Title: 14/15 PSI - Relativity- Lecture 5

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Abstract:

$$S = -m_0 c \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$g_{\mu\nu}(x)$ - metric: used to define lengths on spacetime

SR	GR
$\Lambda, \tilde{\Lambda}$	$\frac{\partial x'}{\partial x}, \frac{\partial x}{\partial x'}$
η	$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$

: line element $dl^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$

used to define lengths on spacetime

element $dl^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$ invariant under general coordinate transformations.

$$S = -m_0 c \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$\eta_{\mu\nu}$ → $g_{\mu\nu}(x)$ - metric: used to define length

$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$

$x'_\mu = \tilde{\Lambda}^\alpha_{\ \mu} x_\alpha$

$x'^\alpha(x)$

$\frac{\partial x'^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta$ | $\frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\beta} = \delta^\alpha_\beta$

$= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\beta}$

SR: $\Lambda, \tilde{\Lambda}$ — η

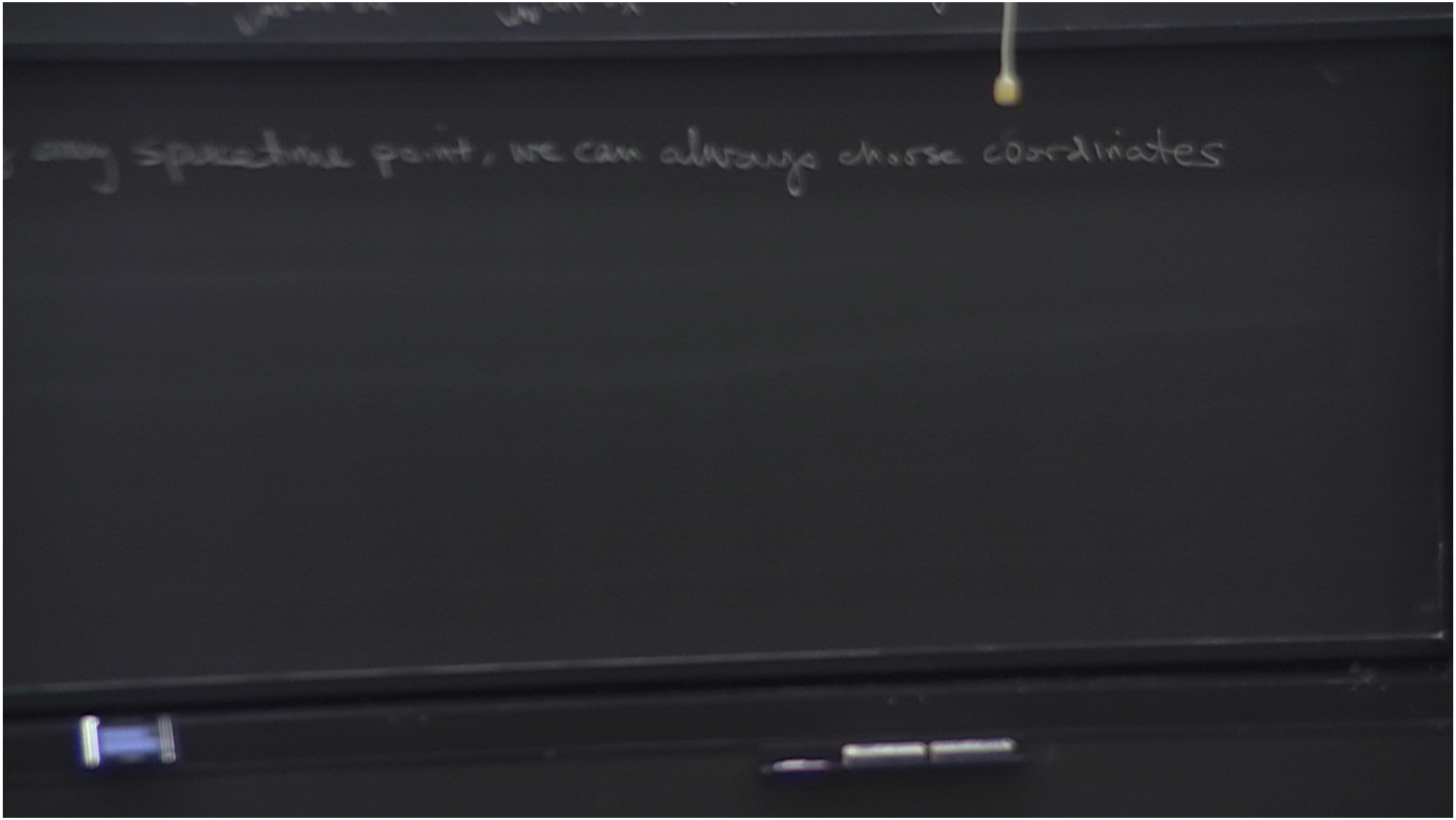
GR: $\frac{\partial x'}{\partial x}, \frac{\partial x}{\partial x'}$

$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$: line element $ds^2 = g_{\mu\nu} dx^\mu$

we will shortly that

$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$: line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx'^\mu dx'^\nu$

We will shortly show that in the vicinity of any spacetime point, we can find coordinates in which $g_{\mu\nu} = \eta_{\mu\nu}$ at that point.

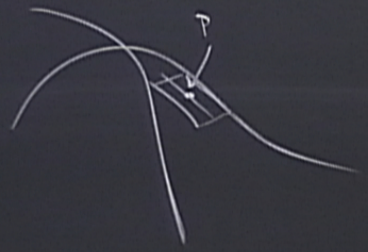

any spacetime point, we can always choose coordinates

$= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\beta}}$ η $g'_{\mu\nu}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$: line element $dl^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g'_{\mu\nu} dx'^{\mu}$

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in which $g_{\mu\nu} = \eta_{\mu\nu}$ at that point.

$$(M)\underline{a} = \underline{F}_g = -\frac{GM(m)}{r^2}\hat{r}$$



Equivalence Principle

$\Lambda, \tilde{\Lambda}$ $\quad \frac{\partial x'}{\partial x}, \frac{\partial x}{\partial x'}$

η $\quad g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$: line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g'_{\mu\nu}dx'^\mu dx'^\nu$ invariant under general coordinate transformation

shortly show that in the vicinity of any spacetime point, we can always choose coordinates

s.t. $g_{\mu\nu} = \eta_{\mu\nu}$ at that point.

