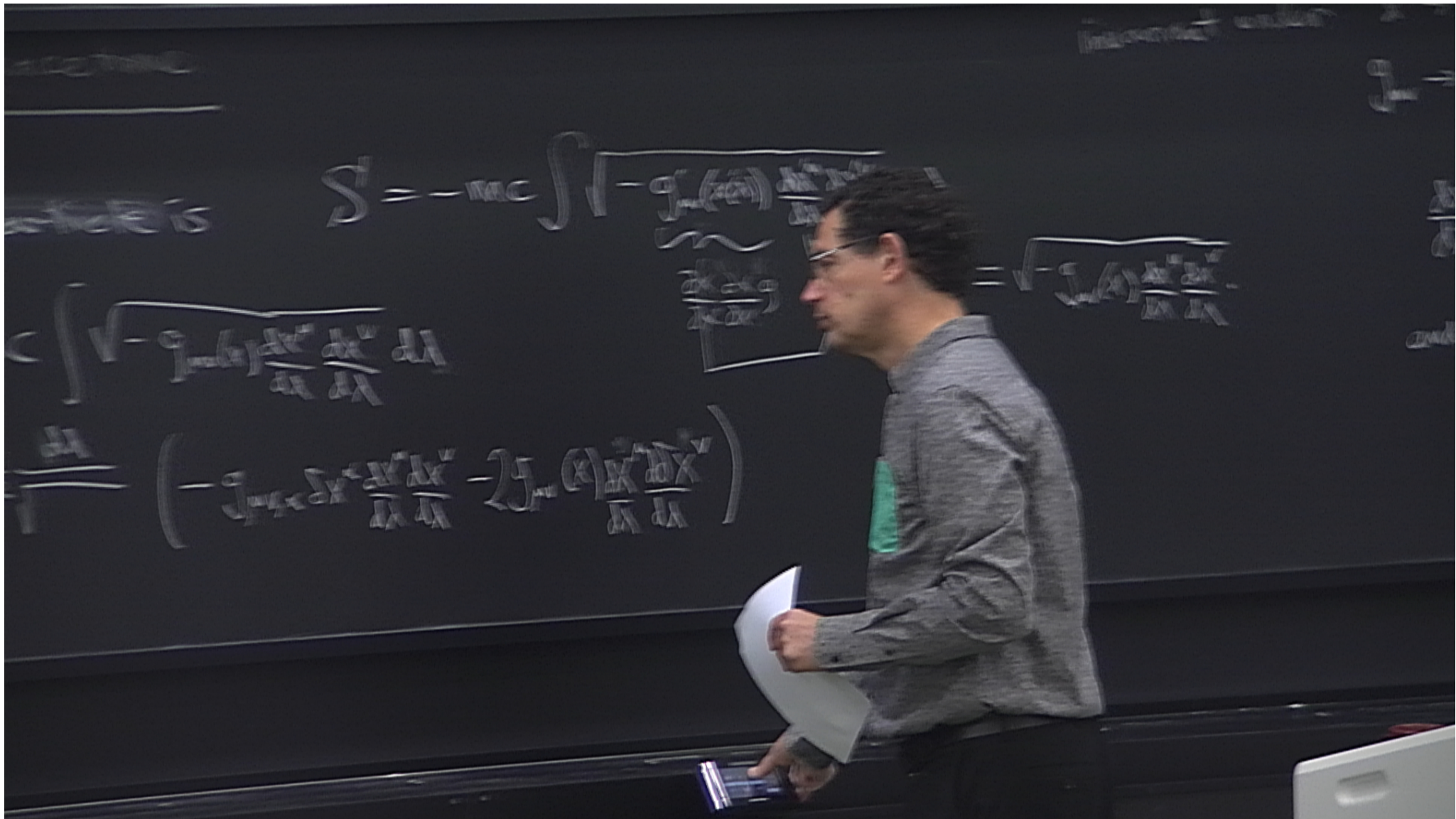
$$x^\mu(\lambda) \rightarrow x^\mu(\lambda) + \delta x^\mu(\lambda)$$

$$\delta S = -mc \int \frac{1}{2} \frac{d\lambda}{\sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} \left(-g_{\mu\nu,x} \delta x^\mu \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} \right)$$

But

$$d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} = d\tau$$

proper time



Geodesics in curved spacetime

Action for a free particle is

$$S = -mc \int \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} g$

$$S = -mc \int \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$x^\mu(\lambda) \rightarrow x^\mu(\lambda) + \delta x^\mu(\lambda)$$

