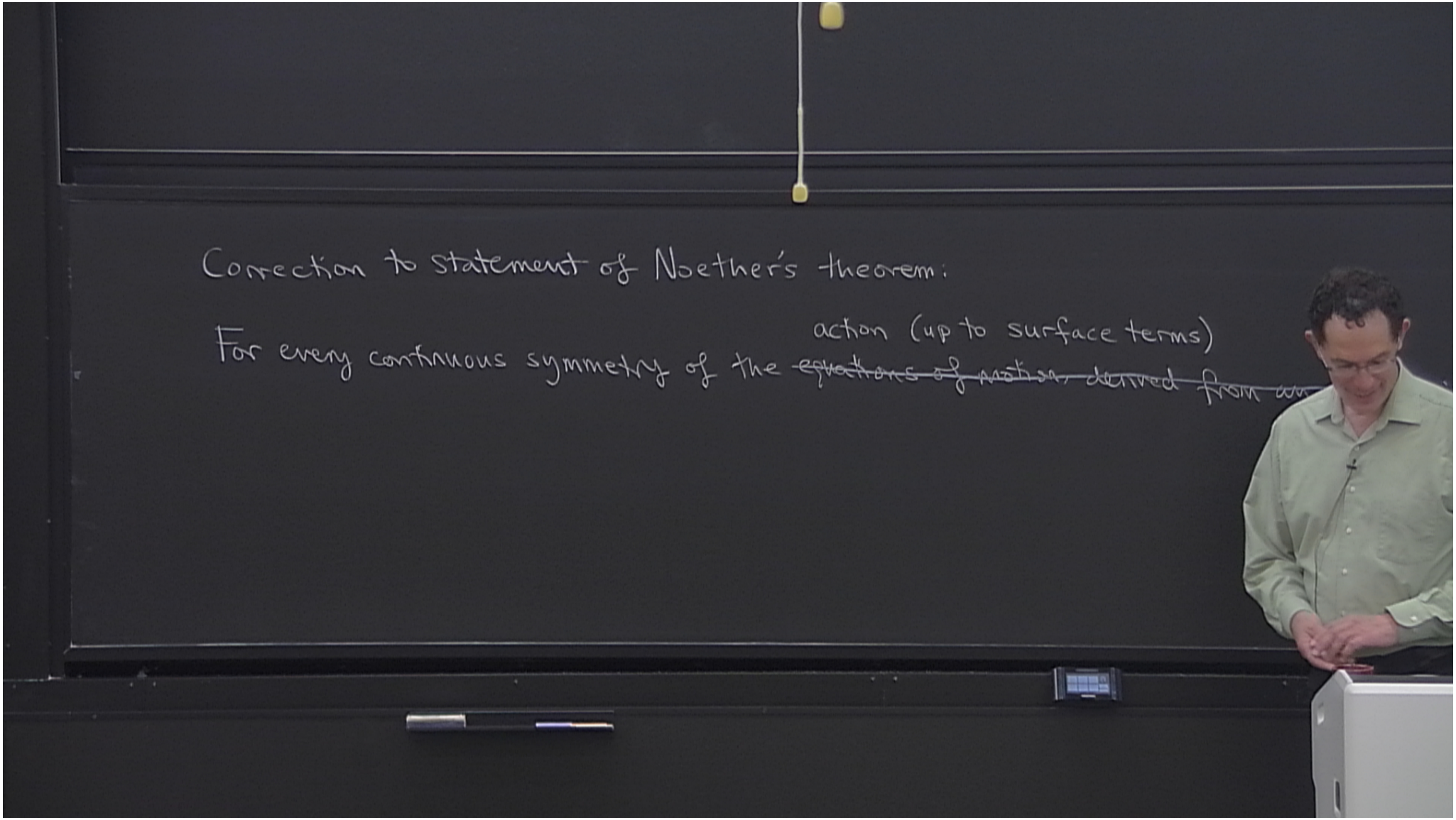
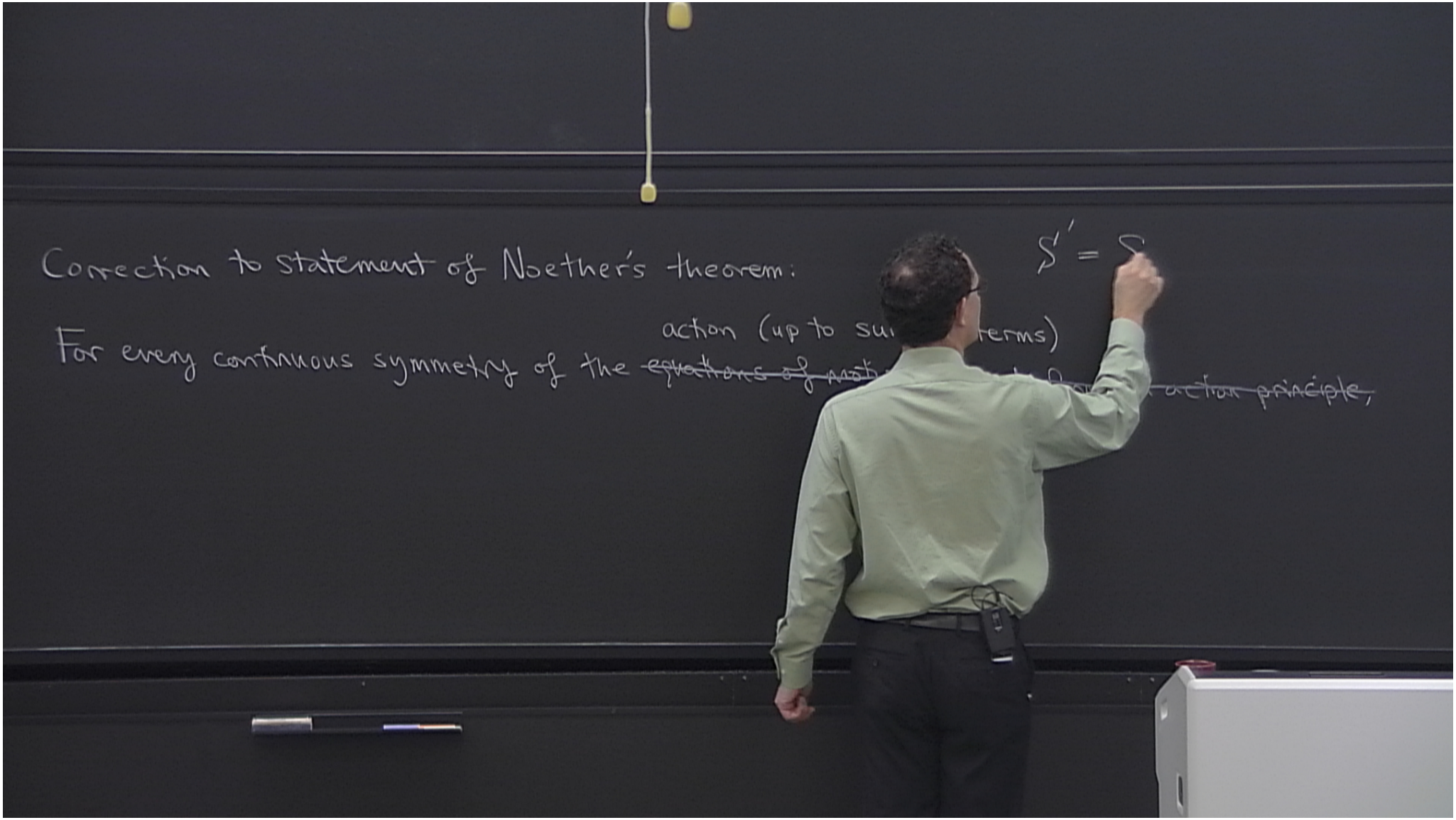
Title: 14/15 PSI - Relativity- Lecture 4

Date: Sep 11, 2014 09:00 AM

URL: http://pirsa.org/14090010

Abstract:


Correction to statement of Noether's theorem:
For every continuous symmetry of the action (up to surface terms)


Correction to statement of Noether's theorem:
$S' = S$
action (up to su
erms)
For every continuous symmetry of the
action principle,

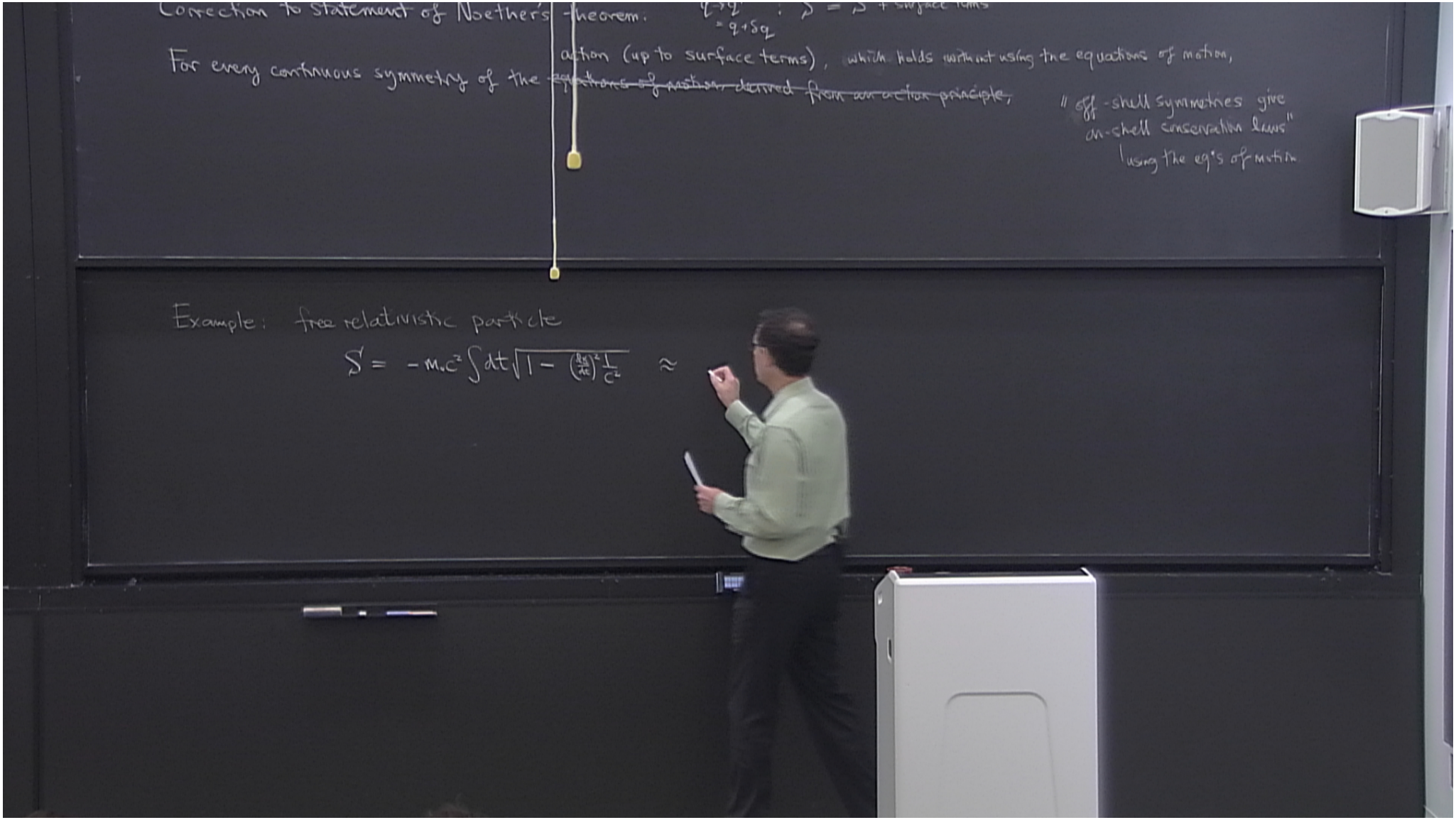
Correction to statement of Noether's theorem.