$$\cancel{m}\,\underline{a} = \underline{F}_g = -\frac{GM\cancel{m}}{r^2}\hat{r}$$



Equivalence Principle : in the vicinity of every point in spacetime, it is possible to choose a coordinate system such that the laws of physics take the same form as they do in an unaccelerated Cartesian coordinate system, in the absence of gravity

i.e. "gravity is the bending of spacetime"

any spacetime point, we can always choose coordinates

$$\cancel{M}\,\underline{a} = \underline{F}_g = -\frac{GM\cancel{m}}{r^2}\hat{r}$$



$$\frac{d}{dt}(\underline{P}) \qquad P^i = \int d^3x\, T^{0i}$$

$$M\ddot{x} = q\underline{E}$$

$T^{\mu}{}_{\nu}$ ∝ curvature of spacetime

of every point in spacetime, it is possible to choose a coordinate system

the same form as they do in an unaccelerated Cartesian coordinate system,

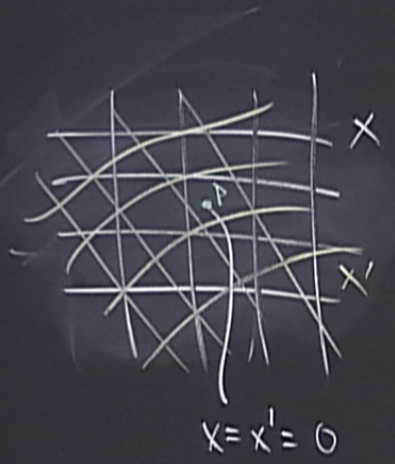
$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

ng of spacetime"

i.e. gravity is the bending of spacetime

Locally flat coordinates

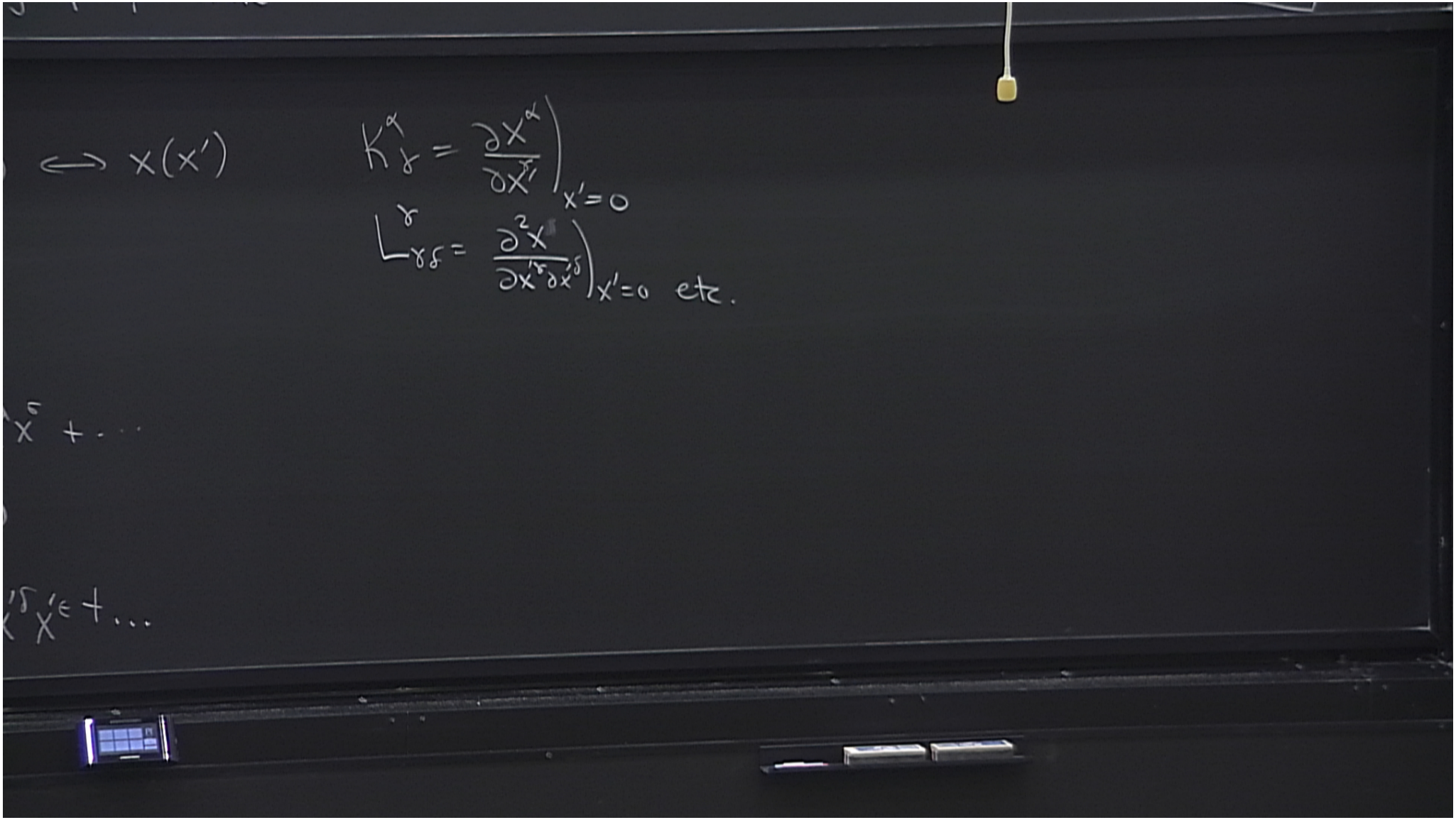
$x'(x) \leftrightarrow x(x')$



$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$$

where $g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + \underbrace{A_{\alpha\beta,\lambda}}_{g_{\alpha\beta,\lambda}(0)}x^\lambda + \frac{1}{2}\underbrace{B_{\alpha\beta,\lambda\sigma}}_{\frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\sigma}(0)}x^\lambda x^\sigma + \dots$

Choose $x^\alpha = K^\alpha_{\ \gamma}x'^\gamma + \frac{1}{2}L^\alpha_{\ \gamma\delta}x'^\gamma x'^\delta + \frac{1}{6}M^\alpha_{\ \gamma\delta\epsilon}x'^\gamma x'^\delta x'^\epsilon$


$\longleftrightarrow x(x')$
$K^\alpha_\gamma = \left.\frac{\partial x^\alpha}{\partial x'^\gamma}\right|_{x'=0}$
$\Gamma^\gamma_{\gamma\delta} = \left.\frac{\partial^2 x^\gamma}{\partial x'^\gamma \partial x'^\delta}\right|_{x'=0}$ etc.
$x^\alpha + \ldots$
$x'^\delta x'^\epsilon + \ldots$

Q: how can I choose K, L, M... to make $g'_{\mu\nu}(x')$ as close as possible to $\eta_{\mu\nu}$.