But

$$d\lambda \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = d\tau$$

proper time

$$\delta S = -mc \int \frac{1}{2} \frac{d\lambda}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left(-g_{\mu\nu,\alpha} \delta x^\alpha \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} \right)$$

$$= -mc \int \frac{1}{2} d\tau \left(-g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu \right) \text{ where } \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

etime

note is

$$\int \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$\frac{d}{d\tau} \left(-g_{\mu\nu, \alpha} \dot{x}^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu}(x) \dot{x}^\mu \ddot{x}^\nu \right)$$

$$\left(-g_{\mu\nu, \alpha} \dot{x}^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu \right) \text{ where } \dot{} = \frac{d}{d\tau}$$

$$S = -mc \int \sqrt{-g'_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

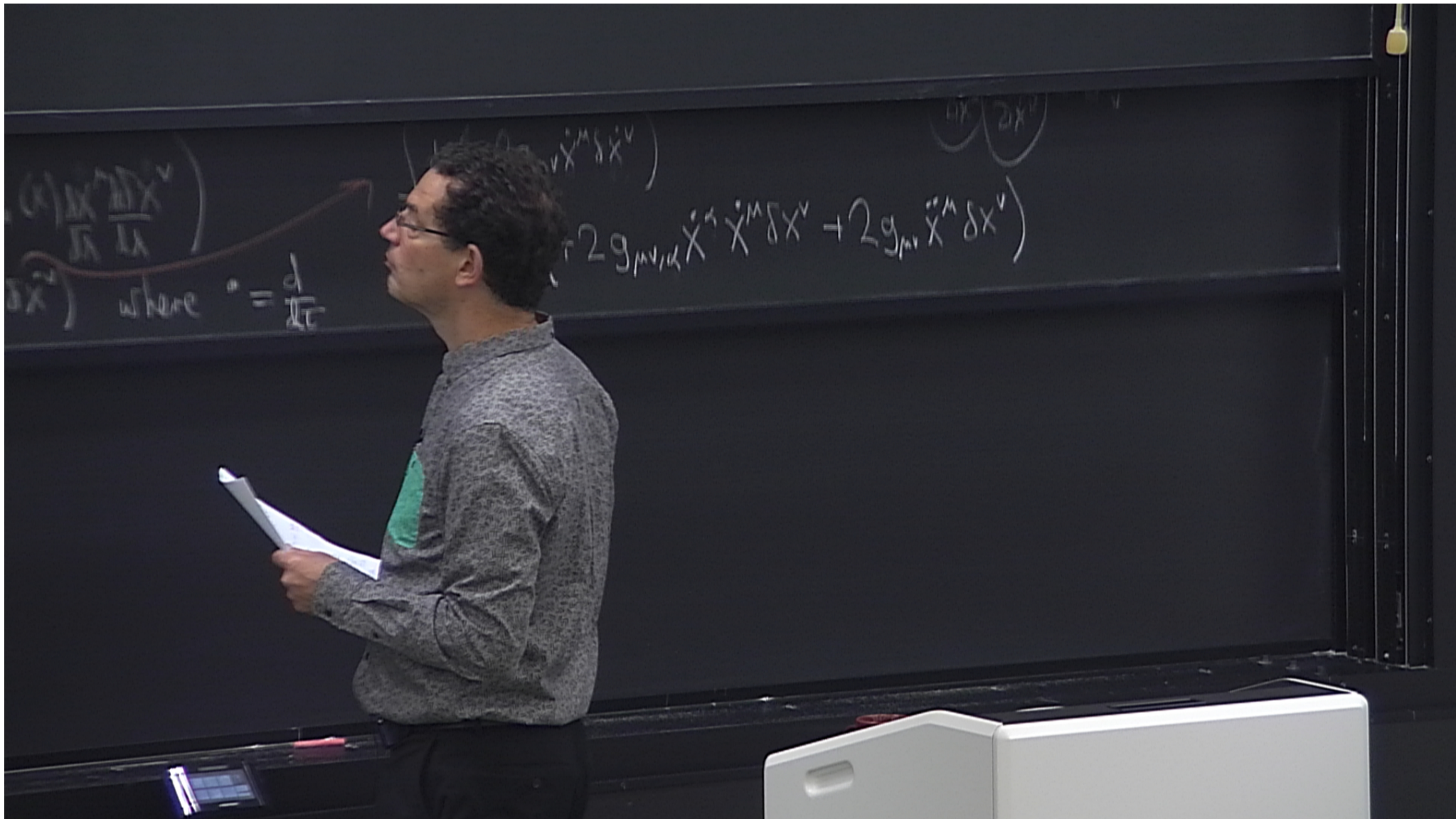
$\underbrace{\frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g}_{\frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g}$

