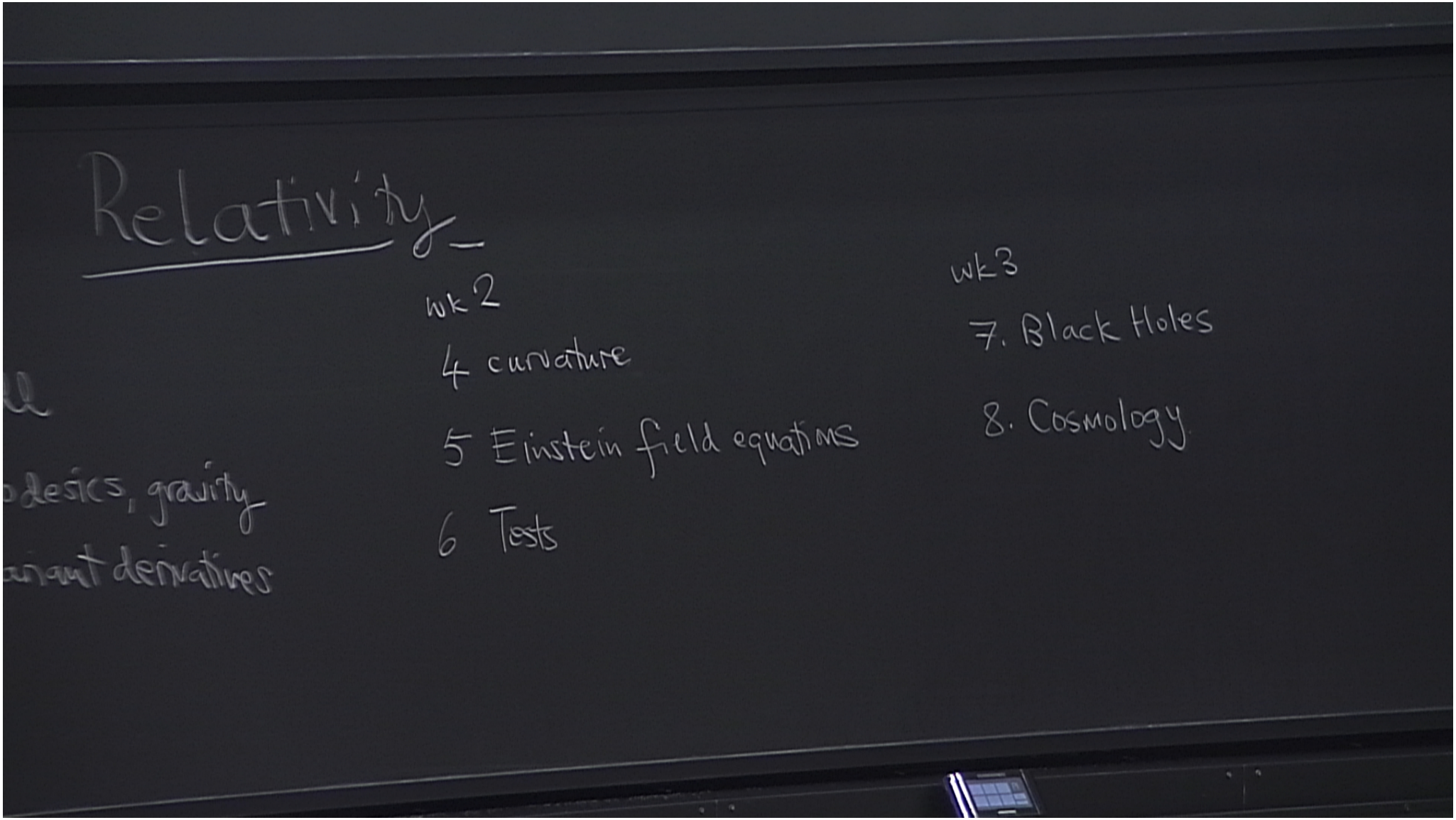
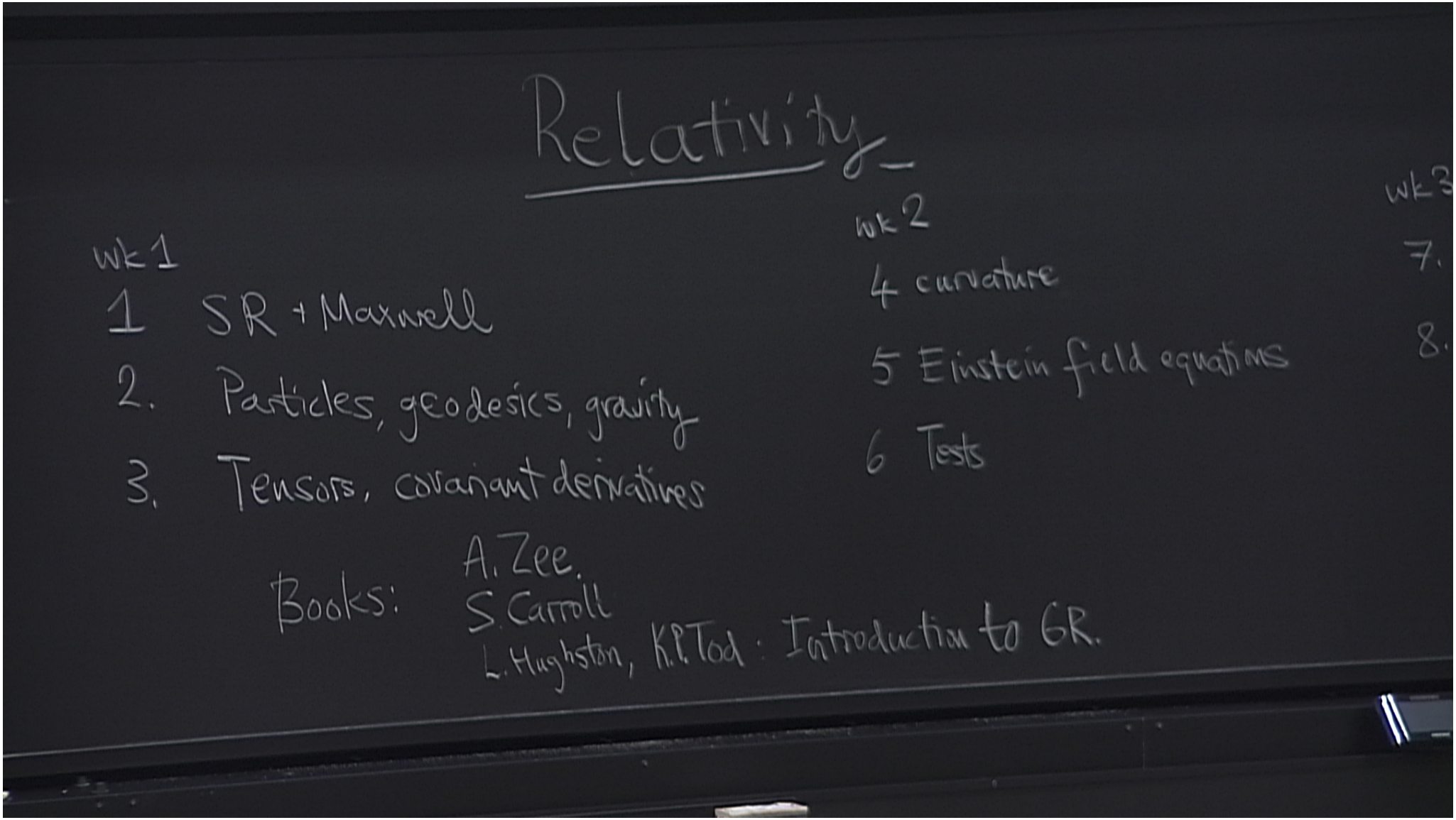
Title: 14/15 PSI - Relativity- Lecture 1