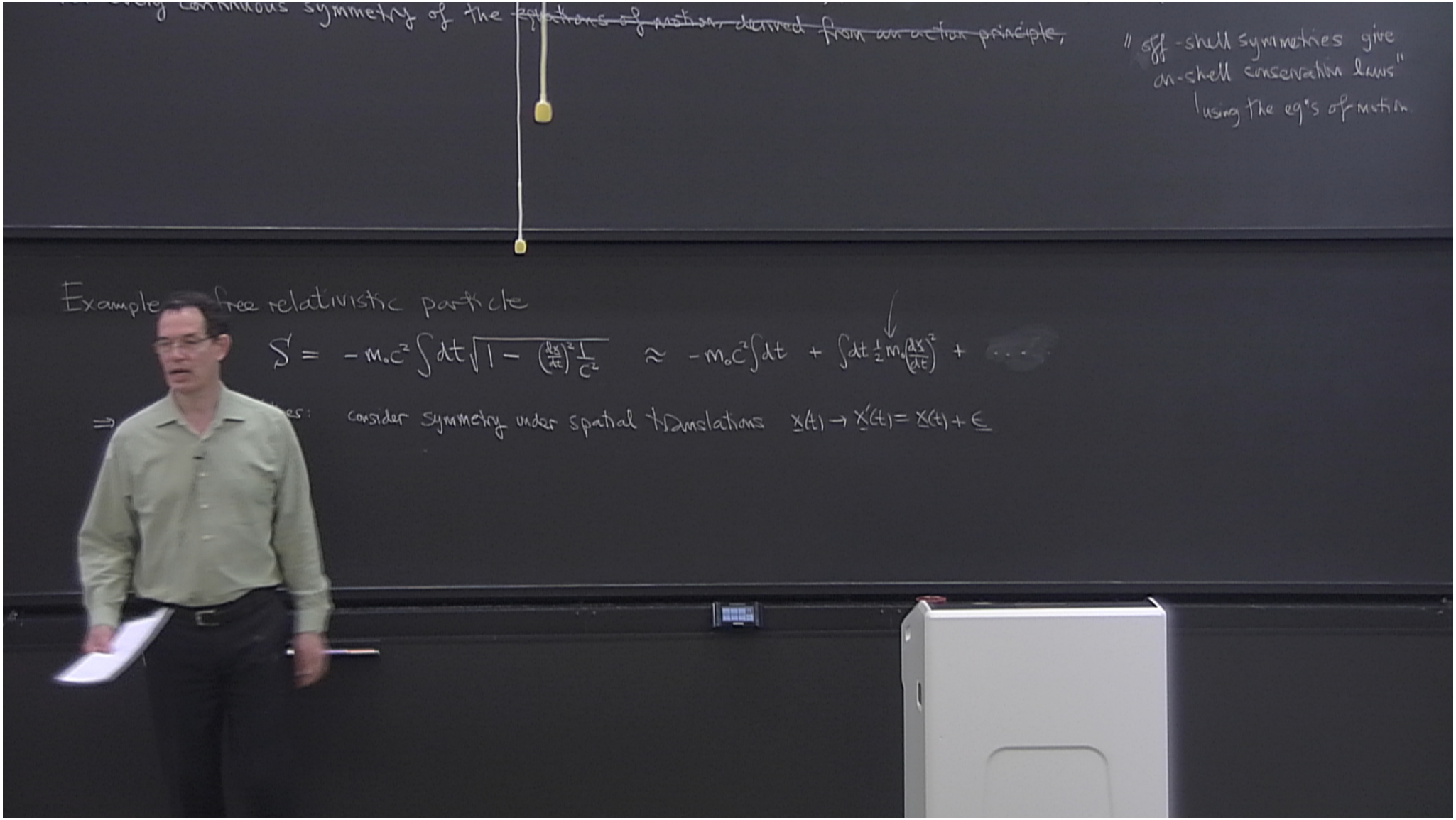
For every continuous symmetry of the ~~equations of motion derived from an action principle~~ action (up to surface terms), which holds without using the equations of motion,

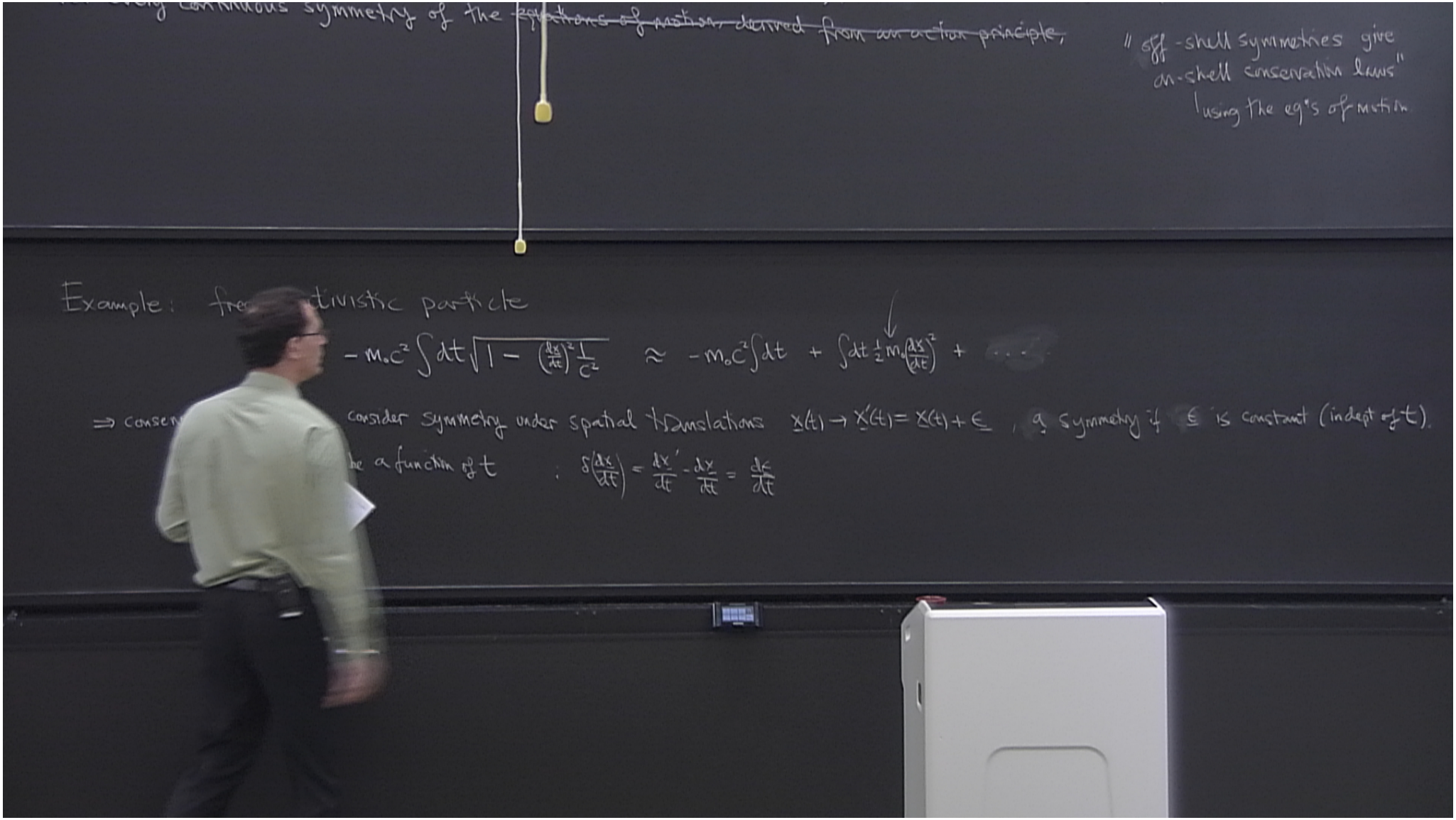
" off-shell symmetries give on-shell conservation laws" (using the eq's of motion)

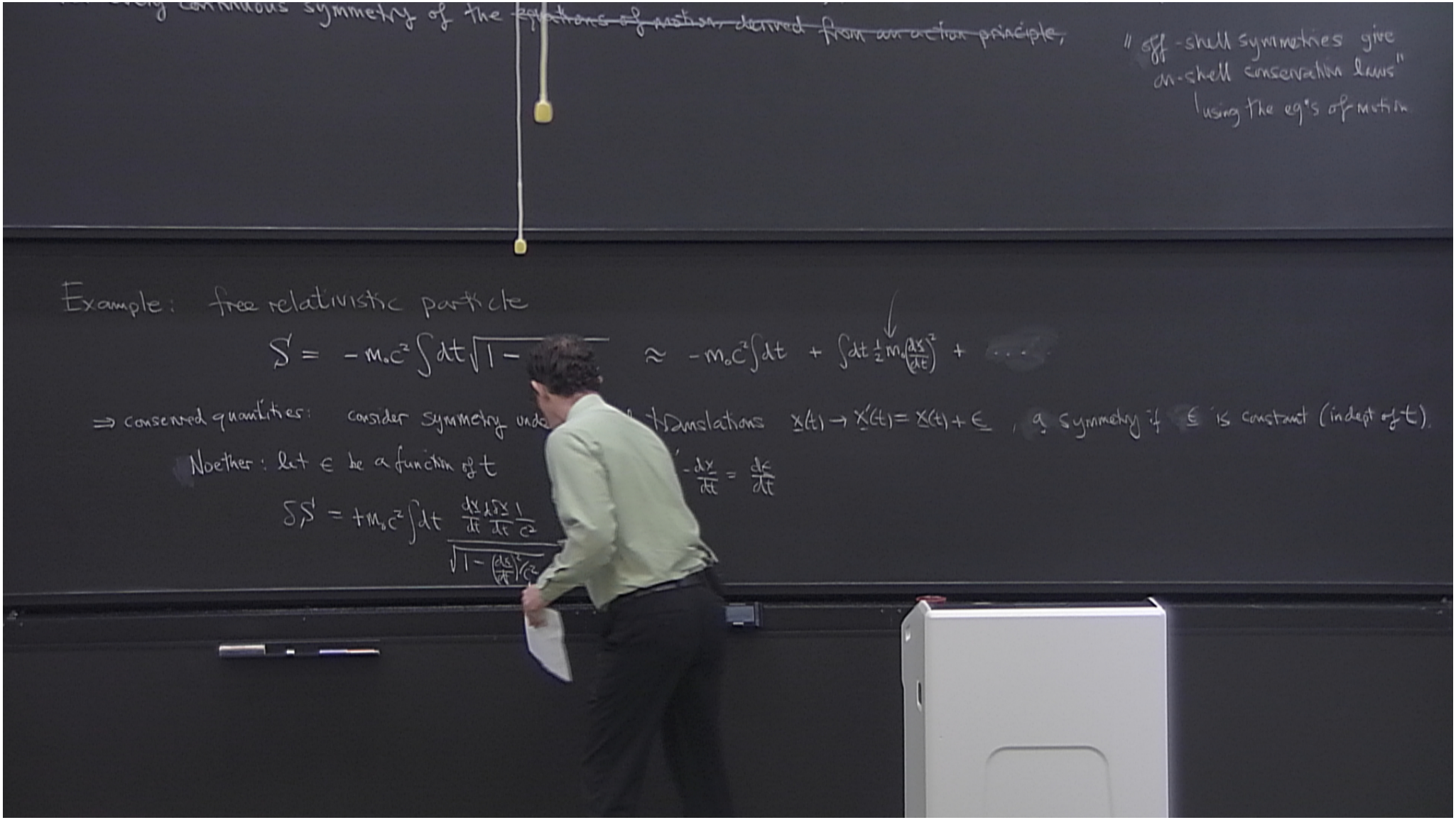
Example: free relativistic particle

$$S = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}} \approx$$




continuous symmetry of the equations of motion, derived from an action principle,
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Example: free relativistic particle
$S = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}} \approx -m_0 c^2 \int dt + \int dt \, \frac{1}{2} m_0 \left(\frac{dx}{dt}\right)^2 + \cdots$
consider symmetry under spatial translations $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \underline{\epsilon}$