$\underbrace{\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}_{\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$

$$= \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

invariant under $x \rightarrow x'$
 $g_{\mu\nu} \rightarrow g'_{\mu\nu}$

and $\frac{d}{d\lambda} \rightarrow \frac{d}{d\lambda'}$



$$\frac{d}{d\tau} g_{\mu\nu}(x(\tau))$$

$$= g_{\mu\nu,\alpha}(x) \frac{dx^\alpha(\tau)}{d\tau}$$

$\frac{\partial}{\partial x^\alpha}$

$$d\tau = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$\delta S = 0 \quad \forall \delta x^\alpha$$

$$\Rightarrow 2 g_{\mu\nu} \ddot{x}^\mu$$

$$\delta S = 0 \quad \forall \delta x^\alpha$$

$$\frac{dx^\alpha}{d\tau}$$

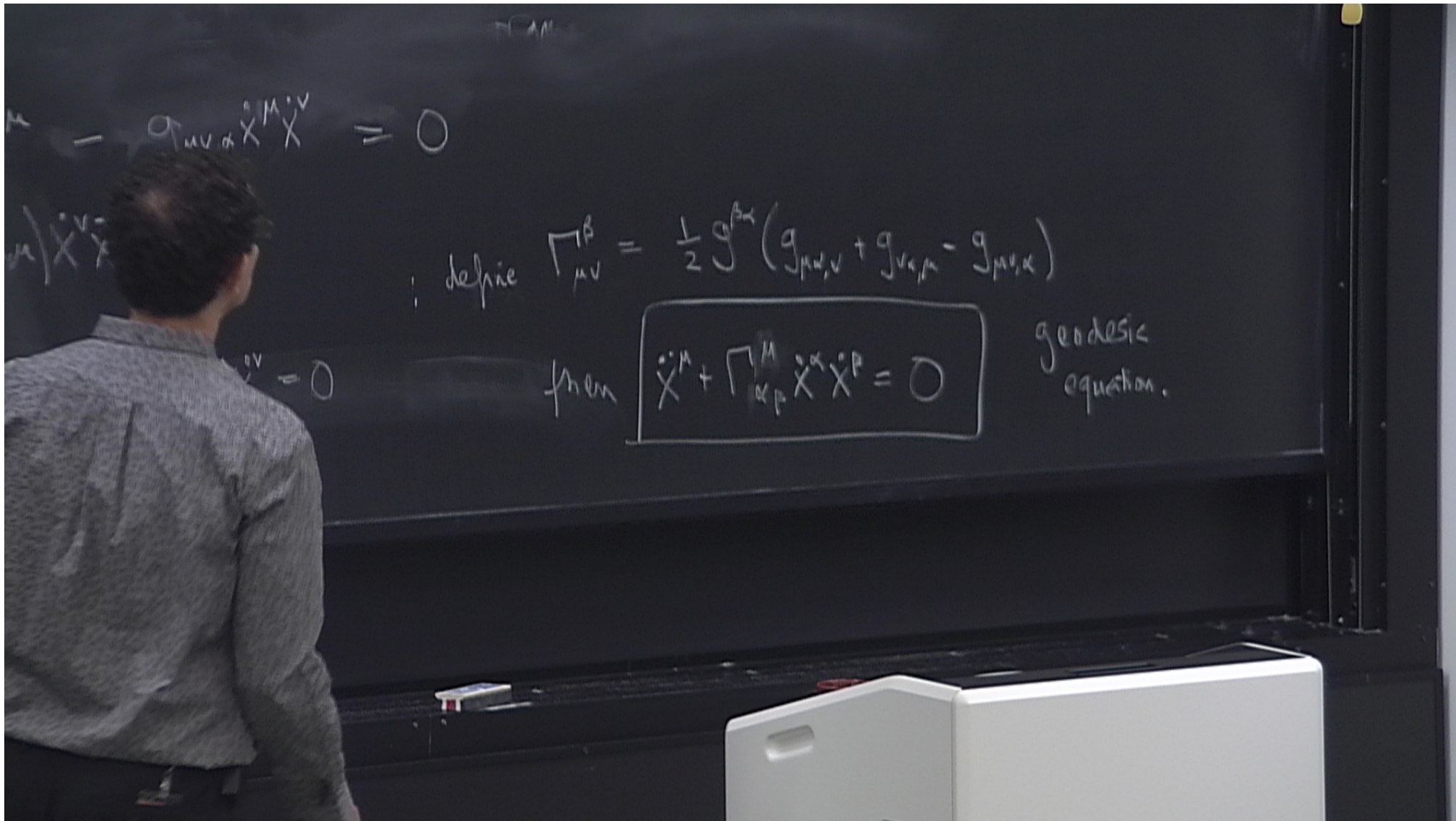
$$\Rightarrow 2g_{\mu\alpha}\ddot{x}^\mu + \underbrace{2g_{\mu\alpha,\nu}\dot{x}^\nu\dot{x}^\mu}_{(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu})\dot{x}^\nu\dot{x}^\mu} - g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu = 0$$

