

Title: 14/15 PSI - Relativity- Lecture 3

Date: Sep 10, 2014 09:00 AM

URL: <http://pirsa.org/14090009>

Abstract:

Recall

e-m tensor

e-m field tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

$$T^{00} = \frac{1}{2\mu_0} \left(\frac{E^2}{c^2} + B^2 \right)$$

$$T^{0i} = \frac{1}{\mu_0 c} (\underline{E} \wedge \underline{B})^i$$

obeys $\partial_{\mu} T^{\mu\nu} = 0$ using Maxwell with $J^{\mu} = 0$.
($\partial_{\mu} J^{\mu} = 0$)

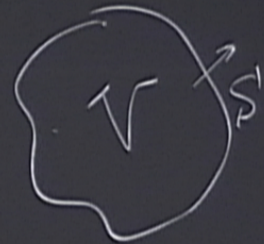
$\Downarrow$

$$\partial_0 T^{00} + \partial_i T^{i0} = 0$$

$$\text{and } \partial_0 T^{0i} + \partial_j T^{ji} = 0$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{using Maxwell with } T^{\mu\nu} = 0.$$

$$(\partial_\mu T^{\mu\nu} = 0)$$



$$\left. \begin{aligned} \partial_t T^{00} + \partial_i T^{i0} &= 0 \\ \partial_t T^{0i} + \partial_j T^{ji} &= 0 \end{aligned} \right\}$$

$$\text{Energy} = \int d^3x T^{00}_{(t,x)}; \quad \frac{d}{dt}(\text{Energy}) = - \int_V d^3x \partial_i T^{i0} = - \int_{S'} T^{i0} dS^i$$

In fact, this is the 0 component of a 4-vector

$$p^\mu = \int_V T^{\mu\nu} \epsilon_{\mu\nu\alpha\beta} dx^\nu dx^\alpha dx^\beta$$

"energy-momentum
4-vector"

Space components $\mathcal{P}^i_C = \int d^3x T^{0i}$

$$\frac{1}{4} \eta^{\mu\nu} F_{\mu\alpha} F_{\nu\beta}]$$

obeys $\partial_\mu T^{\mu\nu} = 0$ using Maxwell with $J^\mu = 0$.

$\Downarrow$

$$(\partial_\mu J^\mu = 0)$$

$$\left. \begin{aligned} \partial_0 T^{00} + \partial_i T^{i0} &= 0 \\ \text{and } \partial_0 T^{0i} + \partial_j T^{ji} &= 0 \end{aligned} \right\}$$

Spatial stresses.

$$\text{Energy} = \int d^3x T_{(t,x)}^{00}; \quad \frac{d}{dt}(\text{Energy})$$

In fact, this is the 0 com

$$p^r = \int_V T^{\gamma\mu} \epsilon_{\mu\alpha\beta} dx^\alpha dx^\beta dx^\gamma$$

$$C.3 \quad T^{0i}$$

$$P^r = (E, pc)$$

energy-momentum 4-v

$$P^\gamma = (E, \underline{p}^c)$$

energy-momentum 4-vector

Space components $\vec{P}_C = \int d^3x T^{0i}$; $\vec{P}^{\gamma} = ($

Example: em wave

$$\begin{aligned} \underline{E} &= (0, 0, E_z(x-ct)) \\ \underline{B} &= (0, -\frac{E_z}{c}(x-ct), 0) \end{aligned} \quad \left. \vphantom{\begin{aligned} \underline{E} &= (0, 0, E_z(x-ct)) \\ \underline{B} &= (0, -\frac{E_z}{c}(x-ct), 0) \end{aligned}} \right\} \begin{aligned} T^{00} &= \frac{1}{\mu_0} \frac{E_z^2}{c^2} \\ T^{01} &= \frac{1}{\mu_0} \frac{E_z^2}{c^2} \end{aligned}$$

$$P^\gamma = (E, \underline{p}c)$$

energy-momentum 4-vector:

for an em wave

$$\left. \begin{aligned} T^{00} &= \frac{1}{\mu_0} E_z^2 / c^2 \\ T^{01} &= \frac{1}{\mu_0} E_z^2 / c^2 \end{aligned} \right\}$$

$$\left. \begin{aligned} \text{Energy} &= \text{Area} \cdot \int dx E_z^2 \epsilon_0 \\ \text{Momentum} &= \frac{\text{Area}}{c} \int dx E_z^2 \epsilon_0 \end{aligned} \right\}$$

$$\underline{|\underline{p}|} = \underline{E/c}$$

Space components $P^i_c = \int d^3x T^{0i}$

Example: em wave $\underline{E} = (0, 0, E_z(x-ct))$
 $\underline{B} = (0, -\frac{E_z}{c}(x-ct), 0)$ $\left. \begin{array}{l} T^{00} = \frac{1}{\mu_0} E_z^2 \\ T^{01} = \frac{1}{\mu_0} E_z^2 \end{array} \right\}$

$$\eta_{rs} P^r P^s = -E^2 + c^2 P^2 = 0.$$

$$P^\gamma = (E, \underline{p}c)$$

energy-momentum 4-vector

for an em wave

$$\left. \begin{aligned} T^{00} &= \frac{1}{\mu_0} E_z^2 / c^2 \\ T^{01} &= \frac{1}{\mu_0} E_z^2 / c^2 \end{aligned} \right\}$$

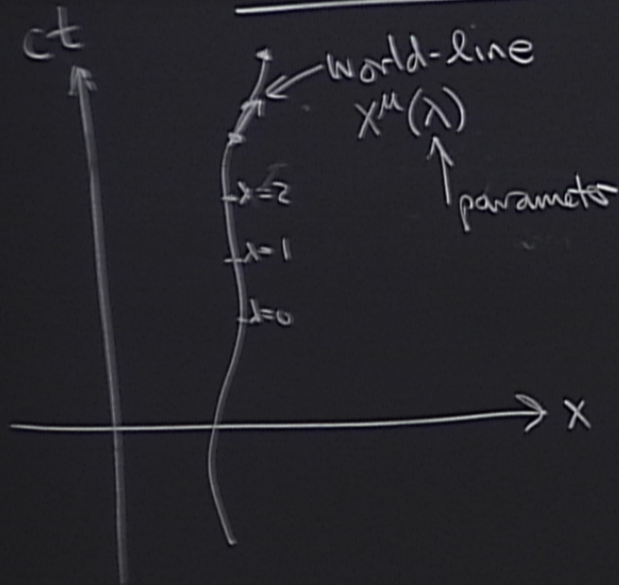
$$\text{Energy} = \text{Area} \cdot \int dx E_z^2 \epsilon_0$$

$$\text{Momentum} = \frac{\text{Area}}{c} \int dx E_z^2 \epsilon_0$$

$$|\underline{p}| = E/c$$

+ energy + momentum consⁿ
 $\Rightarrow$ prove $E = mc^2$

Particles



; tangent 4-vector to world line is $\frac{dx^\mu}{d\lambda}$

$$\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \begin{matrix} \leq 0 & \text{timelike} \\ = 0 & \text{null (like light)} \\ > 0 & \text{spacelike (tachyons)} \end{matrix}$$

$$\frac{dx^\mu}{d\lambda}$$

like

u (like light wave)

unlike

(tachyons, move faster than light, do not seem to exist).

me like world lines ("massive" particles), can define a proper time

$$c^2 d\tau^2 = c^2 dt^2 - d\underline{x}^2 \quad , \quad \text{or} \quad d\tau = dt \sqrt{1 - \frac{\underline{v}^2}{c^2}} \quad \underline{v} = \frac{d\underline{x}}{dt}$$

τ is then a ^{Lorentz-invariant} unique parameterisation of world-line
(up to a constant) $\tau = \int dt \sqrt{1 - \frac{\underline{v}^2}{c^2}}$

