

Title: 14/15 PSI - Relativity- Lecture 2

Date: Sep 09, 2014 09:00 AM

URL: <http://pirsa.org/14090008>

Abstract:

Last time

Last time

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

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$$\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) = \eta^{\alpha\beta}$$

last time

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$$\alpha, \beta = \overset{ct}{\downarrow} \overset{x^i}{\sim} 0, 1, 2, 3$$

$$\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) = \eta^{\alpha\beta}$$

$$\partial_\alpha = \left(\frac{1}{c} \partial_t, \underline{\nabla} \right)$$

$$J^\alpha = (\rho c, \underline{J})$$

$$\epsilon^{\alpha\beta\gamma\delta}, \text{ totally antisymm tensor } \epsilon^{0123} = 1$$

These equations possess a symmetry: Lorentz invariance.

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c.f. classical mechanics $m\ddot{\underline{x}} = -\underline{\nabla}V$

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actions possess symmetry: Lorentz invariance.

classical mechanics

$$= -\underline{\nabla} V$$

obviously invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O} \underline{x}$

3x3 rotations
 $\underline{O}^T \underline{O} = \underline{1}$

actions possess a symmetry: Lorentz invariance.

classical mechanics $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O}\underline{x}$

$$\Rightarrow m\ddot{\underline{x}}' + \underline{\nabla}'V = \underline{O}(m\ddot{\underline{x}} + \underline{\nabla}V) = 0$$

↙ 3x3 rotations
 $\underline{O}^T \underline{O} = \underline{1}$

These equations possess a symmetry: Lorentz invariance.

c.f. classical mechanics $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}'$
 $\Rightarrow m\ddot{\underline{x}}' + \underline{\nabla}'V = O(m\ddot{\underline{x}} + \underline{\nabla}V) = 0$

Lorentz transformation: $x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta}$

These equations possess a symmetry: Lorentz invariance.

c.f. classical mechanics $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}' = \underline{\Lambda} \underline{x}$ 
 $\Rightarrow m\ddot{\underline{x}}' + \underline{\nabla}'V = \underline{\Lambda} (m\ddot{\underline{x}} + \underline{\nabla}V) = 0$

Lorentz transformation: $x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} \rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}}$

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classical mechanics $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}' =$

$$\Rightarrow m\ddot{\underline{x}}' + \underline{\nabla}'V = O(m\ddot{\underline{x}} + \underline{\nabla}V) = 0$$

Lorentz transformation: $x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} \rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x'^{\beta}}$

These equations possess a symmetry: Lorentz invariance.

c.f. classical mechanics $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}' =$
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Lorentz transformation: $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \rightarrow \frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}} = \Lambda^{\beta}_{\alpha} \frac{\partial}{\partial x'^{\beta}}$

invariance

3x3 rotations

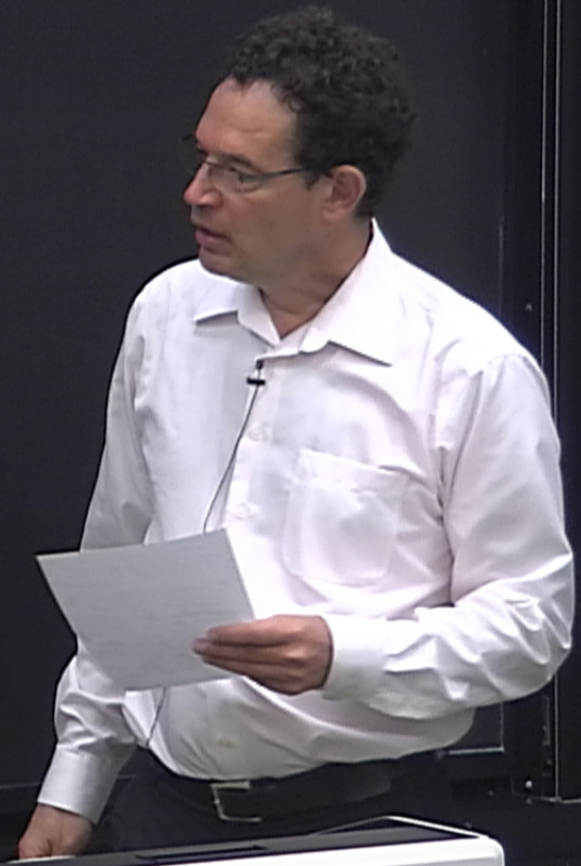
$$O^T O = \mathbb{1}$$

under $\underline{x} \rightarrow \underline{x}' = \underline{O} \underline{x}$

inverse

$$\Lambda^\alpha_\beta \tilde{\Lambda}^\beta_\gamma \equiv \delta^\alpha_\gamma$$

$$\frac{\partial}{\partial x'^\alpha}$$



invariance

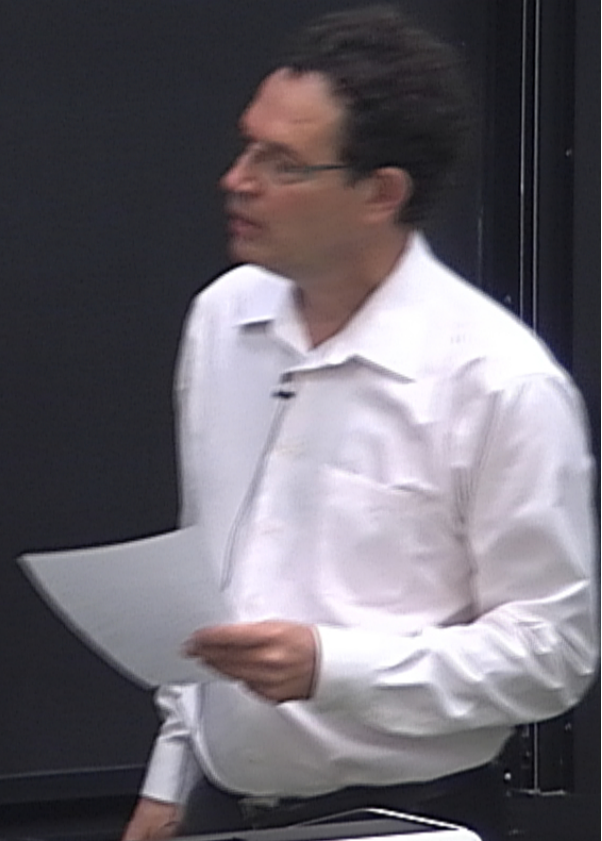
3x3 rotations
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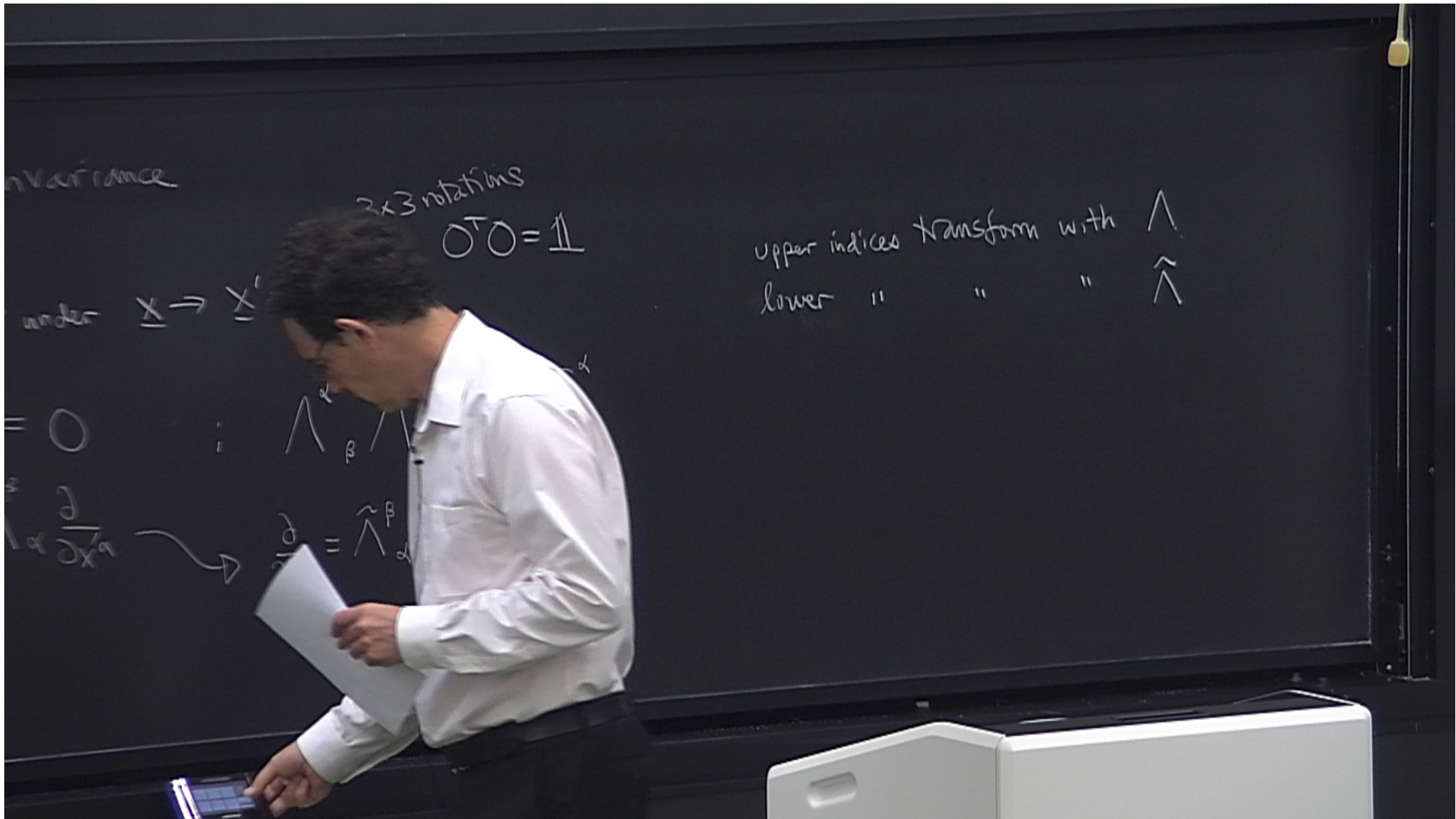
under $\underline{x} \rightarrow \underline{x}' = \underline{O} \underline{x}$