$$g^{\beta\mu}g_{\mu\alpha} = \delta^\beta_\alpha$$

: define

$$\frac{d}{d\lambda}$$

$$\Rightarrow \ddot{x}^\beta + \frac{1}{2}g^{\beta\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})\dot{x}^\mu\dot{x}^\nu = 0$$



$$dV = d\hat{x}^0 d\hat{x}^1 d\hat{x}^2 d\hat{x}^3 = d^4\hat{x}$$

d^4x for arbitrary coords x^μ

$$\frac{\partial \hat{x}^1}{\partial x^0} \frac{\partial \hat{x}^2}{\partial x^0} \frac{\partial \hat{x}^3}{\partial x^0} d^4x$$

(invariant under Lorentz transformations)

$$\hat{x}^a = \Lambda^a_b \hat{x}^b$$

$$d^4\hat{x}' = \det\left(\frac{\partial \hat{x}'}{\partial \hat{x}}\right) d^4\hat{x}$$

$$= \det \Lambda d^4\hat{x} = \pm d^4\hat{x}$$

$= d^4\hat{x}$ for Λ 's connected to the identity.

ed the "connection" : geodesic equation is invariant under general
transformations, but neither $\frac{d^2 x^M}{d\tau^2}$ nor $\Gamma^M_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$ transform as tensors!

$\Gamma^{\lambda}_{\mu\nu}$ is called the "connection" : geodesic equation is invariant

coordinate transformations, but neither $\frac{d^2 x^{\mu}}{d\tau^2}$ nor $\Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}$ transform

$$\frac{dx^{\mu}}{d\tau} = \frac{\partial x^{\mu}}{\partial x^{\beta}} \frac{dx^{\beta}}{d\tau} \quad \text{transforms as a 4-vector}$$

$$\text{but } \frac{d^2 x^{\mu}}{d\tau^2} = \frac{\partial^2 x^{\mu}}{\partial x^{\beta} \partial x^{\delta}} \frac{dx^{\beta}}{d\tau} \frac{dx^{\delta}}{d\tau} + \frac{\partial x^{\mu}}{\partial x^{\beta}} \frac{d^2 x^{\beta}}{d\tau^2}$$

wrong transformation for a 4-vector

$\Gamma^{\lambda}_{\mu\nu}$ is called the "connection" : geodesic equation is invariant under coordinate transformations, but neither $\frac{d^2 x^{\mu}}{d\tau^2}$ nor $\Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}$ transforms

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but $\frac{d^2 x^{\mu}}{d\tau^2} = \underbrace{\frac{\partial^2 x^{\mu}}{\partial x^{\beta} \partial x^{\delta}} \frac{dx^{\beta}}{d\tau} \frac{dx^{\delta}}{d\tau}}_{\text{wrong transformation for a 4-vector}} + \frac{\partial x^{\mu}}{\partial x^{\beta}} \frac{d^2 x^{\beta}}{d\tau^2}$; can check that

$$\Gamma^{\mu}_{\nu\alpha} = \frac{\partial x^{\delta}}{\partial x^{\nu}} \frac{\partial x^{\epsilon}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\delta}} \Gamma^{\delta}_{\epsilon\gamma} + \frac{\partial x^{\mu}}{\partial x^{\beta}} \frac{\partial^2 x^{\beta}}{\partial x^{\nu} \partial x^{\alpha}}$$

the "connection" : geodesic equation is invariant under general
 motions, but neither $\frac{d^2 x^M}{d\tau^2}$ nor $\Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$ transform as tensors!