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Example: free relativistic particle
$-m_0c^2\int dt\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}} \approx -m_0c^2\int dt + \int dt\,\frac{1}{2}m_0\left(\frac{dx}{dt}\right)^2 + \cdots$
consider symmetry under spatial translations $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \underline{\epsilon}$, a symmetry if $\underline{\epsilon}$ is constant (indept of $t$).
be a function of $t$ : $\delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$


symmetry of the equations of motion, derived from an action principle,
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Example: free relativistic particle
$S = -m_0 c^2 \int dt \sqrt{1 - \quad} \approx -m_0 c^2 \int dt + \int dt \, \frac{1}{2} m_0 \left(\frac{dx}{dt}\right)^2 + \ldots$
⇒ conserved quantities: consider symmetry und
translations $x(t) \to x'(t) = x(t) + \epsilon$, a symmetry if $\epsilon$ is constant (indept of $t$).
Noether: let $\epsilon$ be a function of $t$
$\delta S = +m_0 c^2 \int dt \, \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}\frac{1}{c^2}}{\sqrt{1 - \left(\frac{dx}{dt}\right)^2/c^2}}$
$-\frac{dx}{dt} = \frac{d\epsilon}{dt}$

$$\delta S = +m_0 c^2 \int dt \; \frac{\frac{dx}{dt}\frac{d\delta x}{dt}\frac{1}{c^2}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}} = m_0 \int dt \; \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}}$$

by Noether argument, $\delta S = 0$ on shell (i.e. using the eq$^{n}$s of motion) for arbitrary $\epsilon(t)$

$$= - m_0 \int dt \; \frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}}\right)\epsilon(t) \qquad \ldots\left(\frac{\frac{dx}{dt}}{\sqrt{\quad}}\right) = 0$$

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Noether: Let $\epsilon$ be a function of $t$ $\quad \delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$

$$\delta S = +m_0 c^2 \int dt \, \frac{\frac{dx}{dt}\frac{d\delta x}{dt}\frac{1}{c^2}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}} = m_0 \int dt \, \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}}$$

by Noether argument, $\delta S = 0$ on shell (i.e. using the eq$^{n}$s of motion) for arbitrary $\underline{\epsilon}(t)$

$$= - m_0 \int dt \, \frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}}\right)\epsilon(t) \quad \Rightarrow \frac{d}{dt}\left(\frac{m_0 \frac{dx}{dt}}{\sqrt{\quad}}\right) = 0 \quad \Rightarrow \underline{P} = \frac{m_0 \frac{dx}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2\frac{1}{c^2}}} \text{ is conserved}$$

⇒ Conserved quantities: Consider symmetry under spatial translations $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \underline{\epsilon}$, a symmetry if $\underline{\epsilon}$ is const

Noether: let $\epsilon$ be a function of $t$ : $\delta\left(\frac{d\underline{x}}{dt}\right) = \frac{d\underline{x}'}{dt} - \frac{d\underline{x}}{dt} = \frac{d\underline{\epsilon}}{dt}$

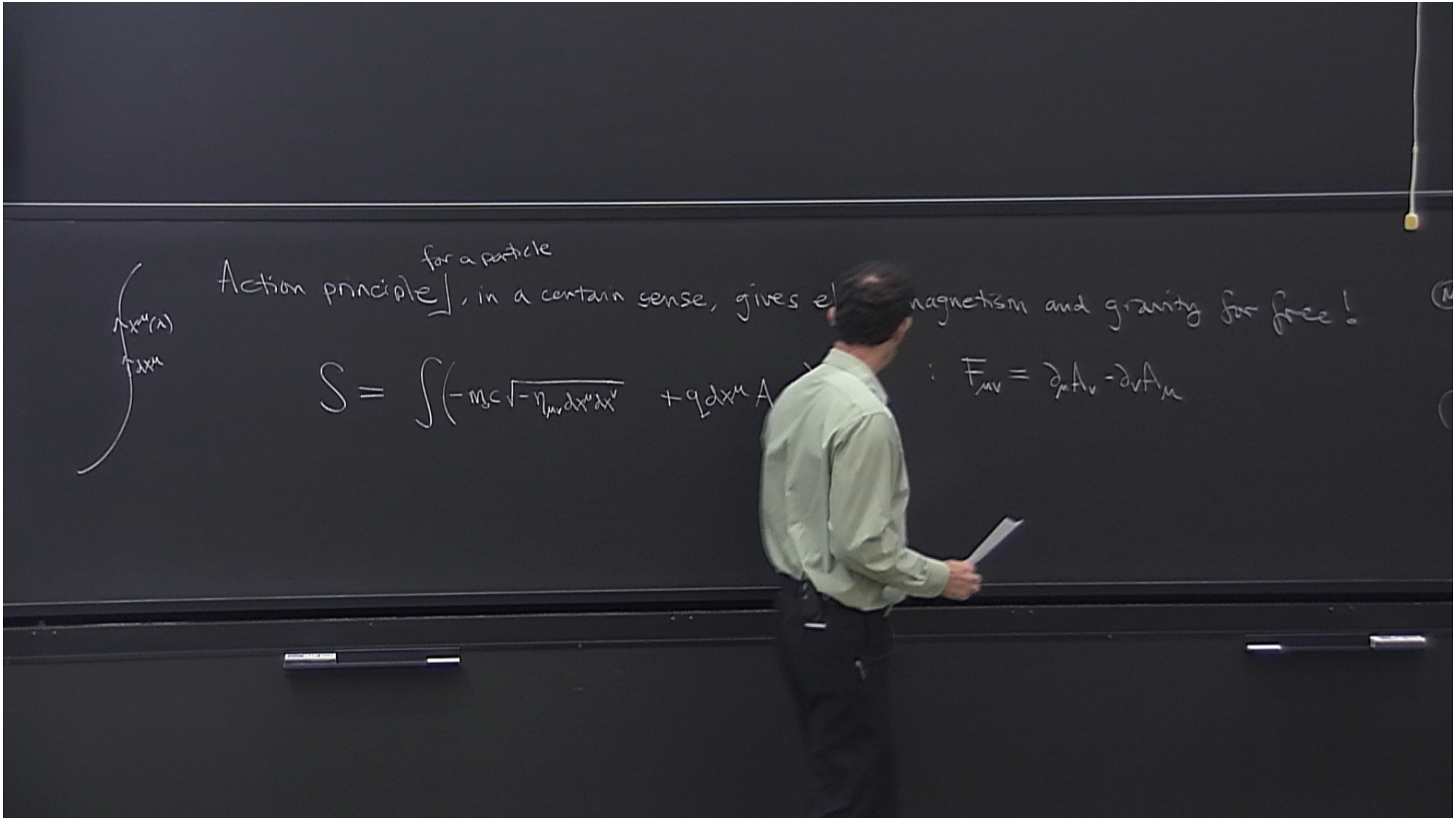
$$\delta S = +m_0 c^2 \int dt \frac{\frac{d\underline{x}}{dt}\cdot\frac{d\underline{\epsilon}}{dt}\frac{1}{c^2}}{\sqrt{1-\left(\frac{d\underline{x}}{dt}\right)^2\frac{1}{c^2}}} = m_0 \int dt \frac{\frac{d\underline{x}}{dt}\cdot\frac{d\underline{\epsilon}}{dt}}{\sqrt{1-\left(\frac{d\underline{x}}{dt}\right)^2\frac{1}{c^2}}}$$

