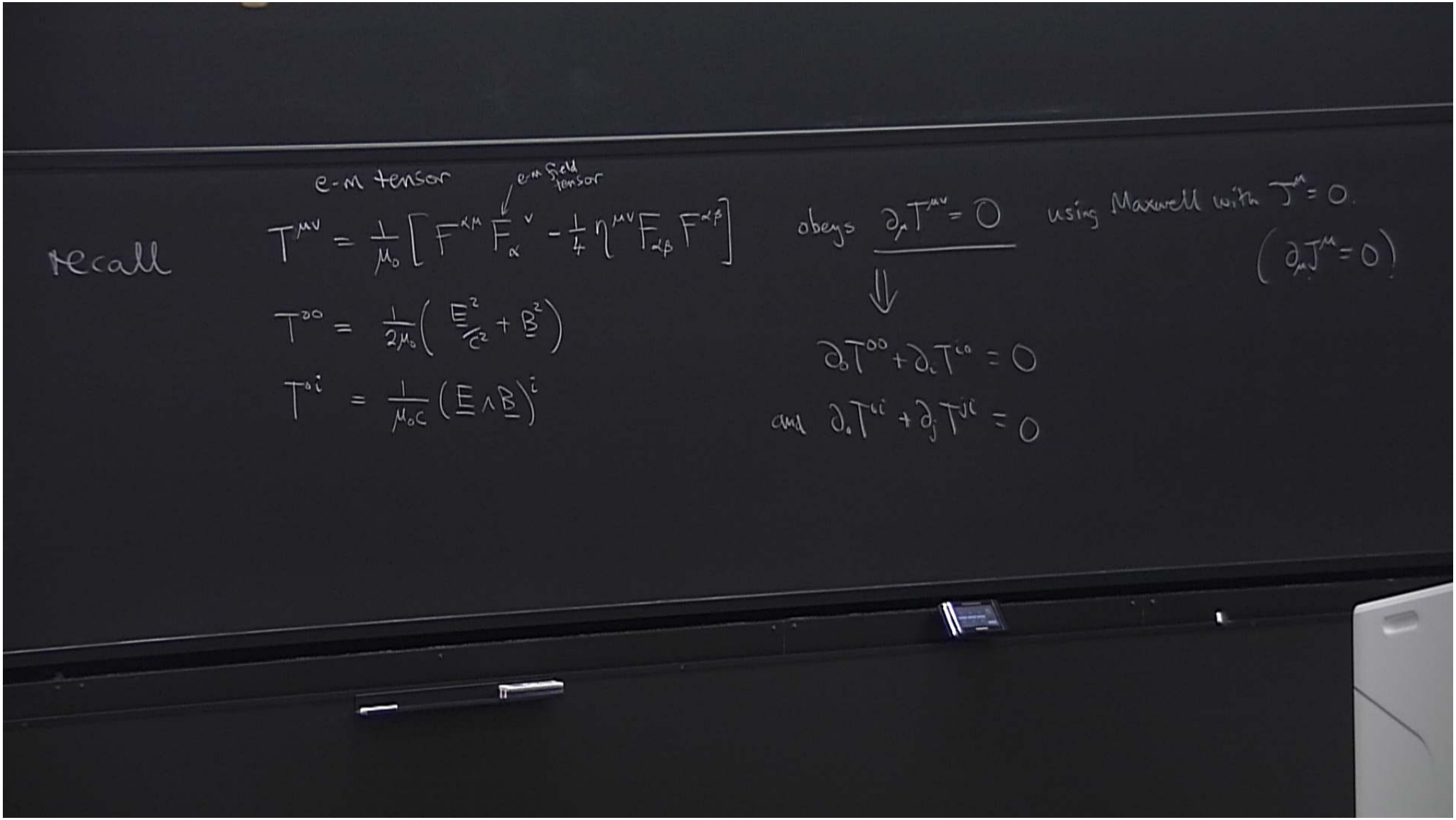
Title: 14/15 PSI - Relativity- Lecture 3

Date: Sep 10, 2014 09:00 AM

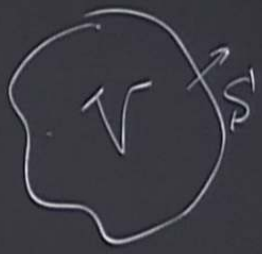
URL: http://pirsa.org/14090009

Abstract:


e-m tensor
e-m field tensor
recall
$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\alpha\mu}F_{\alpha}{}^{\nu} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]$
obeys $\partial_\mu T^{\mu\nu} = 0$ using Maxwell with $J^\mu = 0$.
$(\partial_\mu J^\mu = 0)$
$T^{00} = \frac{1}{2\mu_0}\left(\frac{\underline{E}^2}{c^2} + \underline{B}^2\right)$
$T^{0i} = \frac{1}{\mu_0 c}(\underline{E}\wedge\underline{B})^i$
$\partial_0 T^{00} + \partial_i T^{i0} = 0$
and $\partial_0 T^{0i} + \partial_j T^{ji} = 0$

$\partial_\mu T^{\mu\nu} = 0$ using Maxwell with $J^\mu = 0$.