$\frac{dx^\mu}{d\tau}$ transforms as a 4-vector

$$\frac{\partial x^M}{\partial x'^A} \frac{\partial x'^B}{\partial x'^V} = \delta_V^M$$

$$\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{\partial x'^\mu}{\partial x^\nu} \frac{d^2 x^\nu}{d\tau^2}$$

condition for a 4-vector

can check that

$$\Gamma_{\nu\alpha}^{\mu} = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\alpha}{\partial x'^\beta} \frac{\partial x'^\beta}{\partial x^\gamma} \Gamma_{\alpha\gamma}^{\beta} + \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial^2 x^\alpha}{\partial x'^\beta \partial x'^\gamma} \frac{\partial x'^\beta}{\partial x^\gamma}$$

ction": geodesic equation is invariant under general

neither $\frac{d^2 x^M}{d\tau^2}$ nor $\Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$ transform as tensors!

as a 4-vector

$$\frac{dx^\alpha}{d\tau} \frac{d^2 x^\beta}{d\tau^2}$$

for a 4-vector

can check that

$$\Gamma_{\nu\alpha}^{\mu} = \frac{\partial x^\sigma}{\partial x^{\nu'}} \frac{\partial x^\epsilon}{\partial x^{\alpha'}} \frac{\partial x^{\mu'}}{\partial x^\sigma} \Gamma_{\epsilon\sigma}^{\mu} + \left(\frac{\partial x^{\mu'}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^{\nu'} \partial x^{\alpha'}} \right)$$

$$\frac{\partial x^M}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\nu} = \delta_\nu^M$$

$$\frac{\partial}{\partial x'^\alpha} \rightarrow \frac{\partial^2 x^M}{\partial x'^\beta \partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\nu} + \frac{\partial x^M}{\partial x'^\beta} \underbrace{\frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\alpha} \frac{\partial x'^\sigma}{\partial x^\nu}}_{\frac{\partial}{\partial x'^\alpha} \left(\frac{\partial x'^\beta}{\partial x^\nu} \right)} = 0$$

$\Gamma_{\mu\nu}^\lambda$ is called the "connection"

coordinate transformations, either

$$x'(x(\tau))$$

$$\text{So } \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

$$\Rightarrow \frac{d^2 x'^\mu}{d\tau^2} + \Gamma'_{\alpha\beta}^\mu \frac{dx'^\alpha}{d\tau} \frac{dx'^\beta}{d\tau} = 0$$

$$\frac{dx^\mu}{d\tau} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{dx'^\alpha}{d\tau}$$

$$\text{but } \frac{d^2 x'^\mu}{d\tau^2} = \frac{\partial^2 x'^\mu}{\partial x'^\alpha \partial x'^\beta} \frac{dx'^\alpha}{d\tau} \frac{dx'^\beta}{d\tau}$$

wrong to

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System in which $\partial_\alpha g_{\mu\nu} = 0 \Rightarrow \Gamma_{\mu\nu}^\kappa = 0$ (at a point).

it allows you to write derivatives which transform correctly as tensors
covariant derivatives, reduce to ordinary derivatives in a locally flat coordinate system.
 $\Gamma^{\alpha}_{\mu\nu} = 0$ (at a point).

Γ called a "connection" because it allows you to write

derivatives are called covariant derivatives, reduce

system in which $\partial_\alpha g_{\mu\nu} = 0 \Rightarrow \Gamma_{\mu\nu}^\kappa = 0$ (at a point).

"parallel transport" of a vector along a curve

$$\frac{DV^\kappa}{D\lambda} \equiv \frac{dV^\kappa}{d\lambda} + \Gamma_{\nu\sigma}^\kappa V^\sigma \frac{dx^\nu}{d\lambda} \quad \text{transforms as a 4-}$$

Transformation for a 4-vector

Aside: free particles follow geodesics in spacetime. In spacetimes
conserved quantities (Noether).

In particular, if $g_{\mu\nu}(x)$ is independent of one of the spacetime coordinates

ation for a 4-vector

v_α

$\partial x^\alpha \partial x^\alpha \partial x^\alpha$

$\partial x^\alpha \partial x^\alpha \partial x^\alpha$

spacetime. In spacetimes with symmetries, there are corresponding
one of the spacetime coordinates x^α , then the corresponding

Transformation for a 4-vector

Aside: free particles follow geodesics in spacetime. In spacetimes
conserved quantities (Noether).

