Title: Short-Range Entangled Phases and Topology

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Abstract: Recently a new and rather unexpected connection between condensed matter physics and algebraic topology has been noted. Namely, it appears that phases of matter with an energy gap, no long-range entanglement, and fixed symmetry can be classified using cobordism theory. I will exhibit several examples of this connection and describe a possible explanation.

Outline

- Topological phases of matter
- Free fermionic SRE phases
- Thermal Hall response
- Bosonic SRE phases and group cohomology
- Bosonic SRE phases and cobordisms
- Fermionic SRE phases and cobordisms
- Heuristic explanation

Based on arXiv:1403.1467 and 1406.7329.

Gapped phases of matter

In grade school we learn about the following phases: solid, liquid, gas.

Actually, liquid $\simeq$ gas, and there are many crystalline solids distinguished by their symmetries.

At low temperatures, the basic feature is the presence/absence of a gap between the ground state and the 1st excited state.
If the gap is nonzero even for an infinite system, the phase is called gapped.

Examples of gapless phases: crystals, superfluid ^4He, Fermi liquid.

Examples of gapped phases: broken discrete symmetry, Quantum Hall phases, confining and Higgs phases of gauge theories.

Topological Phases of Matter

Old viewpoint: gapped phases of matter are "boring" because they all look the same at long distances/time-scales.

Modern viewpoint: there is a variety of topological phases of matter which are gapped.

To distinguish them one can either consider a nontrivial spatial topology or to look at the edge physics.

Example: Fractional Quantum Hall states. (Ground-state degeneracy on a space of nontrivial topology, gapless edge modes). IR physics is described by a nontrivial 3d TQFT.

Short-Range Entangled Phases

SRE phases are gapped phases such that the vacuum state does not have long-range entanglement.

There is a unique vacuum for any spatial topology, so the TQFT is almost trivial ("invertible").

Nevertheless, there are distinct SRE phases. One would like to classify such phases up to homotopy.

More generally, can study homotopy classes of SRE phases with a fixed symmetry group G.

Free Fermionic SRE phases

SRE phases of free fermions have been classified (Schnyder et al 2008, Kitaev 2009) under a somewhat stronger equivalence relation: two SRE phases are equivalent if they can be deformed into another after perhaps adding some "trivial" electronic states.

The resulting classification is related to K-theory and Clifford algebras.

Case 1: no symmetry except $(-1)^F$.

d	1	2	3	4	5	6	7	8
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

Case 2: time-reversal symmetry T, $T^2 = (-1)^F$.

d	1	2	3	4	5	6	7	8
	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Case 3: time-reversal symmetry T, $T^2 = 1$.

d	1	2	3	4	5	6	7	8
	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

Free fermionic SRE phases, cont

Case 4: $U(1)$ symmetry

d	1	2
	$\mathbb{Z}$	0

Cases 1-3 are related to Clifford algebras over $\mathbb{R}$ and exhibit periodicity modulo 8.

Case 4 is related to Clifford algebras over $\mathbb{C}$ and exhibits periodicity modulo 2.

The $d = 1$ case

The $d = 1$ case can be analyzed "by hand".

- Only $(-1)^F$: ground state can be bosonic $(-1)^F = 1$ or fermionic $(-1)^F = -1$.
- Time-reversal symmetry T, $T^2 = (-1)^F$: fermionic states come in pairs ("Kramers doublets"), hence the ground state must be bosonic.
- $T^2 = 1$: ground state can be bosonic or fermionic.
- $U(1)$: the ground state has electric charge $Q \in \mathbb{Z}$, $(-1)^F = (-1)^Q$.

The $d = 2$ case

- Only $(-1)^F$: the 1d edge may carry a single Majorana fermion (with zero energy) . Two of them can "pair up" to give a trivial SRE phase. Equivalently: a single massive Majorana fermion in 2d can have $m > 0$ or $m < 0$.
- Time-reversal symmetry T, $T^2 = (-1)^F$: a pair of Majorana zero modes (Kramers doublet) on the edge.
- $T^2 = 1$: the 1d edge may carry Majorana zero modes which cannot pair up.
- $U(1)$: edge fermions must be complex (Dirac), can always gap them out.