$(\partial_\mu J^\mu = 0)$



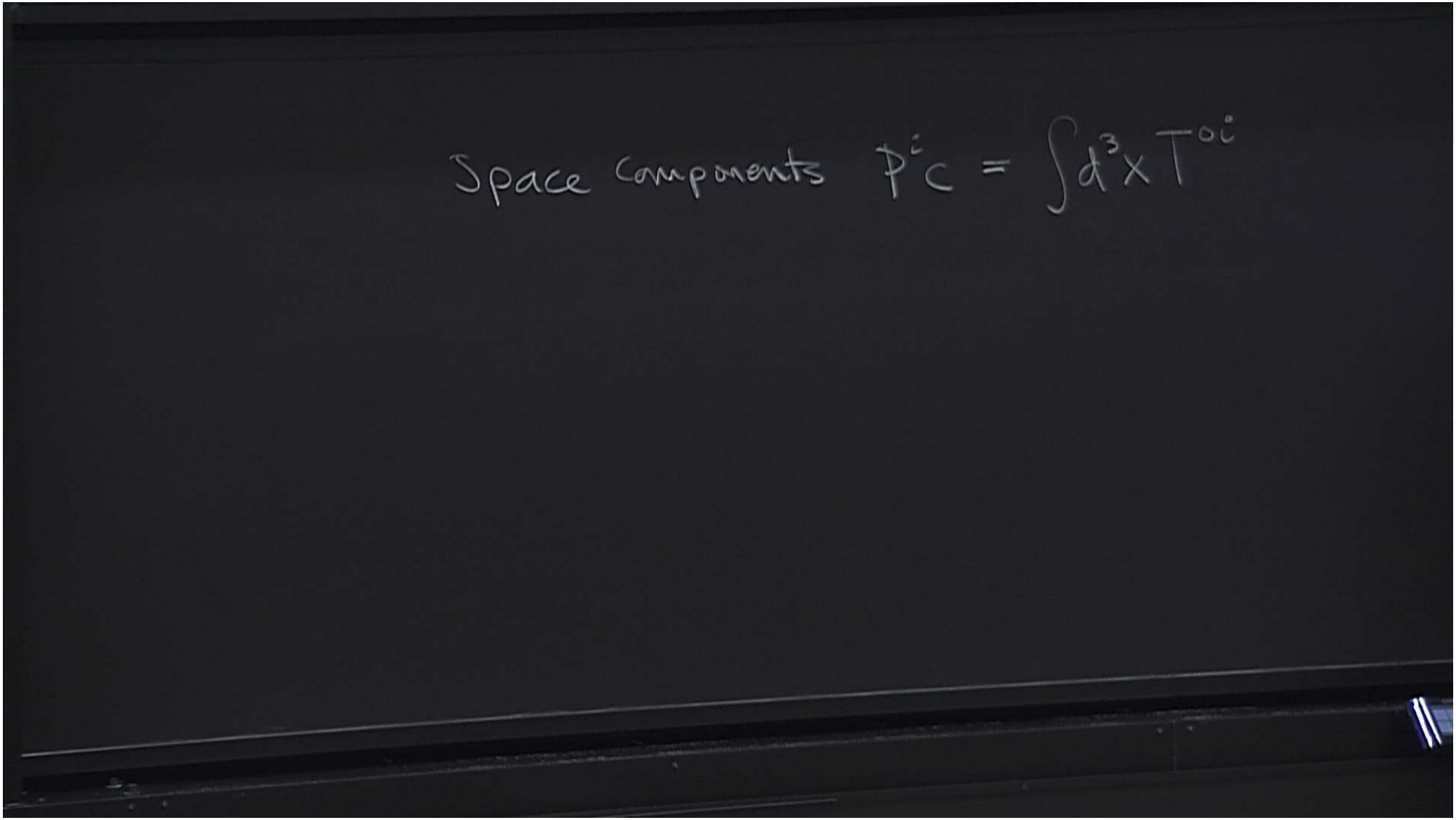
$\Downarrow$

$$\left.\begin{aligned} \partial_0 T^{00} + \partial_i T^{i0} &= 0 \\ \partial_0 T^{ij} + \partial_k T^{kj} &= 0 \end{aligned}\right\}$$

$$\text{Energy} = \underbrace{\int d^3x\, T^{00}_{(t,\vec{x})}}_{\text{in fact, this is the 0 component of a 4-vector}}; \quad \frac{d}{dt}(\text{Energy}) = -\int_V d^3x\, \partial_i T^{i0} = -\int_{S'} T^{i0}\, dS^i$$

$$P^\sigma = \int_V T^{\sigma\mu}\, \epsilon_{\mu\nu\alpha\beta}\, dx^\nu dx^\alpha dx^\beta$$

"energy-momentum 4-vector"


Space components $P^i_C = \int d^3x\, T^{0i}$

$-\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}]$ obeys $\partial_\mu T^{\mu\nu} = 0$ using Maxwell with $J^\mu = 0$.

$(\partial_\mu J^\mu = 0)$

$\Downarrow$

$\partial_0 T^{00} + \partial_i T^{i0} = 0$

and $\partial_0 T^{0i} + \partial_j T^{ji} = 0$

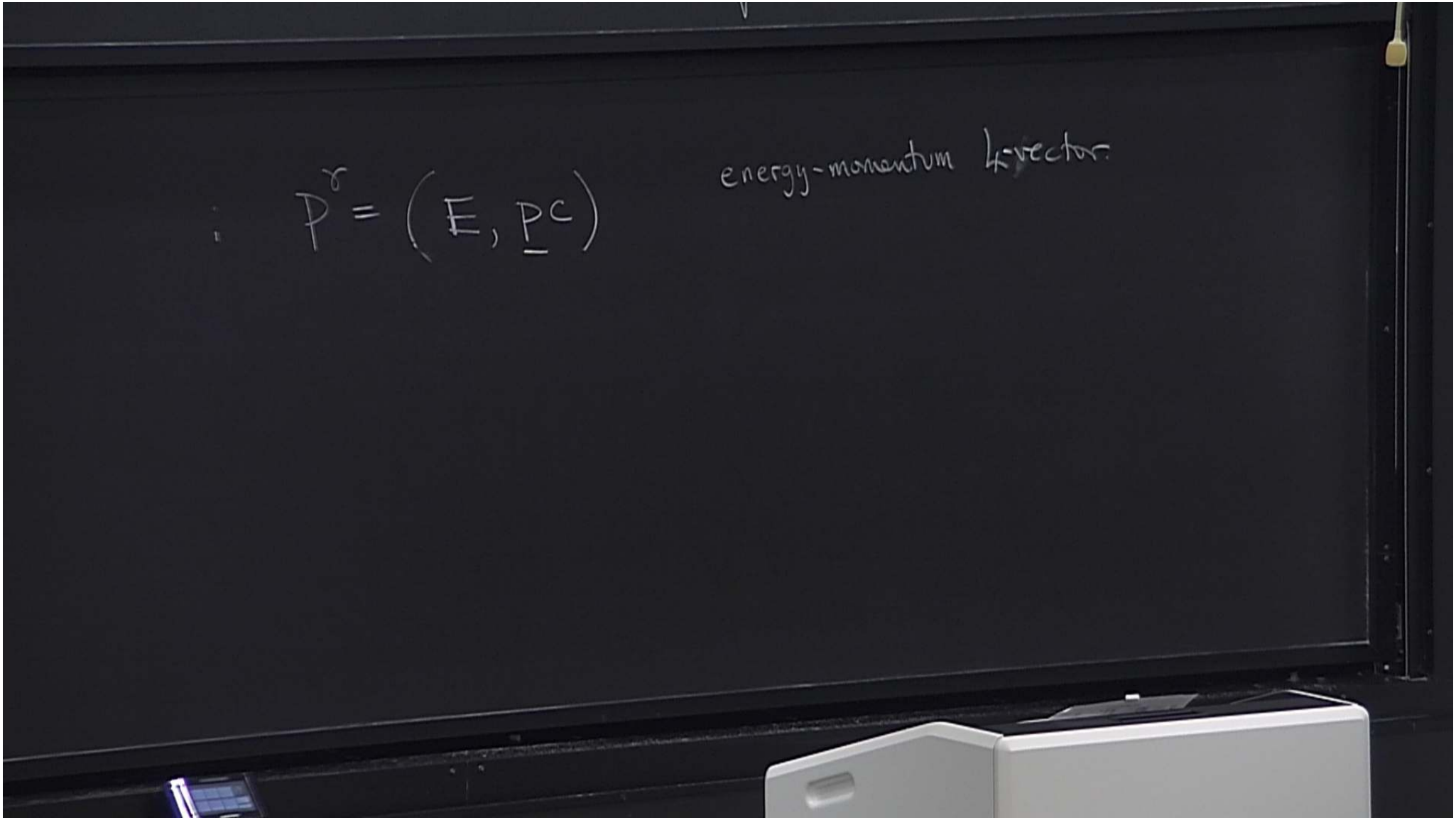
T^{ji}: Spatial stresses.