In particular, if $g_{\mu\nu}(x)$ is independent of one of the spacetime
conjugate momentum is conserved : $S' = -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$:

Transformation for a 4-vector

v_α

$\delta x^\alpha \delta x^\alpha \delta x^\alpha$

$\delta x^\alpha \delta x^\alpha \delta x^\alpha$

metrics in spacetime. In spacetimes with symmetries, there are corresponding

moment of one of the spacetime coordinates x^α , then the corresponding

$$S' = -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \quad ; \quad \delta S = mc \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{\delta x^\nu}{d\tau}$$

Transformation for a 4-vector

v_α

$\delta x^\alpha \delta x^\alpha \delta x^\alpha$

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desics in spacetime. In spacetimes with symmetries, there are corresponding

endent of one of the spacetime coordinates x^α , then the corresponding
(no sum on α)

$$S' = -mc \int \underbrace{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}}_{cd\tau} d\lambda : \delta S \sim m \int d\tau g_{\mu\alpha} \frac{dx^\mu}{d\tau} \frac{\delta x^\alpha}{d\tau} \Rightarrow \underline{p_\alpha = m g_{\mu\alpha} \frac{dx^\mu}{d\tau}} \text{ is conserved}$$

$$p_\nu = \frac{\delta S}{\delta \dot{x}^\nu}$$

normalization for a 4-vector

v^μ

$\delta x^\mu \delta x^\mu \delta x^\mu$

$\delta x^\mu \delta x^\mu \delta x^\mu$

in spacetime. In spacetimes with symmetries, there are corresponding

of one of the spacetime coordinates x^μ , then the corresponding

(no sum on μ)

$$-mc \underbrace{\int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda}_{cd\tau} : \delta S \sim m \int d\tau g_{\mu\alpha} \frac{dx^\mu}{d\tau} \frac{\delta x^\alpha}{d\tau} \Rightarrow \underline{P_\alpha = m g_{\mu\alpha} \frac{dx^\mu}{d\tau}} \text{ is conserved.}$$

$$P_\nu = \frac{\delta S}{\delta \dot{x}^\nu}$$

More general definition of geodesics (which also

γ

γ
 x^i

More general definition of geodesics

Consider curve $x^\mu(\lambda)$, with tangent vector



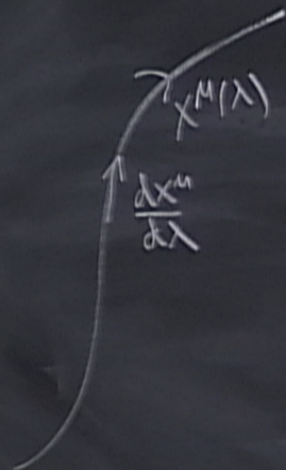
$\frac{dx^\mu}{d\lambda}$

$\frac{dx^\mu}{d\lambda}$

More general definition of geodesics (which also

consider curve $x^\mu(\lambda)$, with tangent vector $\frac{dx^\mu}{d\lambda}$

Geodesic equation is $\frac{D\xi^\mu}{D\lambda} = f(\lambda)\xi^\mu$



More general definition of geodesics (which also applies

consider curve $x^\mu(\lambda)$, with tangent vector $\frac{dx^\mu}{d\lambda}$

Geodesic equation is $\frac{D\xi^\mu}{D\lambda} = f(\lambda)\xi^\mu$ i.e. curve is

of geodesics (which also applies to massless particles) is as follows

with tangent vector $\frac{dx^M}{d\lambda}$

$$\frac{D\vec{\xi}^M}{D\lambda} = f(\lambda)\vec{\xi}^M$$

i.e. curve is parallel to itself.

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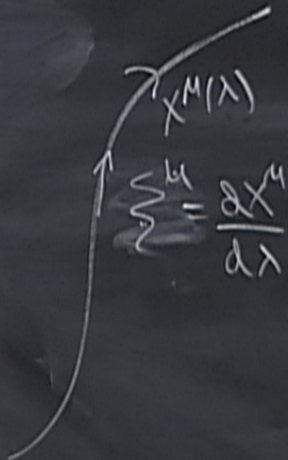
$$\frac{D\vec{\xi}^M}{D\lambda} = f(\lambda)\vec{\xi}^M$$

i.e. curve is parallel to itself.