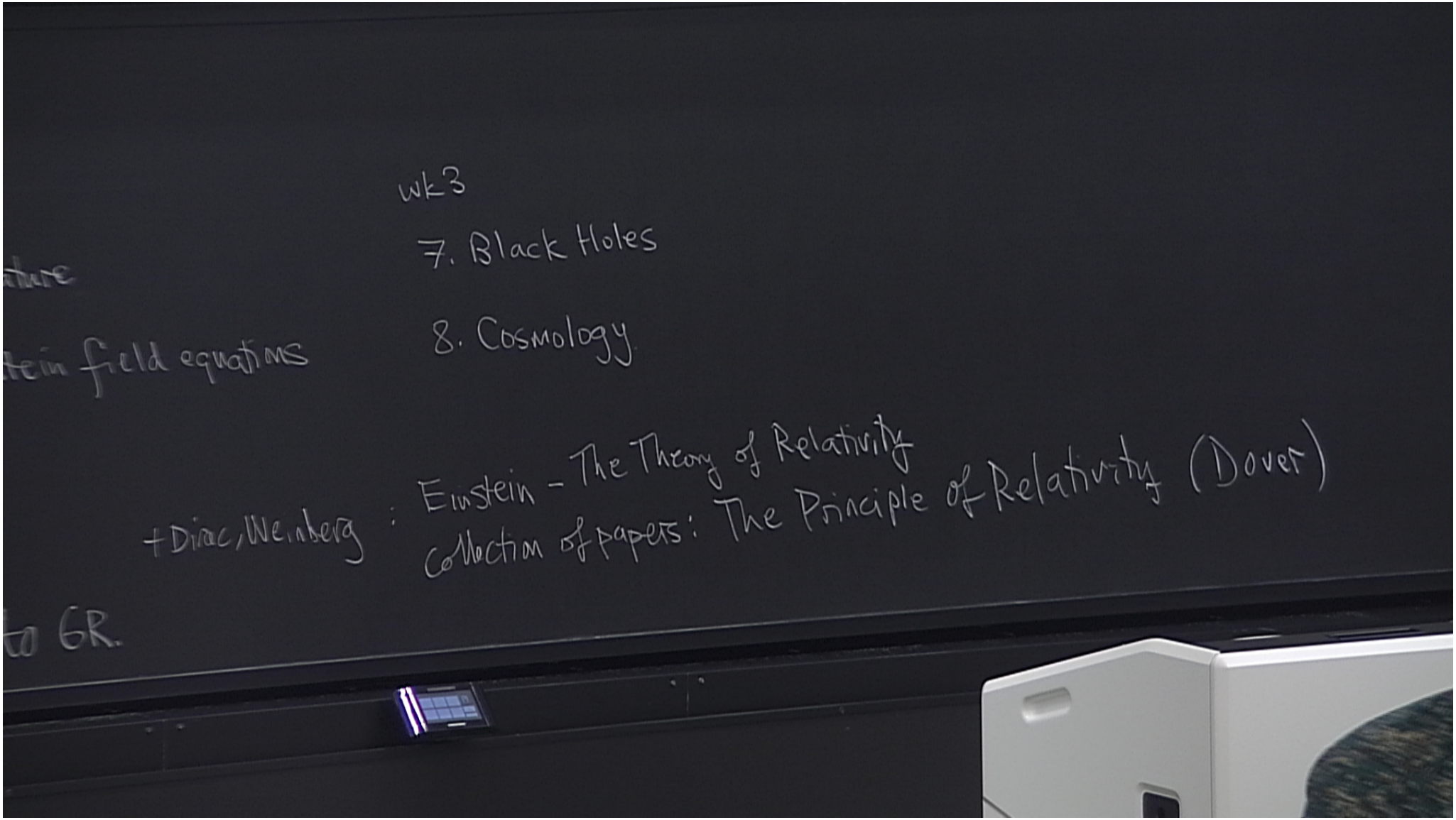
Date: Sep 08, 2014 09:00 AM

URL: http://pirsa.org/14090007

Abstract:


Relativity
wk 2
4 curvature
5 Einstein field equations
6 Tests
wk 3
7. Black Holes
8. Cosmology
desics, gravity
ariant derivatives


Relativity
wk 1
1 SR + Maxwell
2. Particles, geodesics, gravity
3. Tensors, covariant derivatives
wk 2
4 curvature
5 Einstein field equations
6 Tests
wk 3
7.
8.
Books: A. Zee.
S. Carroll
L. Hughston, K.P. Tod: Introduction to GR.


wk3
7. Black Holes
8. Cosmology
tein field equations
+Dirac, Weinberg : Einstein - The Theory of Relativity
Collection of papers: The Principle of Relativity (Dover)
to GR.

Maxwell 1850's + 1860's

Gauss $\nabla \cdot \underline{E} = \rho / \epsilon_0$ ← charge density

$\nabla \cdot \underline{B} = 0$ } consistent

Faraday $\nabla \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

Ampere $\nabla \wedge \underline{B} = \mu_0 \underline{J} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$ ← current density; ← Maxwell's addition

inconsistent! electric charge conservation $\frac{\partial \rho}{\partial t} = -\nabla \cdot \underline{J}$

SI units

$\epsilon_0 = \frac{1}{4\pi} \cdot \frac{1}{9 \times 10^9}$

$\mu_0 \equiv 4\pi \times 10^{-7}$

$\mu_0 \epsilon_0 = \frac{1}{9 \times 10^{16}} = \frac{1}{c^2}$

$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$\underline{E} = (E_x, E_y, E_z)$

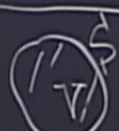
$\nabla \cdot \underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$(\nabla \wedge \underline{E})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} E_k \quad i,j,k = 1,2,3$

$\nabla \cdot (\nabla \wedge \underline{E}) = \partial_i \epsilon_{ijk} \partial_j E_k$



$I = \int_{S'} \underline{J} \cdot d\underline{S} = \frac{dQ}{dt}$



$Q = \int_V d^3x \, \rho$

$\frac{dQ}{dt} = -\int_{S'} \underline{J} \cdot d\underline{S}'$

Maxwell added a term to make equations consistent.

Noticed: there are interesting solutions even when $\rho, \underline{J} = 0$.

$$\frac{\partial^2 \underline{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0}\left(-\underline{\nabla} \times (\underline{\nabla} \times \underline{E})\right)$$

$$= \frac{1}{\mu_0 \epsilon_0}(\underline{\nabla}^2)\,\underline{E}$$

$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$\epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l E_m$$

$$= \partial_i(\underline{\nabla}\cdot\underline{E}) - (\underline{\nabla}^2)E_i$$

$$= 0$$

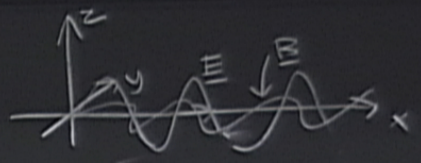
Wave equation for waves with speed $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3\times10^8\ ms^{-1}$!

Wave solutions e.g. plane wave polarized in z dirn

(arb. function)

$$\underline{E} = (0, 0, E_z(x-ct))$$

$$\underline{B} = \left(0, -\frac{E_z}{c}(x-ct), 0\right)$$



Maxwell equations have an amazing symmetry connecting space and time.

$x^\alpha = (ct, x_i)$ $\quad x_1 = x,\ x_2 = y,\ x_3 = z$

$\alpha = 0, 1, 2, 3.$

$\frac{\partial}{\partial x^\alpha} = \partial_\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right)$

$\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$ etc.