$\text{Energy} = \int d^3x\, T^{00}_{(t,\vec{x})}$; $\frac{d}{dt}(\text{Energy})$

(in fact, this is the 0 com

$P^\nu = \int_V T^{\nu\mu}\,\epsilon_{\mu\nu\alpha\beta}\,dx^\nu dx^\alpha dx^\beta$

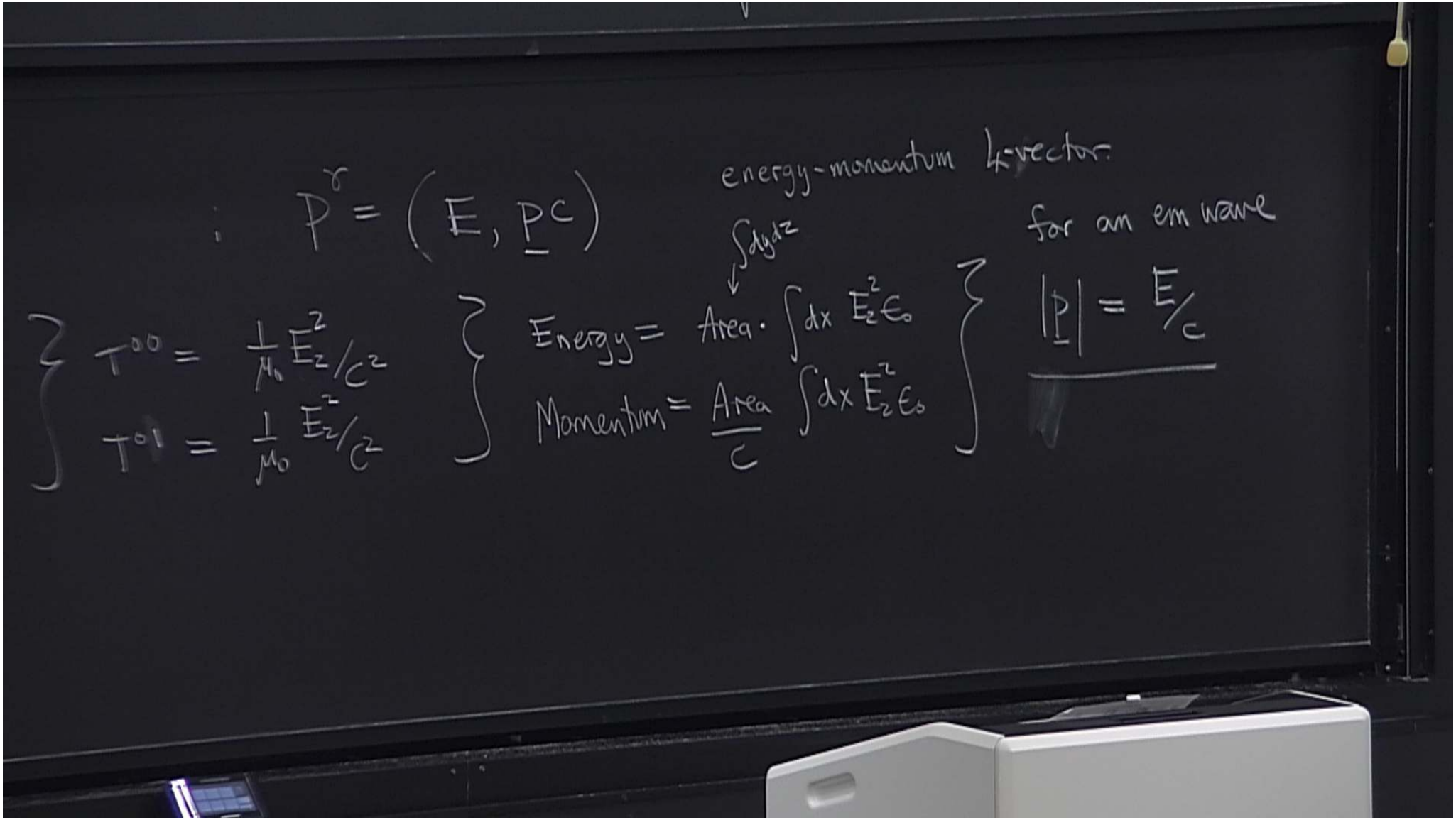
$\int_V T^{0i}$; $P^\nu = (E, pc)$ energy-momentum 4-v


$P^\mu = (E, \underline{P}c)$
energy-momentum 4-vector.

Space components $P^i c = \int d^3x\, T^{0i}$; $P^\alpha =$

Example: em wave $\underline{E} = (0, 0, E_z(x-ct))$ $\quad \underline{B} = (0, -\frac{E_z}{c}(x-ct), 0)$