, satisfies $\eta_{\mu\nu} \frac{u^\mu u^\nu}{c^2} = -1$

$$\left(\frac{dct}{d\tau}, \frac{dx}{d\tau} \right) = \left(\frac{c}{\sqrt{1-\beta^2}}, \frac{\frac{dx}{dt}}{\sqrt{1-\beta^2}} \right)$$

4-velocity is then defined as $u^\mu = \frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu} \frac{u^\mu u^\nu}{c^2} = -1$

4-momentum $p^\mu = \overset{\text{test-mass}}{m_0} u^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$

$$P^\alpha = (E, \underline{P}c)$$

energy-momentum 4-vector.

for an em wave

$$\frac{1}{\mu_0} E_z^2 / c^2$$

$$= \frac{1}{\mu_0} E_z^2 / c^2$$

}

$$\text{Energy} = \text{Area} \cdot \int dx E_z^2 \epsilon_0$$

$$\text{Momentum} = \frac{\text{Area}}{c} \int dx E_z^2 \epsilon_0$$

}

$$|\underline{P}| = E/c$$

+ energy + momentum c
 $\Rightarrow$ prove $E = mc^2$

$$P^\alpha = \left(\frac{E}{c}, \underline{P} \right)$$

4-momentum.

$\frac{x^M}{c}$, satisfies

$$\eta_{\mu\nu} \frac{U^\mu U^\nu}{c^2} = -1$$

$$\left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau} \right) = \left(\frac{c}{\sqrt{1-\beta^2}}, \frac{\frac{dx}{dt}}{\sqrt{1-\beta^2}} \right)$$

$$p^\mu p^\nu = -m_0^2 c^2$$

$$\text{Energy} = p^0 c = \sqrt{p^2 c^2 + m_0^2 c^4}$$

$$\text{If } \underline{p} = 0, \text{ then} \\ \underline{E = m_0 c^2}$$

4-velocity is then defined as $u^\mu = \frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu} \frac{u^\mu u^\nu}{c^2} = -1$

4-momentum $p^\mu = \overset{\text{rest-mass}}{m_0} u^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$ Energy = $p^0 c$
 $= \left(\frac{E}{c}, \underline{p} \right)$

en defined as $u^M = \frac{dx^M}{d\tau}$, satisfies

$$\eta_{\mu\nu} \frac{u^\mu u^\nu}{c^2} = -1$$

$$\left(\frac{dct}{d\tau}, \frac{dx}{d\tau} \right) = \left(\frac{c}{\sqrt{1-\beta^2}}, \frac{v}{\sqrt{1-\beta^2}} \right)$$

$\downarrow$ rest-mass

$$p^\mu = m_0 u^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$$

$$\text{Energy} = p^0 c = \sqrt{\underline{p}^2 c^2 + m_0^2 c^4}$$

$$\text{If } \underline{p} = 0 \Rightarrow E = m_0 c^2$$

$$= \left(\frac{E}{c}, \underline{p} \right)$$

velocity of particle

$$\frac{dx}{dt} = \frac{m_0 \frac{dx}{d\tau}}{\frac{m_0 d(ct)}{c d\tau}} = \frac{\underline{p}}{p^0/c} = c \frac{\underline{p}}{\sqrt{\underline{p}^2 + m_0^2 c^2}} \Rightarrow |\underline{v}|$$

$$\text{simy } \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{\sqrt{\underline{p}^2 + m_0^2 c^2}}{m_0 c} \geq 1, \text{ for a moving particle Energy } E = \gamma m_0 c^2.$$

$$\text{simy } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\sqrt{p^2 + m_0^2 c^2}}{m_0 c} \geq 1, \text{ for a moving particle Energy}$$

How does this connect to classical (Newtonian) mechanics?