$\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad (0, 1, 2, 3)$

$\eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \underline{\nabla}^2$ wave operator

inverse of $\eta^{\alpha\beta}$: $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad \eta^{\alpha\beta}\eta_{\beta\delta} = \delta^\alpha_\delta$

Combine $\underline{E}$ and $\underline{B}$ into one object

$$F_{\alpha\beta} = -F_{\beta\alpha}$$

6 indept components



$$F_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

i.e. $F_{oi} = -E_i/c$

$F_{ij} = \epsilon_{ijk} B_k$

field strength tensor

Relativity

wk 1

wk 2

wk 3

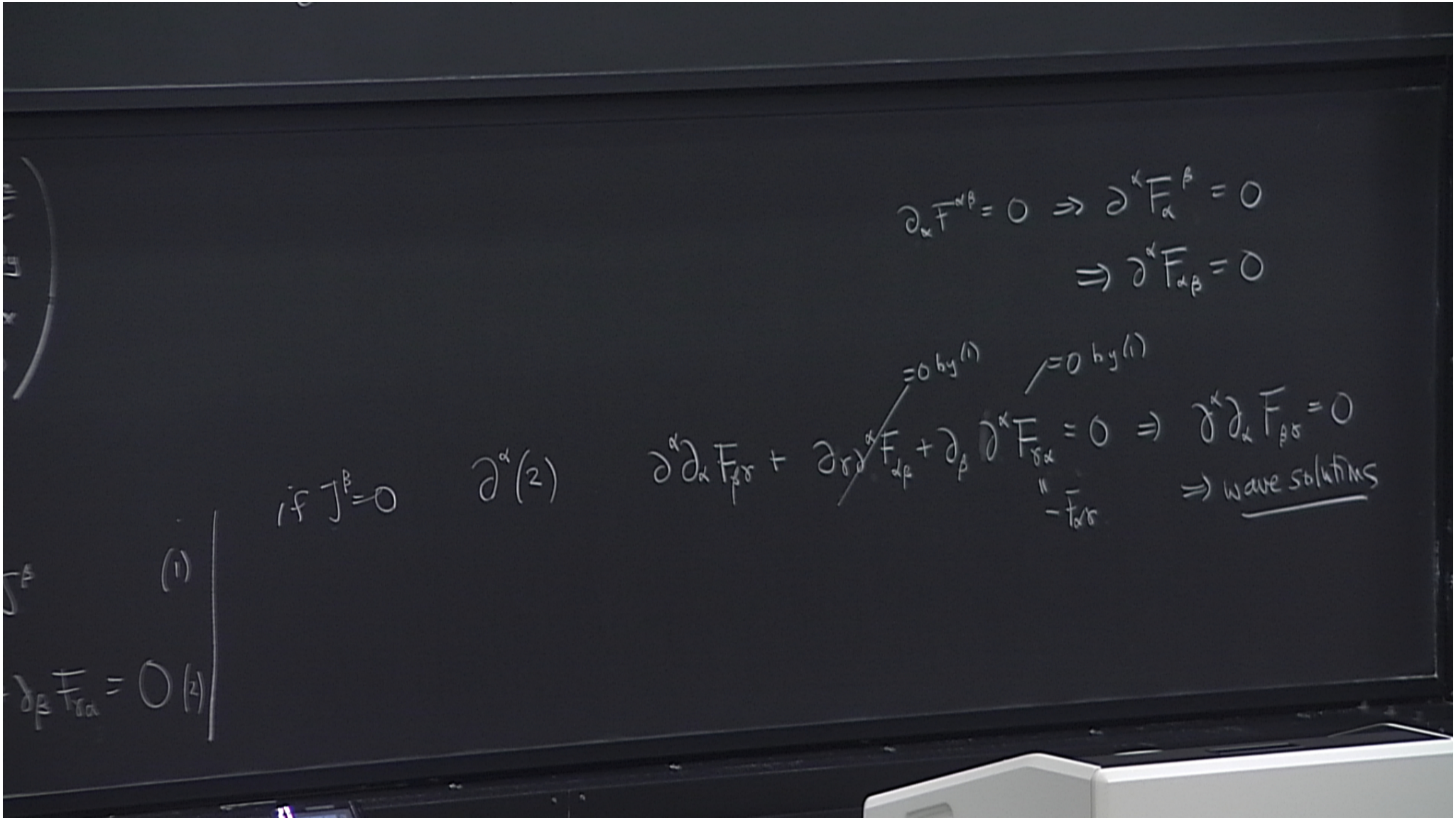
7 Black Holes

likewise $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$

define 4-vector current $J^\alpha = (\rho c, \underline{J})$

Then Maxwell's equations are: $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$

$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$


$\partial_\alpha F^{\alpha\beta} = 0 \Rightarrow \partial^\alpha F_\alpha{}^\beta = 0$
$\Rightarrow \partial^\alpha F_{\alpha\beta} = 0$
$\partial^\alpha (2)$
$\partial^\alpha \partial_\alpha F_{\beta\gamma} + \partial_\gamma \partial^\alpha F_{\alpha\beta} + \partial_\beta \partial^\alpha F_{\gamma\alpha} = 0 \Rightarrow \partial^\alpha \partial_\alpha F_{\beta\gamma} = 0$
$= 0$ by (1)
$= 0$ by (1)
$= -F_{\alpha\gamma}$
$\Rightarrow$ wave solutions
if $J^\beta = 0$
(1)
$\partial_\beta F_{\gamma\alpha} = 0$ (2)