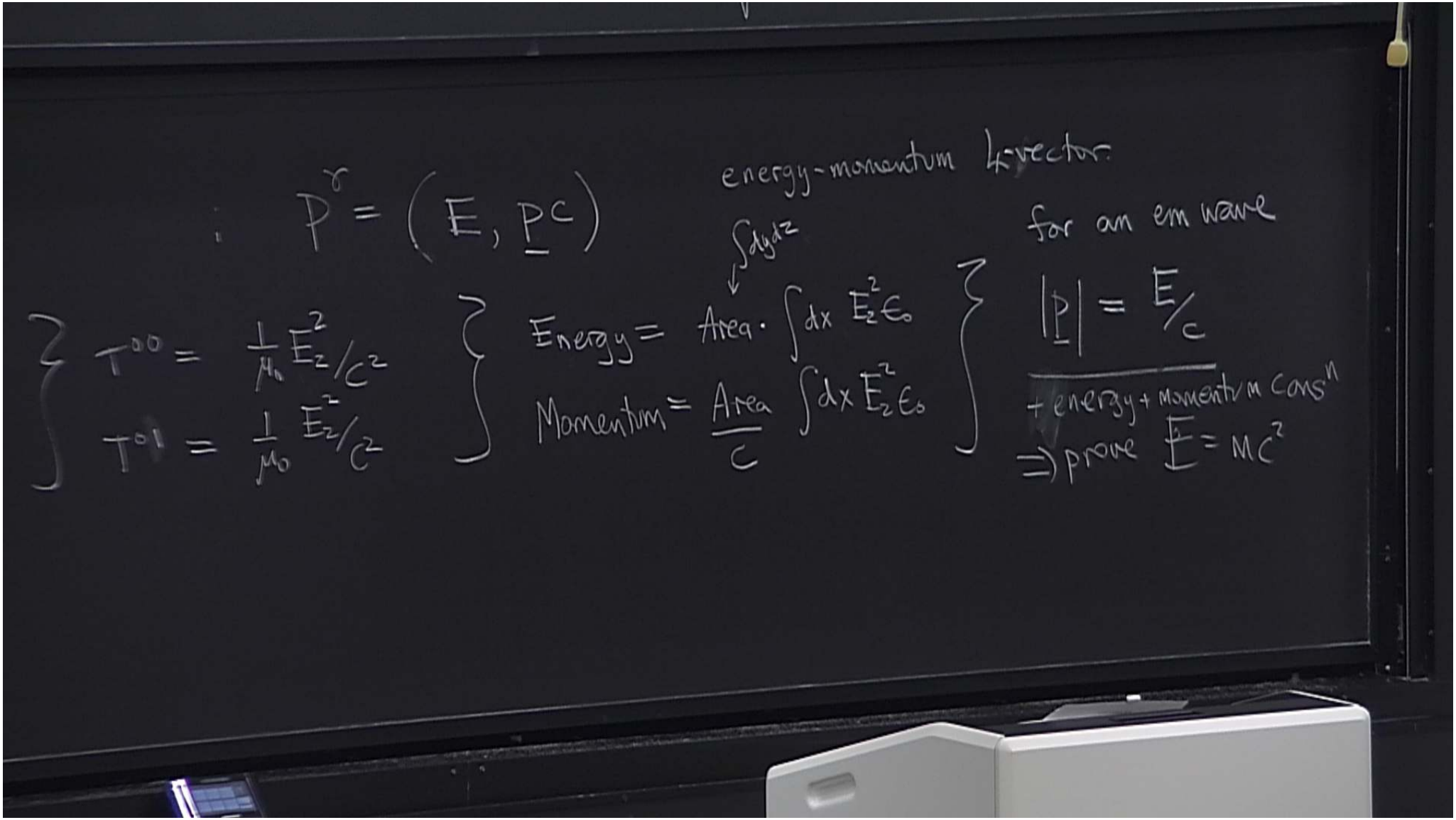
$$\left.\begin{aligned} T^{00} &= \frac{1}{\mu_0} E_z^2/c^2 \\ T^{01} &= \frac{1}{\mu_0} E_z^2/c^2 \end{aligned}\right\}$$


$P^\gamma = (E, \underline{P}c)$
energy-momentum 4-vector
for an em wave
$\int dy dz$
$T^{00} = \frac{1}{\mu_0} E_z^2 / c^2$
$T^{01} = \frac{1}{\mu_0} E_z^2 / c^2$
Energy = Area · $\int dx\, E_z^2 \epsilon_0$
Momentum = $\frac{\text{Area}}{c} \int dx\, E_z^2 \epsilon_0$
$|\underline{P}| = E/c$

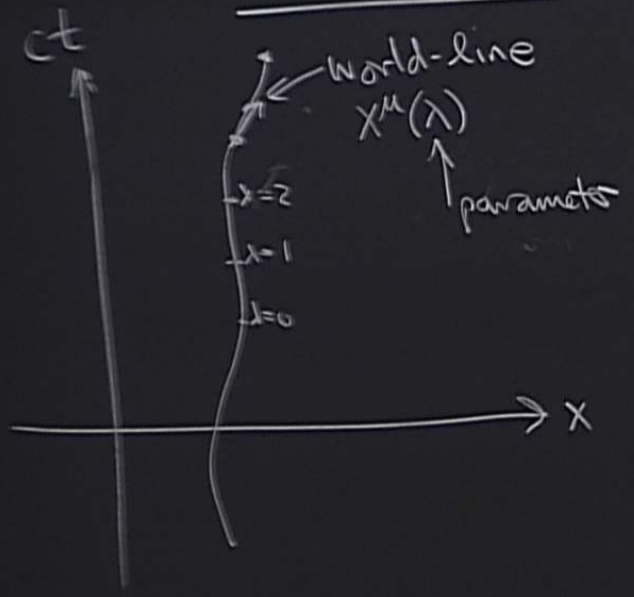
Space components $P^i c = \int d^3x\, T^{0i}$; P

Example: em wave $\underline{E} = (0, 0, E_z(x-ct))$ $\quad \underline{B} = (0, -\frac{E_z}{c}(x-ct), 0)$ $\quad \left.\begin{array}{l} T^{00} = \frac{1}{\mu_0} E_z^2 \\ T^{01} = \frac{1}{\mu_0} E_z \end{array}\right.$

$$\eta_{\gamma\delta} P^\gamma P^\delta = -E^2 + c^2 \underline{P}^2 = 0.$$


$P^\nu = (E, \underline{P}c)$
energy-momentum 4-vector
$\int dy dz$
$T^{00} = \frac{1}{\mu_0} E_z^2/c^2$
$T^{01} = \frac{1}{\mu_0} E_z^2/c$
Energy = Area · $\int dx\, E_z^2 \epsilon_0$
Momentum = $\frac{\text{Area}}{c} \int dx\, E_z^2 \epsilon_0$
for an em wave
$|\underline{P}| = E/c$
+ energy + momentum cons^n
$\Rightarrow$ prove $E = Mc^2$