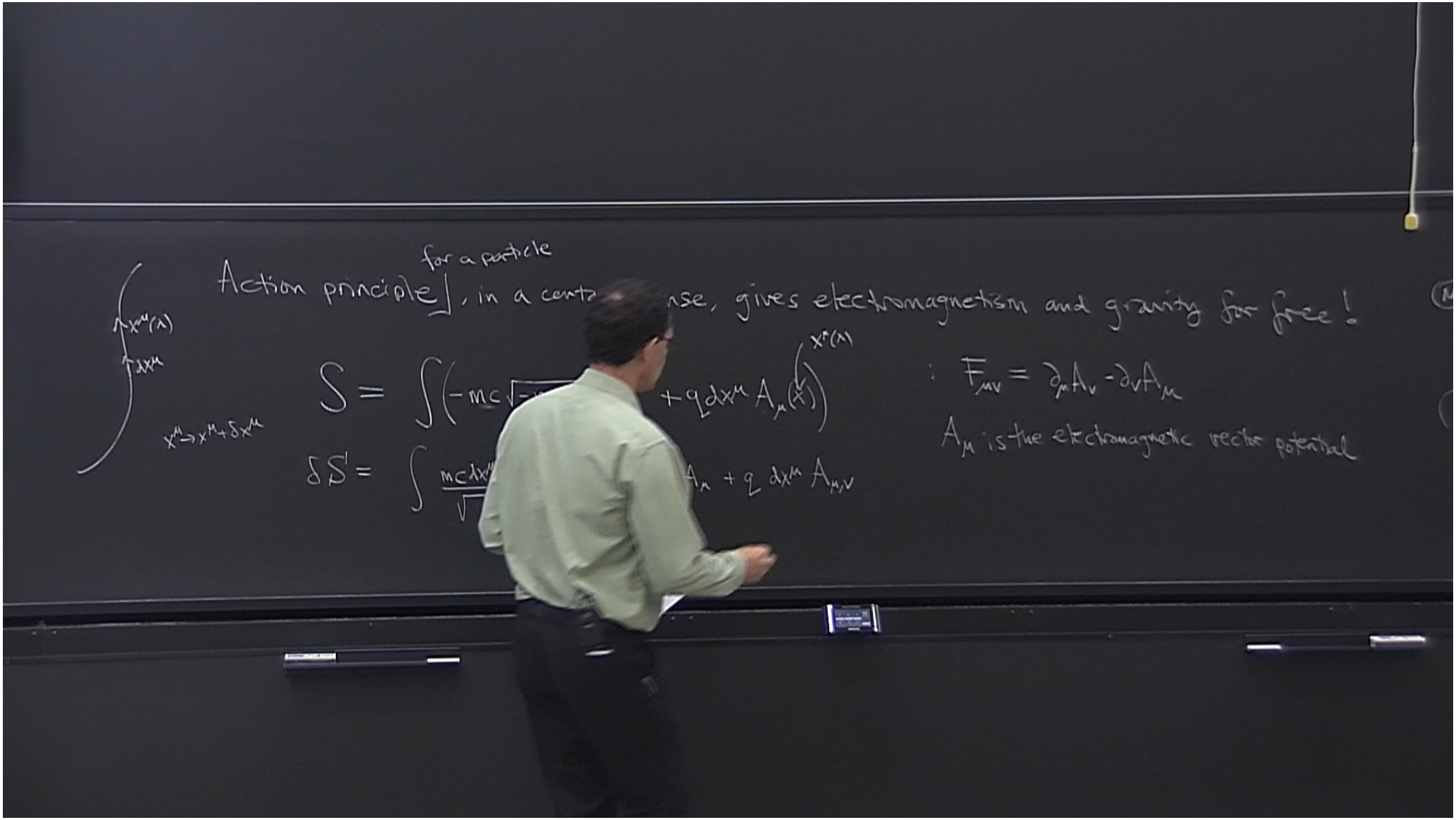
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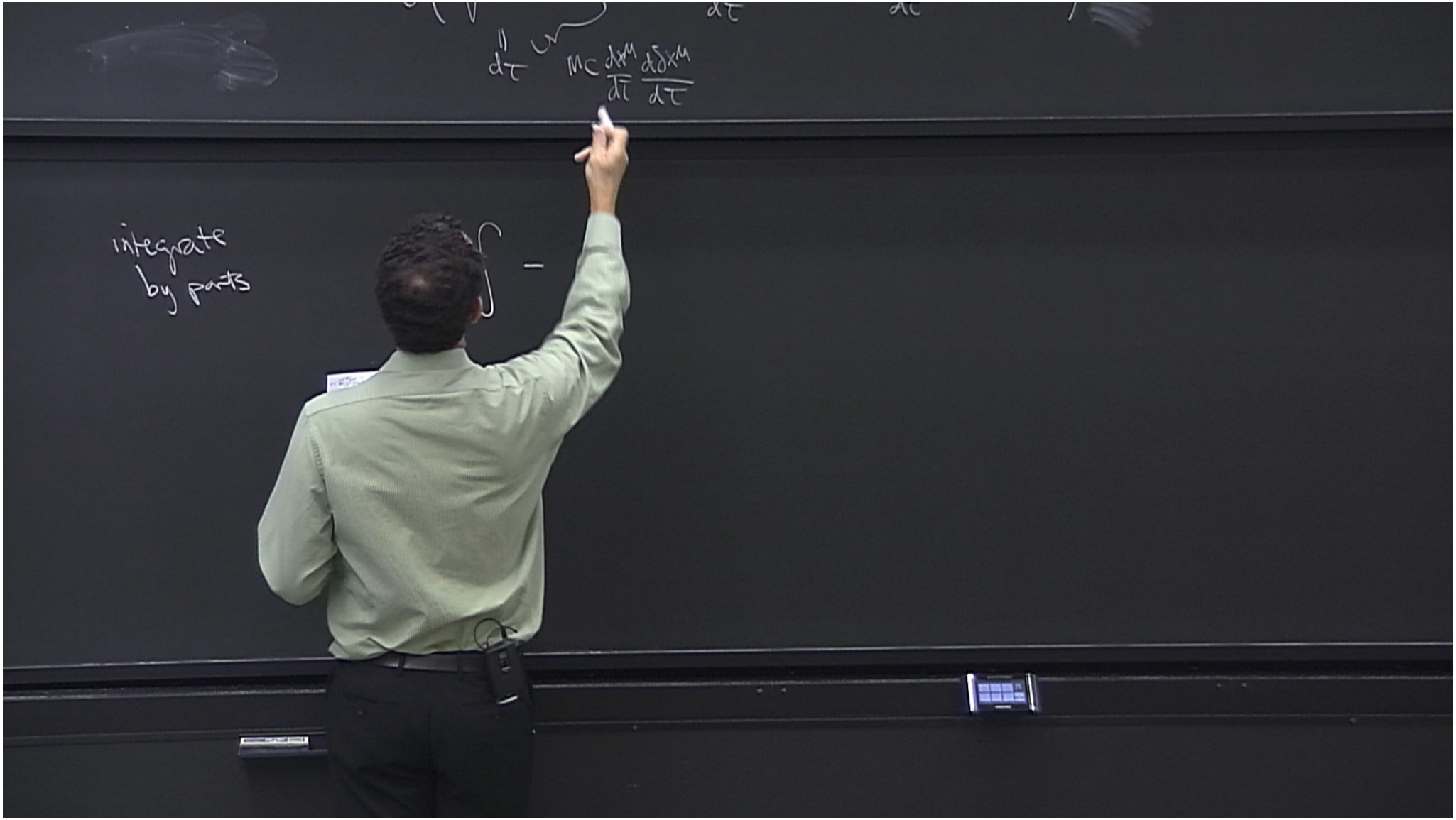
$$= -\, m_0 \int dt \, \frac{d}{dt}\left(\frac{\frac{d\underline{x}}{dt}}{\sqrt{1-\left(\frac{d\underline{x}}{dt}\right)^2\frac{1}{c^2}}}\right)\epsilon(t) \Rightarrow \frac{d}{dt}\left(\frac{m_0\frac{d\underline{x}}{dt}}{\sqrt{\quad}}\right) = 0 \to \underline{p} = \frac{m_0\frac{d\underline{x}}{dt}}{\sqrt{1-(}}$$

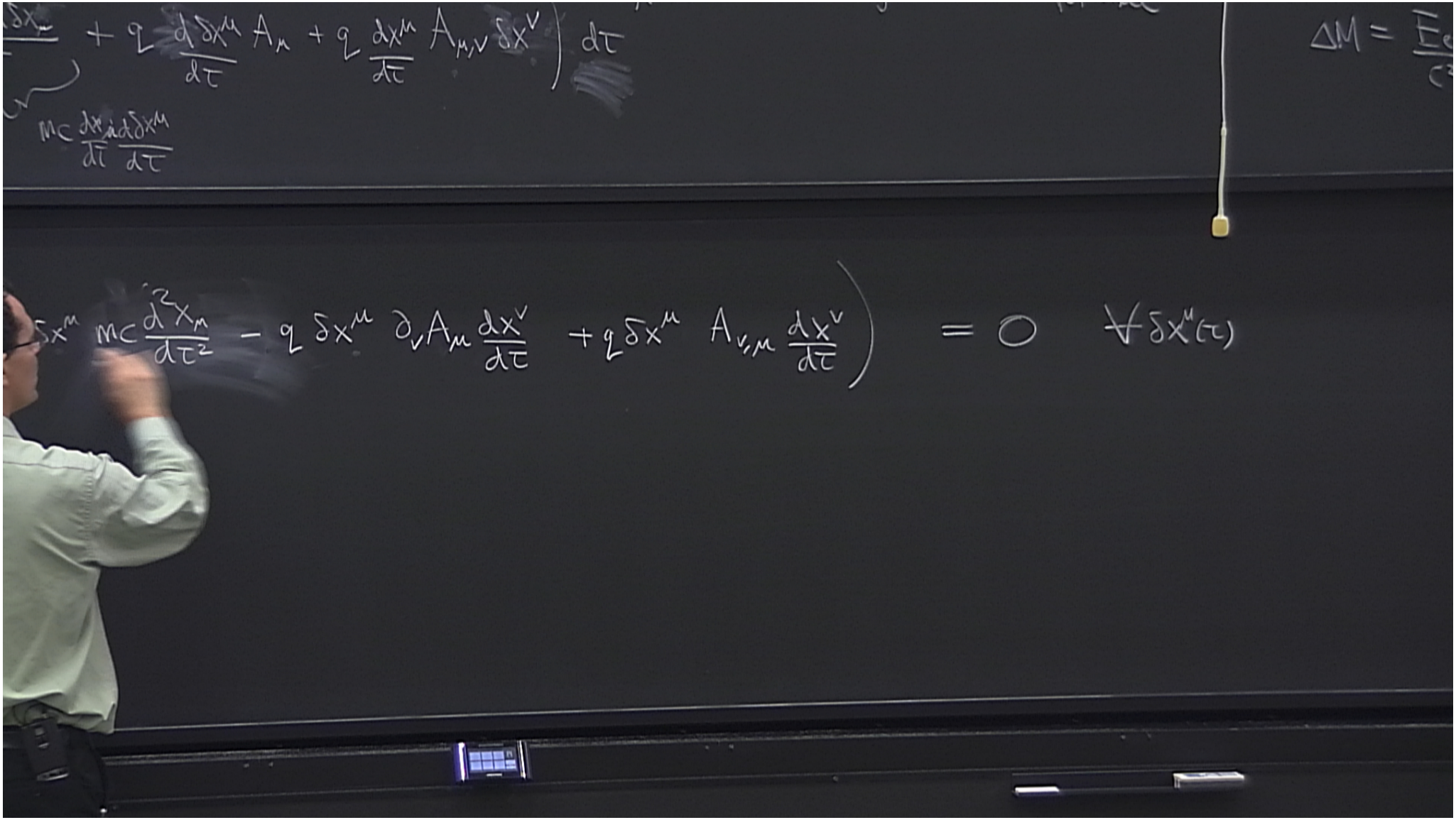
Now consider symmetry under time translations: $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \epsilon\frac{d\underline{x}}{dt}$ : $\delta\underline{x}(t) = \epsilon\frac{d\underline{x}}{dt}$ : $\delta\frac{d\underline{x}}{dt} =$

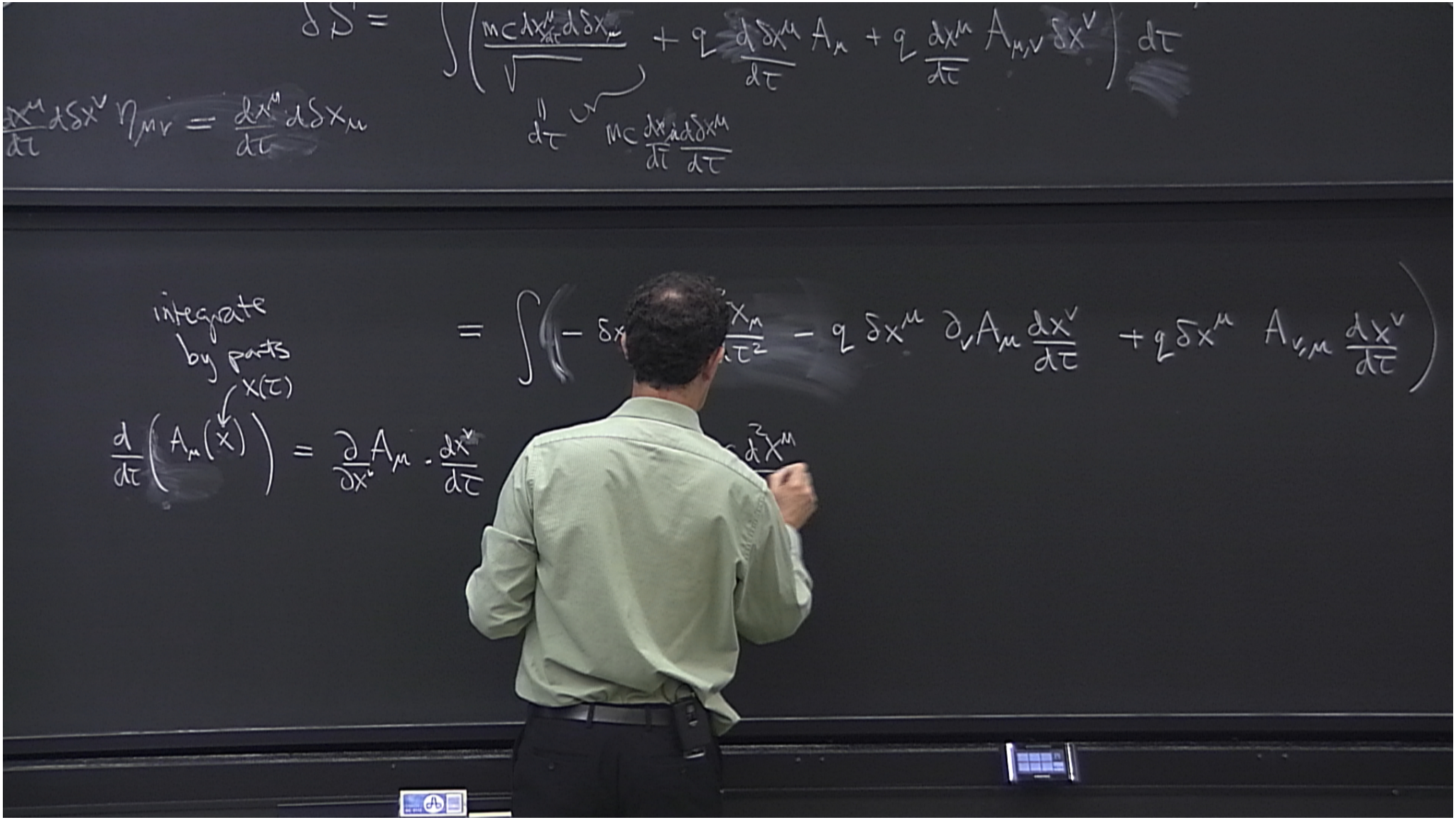
$$\delta S = m_0 c^2 \int \frac{dt \frac{1}{c^2}\frac{d\underline{x}}{dt}\frac{d}{dt}\overbrace{\left(\epsilon\frac{d\underline{x}}{dt}\right)}^{\delta\underline{x}}}{\sqrt{1-\left(\frac{d\underline{x}}{dt}\right)^2\frac{1}{c^2}}}$$