Calculate $g'_{\mu\nu}(0) = g_{\alpha\beta}(0) K^{\alpha}_{\mu} K^{\beta}_{\nu} \stackrel{?}{=} \eta_{\mu\nu}$

$= K^T g(0) K$: if we choose $K = OL$ (O: orthogonal 4×4, L: diagonal 4×4 matrix)

$g_{\mu\nu}$ is symmetric, must have one negative eigenvalue and three positive eigenvalues

Since $g(0)$ is symmetric, can choose O so that $O^T g(0) O = \begin{pmatrix} -l_0^2 & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \ddots \end{pmatrix}$

x') as close as possible to $\eta_{\mu\nu}$.

choose $K = OL$ (orthogonal 4×4; diagonal 4×4 matrix)

so that $O^T g(0) O = \begin{pmatrix} -\ell_0^2 & & & \\ & \ell_1^2 & & \\ & & \ell_2^2 & \\ & & & \ell_3^2 \end{pmatrix}$; then choose $L = \begin{pmatrix} \frac{1}{\ell_0} & & & \\ & \frac{1}{\ell_1} & & \\ & & \frac{1}{\ell_2} & \\ & & & \frac{1}{\ell_3} \end{pmatrix}$

so that $g'_{\mu\nu}(0) = \eta_{\mu\nu}$.

Furthermore, can choose $L^{\alpha}_{\beta\delta}$ so that $g'_{\mu\nu,\lambda}(0) = 0$

$$g'_{\mu\nu,\lambda}(0) = \frac{\partial}{\partial x'^{\lambda}} g'_{\mu\nu}(x') \Big|_0$$

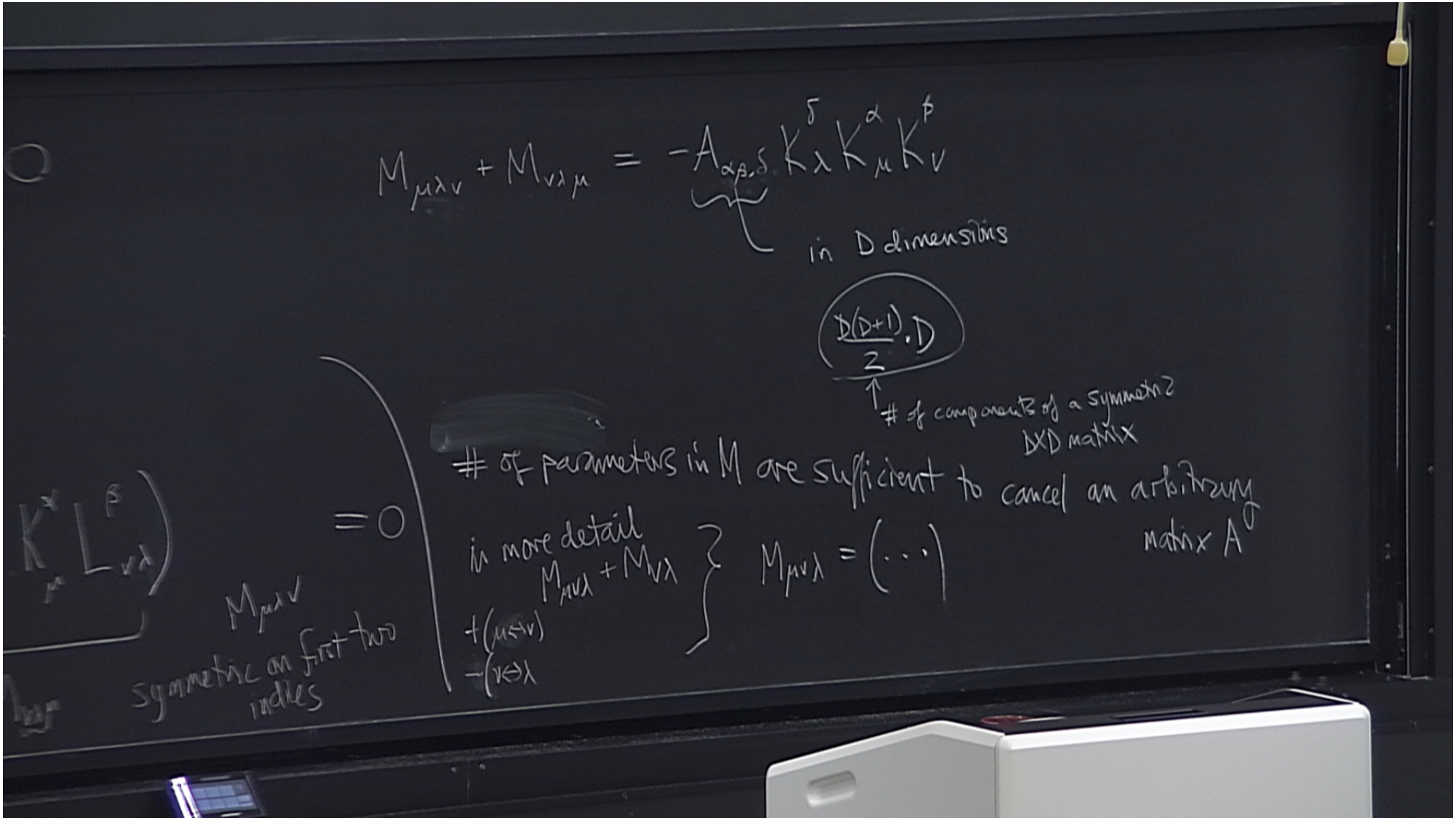
$$\frac{\partial}{\partial x'^{\lambda}} = \frac{\partial x^{\alpha}}{\partial x'^{\lambda}} \frac{\partial}{\partial x^{\alpha}}$$