Check (2), $\alpha\beta\gamma = i,j,k$

$$\frac{\partial}{\partial x_j} A = A_{,j}$$

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0$$

$$\frac{\partial}{\partial x_k} F_{ij}$$

contract w/ ϵ_{ijk} : 1st term $\epsilon_{ijk} F_{ij,k} = \epsilon_{ijk}\epsilon_{ije} B_{e,k}$

$$= 2\nabla\cdot\underline{B}$$

all three terms give same

$$\Rightarrow \nabla\cdot\underline{B} = 0$$

$$\epsilon_{ijk} F_{ki,j} = -\epsilon_{ikj} F_{ki,j} = +\epsilon_{ikj} F_{ik,j}$$

$$\epsilon_{ijk}\epsilon_{ije} = \delta_{jj}\delta_{ke} - \delta_{je}\delta_{kj}$$

$$= 3\delta_{ke} - \delta_{ke}$$

$$= 2\delta_{ke}$$

Check (2), $\alpha\beta\gamma = ijk$

$$\frac{\partial}{\partial x^i} A = A_{,i}$$

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0$$

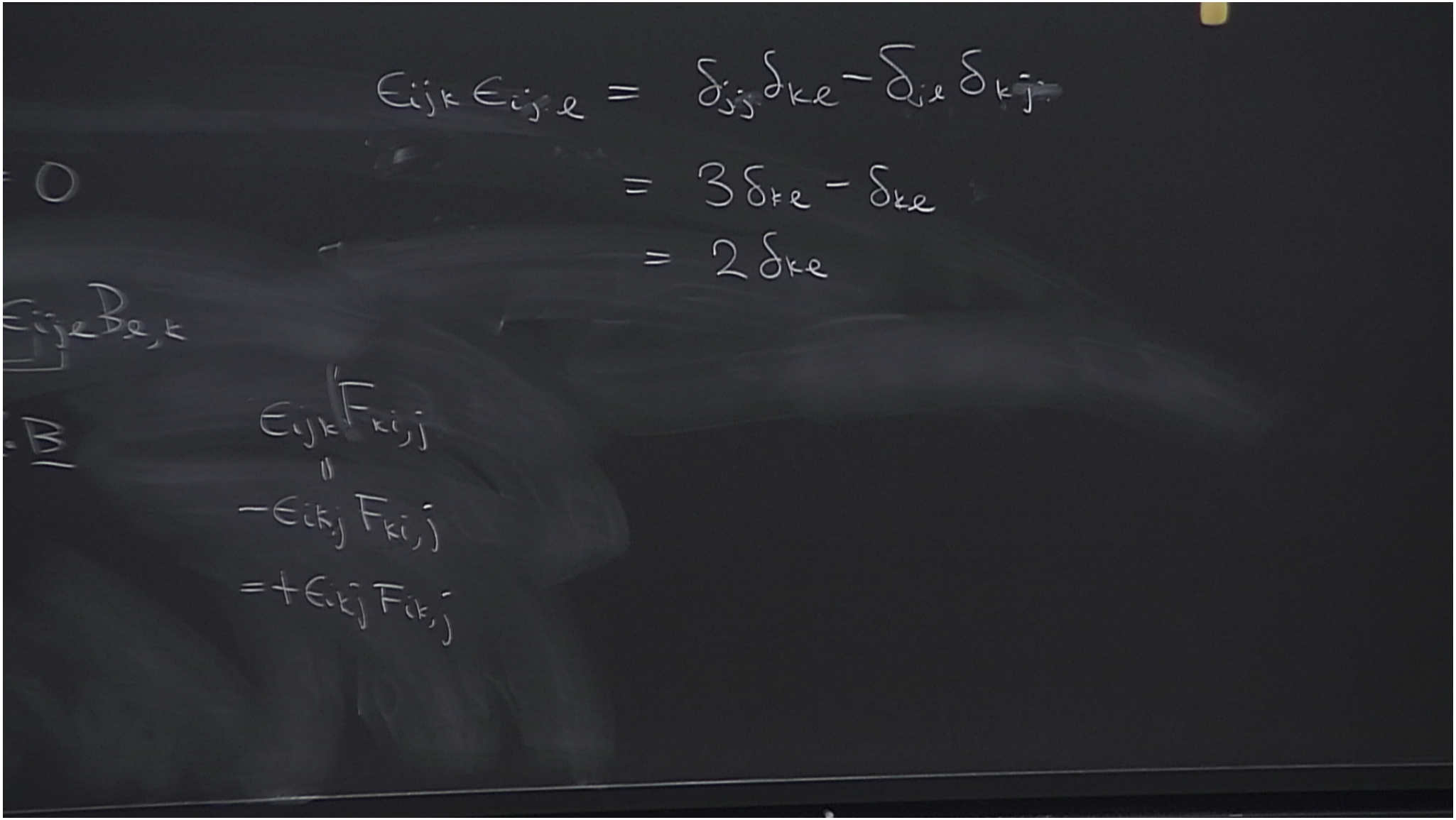
$\frac{\partial}{\partial x^k} F_{ij}$ "

contract w/ ϵ_{ijk} : 1st term. $\epsilon_{ijk} F_{ij,k} = \epsilon_{ijk}\epsilon_{ije} B_{e,k}$