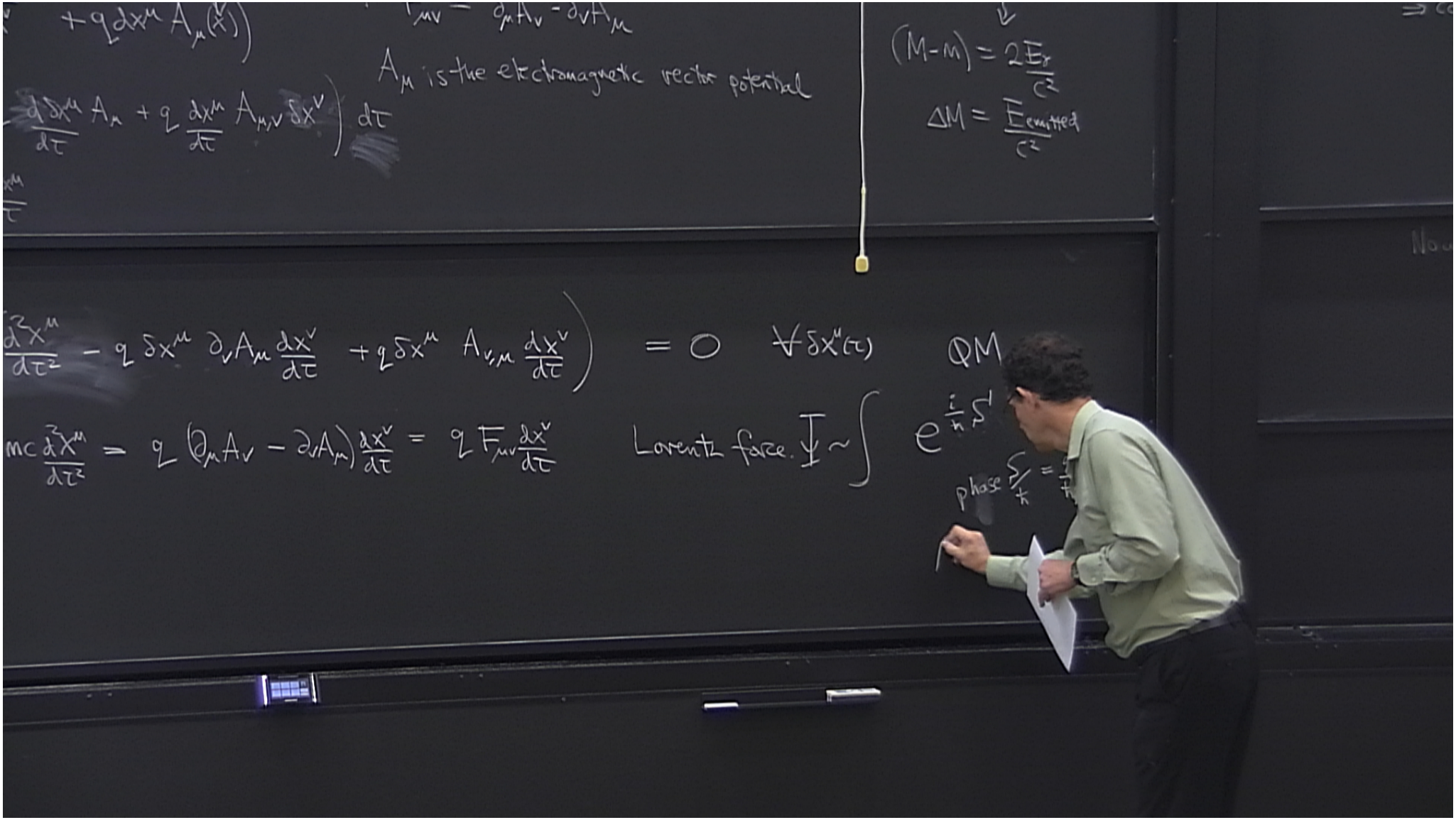

Action principle for a particle, in a certain sense, gives e nagnetism and gravity for free!
$S = \int \left( -mc\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + q dx^\mu A \right.$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

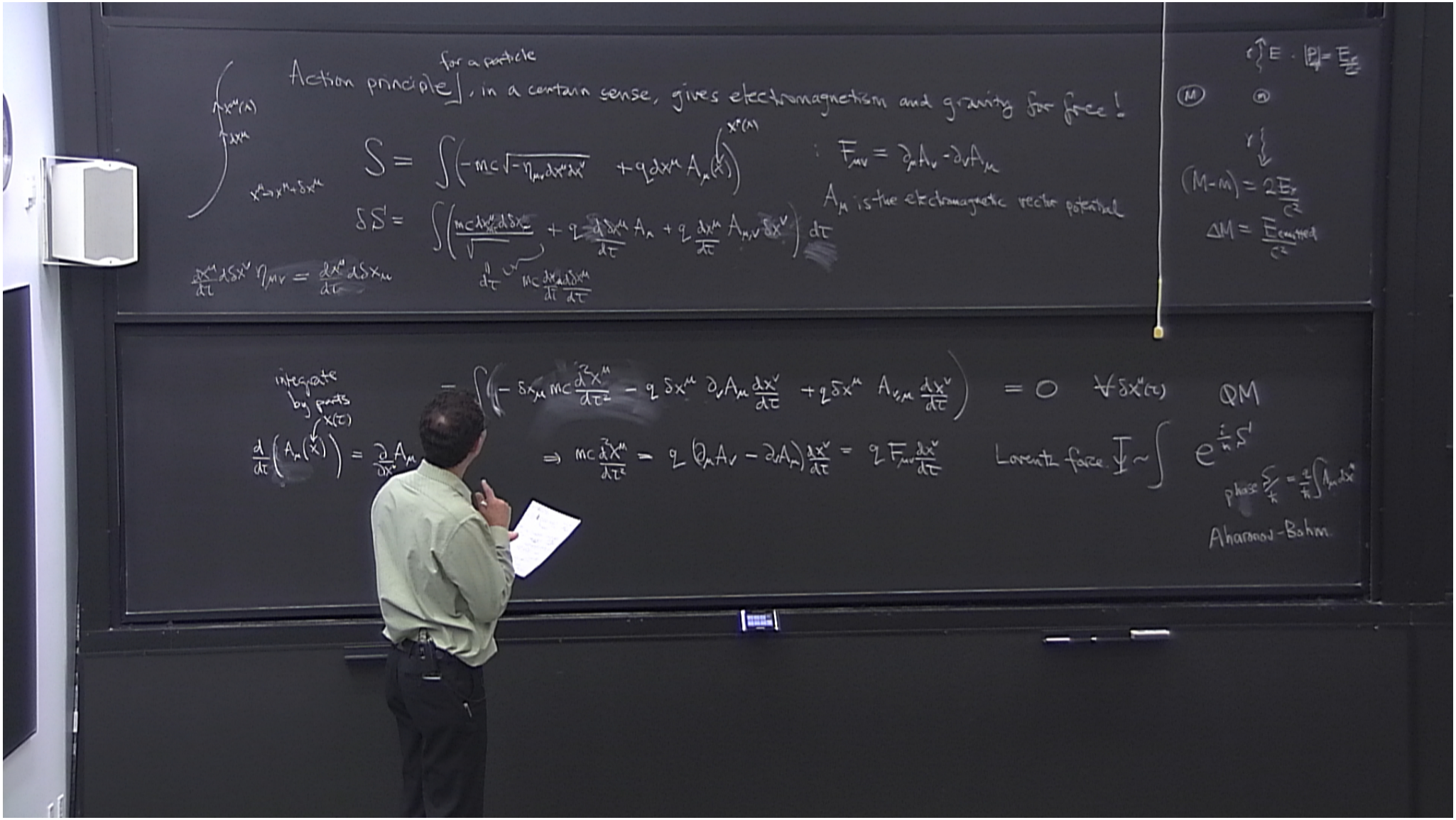

Action principle for a particle, in a cent nse, gives electromagnetism and gravity for free!
$x^\mu(\lambda)$
$x^\mu \to x^\mu + \delta x^\mu$
$S = \int (-mc\sqrt{-} + q dx^\mu A_\mu(x))$
$x^\mu(\lambda)$
$\delta S = \int \frac{mc dx^\mu}{\sqrt{}} A_\mu + q\, dx^\mu A_{\mu,\nu}$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$A_\mu$ is the electromagnetic vector potential