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{x}{2} + o(x^2) \quad E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx m_0 c^2 + \frac{1}{2} m_0 c^2 \cdot \frac{v^2}{c^2} + m_0 c^2 o\left(\frac{v^4}{c^4}\right)$$

$\downarrow$ const. $\downarrow$ $\frac{1}{2} m_0 v^2$, usual formula for kinetic energy

small for $v \ll c$

Lorentz Force

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau} \right)$$

charge of particle

$$m_0 \frac{dU^\mu}{d\tau} = q F^\mu{}_\nu U^\nu$$

4-acceleration

$u=i$

$$m_0 \frac{d^2 x^i}{d\tau^2} = q F_{i0} \frac{d(ct)}{d\tau} +$$

of particle

$u=i$

$$m_0 \frac{d^2 x^i}{d\tau^2} = q F_{i0} \frac{d(ct)}{d\tau} + q F_{ij} u_j$$

$$= \gamma q (E_i + (v \wedge B)_i)$$

$$u_j = \gamma v_j$$

$$u_j = \frac{dx_j}{d\tau}$$

$$d\tau = \gamma dt$$

$$\epsilon_{ijk} B_k u_j$$

$$\gamma \frac{d(p_i)}{dt}$$

$$\Rightarrow \frac{d\mathbf{p}}{dt} = q (\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

Hamilton's action principle

Newtonian mechanics, $S = \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 - V(x) \right)$, Newton's equation

$$= \int_{t_i}^{t_f} dt L(t)$$

↑
Lagrangian.

Consider free particle, in Newtonian mechanics $S = \int dt \frac{1}{2} m \dot{x}^2$

In relativity, only one candidate for the action

Consider free particle, in Newtonian mechanics $S = \int dt \frac{1}{2} m \dot{x}^2$

In relativity, only one candidate for the action

- proper length of world line

$$S \propto \int d\tau = \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \int \sqrt{c^2 dt^2 - d\mathbf{x}^2} = \int dt \sqrt{1 - \dot{\mathbf{x}}^2/c^2}$$

reproduce Newton when $|\frac{d\mathbf{x}}{dt}| \ll c$, by taking

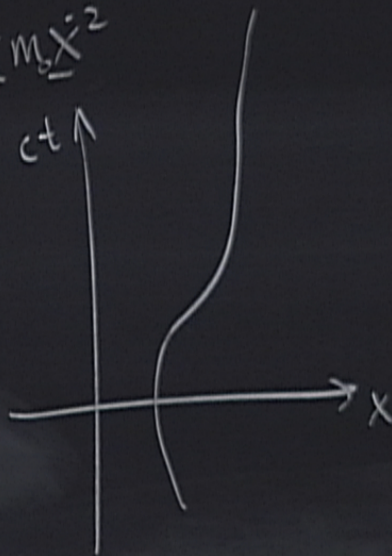
$$S = -m_0 c^2 \int dt \sqrt{1 - \dot{\mathbf{x}}^2/c^2}$$

mechanics $S' = \int dt \frac{1}{2} m_0 \dot{x}^2$

for the action

$$\int \sqrt{c^2 dt^2 - d\vec{x}^2} = \int dt \sqrt{1 - \dot{x}^2/c^2}$$

$$S' = -m_0 c^2 \int dt \sqrt{1 - \dot{x}^2/c^2} \approx -m_0 c^2 \int dt + \int dt \frac{1}{2} m_0 \dot{x}^2$$



Newton

$$\text{simy } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\sqrt{p^2 + m_0^2 c^2}}{m_0 c} \geq 1, \text{ for a}$$

Noether theorem

For every continuous symmetry of the equations of motion, derived from conserved quantity.