$= 2\,\underline{\nabla}\cdot\underline{B}$

all three terms give same

$\Rightarrow \underline{\nabla}\cdot\underline{B} = 0$


$\epsilon_{ijk}\epsilon_{ij\ell} = \delta_{jj}\delta_{k\ell} - \delta_{j\ell}\delta_{kj}$
$= 3\delta_{k\ell} - \delta_{k\ell}$
$= 2\delta_{k\ell}$
$= 0$
$\epsilon_{ij\ell}B_{\ell,k}$
$\cdot B$
$\epsilon_{ijk}F_{ki,j}$
$-\epsilon_{ikj}F_{ki,j}$
$= +\epsilon_{ikj}F_{ik,j}$

Hughston, K.T. 06d: Introduction to GR

(2) again, but $\alpha\beta\gamma = 0jk$

$$F_{0j,k} + F_{k0,j} + F_{jk,0} = 0$$

$$\frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$$

$$-\frac{E_{j,k}}{c} + \frac{F_{kj}}{c} + \epsilon_{jkl}\frac{\partial B_l}{c\,\partial t}$$

$\times \epsilon_{ijk}$

$$2\frac{(\nabla \wedge E)_i}{c} + \frac{2}{c}\frac{\partial B_i}{\partial t} \Rightarrow \quad \nabla \wedge E = -\frac{\partial B}{\partial t}$$

4-vector potential $A^\mu = \left(\frac{\phi}{c}, \underline{A}\right)$; $A_\mu = \left(-\frac{\phi}{c}, \underline{A}\right)$

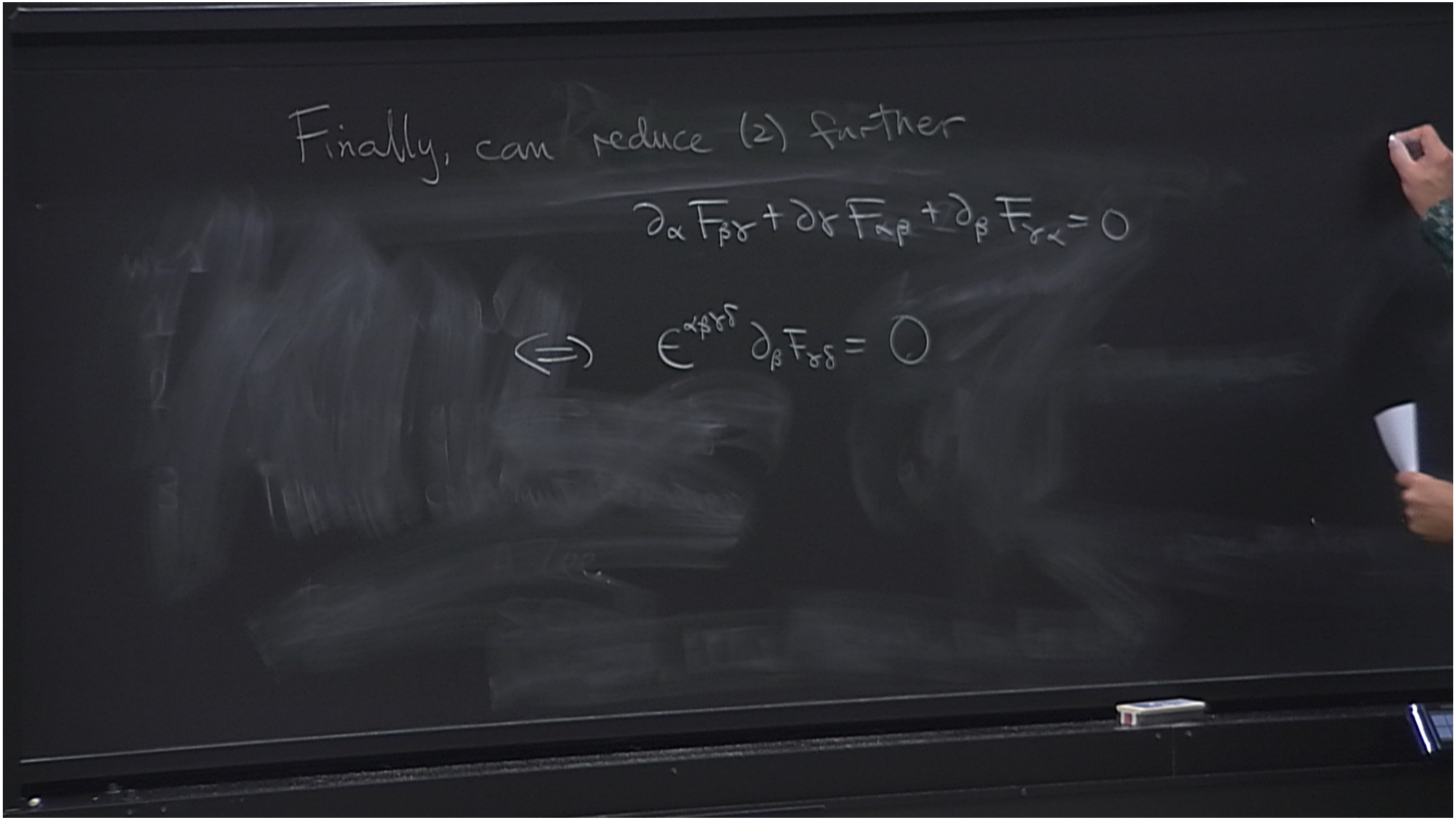
then $\underline{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$