$\downarrow$ inverse

$$\Lambda^\alpha_\beta \tilde{\Lambda}^\beta_\gamma \equiv \delta^\alpha_\gamma$$

$$\frac{\partial}{\partial x'^\alpha} \rightarrow \frac{\partial}{\partial x'^\alpha} = \tilde{\Lambda}^\beta_\alpha \frac{\partial}{\partial x^\beta}$$





ess a Symmetry: Lorentz invariance

cs $m\ddot{\underline{x}} + \underline{\nabla}V = 0$ obviously invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O}\underline{x}$

$$\Rightarrow m\ddot{\underline{x}}' + \underline{\nabla}'V = \underline{O}(m\ddot{\underline{x}} + \underline{\nabla}V) = 0$$

3x3 rotations
 $\underline{O}^T \underline{O} = \underline{1}$

↓ inverse

$$\Lambda^\alpha_\beta \tilde{\Lambda}^\beta_\gamma \equiv \delta^\alpha_\gamma$$

$$\underline{x}^\alpha = \Lambda^\alpha_\beta x^\beta \rightarrow \frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} = \Lambda^\beta_\alpha \frac{\partial}{\partial x'^\beta} \rightarrow \frac{\partial}{\partial x'^\alpha} = \tilde{\Lambda}^\beta_\alpha \frac{\partial}{\partial x^\beta}$$

upper i
lower

invariant under $\underline{x} \rightarrow \underline{x}' = \underline{O} \underline{x}$

upper indices transform with $\underline{\Lambda}$
lower " " " $\underline{\hat{\Lambda}}$

$$O(m\ddot{\underline{x}} + \nabla V) = 0 \quad ; \quad \Lambda^\alpha_\beta \tilde{\Lambda}^\beta_\gamma \equiv \delta^\alpha_\gamma$$

$$\frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} = \Lambda^\beta_\alpha \frac{\partial}{\partial x'^\beta} \quad \rightarrow \quad \frac{\partial}{\partial x'^\alpha} = \tilde{\Lambda}^\beta_\alpha \frac{\partial}{\partial x^\beta}$$

$$\underline{\Lambda} \underline{\hat{\Lambda}} = \underline{1} \Rightarrow \tilde{\Lambda}^\beta_\gamma \Lambda^\gamma_\alpha = \delta^\beta_\alpha$$

$$J^{\alpha'} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$F'^{\alpha\beta} = \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} F^{\gamma\delta}$$

$$J^{\alpha'} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$F^{\alpha'\beta'} = \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} F^{\gamma\delta}$$

$$F^{\alpha'\beta'} = \hat{\Lambda}^{\alpha}_{\gamma} \hat{\Lambda}^{\beta}_{\delta} F^{\gamma\delta}$$

two contracted indices $\cdot$ transform trivially

$$\text{eg } A'^{\alpha} B'_{\alpha} = \Lambda^{\alpha}_{\beta} A^{\beta} \tilde{\Lambda}^{\gamma}_{\alpha} B_{\gamma} = A^{\gamma} B_{\gamma}$$

$$J^{\alpha'} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$F^{\alpha'\beta'} = \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} F^{\gamma\delta}$$

$$F^{\alpha'}_{\beta'} = \hat{\Lambda}^{\alpha}_{\gamma} \hat{\Lambda}^{\delta}_{\beta} F^{\gamma}_{\delta}$$

$$\in \alpha' \beta' \gamma \delta$$

two contracted indices

eg $A^{\alpha'} B_{\alpha'} =$

$$J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

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$$F'_{\alpha\beta} = \hat{\Lambda}^{\gamma}_{\alpha} \hat{\Lambda}^{\delta}_{\beta} F_{\gamma\delta}$$

$$\epsilon'^{\alpha\beta\gamma\delta} = \epsilon^{\chi\psi\eta\xi} \Lambda^{\alpha}_{\chi} \Lambda^{\beta}_{\psi} \Lambda^{\gamma}_{\eta} \Lambda^{\delta}_{\xi}$$

two contracted ind

eg $A'^{\alpha} B'_{\alpha}$

$A^{\alpha} \tilde{\Lambda}^{\gamma}_{\alpha} B_{\gamma}$

transf

$$J^{\alpha'} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

$$F^{\alpha\beta} = \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} F^{\gamma\delta}$$

$$F'_{\alpha\beta} = \tilde{\Lambda}^{\gamma}_{\alpha} \tilde{\Lambda}^{\delta}_{\beta} F_{\gamma\delta}$$

$$\epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\chi\psi\eta\xi} \Lambda^{\alpha}_{\chi} \Lambda^{\beta}_{\psi} \Lambda^{\gamma}_{\eta} \Lambda^{\delta}_{\xi} = \epsilon^{\alpha\beta\gamma\delta} \det(\Lambda)$$

two contracted indices transf

$$\text{eg } A^{\alpha'} B_{\alpha'} = \Lambda^{\alpha}_{\beta} A^{\beta} \tilde{\Lambda}^{\gamma}_{\alpha} B_{\gamma}$$

lin algebra

$$J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

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two contracted indices

transf

$$\text{eg } A'^{\alpha} B'_{\alpha} = \Lambda^{\alpha}_{\beta} A^{\beta} \tilde{\Lambda}^{\gamma}_{\alpha} B_{\gamma}$$

lin algebra

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$J'^{\alpha} = \Lambda^{\alpha}_{\beta} J^{\beta}$$

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$$F'_{\alpha\beta} = \hat{\Lambda}^{\gamma}_{\alpha} \hat{\Lambda}^{\delta}_{\beta} F_{\gamma\delta}$$

$$\epsilon'^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta} \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \Lambda^{\gamma}_{\rho} \Lambda^{\delta}_{\sigma} = \epsilon^{\alpha\beta\gamma\delta} \det(\Lambda)$$

two contracted indices

$$\text{eg } A'^{\alpha} B'_{\alpha} = \Lambda^{\alpha}_{\beta} A^{\beta} \Lambda^{\gamma}_{\alpha} B_{\gamma}$$

lin algebra

$$\det(\Lambda)$$

Most important: $\eta'^{\alpha\beta} = \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} \eta^{\gamma\delta} = \eta^{\alpha\beta}$: η is invariant under

$$\Lambda^\alpha_\beta J^\beta$$

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta F^{\gamma\delta}$$

$$\tilde{\Lambda}^\gamma_\alpha \tilde{\Lambda}^\delta_\beta F_{\gamma\delta}$$

$$= \epsilon^{\chi\psi\eta\xi} \Lambda^\alpha_\chi \Lambda^\beta_\psi \Lambda^\gamma_\eta \Lambda^\delta_\xi$$

two contracted indices

transform trivially

$$\text{eg } A'^\alpha B'_\alpha = \Lambda^\alpha_\beta A^\beta \tilde{\Lambda}^\gamma_\alpha B_\gamma = A^\gamma B_\gamma$$

lin algebra

$$= \epsilon^{\alpha\beta\gamma\delta} \det(\Lambda)$$

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$\eta'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta^{\gamma\delta} = \eta^{\alpha\beta} ; \eta \text{ is invariant under L.T.s.}$$

assess a symmetry: Lorentz invariance

3x3 rotations

This is the defining property of Lorentz transformations:

erty of Lorentz transformations:

$$\Lambda \eta \Lambda^T = \eta$$

$$\Lambda^\alpha_\delta \Lambda^\beta_\sigma \eta^{\sigma\delta}$$

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$$\Lambda \eta \Lambda^T = \eta$$

$$\Rightarrow (\det \Lambda)^2 \det \eta = \det \eta$$

$$\Rightarrow \underline{\det \Lambda = \pm 1}$$

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta^{\gamma\delta}$$

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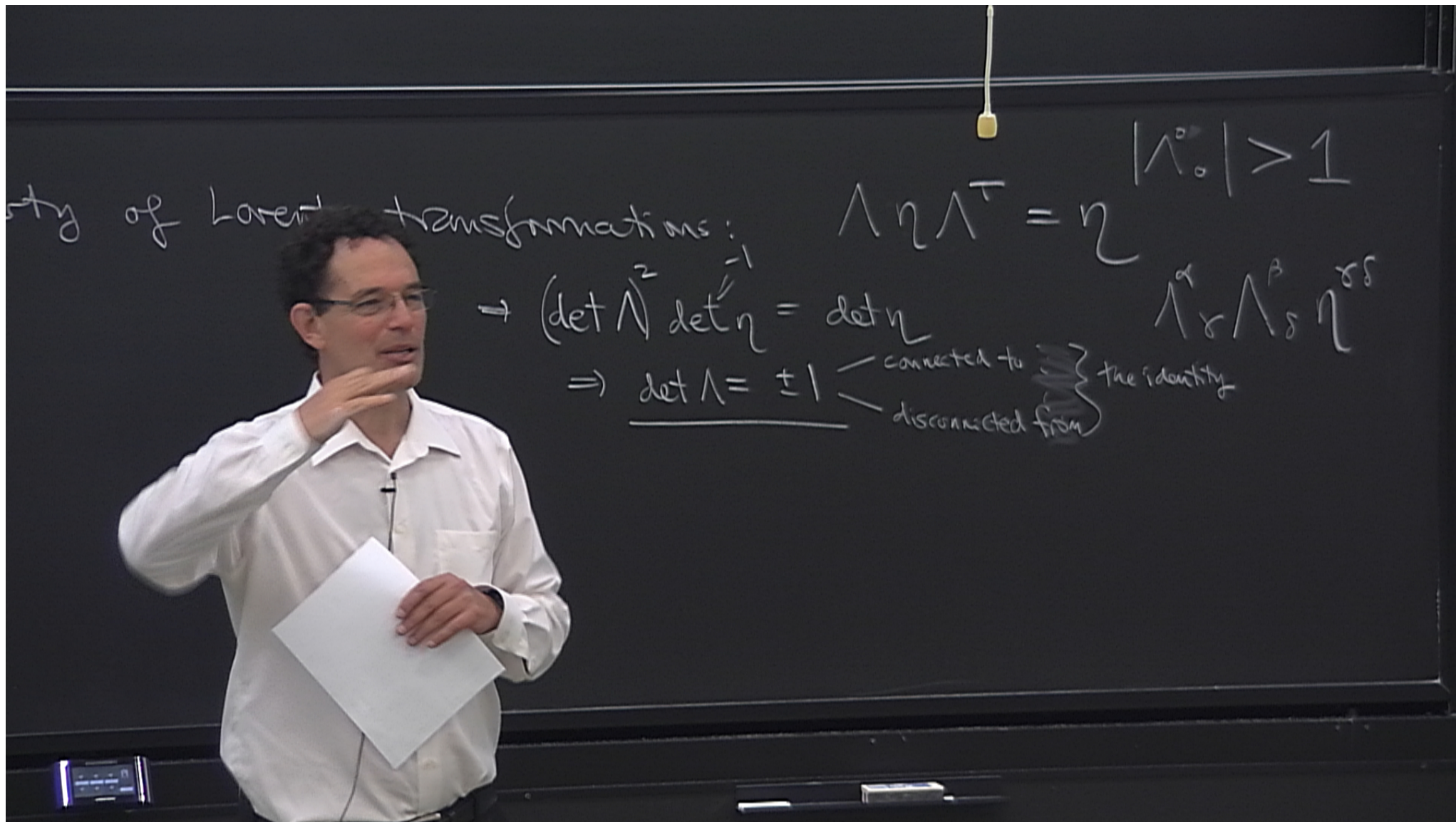
$$\Rightarrow \underline{\det \Lambda = \pm 1} \left\{ \begin{array}{l} \text{connected} \\ \text{disconnected} \end{array} \right\} \text{ to the identity}$$