WEC $\Rightarrow$ 1. , for a moving particle Energy $E = \gamma M_0 c^2$

s of motion, derived from an action principle, there is a corresponding

Proof: if the parameters ϵ^{ab} are constant, we must have $\delta S = 0$ up
 think instead of making ϵ^{ab} a function of time, consider variation
 This is an allowed variation and therefore we must have $\delta S = 0$
 This provides an identity which provides the conservation law.

$$\delta S' = S'' - S' = \int dt \left(\dot{\epsilon}^a J^a(q, \dot{q}, t) + \epsilon^a \frac{dK(q, \dot{q}, t)}{dt} \right) \quad (1)$$

What about classical (Newtonian) mechanics?

$$(x^2) \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \approx \underbrace{m_0 c^2}_{\downarrow} + \frac{1}{2} m_0 c^2 \cdot \underbrace{\frac{v^2}{c^2}}_{\downarrow} + m_0 c^2 \circ \left(\frac{v^4}{c^4} \right) \quad \uparrow$$

parameters generators

we must have $\delta S = 0$ up to surface terms.

tion of time, consider variation (*) with $\epsilon^{ab}(t)$.

Therefore we must have $\delta S = 0$ (up to surface terms) using the equations of motion as the conservation law.

$$(q, \dot{q}, t) + \epsilon^a \frac{dK(q, \dot{q}, t)}{dt}$$

(because must vanish (up to surface terms) when ϵ^a is constant)

coordinates

parameters 'general

if the parameters ϵ^{ab} are constant, we must have $\delta S = 0$

think instead of making ϵ^{ab} a function of time, consider va

This is an allowed variation and therefore we must have δS

This provides an identity which provides the conservation law.

$$\delta S' = S'' - S' = \int dt \left(\dot{\epsilon}^a J^a(q, \dot{q}, t) + \epsilon^a \frac{dK(q, \dot{q}, t)}{dt} \right)$$

has an identity which provides the conservation law.

$$\delta S' = S' - S = \int dt \left(\epsilon^{ab} T^{ab}(q, \dot{q}, t) + \epsilon^{ab} \frac{dK^{ab}(q, \dot{q}, t)}{dt} \right)$$

(because must vanish (up to surface terms) when, ϵ

$$= 0 \quad \forall \epsilon^{ab}(t) \quad \text{if equations of motion are satisfied}$$

$$= - \int dt \epsilon^{ab}(t) \frac{d}{dt} (T^{ab} - K^{ab}) \Rightarrow \underline{\frac{d}{dt} (T^{ab} - K^{ab}) = 0}$$

so $T^{ab} - K^{ab}$ is a
conserved quantity

e.g. action for relativistic particle

$$S' = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}/c\right)^2}$$

e.g. action for relativistic particle

$$S = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}/c\right)^2}$$

spatial translation invariance: $\underline{x}(t) \rightarrow \underline{x}'(t) = \underline{x}(t) + \underline{a}$

↙ arb. const vector

show that the conserved quantity is

time translation invariance: $\underline{x}(t) \rightarrow \underline{x}'(t) = \underline{x}(t + \epsilon) \approx \underline{x}(t) + \epsilon \dot{\underline{x}}(t)$

$\delta \underline{x}(t)$

taking $\left| S' = -m_0 c^2 \int dt \sqrt{1 - \frac{\dot{\underline{x}}^2}{c^2}} \right| \approx -m_0 c^2 \int dt + \int dt \frac{1}{2} m_0 \dot{\underline{x}}^2$

the particle

$$c^2 \int dt \sqrt{1 - \left(\frac{d\underline{x}}{dt} / c \right)^2}$$

invariance: $\underline{x}(t) \rightarrow \underline{x}'(t) = \underline{x}(t) + \underline{q}$ ↙ arb. const vector

show that the conserved quantity is $\underline{p} = m_0 \gamma \frac{d\underline{x}}{dt}$

$$\underline{x}(t) \rightarrow \underline{x}'(t) = \underline{x}(t + \epsilon) \approx \underline{x}(t) + \epsilon \dot{\underline{x}}(t)$$

$$\delta \underline{x}(t) = \epsilon \dot{\underline{x}}(t) \quad \left. \vphantom{\delta \underline{x}(t)} \right\} \text{ conserved quantity is } \underline{p} = m_0 \gamma \underline{c}$$