Particles

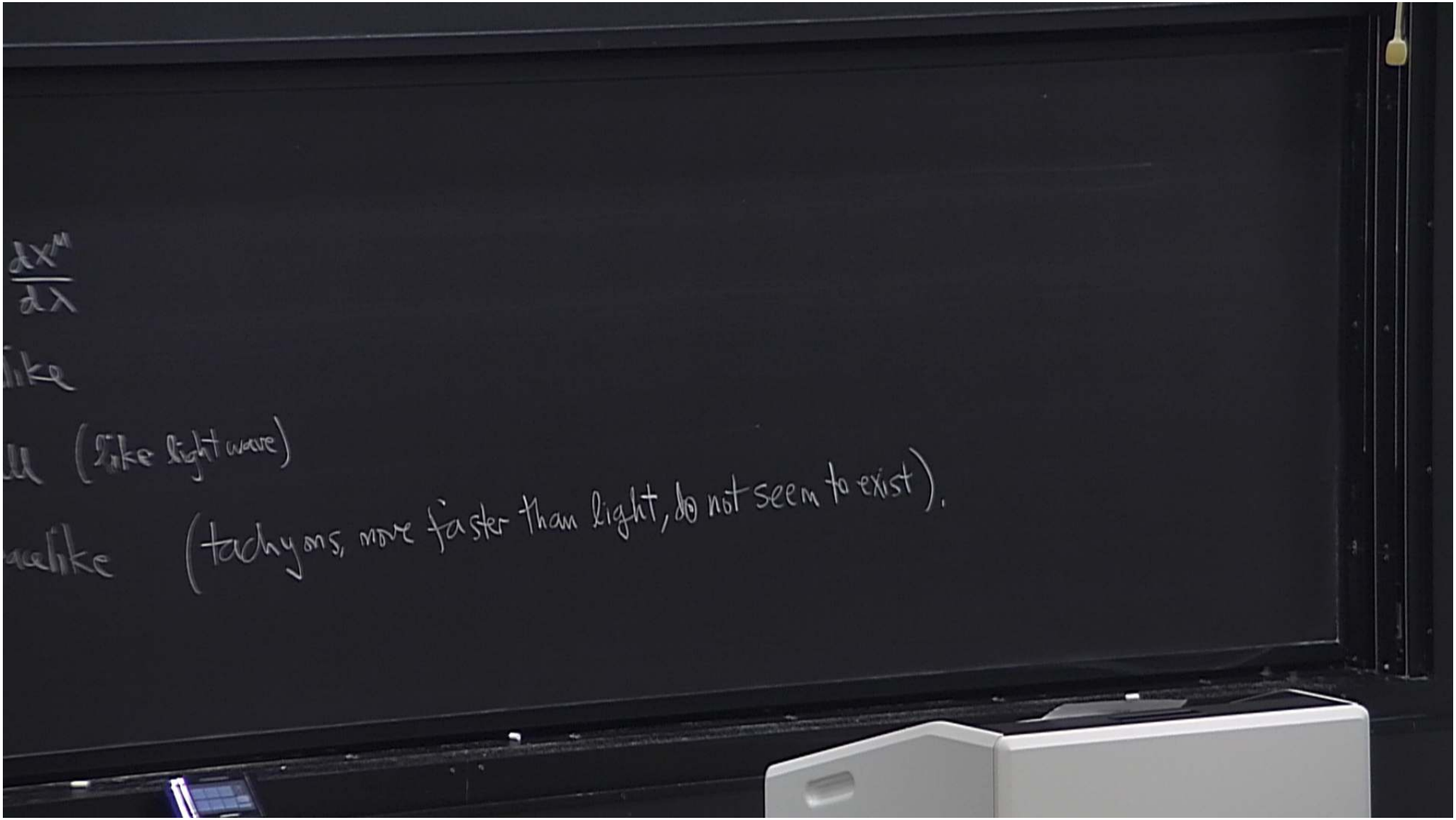


; tangent 4-vector to world line is $\frac{dx^\mu}{d\lambda}$

$\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0$ timelike

$= 0$ null (like light

> 0 spacelike (ta


$\frac{dx^\mu}{d\lambda}$
ike
ll (like light wave)
acelike (tachyons, move faster than light, do not seem to exist).

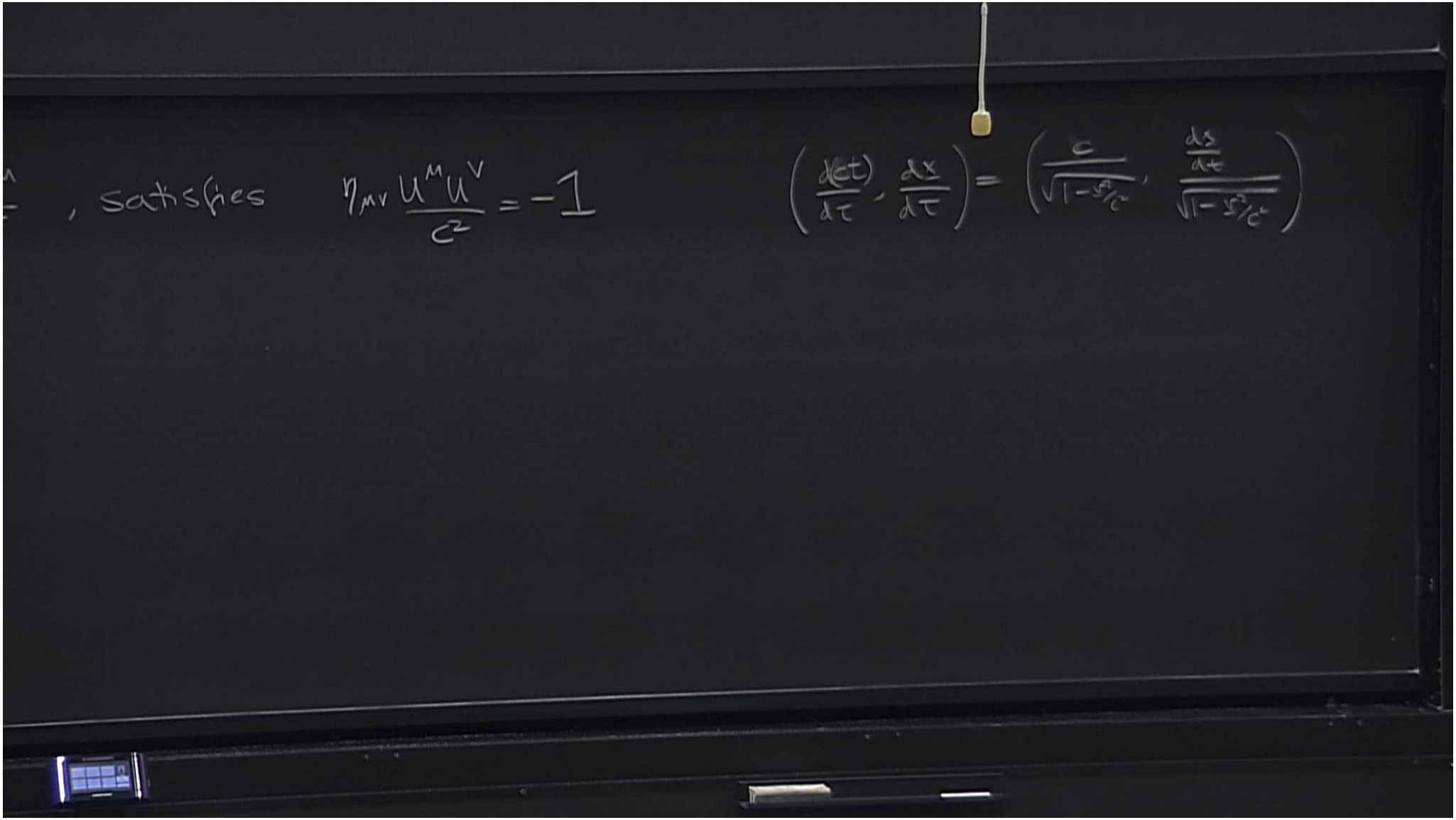
me like world lines ("massive" particles), can define a proper ti