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta^{\gamma\delta}$$

This is the defining property of Lorentz transformations:

$$\rightarrow \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\rightarrow (\det \Lambda)^2 \det \eta = 1 \\ \Rightarrow \underline{\det \Lambda = \pm 1}$$



$$\det \Lambda = \pm 1$$

This is the defining property of Λ

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

This is the defining property of L

$$\det \Lambda = \pm 1$$

(Lorentz group has 4 connected components)

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

property of Lorentz transformations: Λ^α_β

$$\Lambda \eta \Lambda^T = \eta$$

$$|\Lambda^0_0| > 1$$

$$\Rightarrow (\det \Lambda)^2 \det \eta = \det \eta$$

$$\Rightarrow \det \Lambda = \pm 1$$

— connected to the identity
 — disconnected from the identity

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta^{\gamma\delta}$$

last time

$$\partial'_\alpha F'^{\alpha\beta} = -\mu_0 J'^\beta$$

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$$\alpha, \beta = \overset{ct}{\downarrow} \underbrace{0, 1, 2, 3}_{x^i}$$

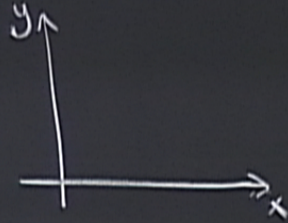
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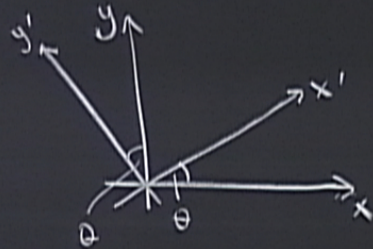
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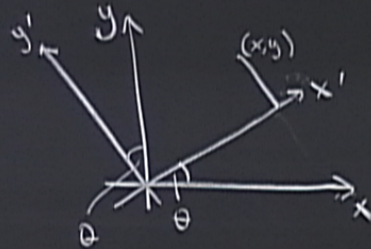
Simple example: 2d rotation



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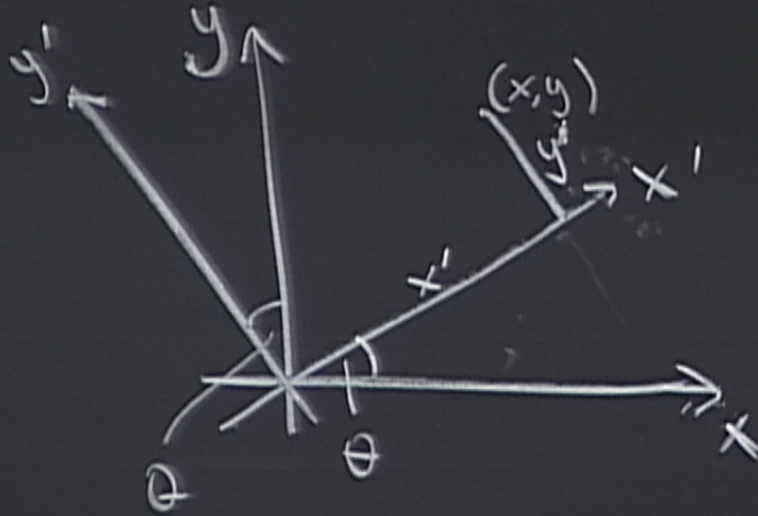


Simple example: 2d rotation

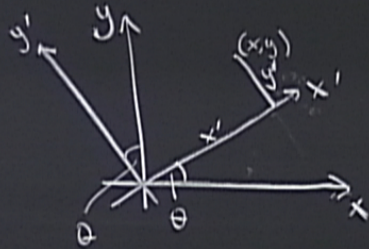


Simple example:

2d rotation

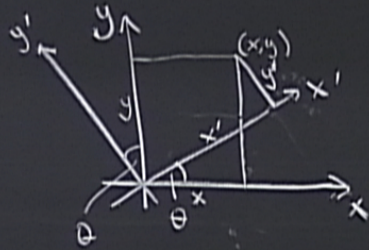


Simple example: 2d rotation



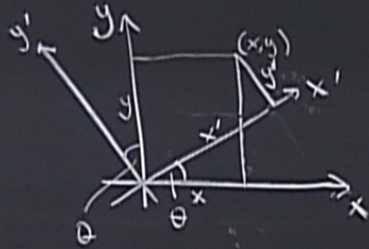
$$\Rightarrow \begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= y \cos \theta - x \sin \theta \end{aligned}$$

Simple example: 2d rotation



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Simple example:



2d rotation

Exercise: show that

$$\Rightarrow x' = x \cos \theta + y \sin \theta$$

$$y' = y \cos \theta - x \sin \theta$$

$$\Rightarrow x'^2 + y'^2 = x^2 + y^2$$

2d rotation

Exercise: show that

$$\Rightarrow x' = x \cos \theta + y \sin \theta$$

$$y' = y \cos \theta - x \sin \theta$$

$$\Rightarrow x'^2 + y'^2 = x^2 + y^2$$

$$\Rightarrow \Lambda^2_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \Lambda_{\beta}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

check that $\Lambda_{\eta} \Lambda^T = \eta$

If consider L.T. involving time, minus sign in η changes everything:

$$\Lambda_{\beta}^{\alpha} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

also satisfies $\Lambda\eta\Lambda^T = -\eta$

changes everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

$$\begin{aligned} ct' &= ct \cosh \theta - x \sinh \theta \\ x' &= x \cosh \theta - ct \sinh \theta \end{aligned}$$

check everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

$$ct' = ct \cosh \theta - x \sinh \theta$$

$$; \quad x' = x \cosh \theta - ct \sinh \theta$$

$$(ct')^2 - x'^2 = (ct)^2 - x^2$$

changes everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

$$ct' = ct \cosh \theta - x \sinh \theta$$

$$; \quad x' = x \cosh \theta - ct \sinh \theta$$

$$-(ct')^2 + x'^2 = -(ct)^2 + x^2$$

changes everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

$$\begin{aligned} ct' &= ct \cosh \theta - x \sinh \theta \\ x' &= x \cosh \theta - ct \sinh \theta \\ -(ct')^2 + x'^2 &= -(ct)^2 + x^2 \end{aligned}$$

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

changes everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

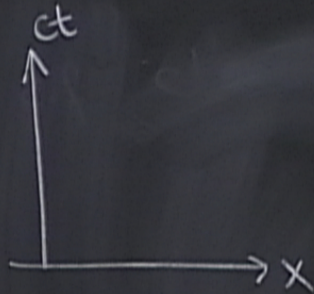
$$\begin{aligned} ct' &= ct \cosh \theta - x \sinh \theta \\ x' &= x \cosh \theta - ct \sinh \theta \\ -(ct')^2 + x'^2 &= -(ct)^2 + x^2 \\ \uparrow \end{aligned}$$

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

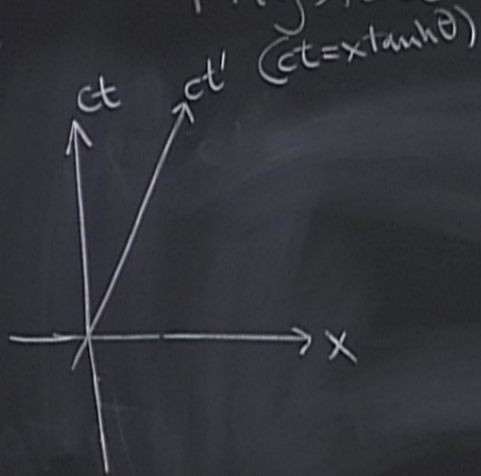
Physical meaning (Einstein 1905)

Physical meaning (Einstein 1905)

Spatetime

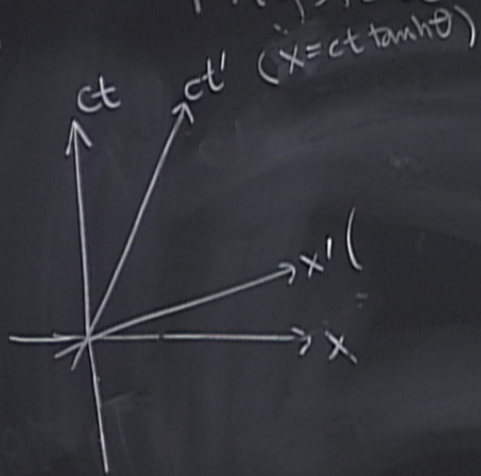


Spacetime Physical meaning (Einstein 1905)



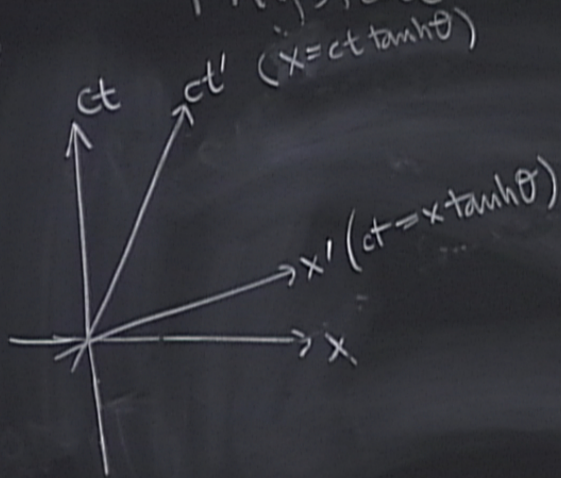
Spacetime

Physical meaning (Einstein 1905)