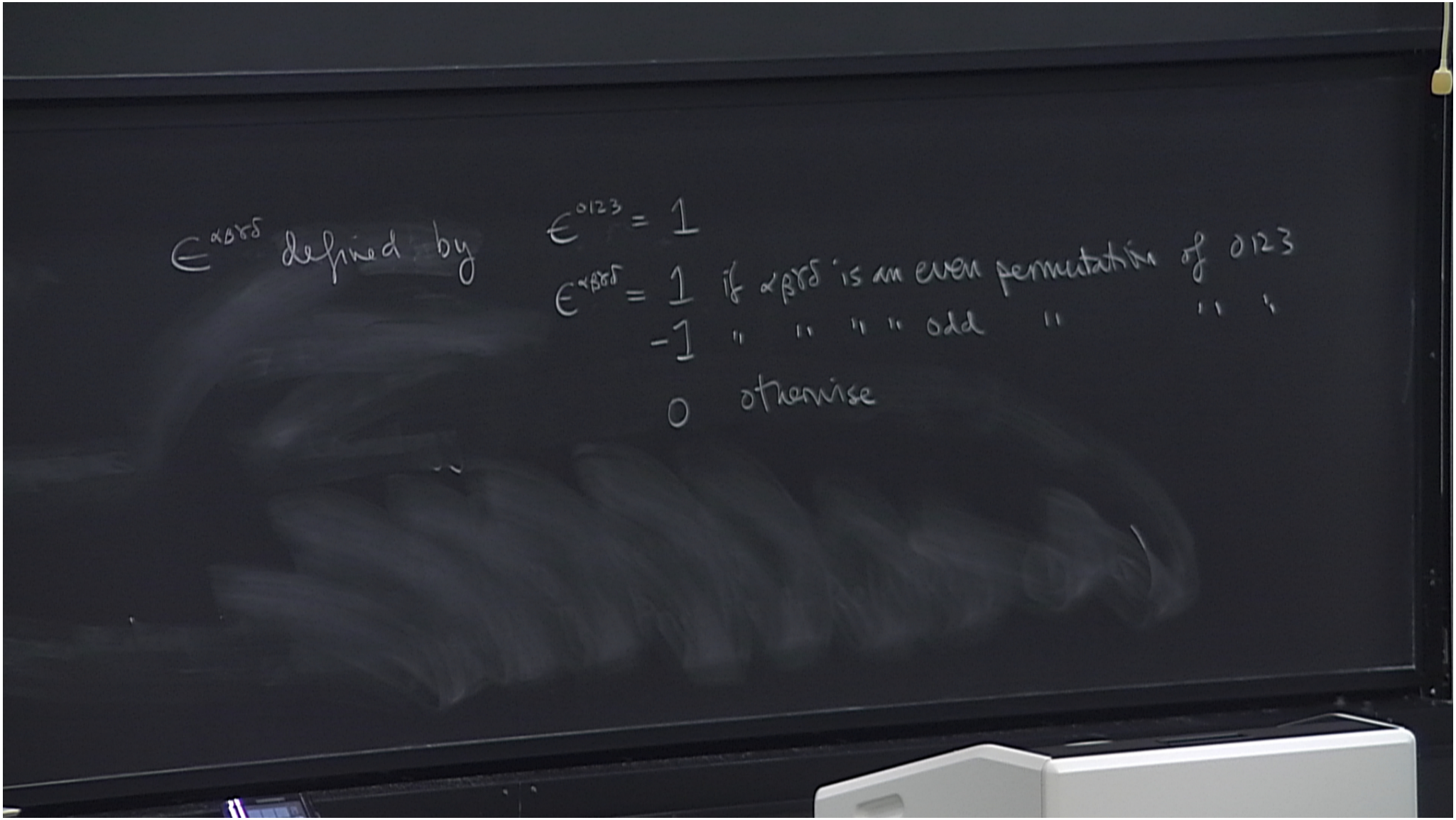
$F_{ij} = \epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i$

$\times \epsilon_{ije}$ $\quad 2B_\ell = \epsilon_{ije}(\partial_i A_j - \partial_j A_i) = 2\epsilon_{\ell ij}\partial_i A_j$

$\Rightarrow \underline{\underline{B} = \nabla \wedge \underline{A}}$

likewise can show that $\underline{\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t}}$


Finally, can reduce (2) further
$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$
$\Leftrightarrow \quad \epsilon^{\alpha\beta\gamma\delta}\, \partial_\beta F_{\gamma\delta} = 0$


$\epsilon^{\alpha\beta\gamma\delta}$ defined by
$\epsilon^{0123} = 1$
$\epsilon^{\alpha\beta\gamma\delta} = 1$ if $\alpha\beta\gamma\delta$ is an even permutation of 0123
-1 " " " " odd " " "
0 otherwise