$c^2 d\tau^2 = c^2 dt^2 - d\underline{x}^2$, or $d\tau = dt\sqrt{1 - \underline{v}^2/c^2}$ $\quad \underline{v} = \frac{d\underline{x}}{dt}$

Lorentz-invariant

τ is then a unique parameterisation of world-line $\tau = \int dt \sqrt{1 - \underline{v}^2/c^2}$

(up to a constant)


, satisfies $\eta_{\mu\nu}\frac{U^\mu U^\nu}{c^2} = -1$
$\left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau}\right) = \left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{\frac{dx}{dt}}{\sqrt{1-v^2/c^2}}\right)$

4-velocity is then defined as $U^\mu = \frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu}\frac{U^\mu U^\nu}{c^2} = -1$

4-momentum $p^\mu = m_0 U^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$ (rest-mass)

$P^\mu = (E, \underline{P}c)$ energy-momentum 4-vector

$\frac{1}{\mu_0} E_z^2/c^2$

$= \frac{1}{\mu_0} E_z^2/c^2$

$\int dy\,dz$

$$\left.\begin{aligned} \text{Energy} &= \text{Area} \cdot \int dx\, E_z^2 \epsilon_0 \\ \text{Momentum} &= \frac{\text{Area}}{c} \int dx\, E_z^2 \epsilon_0 \end{aligned}\right\}$$

for an em wave

$$|\underline{P}| = E/c$$

$\Rightarrow$ energy + momentum

$\Rightarrow$ prove $E = Mc^2$

$P^\mu = \left(\frac{E}{c}, \underline{P}\right)$ 4-momentum.

$\frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu}\frac{u^\mu u^\nu}{c^2} = -1$

$\left(\frac{d(ct)}{d\tau}, \frac{d\underline{x}}{d\tau}\right) = \left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{\frac{d\underline{x}}{dt}}{\sqrt{1-v^2/c^2}}\right)$

$p^\mu p^\nu = -m_0^2 c^2$

Energy $= p^0 c = \sqrt{\underline{p}^2 c^2 + m_0^2 c^4}$ $\Rightarrow$ If $\underline{p} = 0$, then

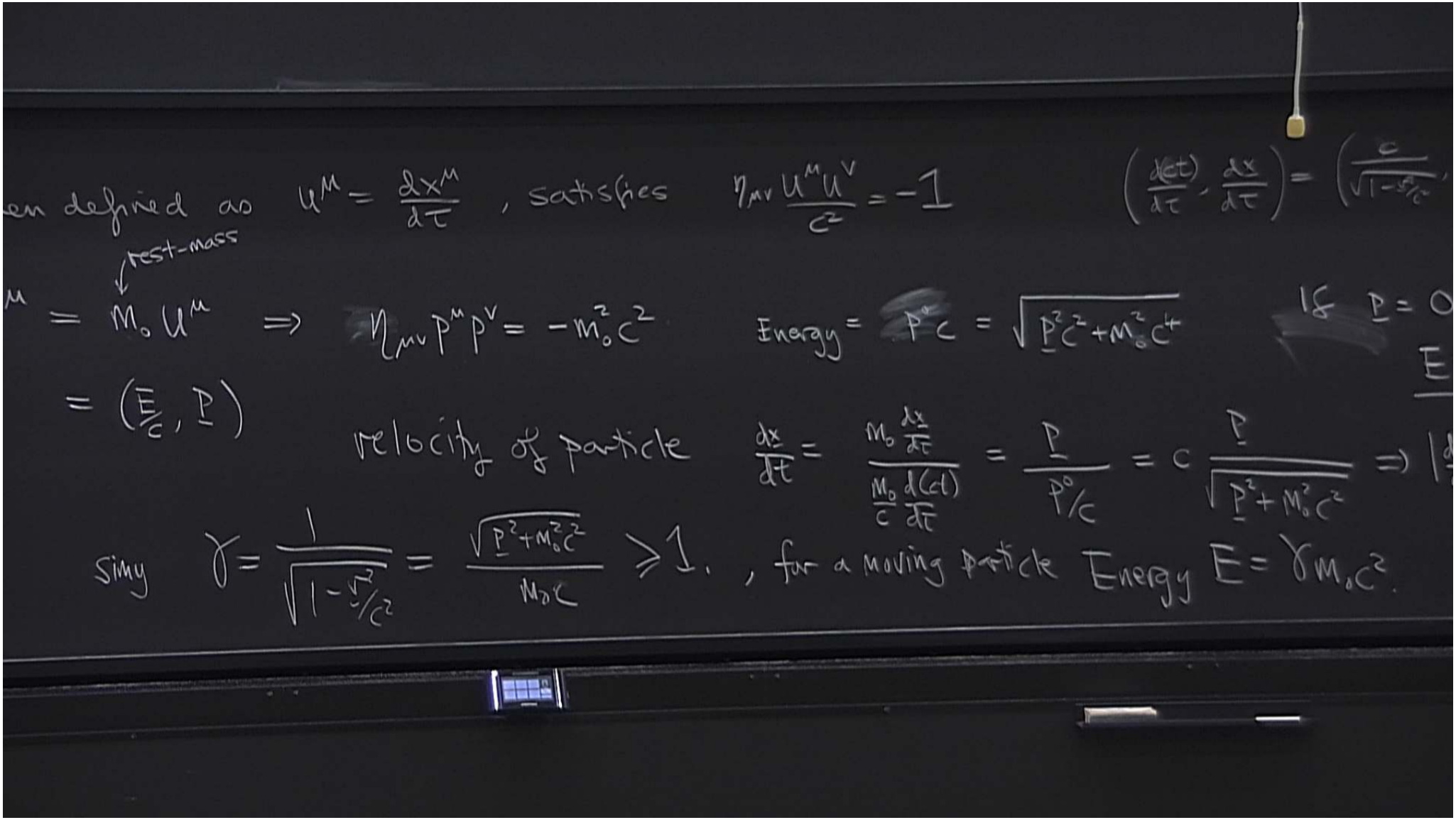
$\underline{E = m_0 c^2}$

4-velocity is then defined as $u^\mu = \frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu}\frac{u^\mu u^\nu}{c^2} = -1$

4-momentum $p^\mu = m_0 u^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$ (rest-mass)

Energy = $p^0 c$

$= \left(\frac{E}{c}, \underline{p}\right)$


en defined as $u^\mu = \frac{dx^\mu}{d\tau}$, satisfies $\eta_{\mu\nu}\frac{u^\mu u^\nu}{c^2} = -1$
$\left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau}\right) = \left(\frac{c}{\sqrt{1-v^2/c^2}}\right.$
rest-mass
$\mu = m_0 u^\mu \Rightarrow \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$
Energy $= p^0 c = \sqrt{\underline{p}^2 c^2 + m_0^2 c^4}$
If $\underline{p} = 0$
$= \left(\frac{E}{c}, \underline{p}\right)$
E
velocity of particle $\frac{dx}{dt} = \frac{m_0 \frac{dx}{d\tau}}{\frac{m_0}{c}\frac{d(ct)}{d\tau}} = \frac{\underline{p}}{p^0/c} = c\frac{\underline{p}}{\sqrt{\underline{p}^2 + m_0^2 c^2}} \Rightarrow$
Simy $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{\sqrt{\underline{p}^2 + m_0^2 c^2}}{m_0 c} \geq 1$., for a moving particle Energy $E = \gamma m_0 c^2$.