More general definition of geodesics (v)

Consider curve $x^\mu(\lambda)$, with tangent vector

Geodesic equation is $\frac{D\dot{x}^\mu}{D\lambda} = f(\lambda)\dot{x}^\mu$


$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

More general definition of geodesics (which also applies)

consider curve $x^\mu(\lambda)$, with tangent vector $\frac{dx^\mu}{d\lambda}$

Geodesic equation is $\frac{D\tilde{e}^\mu}{D\lambda} = f(\lambda)\tilde{e}^\mu$

ie. curve

Affine parametrisation of curve.

$$\xi^M = \frac{dx^M}{d\lambda}$$

$$\frac{D\xi^M}{D\lambda} = \frac{d^2 x^M}{d\lambda^2} + \Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{d^2 \sigma}{d\lambda^2} \frac{dx^M}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right)^2 \left(\frac{d^2 x^M}{d\sigma^2} + \Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{d^2 \sigma}{d\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right)^2 \left(\frac{d^2 x^\mu}{d\sigma^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right) = f(\lambda) \frac{dx^\mu}{d\sigma}$$

$\Rightarrow$ Choose σ so that

$$\frac{d^2 \sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda}$$

$$\Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(\lambda) d\lambda}$$

$$\Rightarrow \sigma = A \int d\lambda e^{\int f(\lambda) d\lambda}$$

want this to be zero

σ is only defined up to affine transformations $\sigma' = a\sigma + b$, where a and

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{d^2 \sigma}{d\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right)^2 \left(\frac{d^2 x^\mu}{d\sigma^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right) = 0$$

$\Rightarrow$ Choose σ so that

$$\frac{d^2 \sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(x) dx} \Rightarrow \sigma =$$

want this to be zero

σ is only defined up to affine transformations

$$\sigma' = a\sigma + b$$

to b, can always arrange that $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} =$

in affine parametrisation,

$$\frac{d^2 x^M}{d\sigma^2} + \Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = 0$$

$$\Rightarrow \frac{D}{D\sigma} \left(\frac{dx^\mu}{d\sigma} \right) = 0 \quad \Rightarrow \quad \frac{D}{D\sigma} \left(g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right) = \left(\nabla_\alpha g_{\mu\nu} \frac{dx^\alpha}{d\sigma} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right)$$

isation,

$$+ \Gamma_{\alpha\beta}^{\mu} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = 0$$

$$) = 0 \Rightarrow \frac{D}{ds} \left(g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \right) = \left(\nabla_{\alpha} g_{\mu\nu} \frac{dx^{\alpha}}{ds} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \right) = 0$$

$= 0$ (will show next time)

non affine parameterisation,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

$$\Rightarrow \frac{D}{Ds} \left(\frac{dx^\mu}{ds} \right) = 0$$

$$\Rightarrow \frac{D}{Ds} \left(g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right) = \left(\nabla_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

$$\Rightarrow g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \text{const along curve}$$

$$= \frac{d}{ds} \left(g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)$$

can choose this to be
 - 1 timelike
 0 null
 + 1 spacelike.

"Polyakov" action

$$S = \frac{1}{2} \int \left(g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) \frac{d\lambda}{e(\lambda)}$$

$\uparrow$
 $e(\lambda)$

$$d\lambda e(\lambda) = dx' e'(\lambda')$$

$$e'(\lambda') = \frac{d\lambda}{d\lambda'} e(\lambda) \quad \text{"einheits"}$$

† for massive particle add $-\frac{1}{2} \int m^2 e d\lambda$

$$\delta S = \int \left(\overset{w/\frac{1}{e} \frac{d}{d\lambda}}{\text{geodesic eq.}} \right) \delta x + \int \left(\underbrace{g_{\mu\nu} \frac{dx^\mu}{e d\lambda} \frac{dx^\nu}{e d\lambda}}_{=0} \right) - \frac{d\lambda}{e} \delta e$$

$$\int \frac{1}{2} \left(\underbrace{g_{\mu\nu} \frac{dx^\mu}{e^2} \frac{dx^\nu}{e^2}}_{\boxed{g_{\mu\nu} \frac{dx^\mu}{e^2} \frac{dx^\nu}{e^2} = -m^2}} - m^2 \right) \delta e$$

$$e = \frac{1}{m} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$