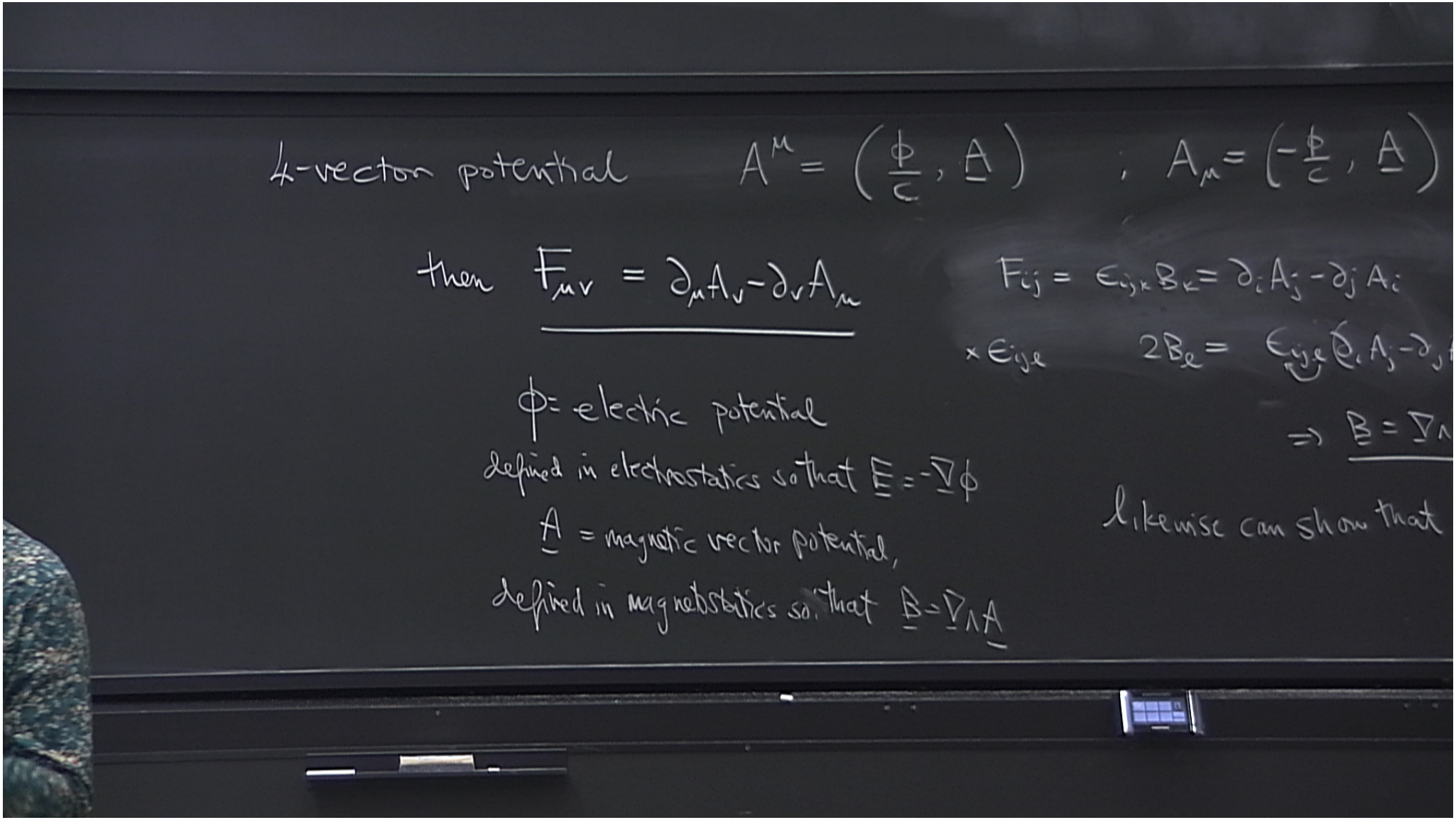
Finally, can reduce (2) further

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$$

$$\Longleftrightarrow \quad \epsilon^{\alpha\beta\gamma\delta}\, \partial_\beta F_{\gamma\delta} = 0$$

Summarise: we have reduced all of electrodynamics to

$$\boxed{\begin{aligned} \partial_\alpha F^{\alpha\beta} &= -\mu_0 J^\beta \\ \epsilon^{\alpha\beta\gamma\delta}\, \partial_\beta F_{\gamma\delta} &= 0 \end{aligned}}$$


4-vector potential $A^\mu = \left(\frac{\phi}{c}, \underline{A}\right)$; $A_\mu = \left(-\frac{\phi}{c}, \underline{A}\right)$
then $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$F_{ij} = \epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i$
$\times \epsilon_{ijl}$ $2B_l = \epsilon_{ijl} \partial_i A_j - \partial_j$
$\Rightarrow \underline{B} = \nabla \wedge$
ϕ = electric potential
defined in electrostatics so that $\underline{E} = -\nabla\phi$
$\underline{A}$ = magnetic vector potential,
defined in magnetostatics so that $\underline{B} = \nabla \wedge \underline{A}$
likewise can show that

Finally, can reduce (2) further

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$$

$$\Leftrightarrow \quad \epsilon^{\alpha\beta\gamma\delta}\,\partial_\beta F_{\gamma\delta} = 0$$

Summarise: we have reduced all of electrodynamics to

$$\boxed{\begin{aligned} &\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \\ &\epsilon^{\alpha\beta\gamma\delta}\,\partial_\beta F_{\gamma\delta} = 0 \end{aligned}} \Longrightarrow \partial_\beta J^\beta = 0$$