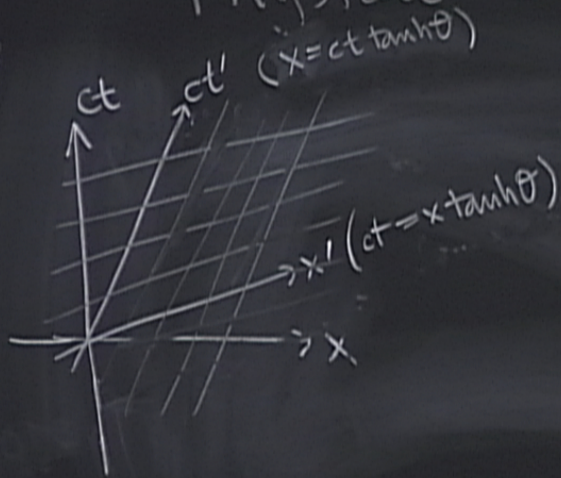


Spacetime

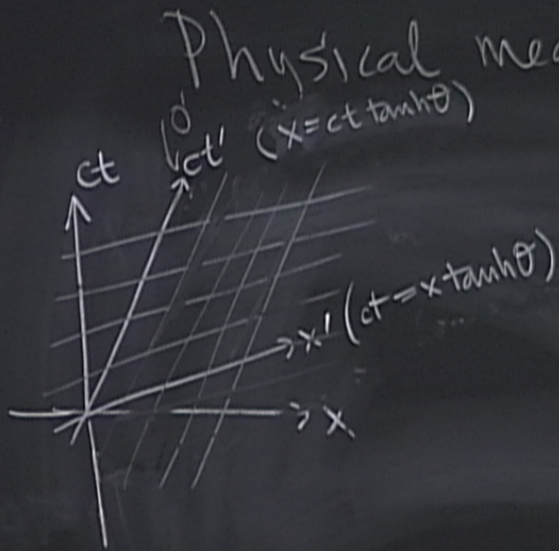
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Spacetime



Physical meaning (Einstein 1905)

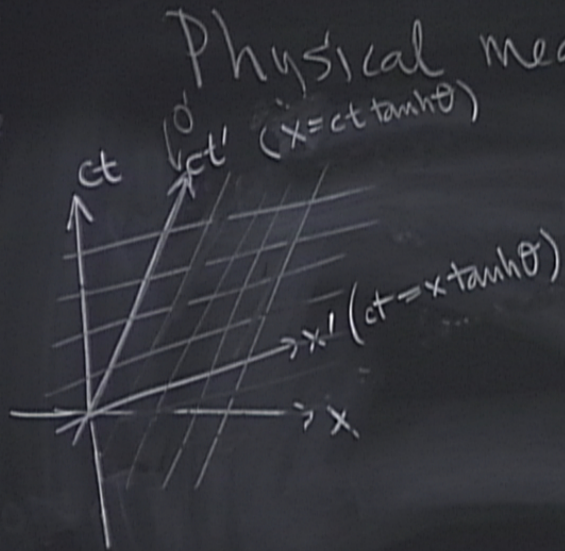
an observer O' using (ct', x') coordinate

O' is moving relative to an observer

(Einstein 1905)'

observer O' using (ct', x') coordinates, is stationary in those coordinates
 O' is moving relative to an observer O using the (ct, x) coordinates,
with a velocity

Spacetime



Physical meaning (Einstein 1905)

an observer O' using (ct', x') coordinates, is

O' is moving relative to an observer O with

with a velocity $c \tanh \theta$

Einstein 1905)

observer O' using (ct', x') coordinates, is stationary in those coordinates

O' is moving relative to an observer O using the (ct, x) coordinates,

with a velocity $c \tanh \theta$ i.e. $v = c \tanh \theta$

(θ is called the "rapidity")

no observer can have $|v| > c$ because $|\tanh \theta| \leq 1$.

(Einstein 1905)

an observer O' using (ct', x') coordinates, is stationary in those coordinates

O' is moving relative to an observer O using the (ct, x) coordinates,

with a velocity $c \tanh \theta$ i.e. $v = c \tanh \theta$ (θ is called the "rapidity")

→ no observer can have $|v| > c$ because $|\tanh \theta| \leq 1$.

transform trivially

$$\cosh \theta = \frac{1}{\sqrt{1 - \tanh^2 \theta}}$$

$$\cosh \theta = \frac{1}{\sqrt{1 - \tanh^2 \theta}} = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$

$$\cosh \theta = \frac{1}{\sqrt{1 - \tanh^2 \theta}} = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$

$$\sinh \theta = \cosh \theta \cdot \tanh \theta = \gamma \frac{v}{c}$$

changes everything:

satisfies $\Lambda \eta \Lambda^T = \eta$

$$\begin{aligned} ct' &= ct \cosh \theta - x \sinh \theta \\ x' &= x \cosh \theta - ct \sinh \theta \\ -(ct')^2 + x'^2 &= -(ct)^2 + x^2 \\ \uparrow \end{aligned}$$

($\neq$)

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

∴ L.T. ($\neq$) is just $t' = \gamma(t - \frac{vx}{c^2})$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

$$\tanh \theta = \gamma \frac{v}{c}$$

, L.T. ($\dagger$) is just

$$\begin{aligned} t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\ x' &= \gamma (x - vt) \end{aligned}$$

$$\Rightarrow t = \gamma(t')$$

If consider L.T. involving time, minus sign in η changes ev

$$\hat{\Lambda}_\beta = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

also satisfies

$$\hat{\Lambda} = \begin{pmatrix} \cosh + \sinh & \\ +\sinh & \cosh \end{pmatrix}$$

L.T. ($\neq$) is just

$$t' = \gamma \left(t - \frac{vx}{c^2} \right)$$
$$x' = \gamma (x - vt)$$

$$\Rightarrow t = \gamma \left(t' + \frac{v}{c^2} x' \right)$$

$$x = \gamma (x' + vt')$$

Notice that if $x = ct$ (trajectory of particle moving at speed of light)

$$\text{then } \left. \begin{aligned} x' &= \gamma(c-v)t \\ t' &= \gamma(1 - \frac{v}{c})t \end{aligned} \right\} x' = ct'$$

$\Rightarrow$ anything moving at speed of light does so for all observers.

Notice that if $x = ct$ (trajectory of particle moving at speed of light)

$$\text{then } \left. \begin{aligned} x' &= \gamma(c-v)t \\ t' &= \gamma(1 - \frac{v}{c})t \end{aligned} \right\} x' = ct'$$

$\Rightarrow$ anything moving at speed of light does so for all observers.

(inconsistency between relativity and Maxwell's derivation of speed of light is resolved.
(going back to Galileo))

$$\hat{\Lambda} = \begin{pmatrix} \cosh + \sinh & \\ +\sinh & \cosh \end{pmatrix}$$

Notice that if $x = ct$ (trajectory of particle moving at speed of light)

$$\text{then } \begin{cases} x' = \gamma(c-v)t \\ t' = \gamma(1 - \frac{v}{c})t \end{cases} \} x' = ct'$$

$\Rightarrow$ anything moving at speed of light does so for all observers.

(inconsistency between relativity and Maxwell's derivation of speed of light is resolved.
(going back to Galileo) (at the cost of revising all of classical mechanics!)

$$\hat{\Lambda} = \begin{pmatrix} \cosh + \sinh & \\ +\sinh & \cosh \end{pmatrix}$$

Notice that if $x=ct$ (trajectory of particle moving at speed of light)

$$\text{then } \left. \begin{array}{l} x' = \gamma(c-v)t \\ t' = \gamma(1-\frac{v}{c})t \end{array} \right\} x' = ct'$$

$\Rightarrow$ anything moving at speed of light does so for all observers.

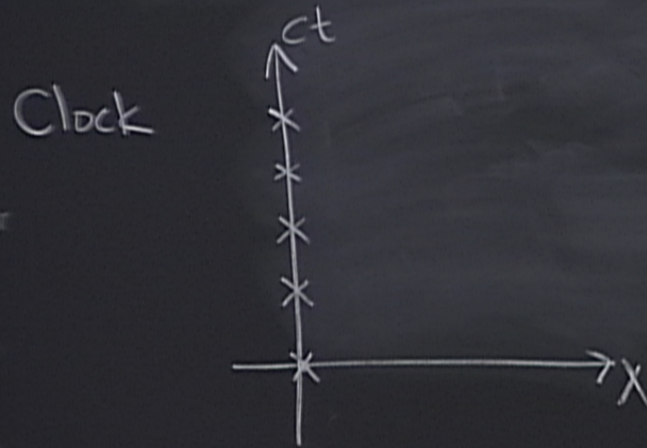
(inconsistency between relativity and Maxwell's derivation of speed of light is resolved.
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Clocks and rulers