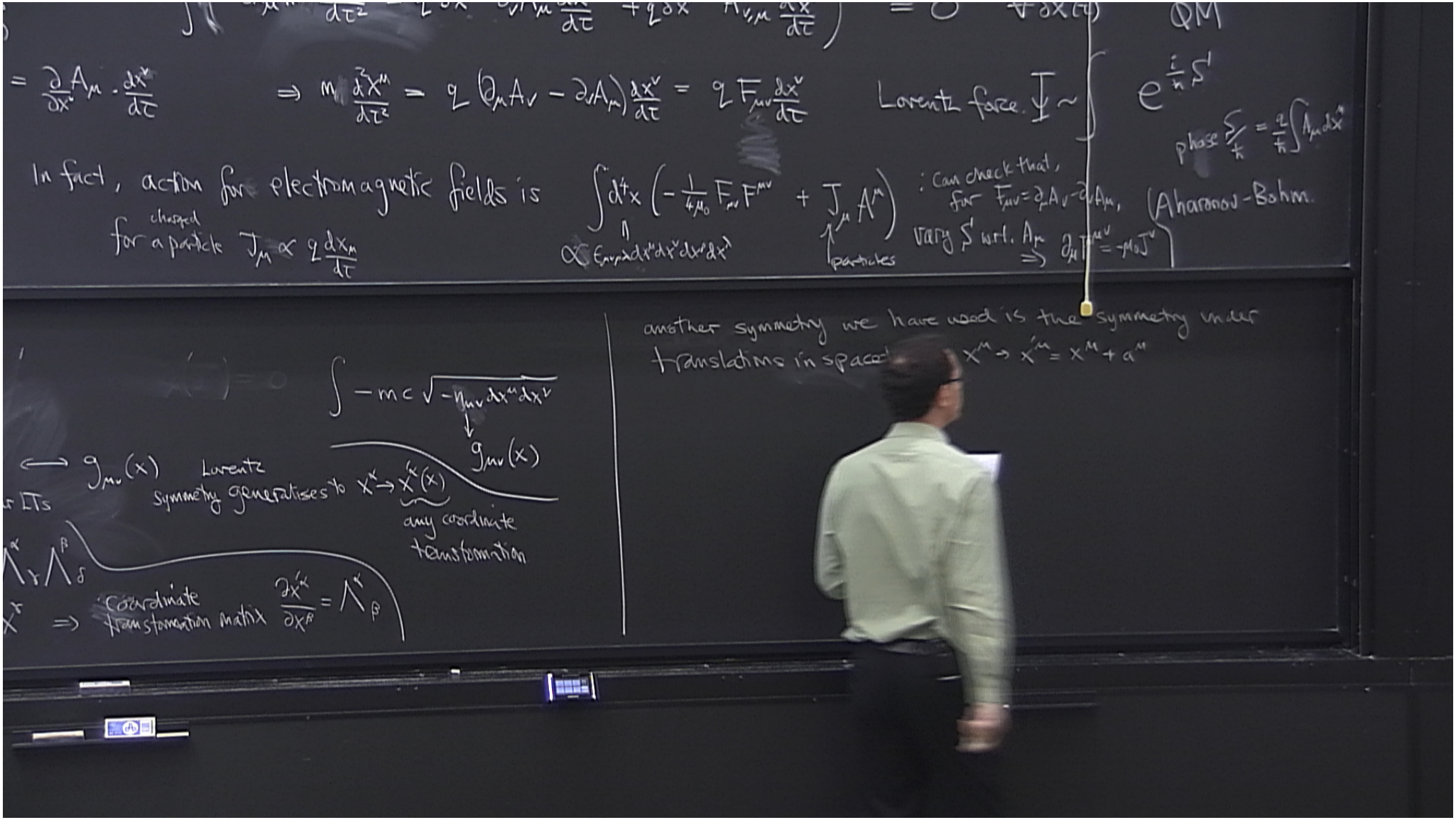

integrate
by parts


$+ q\frac{d\delta x^\mu}{d\tau} A_\mu + q\frac{dx^\mu}{d\tau} A_{\mu,\nu}\delta x^\nu \Big) d\tau$
$mc\frac{dx^\mu}{d\tau}\frac{d\delta x^\mu}{d\tau}$
$\Delta M = \frac{E}{c^2}$
$\delta x^\mu \, mc\frac{d^2x_\mu}{d\tau^2} - q\delta x^\mu \partial_\nu A_\mu \frac{dx^\nu}{d\tau} + q\delta x^\mu A_{\nu,\mu}\frac{dx^\nu}{d\tau}\Big) = 0 \quad \forall \delta x^\mu(\tau)$


integrate by parts
x(τ)


$+q\,dx^\mu A_\mu(x)\Big)$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
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$\frac{d\delta x^\mu}{d\tau} A_\mu + q\frac{dx^\mu}{d\tau} A_{\mu,\nu}\delta x^\nu\Big)\,d\tau$
$(M-m) = 2\frac{E_r}{c^2}$
$\Delta M = \frac{E_{emitted}}{c^2}$
$\frac{d^2x^\mu}{d\tau^2} - q\,\delta x^\mu\,\partial_\nu A_\mu \frac{dx^\nu}{d\tau} + q\,\delta x^\mu\, A_{\nu,\mu}\frac{dx^\nu}{d\tau}\Big) = 0 \quad \forall \delta x^\mu(\tau)$
QM
$mc\frac{d^2x^\mu}{d\tau^2} = q\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\frac{dx^\nu}{d\tau} = q F_{\mu\nu}\frac{dx^\nu}{d\tau}$
Lorentz force. $\Psi \sim \int e^{\frac{i}{\hbar}S}$
phase $S/\hbar =$


Action principle for a particle, in a certain sense, gives electromagnetism and gravity for free!
$S = \int \left( -mc\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + q A_\mu dx^\mu \right)$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$A_\mu$ is the electromagnetic vector potential
$\delta S = \int \left( \frac{mc\, dx^\mu d\delta x^\mu}{d\tau} + q \frac{d\delta x^\mu}{d\tau} A_\mu + q \frac{dx^\mu}{d\tau} A_{\mu,\nu} \delta x^\nu \right) d\tau$
integrate by parts
$= \int \left( -\delta x_\mu mc \frac{d^2 x^\mu}{d\tau^2} - q\, \delta x^\mu\, \partial_\nu A_\mu \frac{dx^\nu}{d\tau} + q \delta x^\mu A_{\nu,\mu} \frac{dx^\nu}{d\tau} \right) = 0 \quad \forall \delta x^\mu(\tau)$
$\Rightarrow mc \frac{d^2 x^\mu}{d\tau^2} = q \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \frac{dx^\nu}{d\tau} = q F_{\mu\nu} \frac{dx^\nu}{d\tau}$
Lorentz force
QM
$\psi \sim \int e^{\frac{i}{\hbar} S'}$
phase $\frac{S}{\hbar} = \frac{q}{i\hbar} \int A_\mu dx^\mu$
Aharonov-Bohm


QM
$= \frac{\partial}{\partial x^\mu} A_\nu \cdot \frac{dx^\nu}{d\tau}$
$\Rightarrow m \frac{d^2 x^\mu}{d\tau^2} = q(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{dx^\nu}{d\tau} = q F_{\mu\nu}\frac{dx^\nu}{d\tau}$
Lorentz force.
$\Psi \sim \int e^{\frac{i}{\hbar}S'}$
phase $\frac{S'}{\hbar} = \frac{q}{\hbar}\int A_\mu dx^\mu$
In fact, action for electromagnetic fields is
$\int d^4x \left(-\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\right)$
$\propto \epsilon_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma$
particles
: can check that, for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,
vary $S'$ wrt. $A_\mu$ $\Rightarrow \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$
(Aharonov-Bohm.
for a charged particle $J_\mu \propto q\frac{dx_\mu}{d\tau}$
$\int -mc\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$
$\downarrow$
$g_{\mu\nu}(x)$
$\longleftrightarrow g_{\mu\nu}(x)$ Lorentz symmetry generalises to $x^\mu \to x'^\mu(x)$
any coordinate transformation
LIs
$\Lambda^\alpha_\gamma \Lambda^\beta_\delta$
$x^\gamma$
$\Rightarrow$ Coordinate transformation matrix $\frac{\partial x'^\alpha}{\partial x^\beta} = \Lambda^\alpha_\beta$
another symmetry we have used is the symmetry under translations in space $x^\mu \to x'^\mu = x^\mu + a^\mu$

$\frac{dx^\nu}{d\tau} \Rightarrow m\frac{d^2x^\mu}{d\tau^2} = q(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{dx^\nu}{d\tau} = qF_{\mu\nu}\frac{dx^\nu}{d\tau}$ Lorentz force. $\Psi \sim e^{\frac{i}{\hbar}S}$

phase $\frac{S}{\hbar} = \frac{q}{\hbar}\int A_\mu dx^\mu$

, action for electromagnetic fields is $\int d^4x \left(-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\right)$ : Can check that, for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, vary $S$ w.r.t. $A_\mu$ $\Rightarrow \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ (Aharonov-Bohm.

$\propto \epsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma$

charged particles

for a particle $J_\mu \propto q\frac{dx_\mu}{d\tau}$

$\int -mc\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$

$\downarrow$

$g_{\mu\nu}(x)$

$(x)$ Lorentz symmetry generalises to $x^\nu \to x'^\nu(x)$ any coordinate transformation

coordinate transformation matrix $\frac{\partial x'^\mu}{\partial x^\beta} = \Lambda^\mu{}_\beta$

another symmetry we have used is the symmetry under translations in spacetime, $x^\mu \to x'^\mu = x^\mu + a^\mu$

Together, Lorentz transformations and spacetime translations form the Poincaré group.

$\frac{dx^\nu}{d\tau} \Rightarrow m\frac{d^2x^\mu}{d\tau^2} = q\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\frac{dx^\nu}{d\tau} = qF_{\mu\nu}\frac{dx^\nu}{d\tau}$ Lorentz force. $\Psi \sim$ $e^{\frac{i}{\hbar}S}$

phase $\frac{S}{\hbar} = \frac{q}{\hbar}\int A_\mu dx^\mu$

, action for electromagnetic fields is $\int d^4x \left(-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\right)$ : Can check that, for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, vary $S$ w.r.t. $A_\mu$ $\Rightarrow \partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ (Aharonov-Bohm.

$\propto \epsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma$

charged for a particle $J_\mu \propto q\frac{dx_\mu}{d\tau}$

particles

$\int -mc\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$

$\downarrow$

$g_{\mu\nu}(x)$

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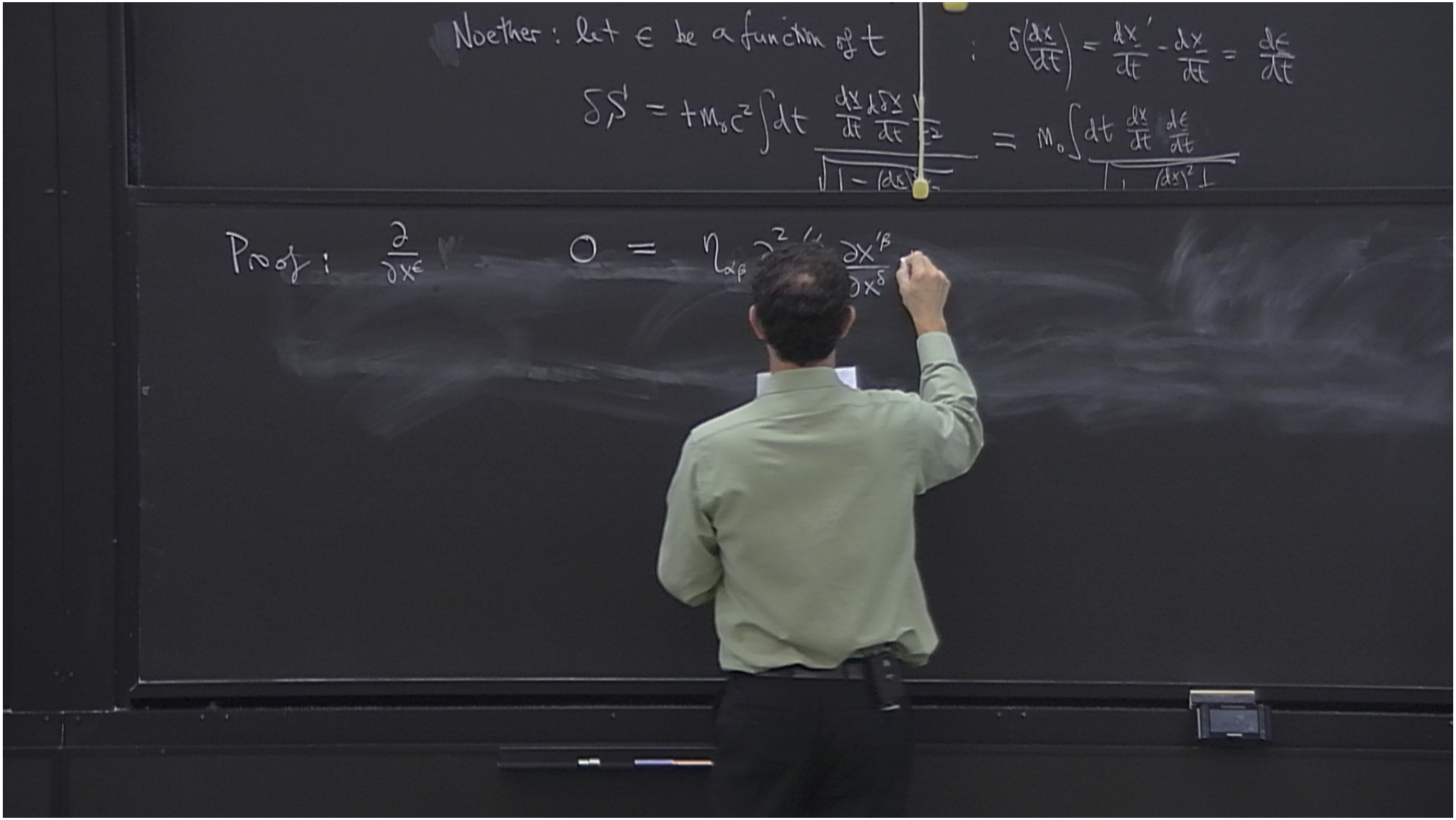
coordinate transformation matrix $\frac{\partial x'^\mu}{\partial x^\beta} = \Lambda^\mu_{\ \beta}$