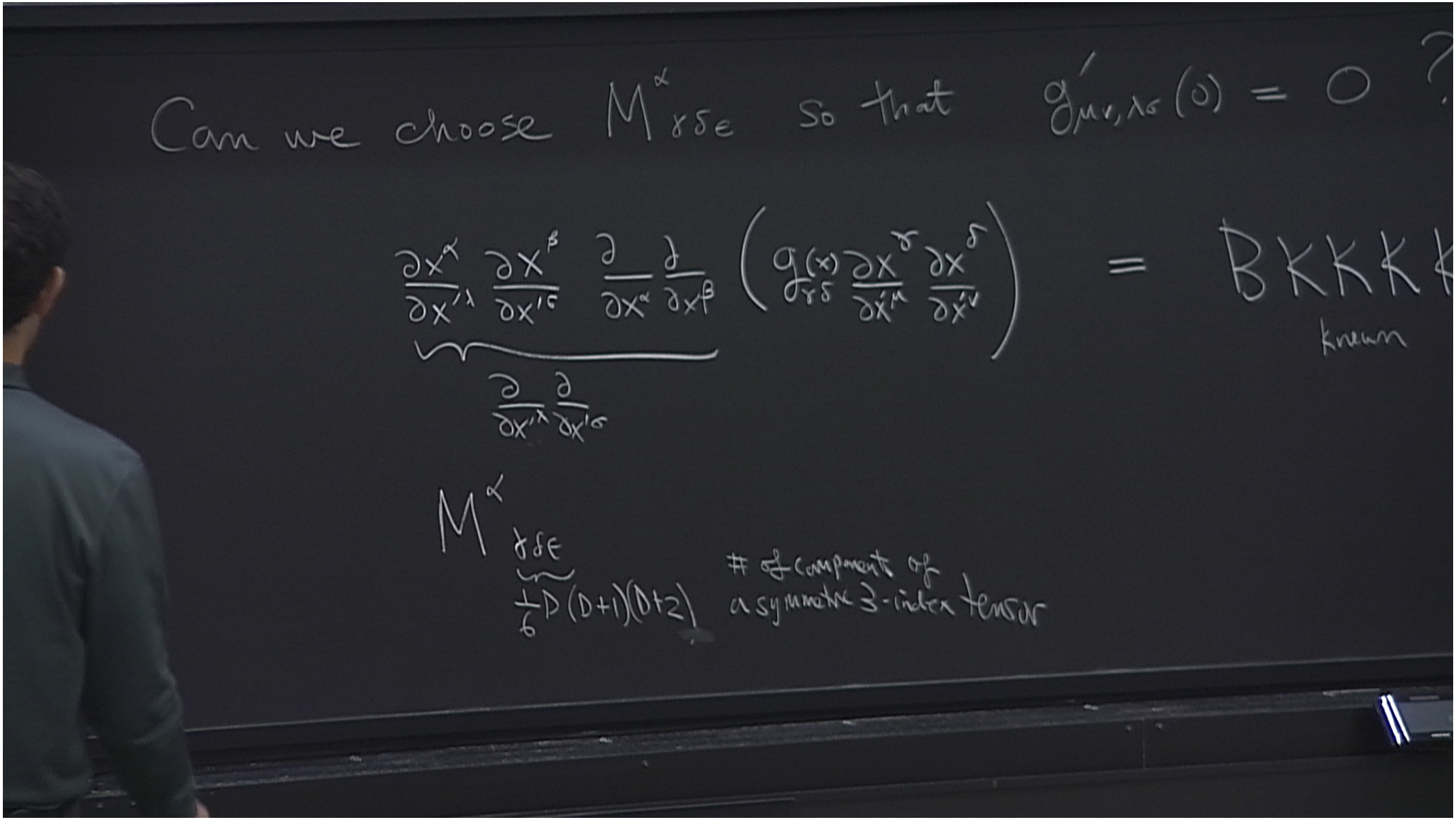
$$\frac{\partial x^{\alpha}}{\partial x'^{\lambda}} \rightsquigarrow K^{\alpha}_{\lambda}$$

$$= A_{\alpha\beta\delta} K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu} + g_{\alpha\beta}(0) \left(L^{\alpha}_{\mu\lambda} K^{\beta}_{\nu} + K^{\alpha}_{\mu} L^{\beta}_{\nu\lambda} \right)$$

$M_{\mu\lambda\nu}$ symmetric on first two indices

$M_{\mu\lambda\nu} + M_{\nu\lambda\mu}$

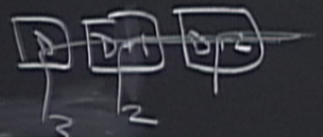

$M_{\mu\lambda\nu} + M_{\nu\lambda\mu} = -A_{\alpha\beta\delta} K^{\delta}_{\lambda} K^{\alpha}_{\mu} K^{\beta}_{\nu}$
in D dimensions
$\frac{D(D+1)}{2} \cdot D$
# of components of a symmetric DxD matrix
# of parameters in M are sufficient to cancel an arbitrary matrix A
in more detail
$M_{\mu\nu\lambda} + M_{\nu\lambda}$ } $M_{\mu\nu\lambda} = (\dots)$
$+(\mu \leftrightarrow \nu)$
$-(\nu \leftrightarrow \lambda$
$M_{\mu\lambda\nu}$
symmetric on first two indices
$=0$


Can we choose $M^{\alpha}_{\gamma\delta\epsilon}$ so that $g'_{\mu\nu,\lambda\sigma}(0) = 0$?
$\frac{\partial x^{\alpha}}{\partial x'^{\lambda}} \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial}{\partial x^{\alpha}} \frac{\partial}{\partial x^{\beta}} \left(g_{\gamma\delta}(x) \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \right) = B K K K K$
known
$\frac{\partial}{\partial x'^{\lambda}} \frac{\partial}{\partial x'^{\sigma}}$
$M^{\alpha}_{\gamma\delta\epsilon}$
$\frac{1}{6} D(D+1)(D+2)$
# of components of a symmetric 3-index tensor

Can we choose $M^{\alpha}_{\gamma\delta\epsilon}$ so that $g'_{\mu\nu,\lambda\sigma}(0) = 0$