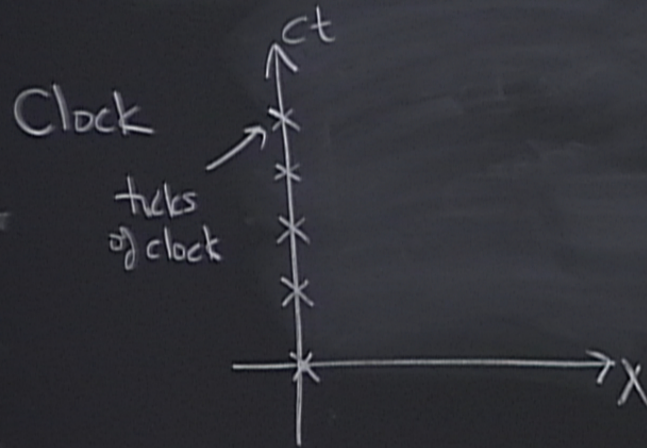
Clocks and rulers



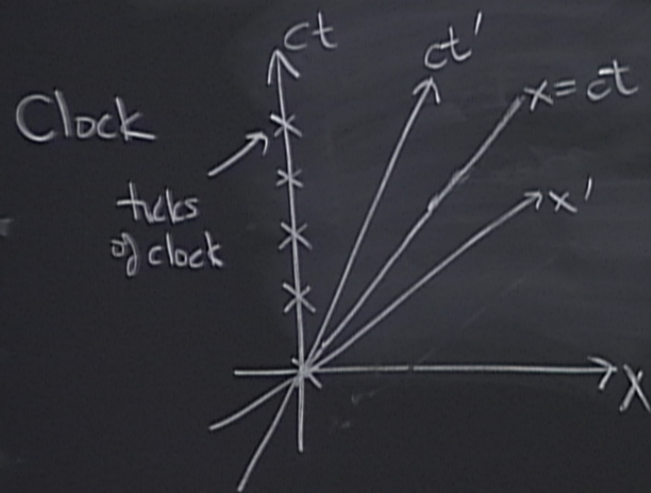
Clocks and rulers



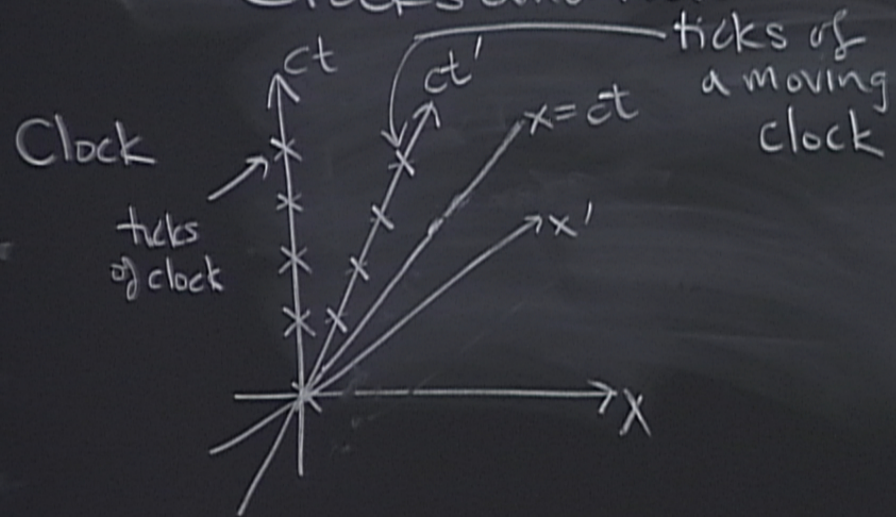
Clocks and rulers



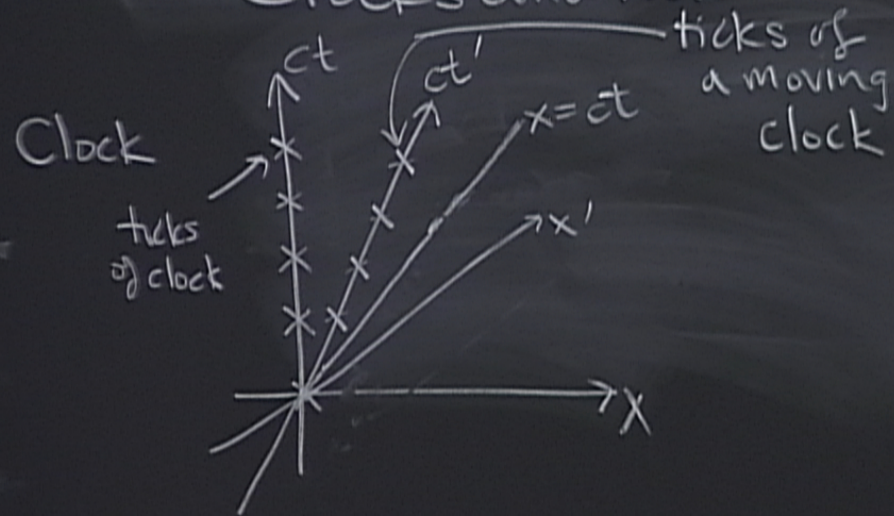
Clocks and rulers



Clocks and rulers



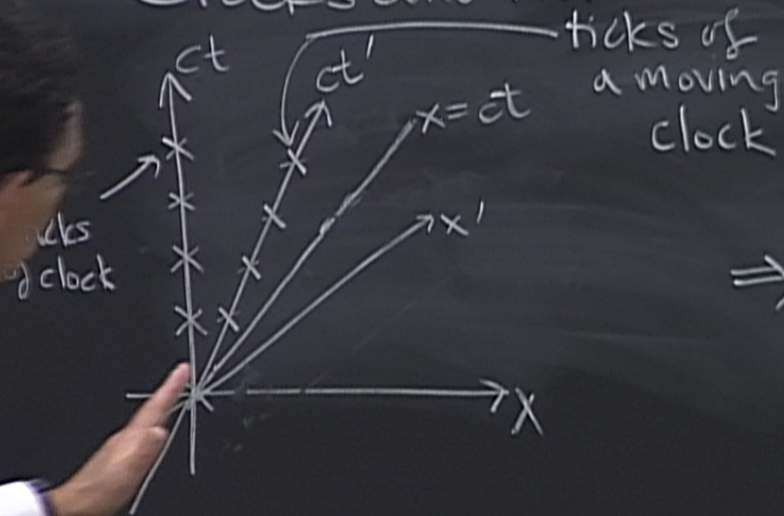
Clocks and rulers



$$x' = 0$$

$$t' = 0, 1, 2, 3 \dots$$

Clocks and rulers



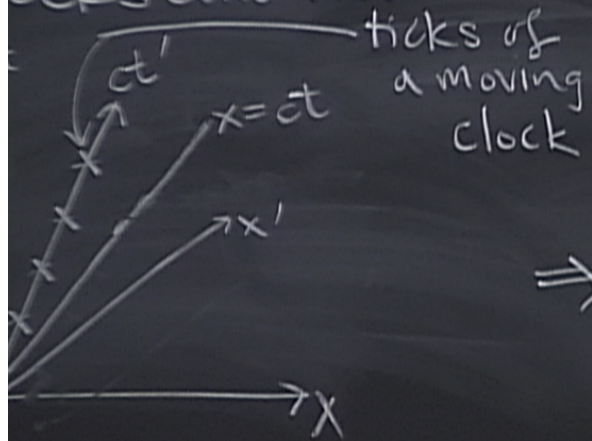
$$x' = 0$$

$$t' = 0, 1, 2, 3 \dots$$

$$\Rightarrow t = 0, \gamma, 2\gamma, 3\gamma \dots$$

$$x = 0, \gamma v, 2\gamma v \dots = vt$$

clocks and rulers



ticks of
a moving
clock

$$x' = 0$$

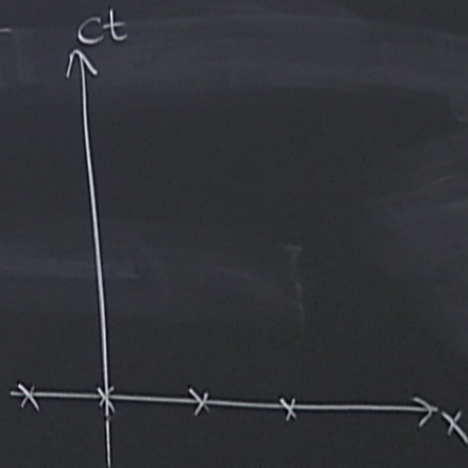
$$t' = 0, 1, 2, 3 \dots$$

$$\Rightarrow t = 0, \gamma, 2\gamma, 3\gamma \dots$$

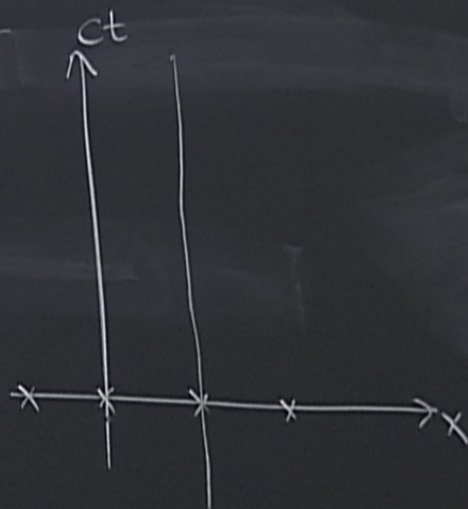
$$x = 0, \gamma v, 2\gamma v \dots = vt$$

"Time dilation"