Periodicity

Free fermionic SRE phases are related to Morita-equivalence classes of real and complex) Clifford algebras (Kitaev 2009).

Real (resp. complex) Clifford algebras exhibit $\mathrm{mod}\ 8$ (resp. $\mathrm{mod}\ 2$) periodicity (Bott periodicity). Hence the classification of free fermionic SRE phases with a fixed symmetry exhibits periodicity in d.

Mendeleev (1869) showed that chemical elements exhibit approximate $\mathrm{mod}\ 8$ periodicity (not related to Bott periodicity, as far as I know). But as the atomic number increases, new phenomena arise (radioactivity, instability, etc.).

We will see that once interactions are included, periodicity in d is destroyed: for large d there are more phases than for small d.

Quantum Hall response

If a system has a $U(1)$ symmetry, it can be coupled to a background electromagnetic field $A = A_\mu dx^\mu$.

In 3d, one can ask if constant electric field induces a current in the perpendicular direction:

$$J_x = \frac{1}{2\pi}\sigma_{xy}E_y, \quad E_y = \partial_0 A_y - \partial_y A_0.$$

σ_{xy} is called Hall conductivity. In an SRE phase it is quantized.

One can encode quantum Hall response by an effective action:

$$J^\mu = \frac{\delta S_{eff}}{\delta A_\mu}, \qquad S_{eff} = \frac{\sigma_{xy}}{4\pi}\int AdA.$$

Similarly, for any odd d we can write down a $U(1)$ Chern-Simons action encoding "quantum Hall response". Its coefficient is an integer.

Thermal Hall response

If $U(1)$ symmetry is absent, σ_{xy} is not defined. But we can always study thermal Hall conductivity: energy flux in the direction perpendicular to the thermal gradient.

This flux is not associated with dissipation and can also be described by S_{eff}. S_{eff} now depends on the metric. In the IR limit, it is invariant under rescaling and depends only on the Levi-Civita connection ω. In 3d the action is

$$S_{eff} = \frac{\kappa}{4\pi} \int \mathrm{Tr} \left(\omega d\omega + \frac{2}{3}\omega^3 \right), \quad \kappa \in \mathbb{Z}.$$

More generally, gravitational Chern-Simons terms can be written if $d = 4k - 1$.

For free fermion SREs in 3d with $U(1)$ symmetry, κ and σ_{xy} are proportional. Equivalently, edge states are free chiral fermions with unit $U(1)$ charge, so a single integer suffices to characterize the phase.

Bosonic SRE phases

Bosons tend to form bose-condensates, to get a nontrivial phase need to understand interacting systems of bosons.

Case-by-case studies suggest the following classification in low dimensions:

Case 1: no symmetry.

d	1	2	3	4
	0	0	$\mathbb{Z}$	0

Case 2: time-reversal symmetry T, $T^2 = 1$.

d	1	2	3	4
	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Case 3: $U(1)$ symmetry

d	1	2	3	4
	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0

Some questions

1. Can one find a classification scheme for bosonic SRE phases valid for all dimensions and all symmetry groups?

2. Can one find a similar classification for fermionic SRE phases which is valid for strongly interacting systems?

3. Are there fermionic SRE phases whose existence depends on interactions?

A partial answer to the first question is provided by the group cohomology classification.

Group cohomology

For any finite group G one can define BG, the classifying space of G-bundles. It is a space of type $K(G, 1)$, that is, $\pi_1(BG) = G$, $\pi_i(BG) = 0$ for $i > 1$.

E.g., a model for $B\mathbb{Z}_2$ is $\mathbb{RP}^\infty$.

A principal G-bundle on any X is a pull-back of a universal principal G-bundle EG on BG.