Since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{\sqrt{p^2+m_0^2c^2}}{m_0 c} \geq 1$, for a moving particle Energy

How does this connect to classical (Newtonian) mechanics?

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{x}{2} + o(x^2)$$

$$E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} \approx m_0 c^2 + \frac{1}{2} m_0 c^2 \cdot \frac{v^2}{c^2} + m_0 c^2 \cdot o\left(\frac{v^4}{c^4}\right)$$

$m_0 c^2$ → Const.

$\frac{1}{2} m_0 v^2$, usual formula for kinetic energy

$o\left(\frac{v^4}{c^4}\right)$ small for $v \ll c$

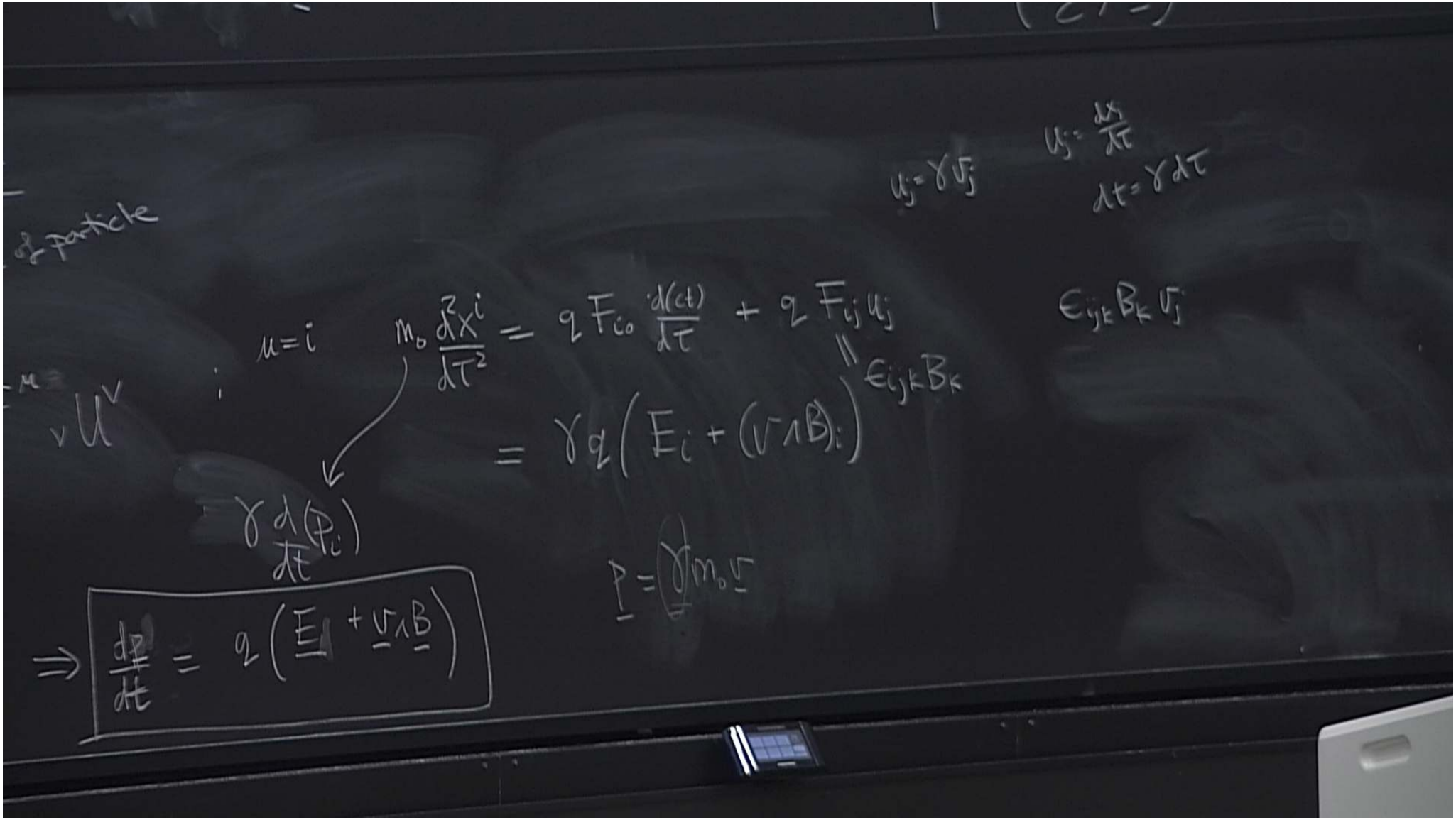
Lorentz Force

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau}\right)$$

charge of particle

$$m_0 \frac{dU^\mu}{d\tau} = q F^\mu{}_\nu U^\nu \quad \therefore \mu = i \qquad m_0 \frac{d^2 x^i}{d\tau^2} = q F_{i0} \frac{d(ct)}{d\tau} +$$

$\frac{dU^\mu}{d\tau}$ ← 4-acceleration


of particle
$u_j = \gamma v_j$
$u_j = \frac{dx_j}{d\tau}$
$dt = \gamma d\tau$
$\mu = i$
$m_0 \frac{d^2 x^i}{d\tau^2} = q F_{i0} \frac{d(ct)}{d\tau} + q F_{ij} u_j$
$\epsilon_{ijk} B_k v_j$
$= \gamma q \left(E_i + (\underline{v} \wedge \underline{B})_i \right)$
$\epsilon_{ijk} B_k$
$\gamma \frac{d}{dt}(p_i)$
$\underline{p} = \gamma m_0 \underline{v}$
$\Rightarrow \frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{v} \wedge \underline{B}\right)$

Hamilton's action principle

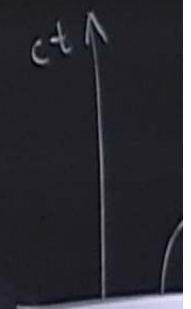
Newtonian mechanics, $S' = \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x}) \right)$, Newton's equa

$= \int_{t_i}^{t_f} dt \, L(t)$

↑ Lagrangian.