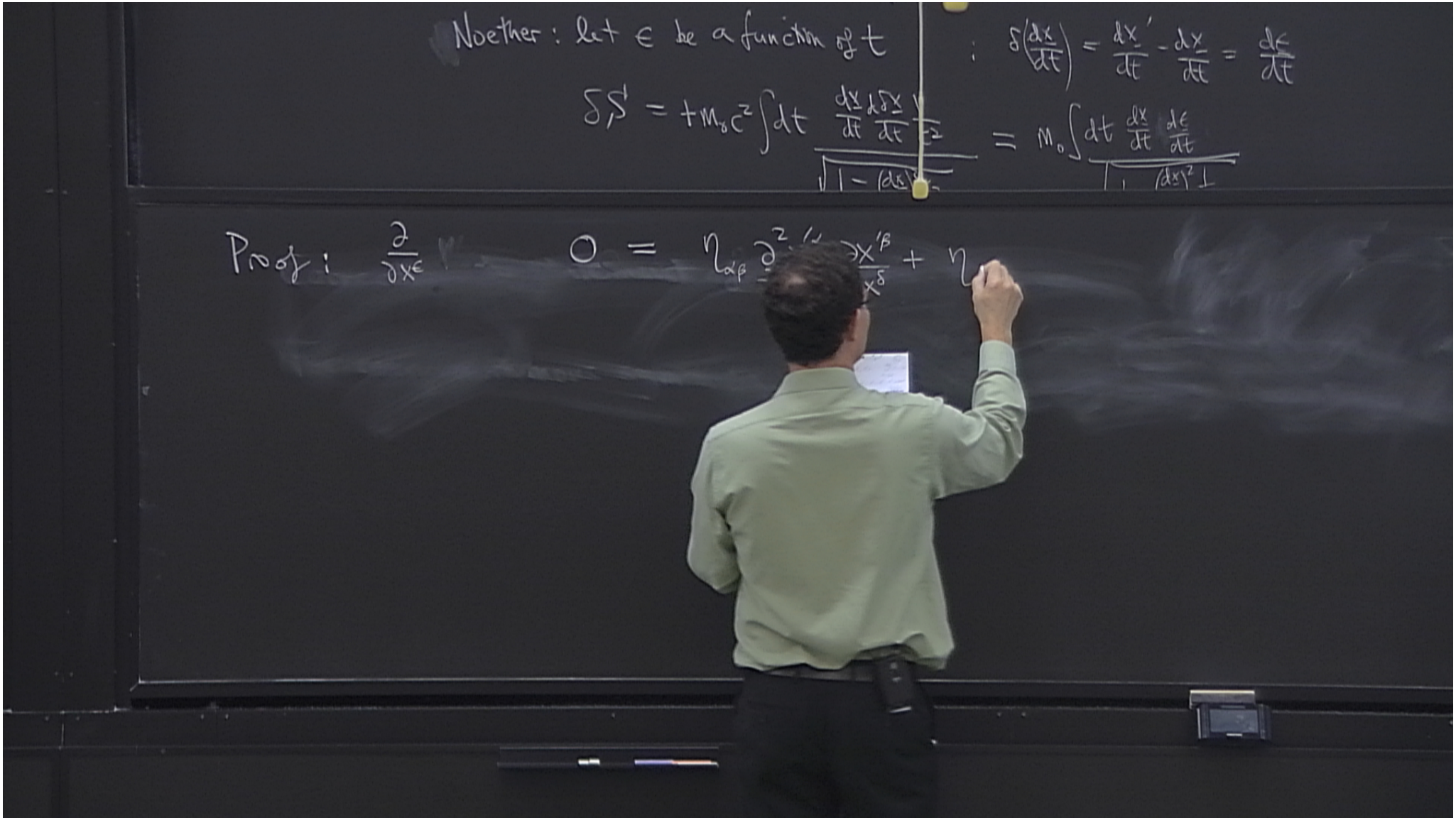
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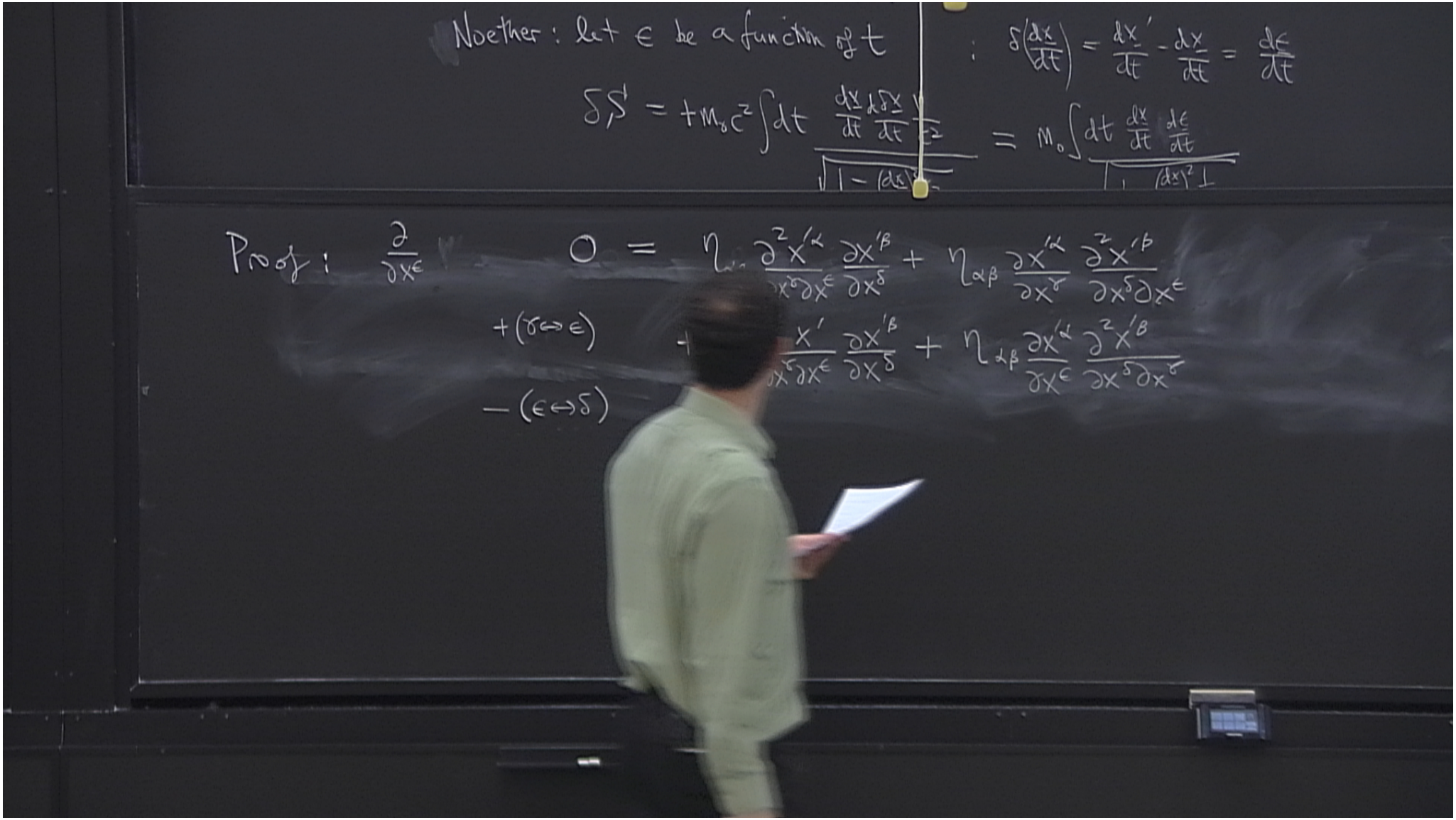
Toget... z transformations and spacetime translations form the Po... group (a 10-parameter group).

In fact, ... the most general continuous transformations $x^\alpha \to x'^\alpha$ ... preserve $\eta_{\mu\nu}$, i.e. for which

$= \eta_{\gamma\beta}\frac{\partial x'^\gamma}{\partial x^\delta}\frac{\partial x'^\delta}{\partial x^\sigma}$


Noether: Let $\epsilon$ be a function of $t$
$\delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$
$\delta S' = +m_0 c^2 \int dt \frac{\frac{dx}{dt}\frac{d\delta x}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}} = m_0 \int dt \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}$
Proof: $\frac{\partial}{\partial x^\epsilon}$
$0 = \eta_{\alpha\beta}$
$\frac{\partial x'^\beta}{\partial x^\delta}$


Noether: Let ε be a function of t
δ(dx/dt) = dx'/dt − dx/dt = dε/dt
δS' = +m₀c² ∫ dt
= m₀ ∫ dt
Proof:
∂/∂xᵉ
0 = η
+ η


Noether: Let $\epsilon$ be a function of $t$
$\delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$
$\delta S' = +m_0 c^2 \int dt \frac{\frac{dx}{dt}\frac{d\delta x}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}} = m_0 \int dt \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}$
Proof: $\frac{\partial}{\partial x^\epsilon}$
$0 = \eta_{\alpha\beta}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\epsilon}$
$+(\gamma \leftrightarrow \epsilon)$
$+ \frac{x'}{\partial x^\delta \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\epsilon}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\gamma}$
$-(\epsilon \leftrightarrow \delta)$