Group cohomology of G with coefficients in an abelian group K is defined to be $H^*(BG, K)$.

More generally, let K be an abelian group with an action ρ of G. It gives rise to a flat bundle K^ρ over BG with fiber K. Group cohomology of G with coefficients in (K, ρ) is $H^*(BG, K^\rho)$.

Group cohomology classification of bosonic SREs

Chen, Gu, Liu & Wen (2012) proposed a general classification scheme for bosonic SRE phases based on group cohomology.

Let G be a finite internal symmetry. Chen et al. proposed that in d space-time dimensions bosonic SRE phases with symmetry G are classified by $H^d(BG, U(1))$.

Let G be a finite symmetry involving time-reversal. Thus we are given a homomorphism $\rho : G \to \mathbb{Z}_2$. Chen et al. proposed that bosonic SRE phases are classified by $H^d(BG, U(1)^\rho)$. Here the action of G on $U(1)$ is obtained by combining the homomorphism $G \to \mathbb{Z}_2$ and the action of $\mathbb{Z}_2$ on $U(1)$ given by complex conjugation.

Why group cohomology?

For G not involving time-reversal, there is a simple explanation.

We can promote G to a gauge symmetry and couple the system to a background gauge field A. It is a (flat) G-connection on a principal G-bundle.

Integrating out the "matter", we get an effective action for A. If we assume that this action is topological, possible actions in dimension d are classified by $H^d(BG, U(1))$ (Dijkgraaf and Witten 1990).

The case with time-reversal symmetry is a bit more complicated.

Comments

1. The assumption that the action is topological only applies to phases with zero thermal Hall response.
2. If G is not finite, Chen et al. proposed to use Borel group cohomology of G. For $G = U(1)$ this is more or less the same as regarding $U(1)$ as a limit of $\mathbb{Z}_N$ for $N \to \infty$.
3. There is a version of the proposal for fermionic systems ("group supercohomology") Gu & Wen 2012.
4. For $d = 1$, the statement is merely that the SRE phase is characterized by the way G acts on the (unique) ground state. Obviously true.
5. For $d = 2$, it has been proved that bosonic SRE phases are labeled by $H^2(BG, U(1)^\rho)$ (Chen, Gu, Wen 2011, Kitaev & Fidkowski 2011).
6. For bosonic SRE phases in $d = 4$ with time-reversal symmetry the group cohomology proposal gives $\mathbb{Z}_2$, while the correct answer appears to be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (Vishwanath & Senthil 2012).

Bosonic SREs with time-reversal symmetry

Let us look again at the bosonic SRE phases with time-reversal symmetry and compare with unoriented bordisms of a point $\Omega_d^O(pt)$:

d	1	2	3	4	5	6	7	8
	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$

Perhaps the correct classification should rely on cobordism groups of BG rather than its cohomology groups (A.K. 2014).

In fact, a footnote in Dijkgraaf-Witten (1990) mentions that if the space-time is a manifold rather than a general topological space, it is more natural to consider cobordisms of BG.

Bordism groups of a point

Two closed d-manifolds M_1 and M_2 are called cobordant if there exists a $d+1$-dimensional compact manifold N with boundary $M_1 \sqcup M_2$. This is an equivalence relation.

The set of cobordism classes of closed d-dimensional manifolds is called the unoriented bordism group of a point, $\Omega_d^O(pt)$. Group operation is given by the disjoint union. These groups have been computed by R. Thom for all d.

If we require everything to be oriented, we get the oriented bordism group $\Omega_d^{SO}(pt)$. In low dimensions they are:

d	1	2	3	4	5	6	7	8
	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

Bordism groups of a space

These are special cases of the following construction (Atiyah 1960). Let X be a topological space with a $\mathbb{Z}_2$ local system ρ. If X is a manifold M, it has a canonical $\mathbb{Z}_2$ local system (the orientation bundle) which we denote ξ_M. If M is oriented, ξ_M is given a trivialization.