Consider free particle, in Newtonian mechanics $S = \int dt \, \frac{1}{2} m \dot{\underline{x}}^2$

In relativity, only one candidate for the action



Consider free particle, in Newtonian mechanics $S = \int dt\, \frac{1}{2} m \dot{\underline{x}}^2$

In relativity, only one candidate for the action

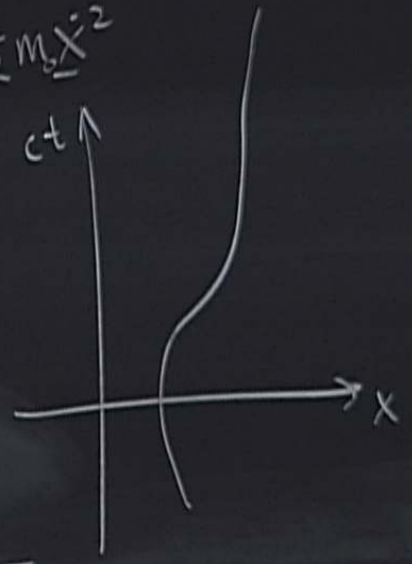
– proper length of world line

$$S \propto \int d\tau = \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \int \sqrt{c^2 dt^2 - d\underline{x}^2} = c \int dt \sqrt{1 - \dot{\underline{x}}^2/c^2}$$

reproduce Newton when $\left|\frac{d\underline{x}}{dt}\right| \ll c$, by taking $S = -m_0 c^2 \int dt \sqrt{1 - \dot{\underline{x}}^2/c^2}$

mechanics $S = \int dt \, \frac{1}{2} m_0 \underline{\dot{x}}^2$

for the action



$$\int \sqrt{c^2 dt^2 - d\underline{x}^2} = c \int dt \sqrt{1 - \underline{\dot{x}}^2/c^2}$$

$$\boxed{S = -m_0 c^2 \int dt \sqrt{1 - \underline{\dot{x}}^2/c^2}} \approx -m_0 c^2 \int dt + \int dt \, \frac{1}{2} m_0 \underline{\dot{x}}^2$$

Newton

Since $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\sqrt{p^2 + m_0^2 c^2}}{m_0 c} \geq 1$, for a

Noether theorem

For every continuous symmetry of the equations of motion, derived from conserved quantity.

$\frac{}{M_0 c} \geq 1$, for a moving particle Energy $E = \gamma M_0 c^2$

s of motion, derived from an action principle, there is a corresponding

$\frac{M_0}{c}\frac{d(ct)}{dt}$ p^0/c

Since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{\sqrt{p^2+M_0^2c^2}}{M_0c} \geq 1$, for a moving particle

Noether theorem

For every continuous symmetry of the equations of motion, derived from an action princi
conserved quantity.

A continuous symmetry is a transformation which, for small values of its parameters, reads

$$q^q(t) \rightarrow q'^q(t) = q^q(t) + \epsilon^{ab} X^b(q, \dot{q}, t) + O(\epsilon^2)$$

coordinates — parameters — generators

Proof: if the parameters ϵ^{ab} are constant, we must have $\delta S = 0$ up

think instead of making ϵ^{ab} a function of time, consider variati

This is an allowed variation and therefore we must have $\delta S = 0$

This provides an identity which provides the conservation law.

$$\delta S' = S'' - S' = \int dt \left(\dot{\epsilon}^a J^a(q,\dot{q},t) + \epsilon^a \frac{dK(q,\dot{q},t)}{dt} \right)$$

to classical (Newtonian) mechanics?

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \approx m_0 c^2 + \frac{1}{2} m_0 c^2 \cdot \frac{v^2}{c^2} + m_0 c^2 \cdot \left(\frac{v^4}{c^4}\right)$$

parameters ↑ generators ↑

we must have $\delta S = 0$ up to surface terms.

ction of time, consider variation (*) with $\epsilon^{ab}(t)$.

erefore we must have $\delta S = 0$ (up to surface terms) using the equations of motion
es the conservation law.

$$(q, \dot{q}, t) + \epsilon^a \frac{dK(q, \dot{q}, t)}{dt}$$

(because must vanish (up to surface terms) when ϵ^a is constant)

coordinates parameters general

if the parameters ϵ^{ab} are constant, we must have $\delta S = 0$

think instead of making ϵ^{ab} a function of time, consider va

This is an allowed variation and therefore we must have δS

This provides an identity which provides the conservation law.

$$\delta S' = S'' - S' = \int dt \left(\dot{\epsilon}^a J^a(q, \dot{q}, t) + \epsilon^a \frac{dK(q, \dot{q}, t)}{dt} \right)$$

...es an identity which provides the conservation law.

$$\delta S = S' - S = \int dt \left(\dot{\epsilon}^{ab} J^{ab}(q,\dot{q},t) + \epsilon^{ab} \frac{d K^{ab}(q,\dot{q},t)}{dt} \right)$$

(because must vanish (up to surface terms) when, ϵ

$= 0 \quad \forall \epsilon^{ab}(t)$ if equations of motion are satisfied

$$= -\int dt \; \epsilon^{ab}(t) \frac{d}{dt}\left(J^{ab} - K^{ab} \right) \quad \Rightarrow \quad \underline{\frac{d}{dt}\left(J^{ab} - K^{ab} \right) = 0}$$

so $J^{ab} - K^{ab}$ is a conserved quantity

e.g. action for relativistic particle

$$S = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}/c\right)^2}$$

e.g. action for relativistic particle

$$S = -m_0 c^2 \int dt \sqrt{1 - \left(\frac{d\underline{x}}{dt}/c\right)^2}$$

spatial translation invariance: $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \underline{a}$ (arb'y const vector)

Show that the conserved quantity is

time translation invariance: $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t+\epsilon) \approx \underline{x}(t) + \epsilon \dot{\underline{x}}(t)$ $\delta\underline{x}(t)$

taking $S = -m_0 c^2 \int dt \sqrt{1 - \dot{\underline{x}}^2/c^2} \approx -m_0 c^2 \int dt + \int dt \, \frac{1}{2} m_0 \dot{\underline{x}}^2$

he particle

$c^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}/c\right)^2}$

riance: $\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t) + \underline{a}$ ← arb'y const vector

Show that the conserved quantity is $\underline{p} = m_0 \gamma \frac{d\underline{x}}{dt}$

$\underline{x}(t) \to \underline{x}'(t) = \underline{x}(t+\epsilon) \approx \underline{x}(t) + \epsilon \dot{\underline{x}}(t)$

$\delta \underline{x}(t) = \epsilon \dot{\underline{x}}(t)$ } conserved quantity is $p^0 = m_0 \gamma c$