Noether: Let $\epsilon$ be a function of $t$

$$\delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$$

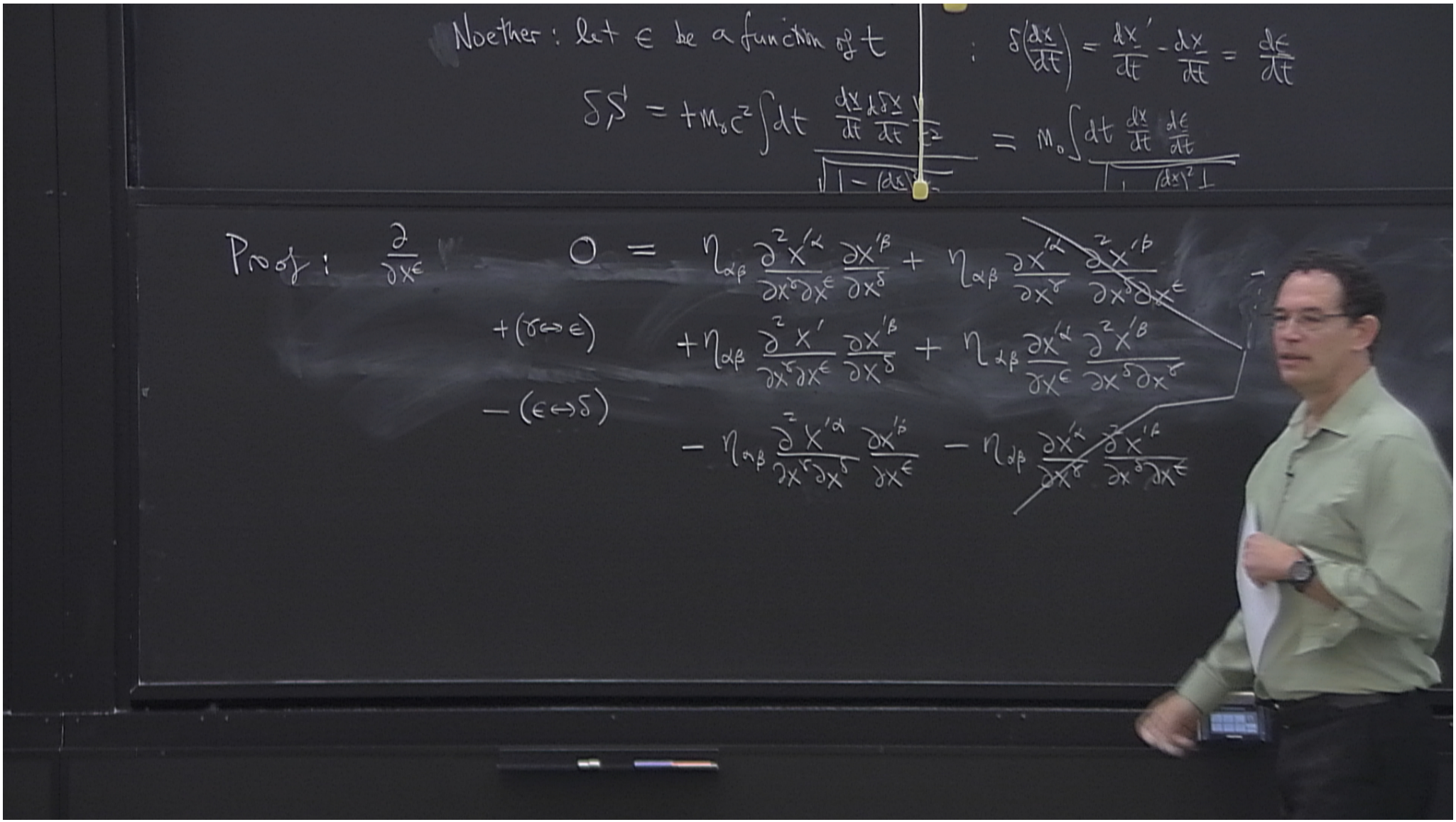
$$\delta S' = +m_0 c^2 \int dt \, \frac{\frac{dx}{dt}\frac{d\delta x}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}} = m_0 \int dt \, \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}$$

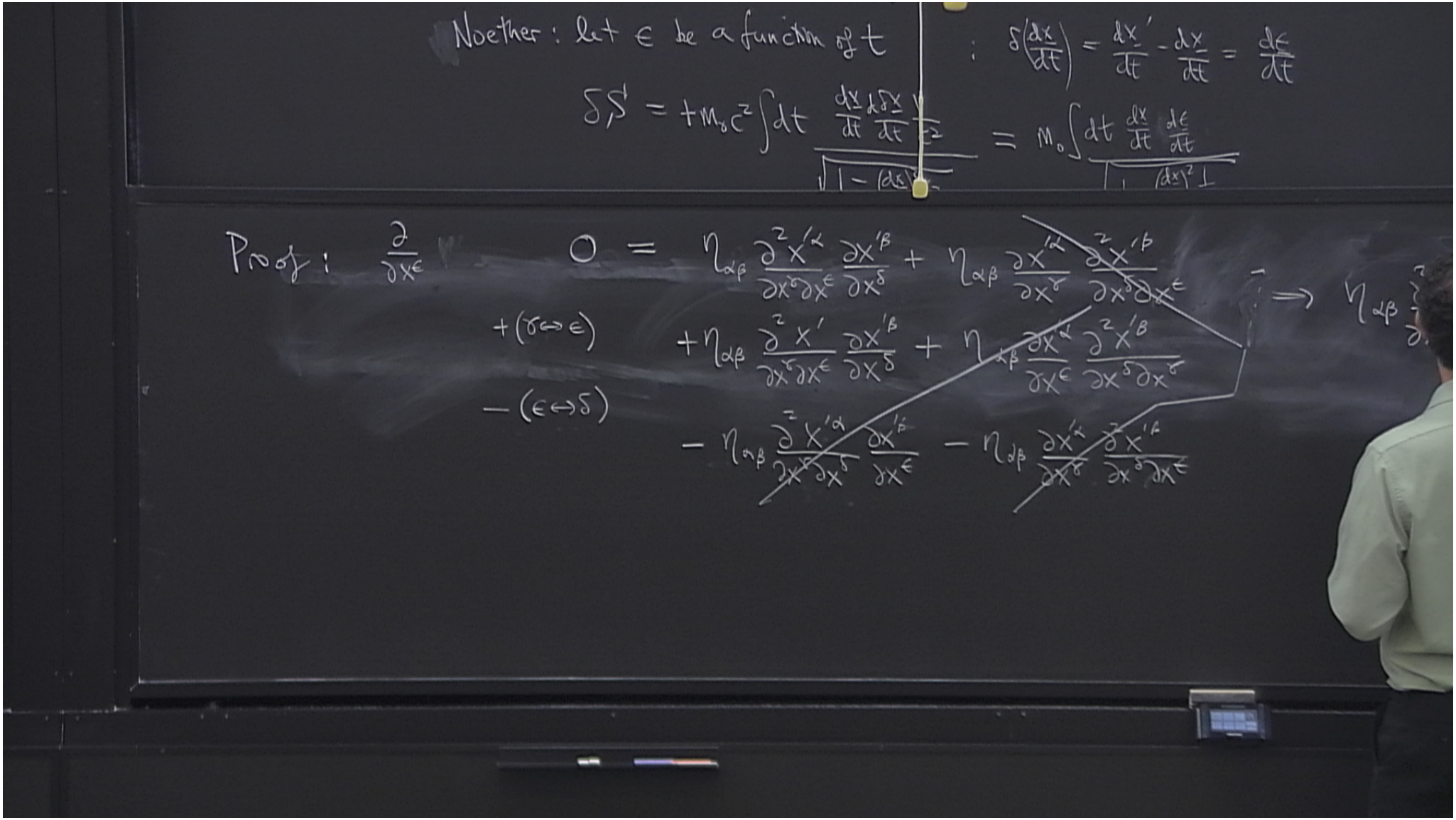
Proof: $\frac{\partial}{\partial x^\epsilon}$

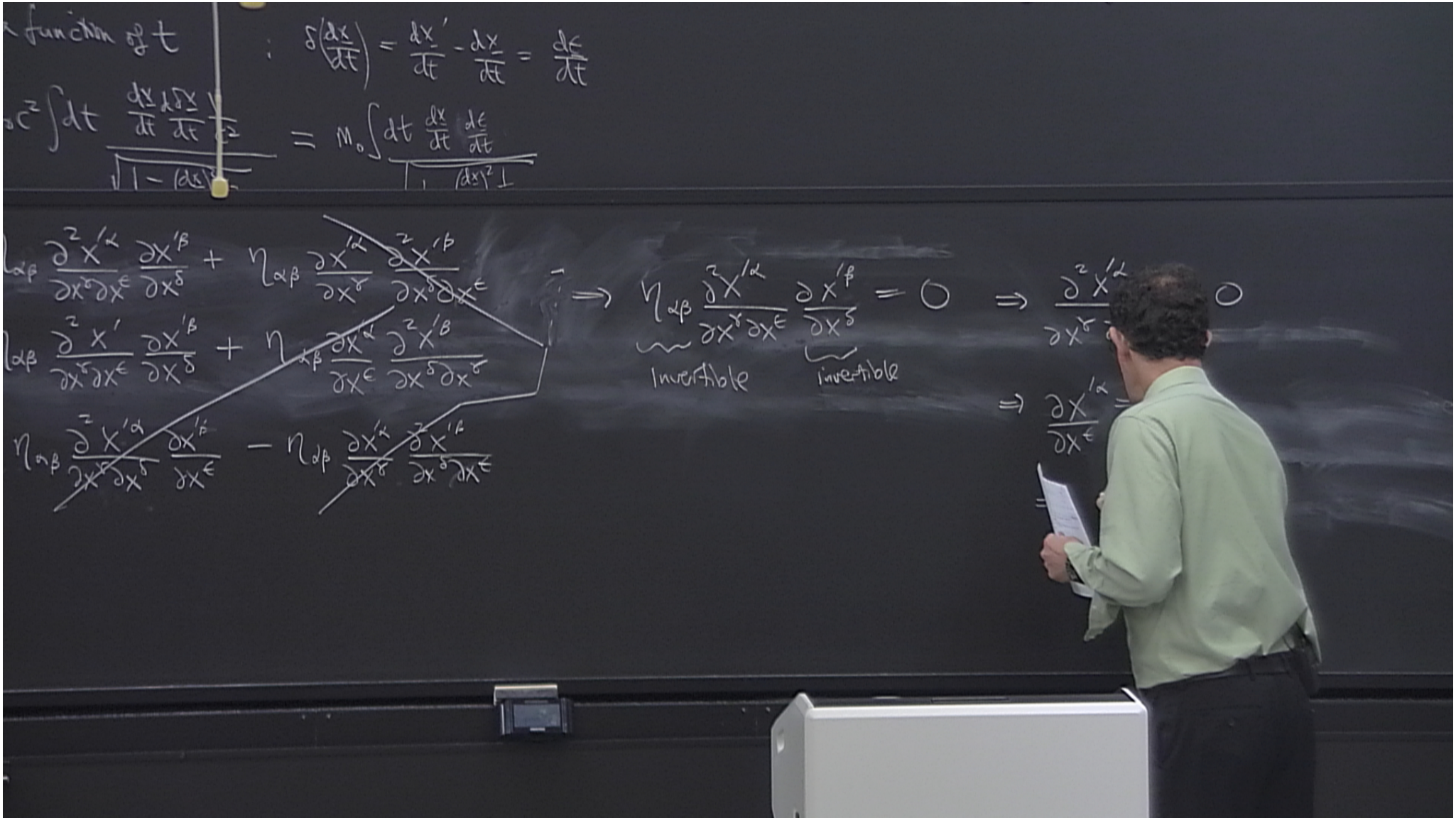
$$0 = \eta_{\alpha\beta}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\epsilon}$$

$$+(\gamma \leftrightarrow \epsilon) \qquad +\eta_{\alpha\beta}\frac{\partial^2 x'}{\partial x^\delta \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\gamma} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\epsilon}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\gamma}$$

$$-(\epsilon \leftrightarrow \delta) \qquad -\eta_{\alpha\beta}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}\frac{\partial x'^\beta}{\partial x^\epsilon} - \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\epsilon}$$




Noether: Let $\epsilon$ be a function of $t$
$\delta\left(\frac{dx}{dt}\right) = \frac{dx'}{dt} - \frac{dx}{dt} = \frac{d\epsilon}{dt}$
$\delta S' = +m_0 c^2 \int dt \frac{\frac{dx}{dt}\frac{d\delta x}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}} = m_0 \int dt \frac{\frac{dx}{dt}\frac{d\epsilon}{dt}}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}$
Proof: $\frac{\partial}{\partial x^\epsilon}$
$0 = \eta_{\alpha\beta}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\epsilon}$
$+(\gamma \leftrightarrow \epsilon)$
$+\eta_{\alpha\beta}\frac{\partial^2 x'}{\partial x^\delta \partial x^\epsilon}\frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\epsilon}\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\gamma}$
$-(\epsilon \leftrightarrow \delta)$
$-\eta_{\alpha\beta}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}\frac{\partial x'^\beta}{\partial x^\epsilon} - \eta_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\epsilon}$
$\Rightarrow \eta_{\alpha\beta}$


Invertible
invertible