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \frac{\partial}{\partial x'^{\lambda}} \frac{\partial}{\partial x'^{\sigma}} \left(g_{\gamma\delta}(x) \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \right) = \text{BKK}$$

known



$$\frac{D(D+1)(D+2)}{\text{hooks}}$$

$$= \frac{D(D+1)(D+2)}{6}$$

$$M^{\alpha}_{\underbrace{\gamma\delta\epsilon}} \quad \frac{1}{6}D(D+1)(D+2)$$

# of components of a symmetric 3-index tensor

of spacetime

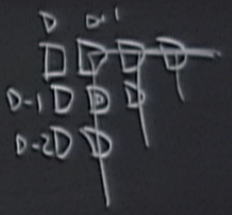
$\longleftrightarrow x(x')$

$$K^{\alpha}_{\gamma} = \left.\frac{\partial x^{\alpha}}{\partial x'^{\gamma}}\right|_{x'=0}$$

$$L^{\gamma}_{\alpha\beta} = \left.\frac{\partial^2 x^{\gamma}}{\partial x'^{\alpha}\partial x'^{\beta}}\right|_{x'=0} \text{ etc.}$$

$+ \ldots$

$x' \in + \ldots$

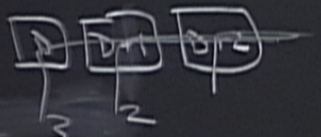


flamemesh

Can we choose $M^{\alpha}_{\gamma\delta\epsilon}$ so that $g'_{\mu\nu,\lambda\sigma}(0) = 0$

$$\underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\lambda}}\frac{\partial x'^{\beta}}{\partial x^{\sigma}}}_{\frac{\partial}{\partial x'^{\lambda}}\frac{\partial}{\partial x'^{\sigma}}}\frac{\partial}{\partial x'^{\alpha}}\frac{\partial}{\partial x'^{\beta}}\left(g^{(x)}_{\gamma\delta}\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) = \text{BKKK}$$

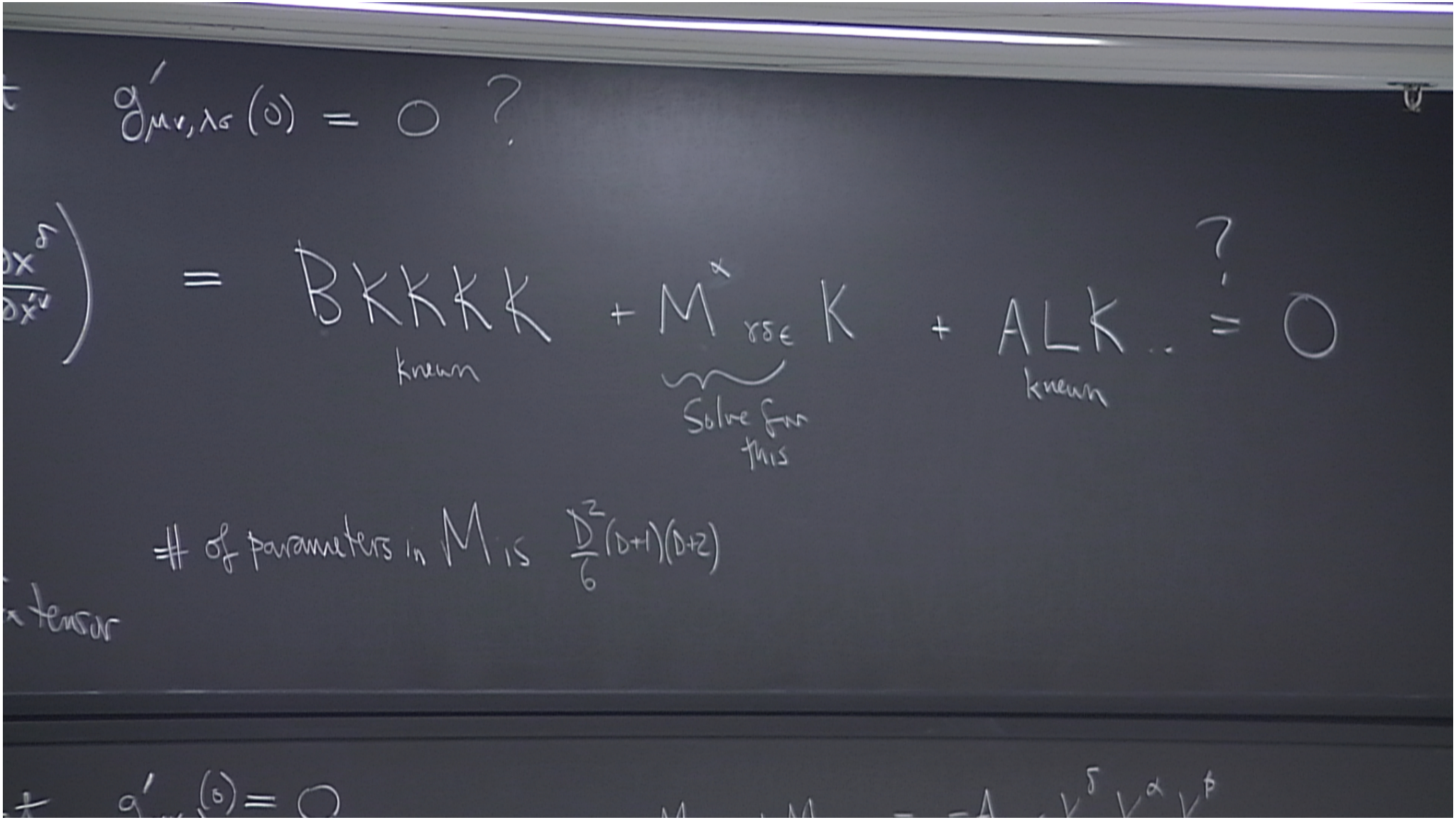
known



$$\frac{D(D+1)(D+2)}{\text{hooks}} = \frac{D(D+1)(D+2)}{6}$$

$$M^{\alpha}_{\underbrace{\gamma\delta\epsilon}_{\frac{1}{6}D(D+1)(D+2)}}$$

# of components of a symmetric 3-index tensor


$g'_{\mu\nu,\lambda\sigma}(0) = 0$?
$= BKKKK + M^{\alpha}{}_{\gamma\delta\epsilon} K + ALK \ldots \stackrel{?}{=} 0$
known
Solve for this
known
# of parameters in M is $\frac{D^2}{6}(D+1)(D+2)$
tensor
$g'_{\ldots}(0) = 0$