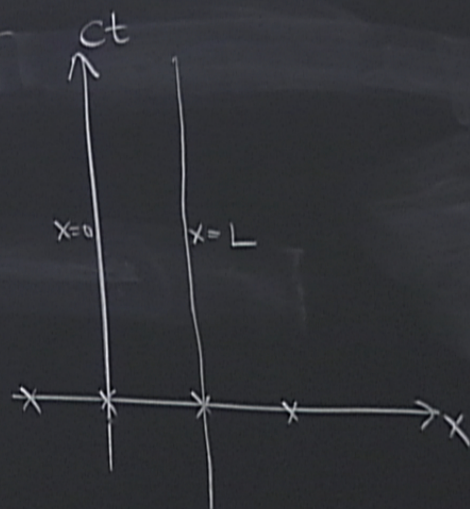
Rulers

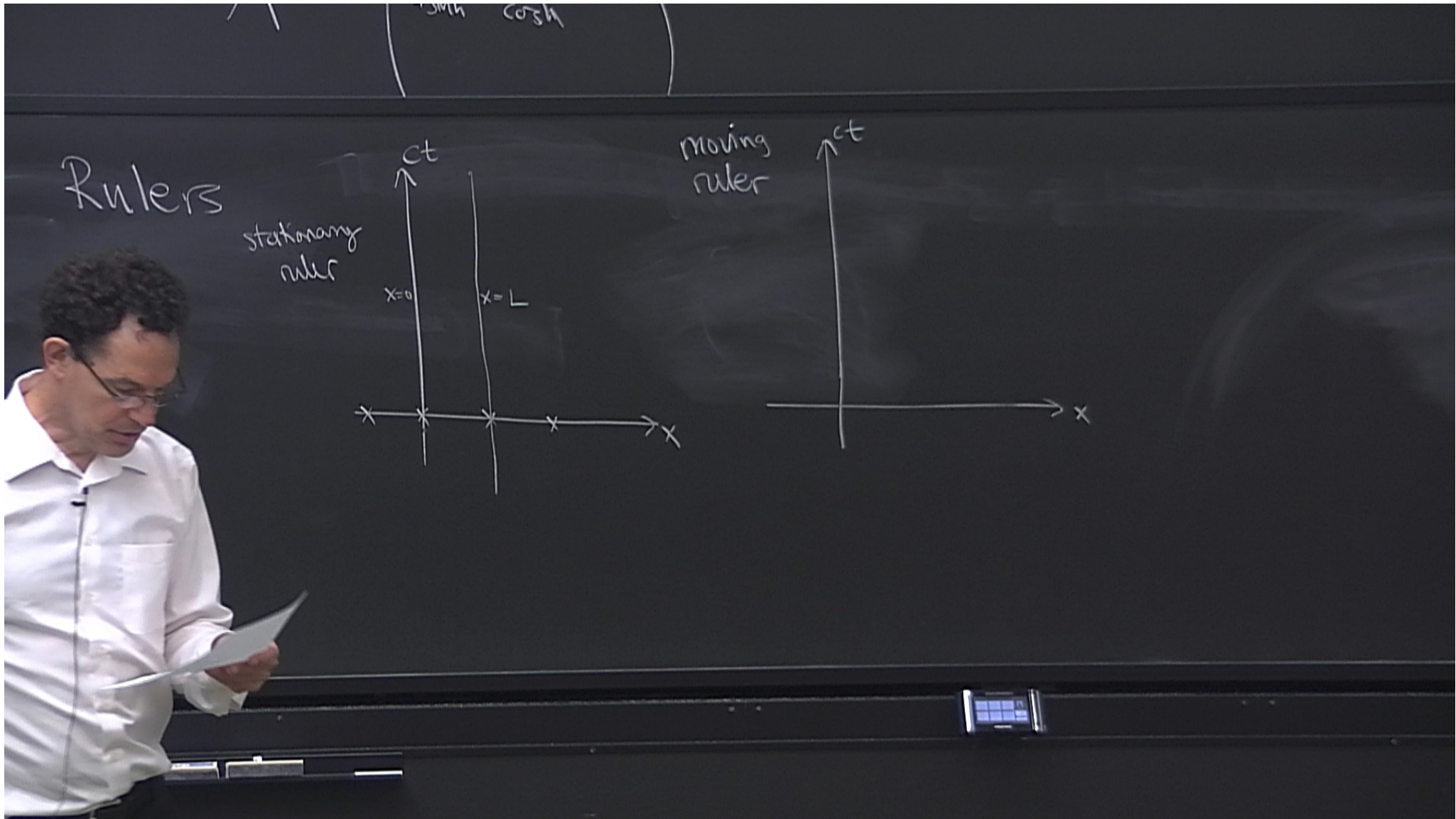


Rulers



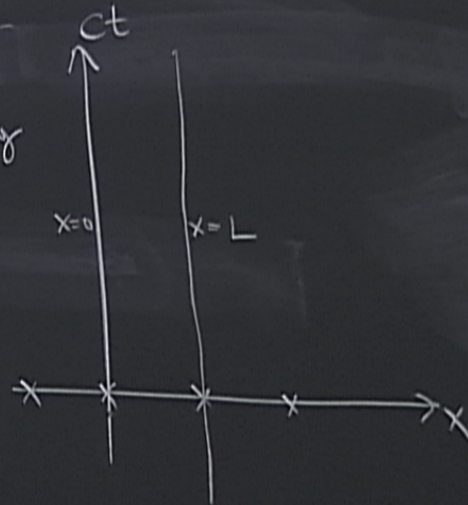
Rulers



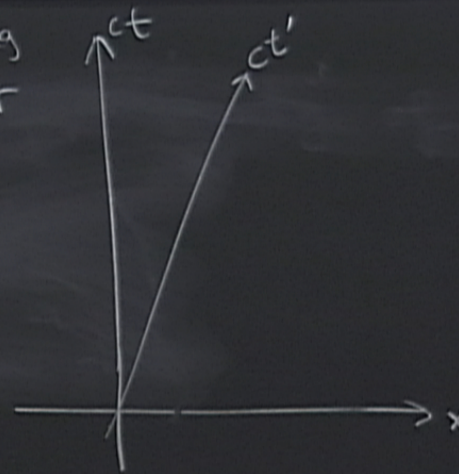


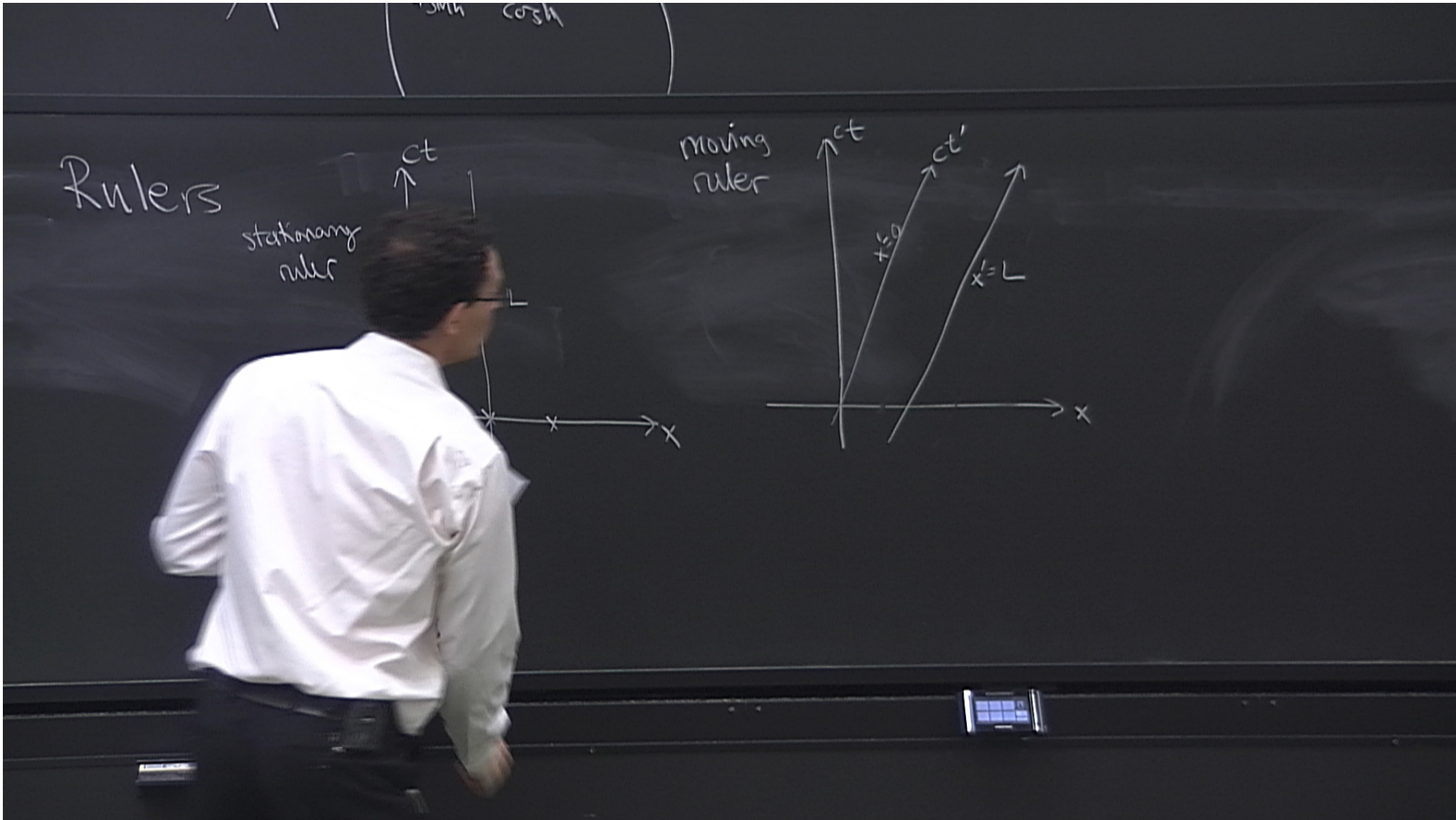
Rulers

stationary
ruler



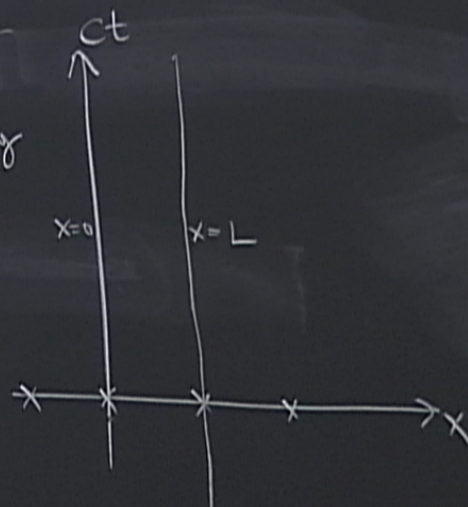
moving
ruler



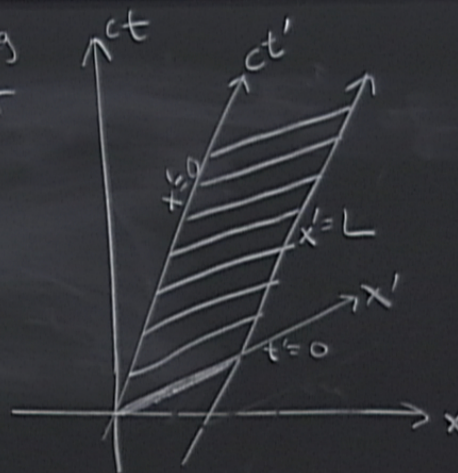


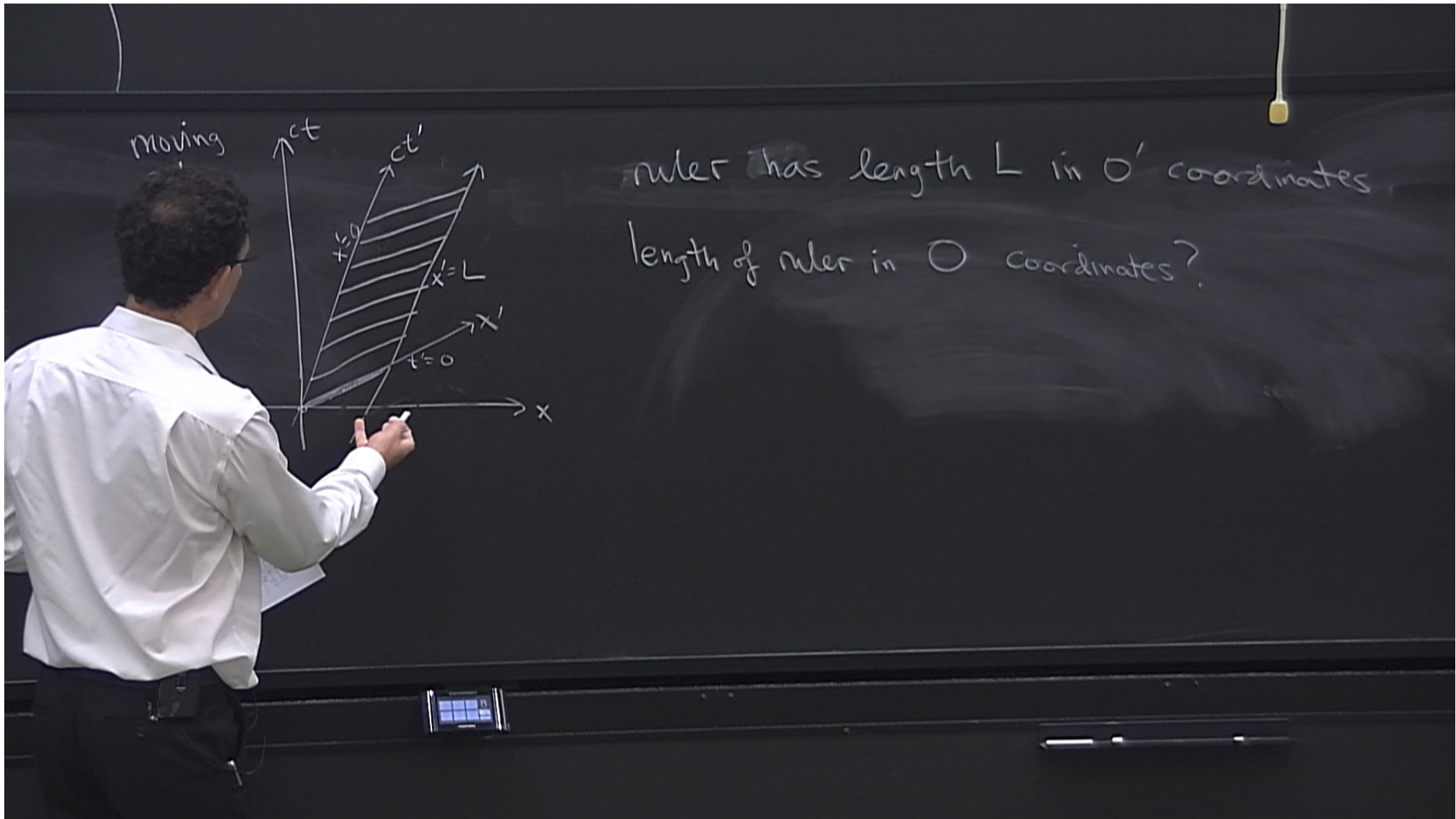
Rulers

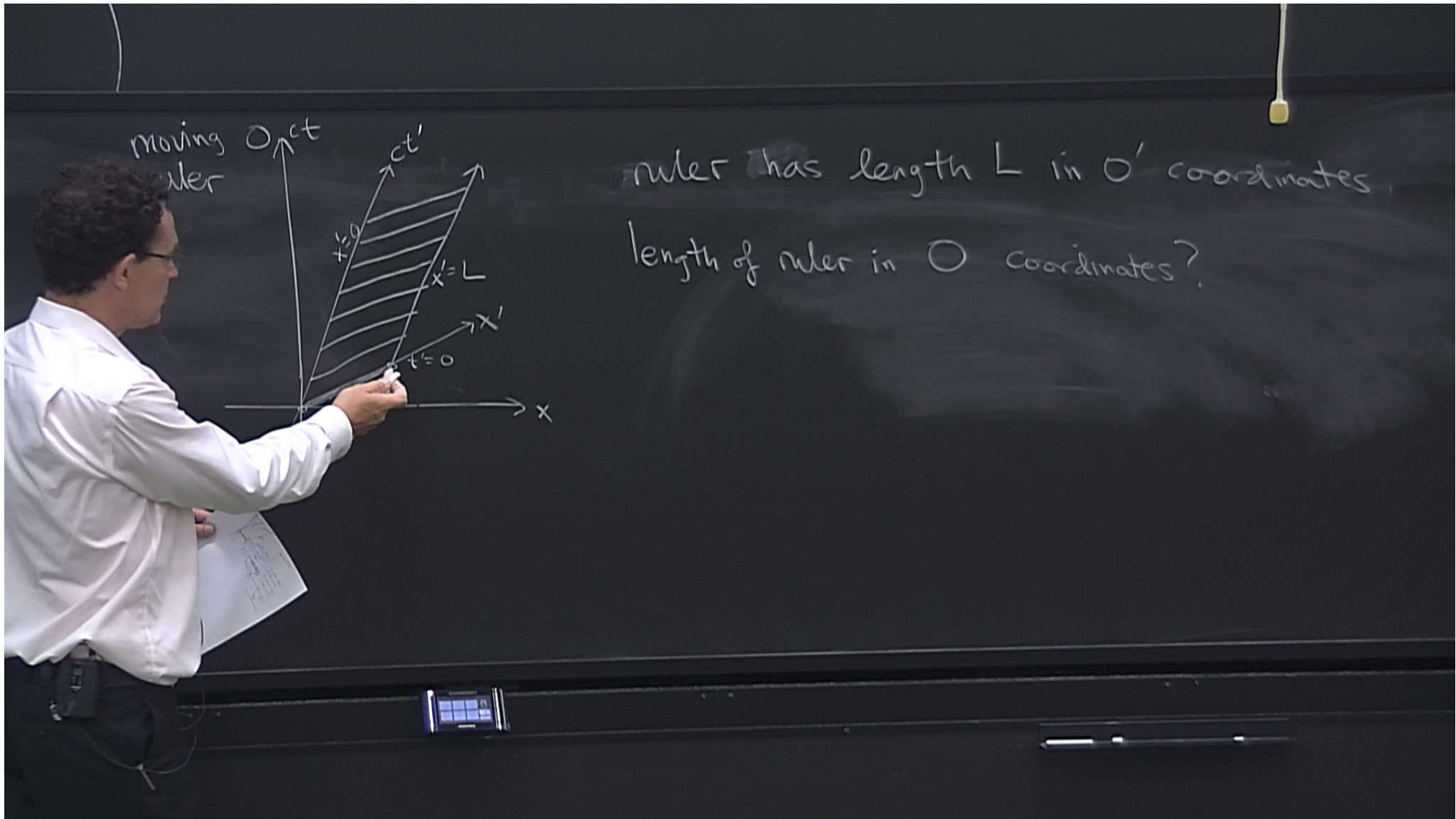
stationary ruler



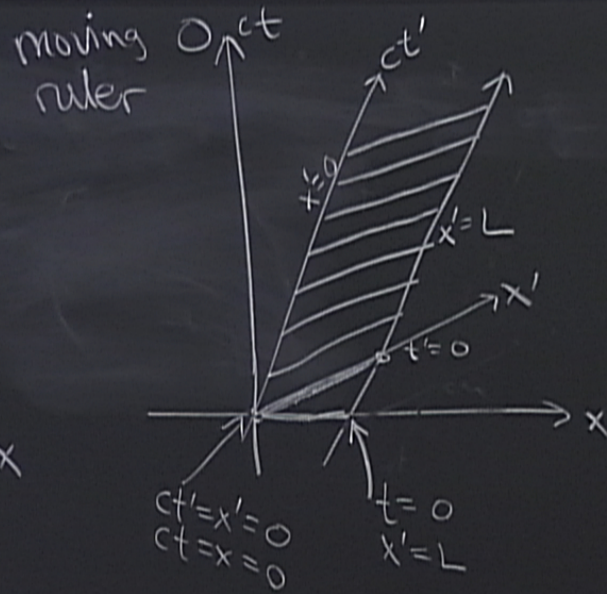
moving ruler





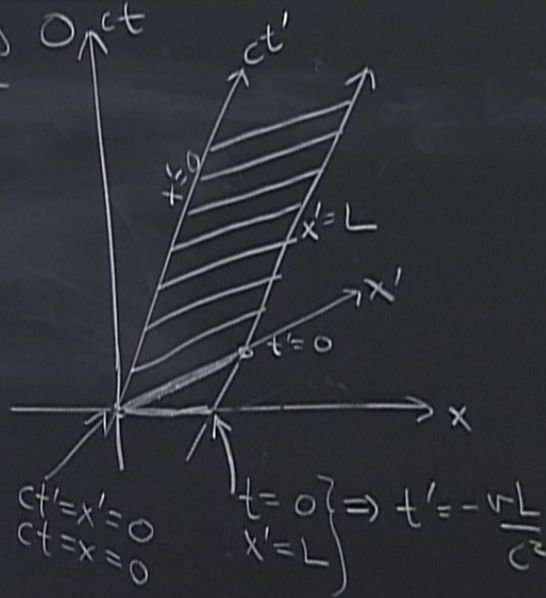


ruler has length L in O' coordinates
length of ruler in O coordinates?

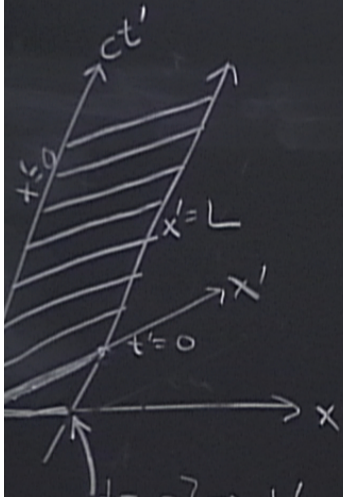


ruler has length L in O' coordinates
length of ruler in O coordinates?

moving ruler



ruler has length L in O' coordinates
length of ruler in O coordinates?



ruler has length L in O' coordinates

length of ruler in O coordinates?

$$\left. \begin{matrix} t=0 \\ x'=L \end{matrix} \right\} \Rightarrow t' = -\frac{vL}{c^2} \quad \left. \begin{matrix} x = \gamma \left(L - \frac{vL}{c^2} \right) = \gamma \left(1 - \frac{v^2}{c^2} \right) L = \frac{1}{\gamma} L \end{matrix} \right\} \Rightarrow \begin{matrix} \text{length of ruler in } O \text{ coordinates} \\ \text{is } \frac{1}{\gamma} L \end{matrix}$$

$\Rightarrow$ Lorentz contraction

Energy and Momentum

Energy and Momentum

— in electromagnetic field.