Define the group $\Omega_d(X,\rho)$ as follows: an element of $\Omega_d(X,\rho)$ is represented by a closed d-dimensional manifold M equipped with a map $f: M \to X$ and an isomorphism $F: f^*\rho \simeq \xi_M$. We consider triples (M,f,F) up to cobordism. Group operation is given by the disjoint union.

Example 1: If ρ is trivial, M must be oriented. Thus $\Omega_d(X,\rho) = \Omega_d^{SO}(X)$.

Example 2: If $X = B\mathbb{Z}_2$ and $\rho = E\mathbb{Z}_2$, then f is determined by M up to homotopy. Thus $\Omega_d(B\mathbb{Z}_2, E\mathbb{Z}_2) = \Omega_d^O(pt)$.

Gauged symmetries and bordisms of BG

Suppose we are given a finite symmetry group G and a homomorphism $\rho : G \to \mathbb{Z}_2$. If ρ is trivial, G consists of internal symmetries only. Since G does not contain orientation-reversing elements, the space-time M on which the SRE phase lives can be given an orientation.

Gauging G means choosing a G gauge field A on M. Equivalently, A is a map $M \to BG$. The pair (M, A) thus defines an element of $\Omega_d^{SO}(BG)$.

If ρ is nontrivial, G contains time-reversing elements. Gauging such a symmetry means allowing M to be unorientable. The G gauge field A now satisfies a constraint: $\rho(A)$ must be a connection on the orientation bundle ξ_M.

Equivalently, if A is regarded as a map $M \to BG$, then $\rho \circ A : M \to B\mathbb{Z}_2$ must be such that $(\rho \circ A)^* E\mathbb{Z}_2 \simeq \xi_M$. Thus (M, A) defines an element of $\Omega_d(BG, \rho)$.

Cobordisms with $U(1)$ coefficients

Group cohomology classification makes use of $H^d(BG, U(1)) = \mathrm{Hom}(H_d(BG), U(1))$.

Similarly, we define oriented cobordisms with $U(1)$ coefficients as

$$\Omega^d_{SO}(BG, U(1)) = \mathrm{Hom}(\Omega^{SO}_d(BG), U(1)).$$

More generally, we define

$$\Omega^d(BG, U(1)^\rho) = \mathrm{Hom}(\Omega_d(BG, \rho), U(1)).$$

Cobordisms with $U(1)$ coefficients form a generalized cohomology theory. There is a map

$$H^d(BG, U(1)^\rho) \to \Omega^d(BG, U(1)^\rho).$$

The cobordism proposal for bosonic SREs

For trivial ρ, we propose that bosonic SRE phases with symmetry G are classified by the Pontryagin dual of the torsion part of $\Omega_d^{SO}(BG)$ (A.K. 2014).

That is, if $\Omega_d^{SO}(BG)$ contains a free part, then $\Omega_{SO}^d(BG, U(1))$ contains $U(1)$ factors, and we quotient by them.

This proposal applies to bosonic SRE phases with trivial thermal Hall response and finite G.

If ρ is nontrivial, $\Omega_d(BG, \rho)$ has no free part, so we propose that bosonic SRE phases are classified by $\Omega^d(BG, U(1)^\rho)$.

Thermal Hall response must vanish in this case because gravitational Chern-Simons terms break T.

The cobordism proposal for fermionic SREs

For simplicity, let us consider the following three cases (R. Thorngren, A. Turzillo, Z. Wang, A.K. 2014).