$M_{\mu\lambda\nu} + M_{\nu\lambda\mu}$ symmetric on first two

endent parameters in $B_{\alpha\beta,\lambda\sigma}$

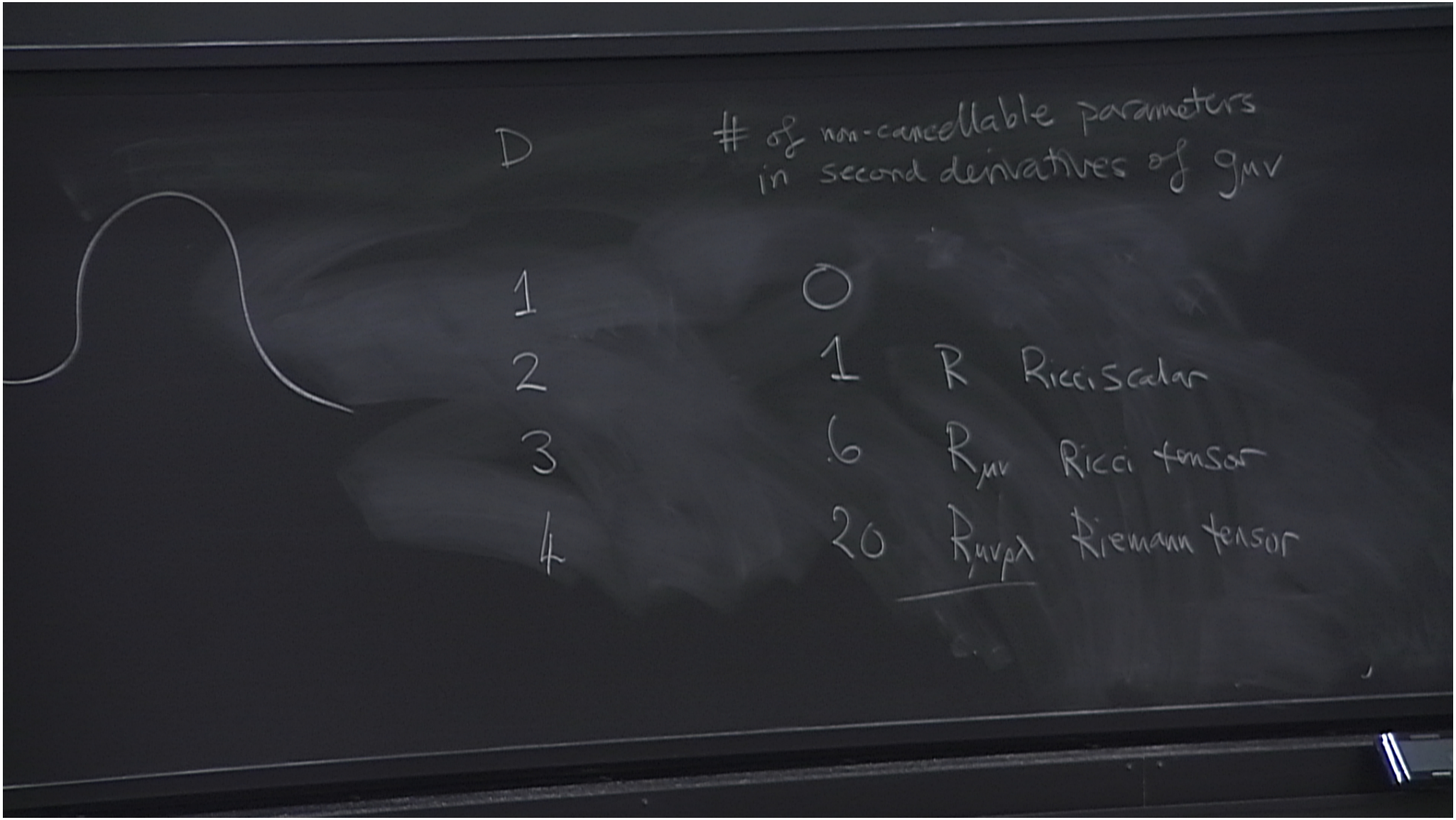
symm symm

$$\frac{D(D+1)}{2}\,\frac{D(D+1)}{2} = \frac{D^2(D+1)^2}{4}$$

$\Rightarrow$ # of parameters in B which cannot be cancelled by M is

$$\frac{D^2(D+1)^2}{4} - \frac{1}{6}D^2(D+1)(D+2) = D(D+1)\left(\frac{1}{4}D(D+1) - \frac{1}{6}D(D+2)\right) = \frac{1}{12}D^2(D^2-1)$$


positive
negative
>0


D
# of non-cancellable parameters in second derivatives of $g_{\mu\nu}$
1 0
2 1 R Ricci Scalar
3 6 $R_{\mu\nu}$ Ricci tensor
4 20 $R_{\mu\nu\rho\lambda}$ Riemann tensor

of spacetime

$\Longleftrightarrow x(x')$

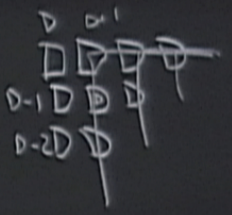
$$K^{\alpha}_{\gamma} = \left.\frac{\partial x^{\alpha}}{\partial x'^{\gamma}}\right|_{x'=0}$$

$$L^{\gamma}_{\alpha\delta} = \left.\frac{\partial^2 x^{\gamma}}{\partial x'^{\alpha}\partial x'^{\delta}}\right|_{x'=0} \text{ etc.}$$