Energy and Momentum

— in electromagnetic field.

energy density ρ_{en}

Energy and Momentum

— in electromagnetic field.

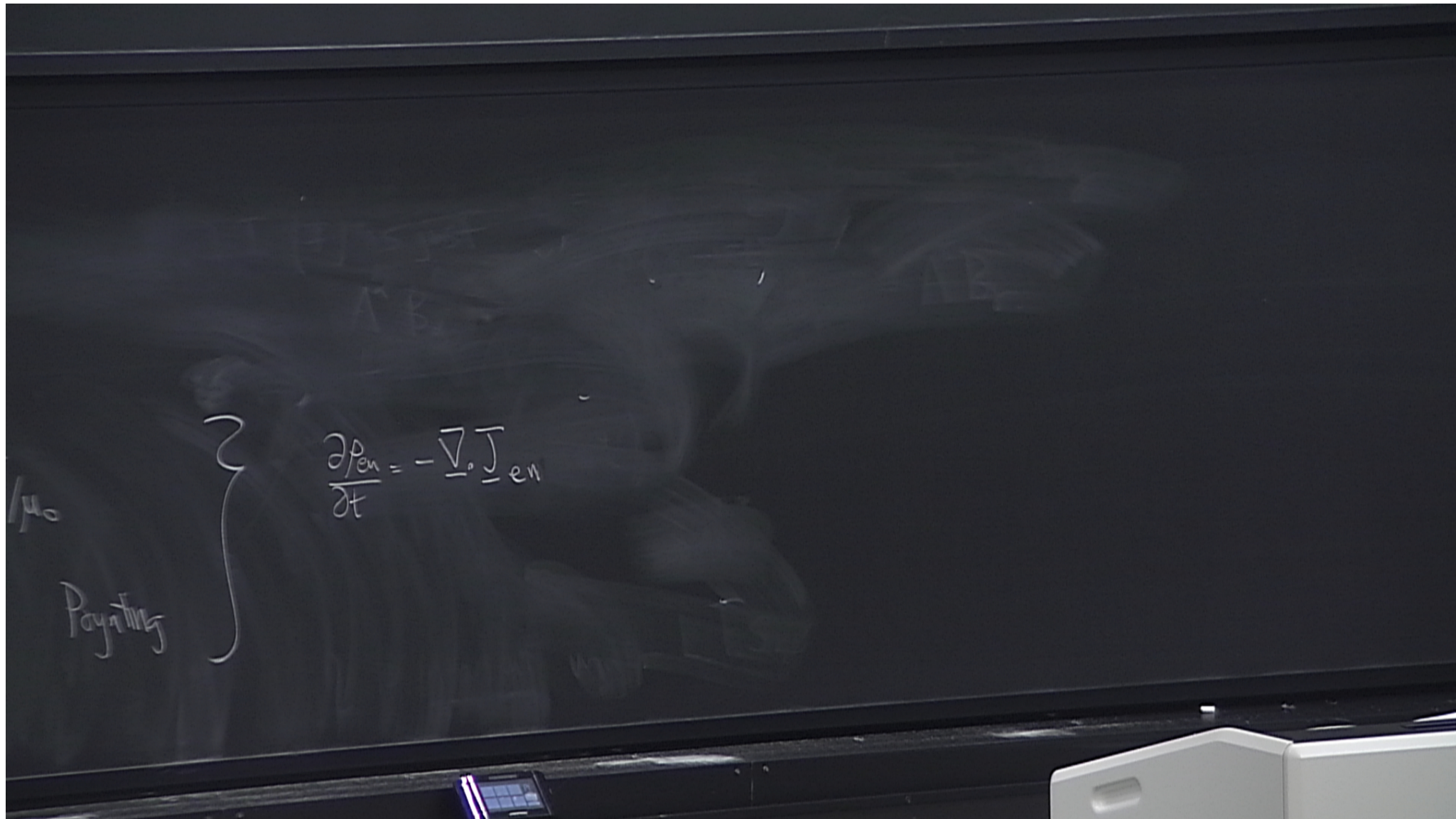
energy density $\rho_{\text{em}} = \frac{1}{2} \epsilon_0 \underline{E}^2 + \frac{1}{2} \underline{B}^2 / \mu_0$

Energy and Momentum

— in electromagnetic field.

energy density $\rho_{\text{en}} = \frac{1}{2} \epsilon_0 \underline{E}^2 + \frac{1}{2} \underline{B}^2 / \mu_0$

flux $\underline{J}_{\text{en}} = \frac{1}{\mu_0} (\underline{E} \wedge \underline{B})$ Poynting



combine to form the energy momentum tensor:

at speed of light c

combine to the energy momentum tensor:

Symmetric,
two-index
tensor

$T_{\mu\nu}$

$[F$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

combine to form the energy momentum tensor:

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$$T^{00} = \rho_{em}$$
$$T^{0i} = J_{em}^i$$

combine to form the energy momentum tensor:

Sym
two-

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$$T^{00} = \rho_{\text{em}}$$

$$T^{0i} = J_{\text{em}}^i$$

$$\partial_{\mu} = \partial$$

$$\partial_{\mu} T^{\mu\nu}$$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$$T^{00} = \rho_{\text{em}}$$

$$T^{0i} = J_{\text{em}}^i$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

Maxwell
with
 $J^{\alpha}=0$

$$\partial_{\mu} T^{\mu\nu} =$$

$$T^{00} = \rho_{\text{em}}$$

$$T^{0i} = J_{\text{em}}^i$$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$\downarrow$
 $\partial_{\mu} F^{\alpha\mu} = 0$
 + speed

$$T^{00} = \rho_{em}$$

$$T^{0i} = J_{em}^i$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

Maxwell
with
 $J^{\alpha} = 0$

$$\partial_{\mu} T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} \partial_{\mu} F_{\alpha}^{\nu} - \frac{1}{2} (\partial^{\nu} F^{\alpha\beta}) F_{\alpha\beta} \right) = \frac{1}{2\mu_0} \left(F^{\alpha\beta} (\partial_{\mu} F^{\mu\nu}) \right)$$

If consider L.T. involving time, minus sign in η changes everything:

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

also satisfies $\Lambda\eta\Lambda^T = \eta$

$$\Lambda^{-1} = \begin{pmatrix} \cosh\theta & +\sinh\theta \\ +\sinh\theta & \cosh\theta \end{pmatrix}$$

$$ct' = ct \cosh\theta - x \sinh\theta$$

$$x' = x \cosh\theta - ct \sinh\theta$$

$$-(ct')^2 + x'^2 = -(ct)^2 + x^2$$

(#)

$$\cosh^2\theta - \sinh^2\theta = 1$$

combine to form the energy momentum tensor:

Symmet
two-ind
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$$T^{00} = \rho_{em}$$

$$T^{0i} = J_{em}^i$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

Max

$$\partial_{\lambda} T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} \partial_{\lambda} F_{\alpha}^{\nu} - \frac{1}{2} (\partial^{\nu} F^{\mu\beta}) F_{\beta\lambda} \right) = \frac{1}{2\mu_0} \left(F^{\mu\beta} (\partial_{\lambda} F_{\alpha}^{\nu} - \partial_{\alpha} F_{\lambda}^{\nu}) - \frac{1}{2} F_{\alpha\beta} \partial^{\nu} F^{\mu\alpha} \right)$$

from the energy momentum tensor:

$$= \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

$$T^{00} = \rho_{em}$$

$$T^{0i} = J_{em}^i$$

$$\partial_\mu F^\mu_\nu - \frac{1}{2} (\partial^\mu F_{\alpha\beta}) F^{\alpha\beta}_\nu = \frac{1}{2\mu_0} \left(F^{\alpha\mu} (\partial_\mu F_\alpha^\nu - \partial_\alpha F_\mu^\nu) - F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right)$$

$$= \frac{1}{2\mu_0}$$

um tensor:

$$T^{00} = \rho_{em}$$

$$T^{0i} = J_{em}^i$$

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= \frac{1}{2\mu_0} \left(F^{\alpha\mu} (\partial_\mu F^\nu_\alpha - \partial_\alpha F^\nu_\mu) - F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right) \\ &= \frac{1}{2\mu_0} \left(F^{\alpha\mu} (\partial_\mu F^\nu_\alpha + \partial_\alpha F^\nu_\mu + \partial^\nu F_{\mu\alpha}) \right) = 0 \text{ by Maxwell's second eq.} \\ &\quad \partial_\alpha F^{\alpha\beta} = 0 \end{aligned}$$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\alpha\mu} F_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$\downarrow$
 $\partial_{\mu} F^{\alpha\mu} = 0$
 (Maxwell eq. 1)

$$T^{00} =$$

$$T^{0i} =$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

Maxwell
with
 $J^{\alpha} = 0$

$$\partial_{\mu} T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\alpha\mu} \partial_{\mu} F_{\alpha}^{\nu} - \frac{1}{2} (\partial^{\nu} F^{\alpha\beta}) F_{\alpha\beta} \right) = \frac{1}{2\mu_0} \left(F^{\alpha\mu} \partial_{\mu} F_{\alpha}^{\nu} + F^{\alpha\mu} \partial_{\mu} F_{\alpha}^{\nu} \right)$$

$$= 0$$

$$\Lambda^\beta_\alpha = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 & 0 \\ \sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

also satisfies $\Lambda^\mu_\mu = 1$

$$-(ct')^2 + x'^2 = -(ct)^2 + x^2$$

combine to form the energy momentum tensor:

Symmetric,
two-index
tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left[F^{\mu\alpha} F^\nu_{\alpha} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$$T^{00} = \rho_{em}$$

$$T^{0i} = J_{em}^i$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

Maxwell
with
 $J^\mu = 0$

$$\partial_\mu T^{\mu\nu}$$

$$= \frac{1}{\mu_0} \left(F^{\mu\alpha} \partial_\mu F^\nu_{\alpha} - \frac{1}{2} (\partial^\nu F^{\alpha\beta}) F_{\alpha\beta} \right) = \frac{1}{2\mu_0} \left(F^{\mu\alpha} (\partial_\mu F^\nu_{\alpha} - \partial_\alpha F^\nu_{\mu}) - F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right)$$

$$= \frac{1}{2\mu_0} \left(F^{\mu\alpha} (\partial_\mu F^\nu_{\alpha} + \partial_\alpha F^\nu_{\mu} + \partial^\nu F_{\mu\alpha}) \right) = 0$$

by Maxwell's second eq.
 $\partial_\mu F^{\mu\nu} = J^\nu = 0$