Case 1: No symmetry (except $(-1)^F$). Use the Pontryagin-dual of the torsion part of $\Omega_d^{Spin}(pt)$ and include thermal Hall conductivities for $d = 3$ and $d = 7$.

d	1	2	3	4	5	6	7	8	9
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$

$$\Downarrow$$

d	1	2	3	4	5	6	7	8	9
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$

Note: in $d = 7$ there are two distinct gravitational Chern-Simons terms, because in $d = 8$ there are two Pontryagin densities:

$$p_2 \sim \mathrm{Tr}\, R^4, \quad p_1^2 \sim (\mathrm{Tr}\, R^2)^2$$

The cobordism proposal for fermionic SREs, cont

Case 2: Time-reversal symmetry T, $T^2 = (-1)^F$. Use $\Omega^d_{Pin^+}(pt)$.

d	1	2	3	4	5	6	7	8	9
	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	0

Case 3: Time-reversal symmetry T, $T^2 = 1$. Use $\Omega^d_{Pin^-}(pt)$.

d	1	2	3	4	5	6	7	8	9
	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$

A reminder on Pin and Spin groups

Spin(d) is an extension of $SO(d)$ by $\mathbb{Z}_2$:

$$\{\pm 1\} \to Spin(d) \to SO(d).$$

There are two natural extensions of $O(d)$ by $\mathbb{Z}_2$ called $Pin^+(d)$ and $Pin^-(d)$.

- Pin^+: a reflection $P \in O(d)$ lifts to $\tilde{P} \in Pin^+(d)$ such that $\tilde{P}^2 = 1$.
- Pin^-: a reflection $P \in O(d)$ lifts to $\tilde{P} \in Pin^-(d)$ such that $\tilde{P}^2 = -1$.

Comments on the fermionic case

1. Reproduces known results about interacting fermionic SRE phases.
2. Predicts new interacting fermionic SRE phases for $d > 6$.
3. Disagrees with "group supercohomology" for $d = 4$ and $T^2 = 1$.
4. Some fermionic SREs are equivalent to bosonic SREs; this corresponds to known maps from SO cobordisms to Spin cobordisms and from O cobordisms to Pin cobordisms.

What does it all mean?

Consider an SRE phase on a closed d-manifold M with a G-gauge field A. The partition function is a complex number $\exp(iS_{eff}(M,A))$, where S_{eff} is real.

- If the thermal Hall response vanishes, S_{eff} depends only on the topology of M and A.
- S_{eff} is "local" and therefore additive under disjoint union.
- S_{eff} changes sign under orientation reversal (CPT).

If S_{eff} is cobordism-invariant, $\exp(iS_{eff})$ becomes a function $\Omega_d(BG,\rho) \to U(1)$

If we classify SRE phases up to homotopy, need to ignore continuous parameters in S_{eff}. This is equivalent to keeping only the torsion part of $\Omega_d(BG,\rho)$.

An effective action from cobordisms

Consider again bosonic SRE phases with time-reversal symmetry. R. Thom showed that the unoriented cobordism class of M is determined by its Stiefel-Whitney numbers. Thus

$$S_{eff} = \pi \int_M P(w_1, \ldots, w_d),$$

where $w_i \in H^i(M, \mathbb{Z}_2)$ is an SW class and P is a polynomial.

- $d = 1$: all SW numbers vanish $\rightarrow$ no nontrivial SRE phases.
- $d = 2$: $w_1^2 = w_2 \rightarrow$ unique nontrivial SRE with $S_{eff} \sim \int w_1^2$.
- $d = 3$: all SW numbers vanish $\rightarrow$ no nontrivial SRE phases.
- $d = 4$: two independent actions, $\int w_1^4$ and $\int w_2^2$.

Comments

- The effective action can be used to analyze the edge physics of the SRE phase.
- What if G is a compact Lie group?. Need to study the classifying space of flat G-connections.
- In the fermionic case the effective action does not look local.
- For example, for $d = 2$ and no symmetry, the only cobordism invariant of a spin structure is the Arf invariant (even vs. odd spin structures). It is not an integral over M of a 2-cochain.
- For $d = 4$ and time-reversal symmetry, $T^2 = (-1)^F$, the only cobordism invariant of a Pin^+ structure is the eta-invariant. It is also nonlocal.
- There is a lot to be understood here!