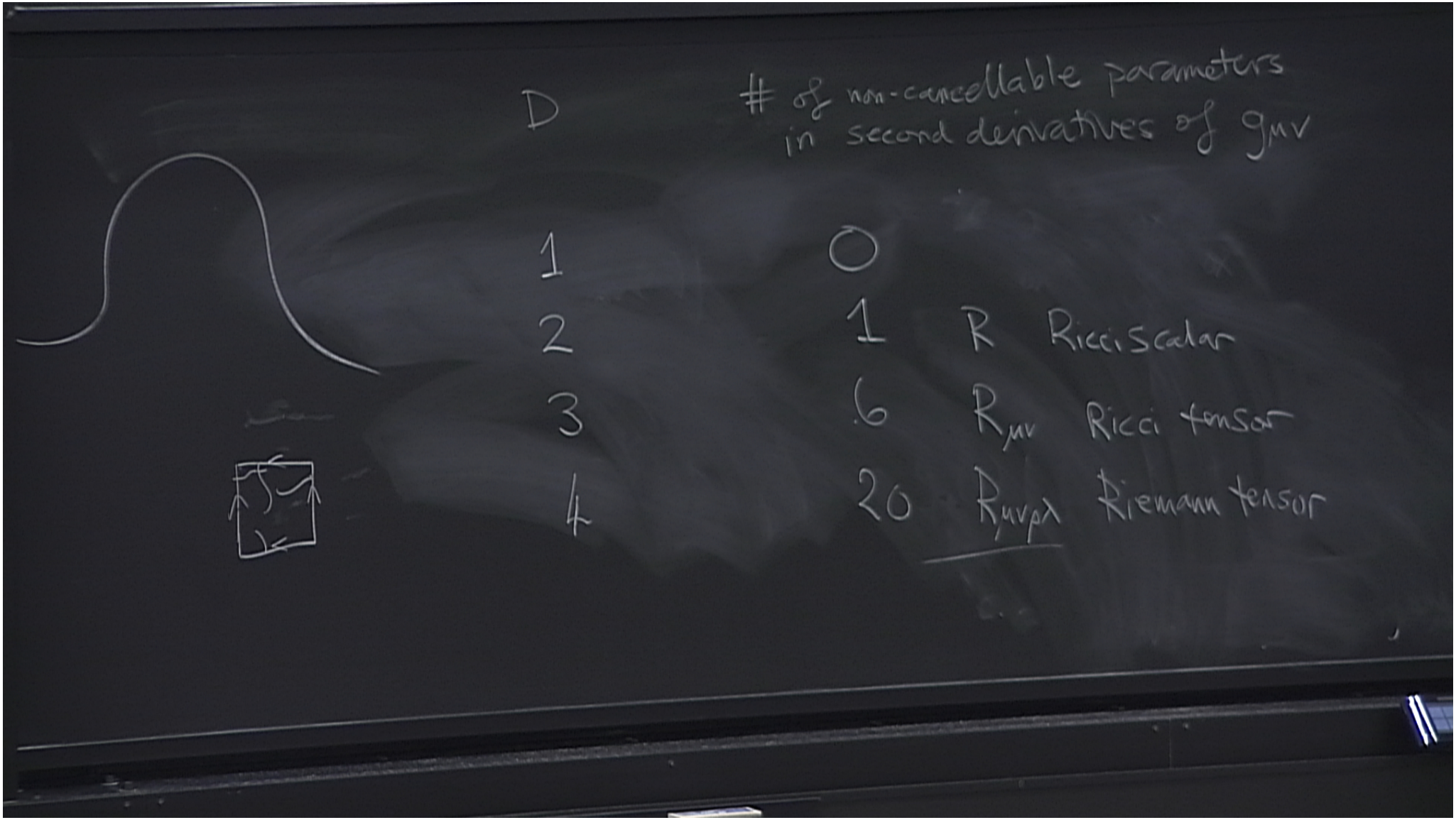
$+ \ldots$

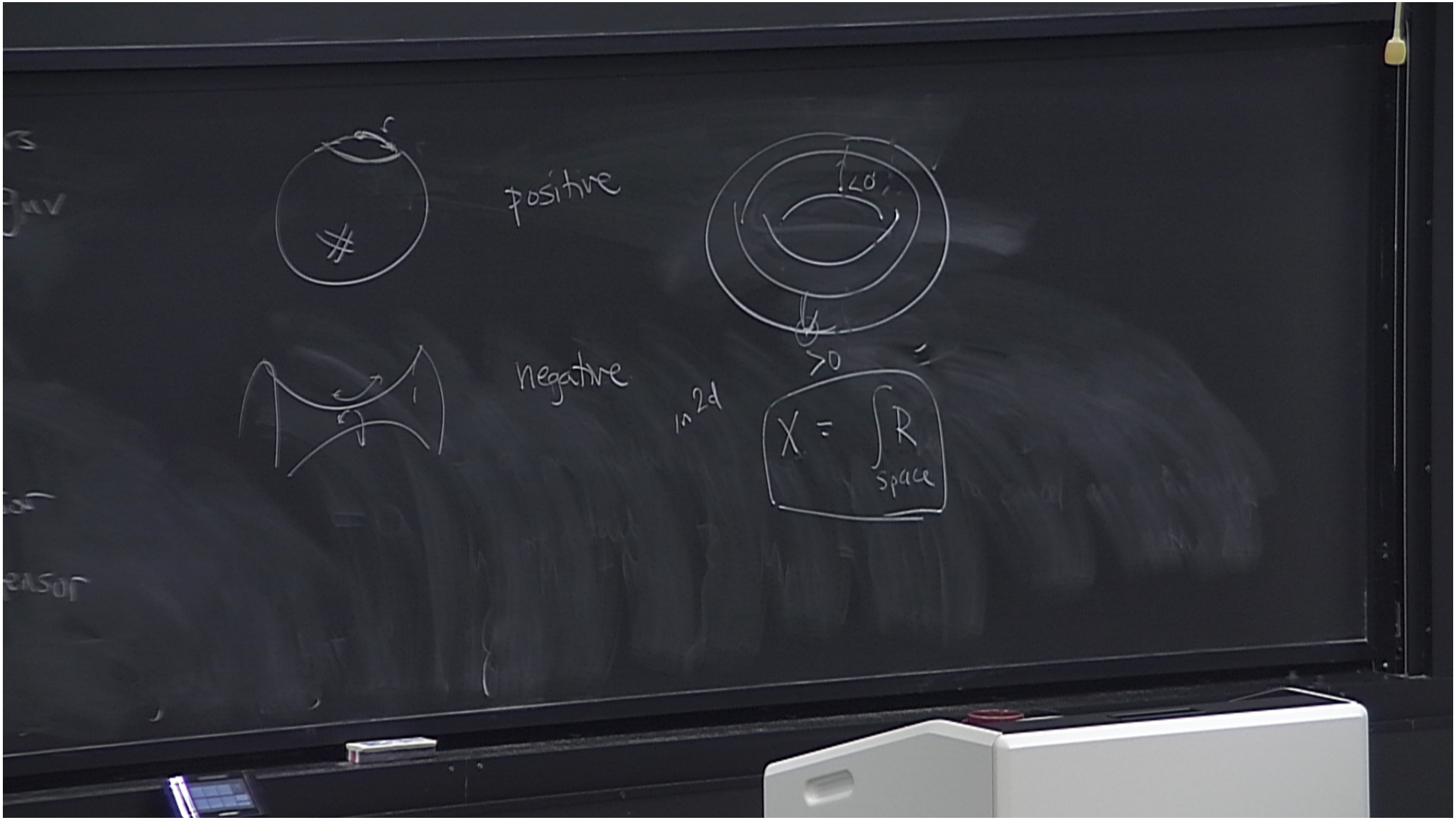
$x' \in + \ldots$

$f^2(x)dx^2 = dx'^2 \qquad x' = \int^{x} f(x)\,dx$



tlamermesh


D
# of non-cancellable parameters in second derivatives of $g_{\mu\nu}$
1 0
2 1 R Ricci scalar
3 6 $R_{\mu\nu}$ Ricci tensor
4 20 $R_{\mu\nu\rho\lambda}$ Riemann tensor


positive
negative
in 2d
<0
>0
X = ∫ R
space

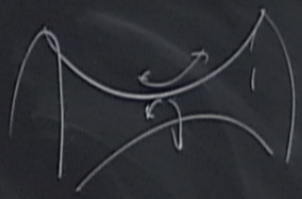
D

# of non-cancellable parameters in second derivatives of $g_{\mu\nu}$



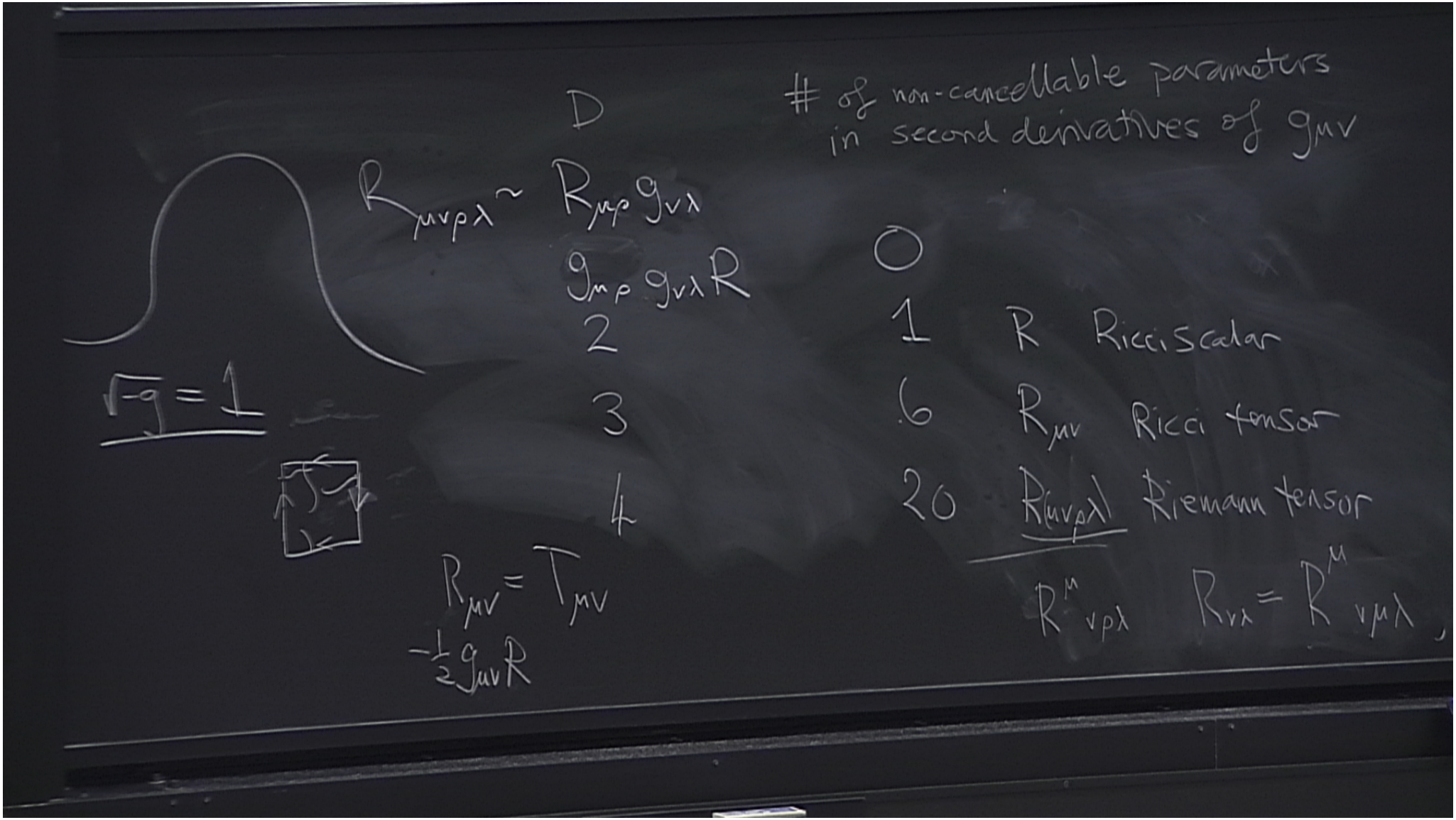
positive

D	#	
1	0	
2	1	R Ricci scalar
3	6	$R_{\mu\nu}$ Ricci tensor
4	20	$R_{\mu\nu\rho\lambda}$ Riemann tensor



negative

$R^{\mu}{}_{\nu\rho\lambda}$ $R_{\nu\lambda} = R^{\mu}{}_{\nu\mu\lambda}$, $R = R^{\mu}{}_{\nu\mu\lambda} g^{\nu\lambda}$


D
# of non-cancellable parameters in second derivatives of $g_{\mu\nu}$
$R_{\mu\nu\rho\lambda} \sim R_{\mu\rho} g_{\nu\lambda}$
$g_{\mu\rho} g_{\nu\lambda} R$
0
1 R Ricci scalar
2
3
6 $R_{\mu\nu}$ Ricci tensor
4
20 $R_{\mu\nu\rho\lambda}$ Riemann tensor
$\sqrt{-g} = 1$
$R_{\mu\nu} = T_{\mu\nu}$
$-\frac{1}{2} g_{\mu\nu} R$
$R^{\mu}{}_{\nu\rho\lambda}$ $R_{\nu\lambda} = R^{\mu}{}_{\nu\mu\lambda}$