Title: Towards construction of a Wightman QFT in four dimensions

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Abstract: We prove that the $\lambda\phi^4_4$ quantum field theory on noncommutative Moyal space is, in the limit of infinite noncommutativity, exactly solvable in terms of the solution of a non-linear integral equation. The proof involves matrix model techniques which might be relevant for 2D quantum gravity and its generalisation to coloured tensor models of rank ≥ 3. Surprisingly, our limit describes Schwinger functions of a Euclidean quantum field theory on standard $\mathbb{R}^4$ which satisfy the easy Osterwalder-Schrader axioms boundedness, covariance and symmetry. We prove that the decisive reflection positivity axiom is, for the 2-point function, equivalent to the question whether or not the solution of the integral equation is a Stieltjes function. The numerical solution of the integral equation leaves no doubt that this is true for coupling constants $\lambda\in[-0.39,0]$.

Towards construction of a Wightman QFT in four dimensions

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

(based on joint work with Harald Grosse,
arXiv: 1205.0465, 1306.2816, 1402.1041 & 1406.7755)

Introduction

axiomatic settings for rigorous quantum field theories by

1. Wightman [1956]
2. Haag-Kastler [1964]
3. Osterwalder-Schrader [1974]

today: numerous examples in dimension 1,2,3;
not a single non-trivial example in 4 dimensions

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We have got a candidate:

- Construction of 4D Euclidean QFT is achieved (2012/13).
  Find phase transitions and critical phenomena.
- Osterwalder-Schrader axioms are under investigation.
  So far everything is OK.
- Non-triviality is open, but not impossible.
  Ideally, we can get the 4D-analogue of factorising S-matrices.
  In 2D, related to integrability [Kulish, 1976] and Yang-Baxter

Historical notes on 4D QFT

1. Perturbative argument that QED cannot exist as 4D QFT [Landau-Abrikosov-Khalatnikov, 1954]

   (this almost killed renormalisation theory)

2. Same argument (sign of β-function) for $\lambda\phi^4_4$.
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3. Asymptotic freedom in QCD
   [Gross-Wilczek, 1973]; [Politzer, 1973]

4. Construction of Yang-Mills theory is Millennium Prize problem.

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Having one example of a rigorously constructed 4D QFT, even with factorisation into 2-particle scattering, would be something. . .

Regularisation & renormalisation

1. We follow the Euclidean track, starting from a partition function.
2. To make this rigorous we need two regulators: finite volume and finite energy density.
3. Pass to quantities (densities and with certain normalised functions) which have infinite volume & energy limits.

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  Search for a regulator which has more (or very different) symmetry, so constraining that it completely solves the model.

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A regularisation of ϕ^4_4

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\;\;(-\Delta \qquad\qquad\qquad + \mu^2\quad)\phi+\frac{\lambda}{4}\phi\;\;\phi\;\;\phi\;\;\phi\Big)(x)$$

A regularisation of ϕ_4^4

$$S[\phi]= \int_{\mathbb{R}^4} \frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\ \ (-\Delta+\Omega^2(\qquad x)^2 + \mu^2 \quad)\phi + \frac{\lambda}{4}\phi\ \ \phi\ \ \phi\ \ \phi\Big)(x)$$

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$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

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matrix basis $f_{\underline{mn}}(x) = f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1) = 2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x) = 64\pi^2 V\delta_{\underline{mn}}$

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$$S[\Phi]=V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Bigg)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big),\qquad |\underline{m}|:=m_1+m_2\le\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.

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- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.
- need $V\to\infty$; stringy [Minwalla, van Raamsdonk & Seiberg, 1999]

More generally: field-theoretical matrix models

Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$
  for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

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Observe: $\mathcal{Z}$ is covariant, but not invariant under $\Phi \mapsto U\Phi U^*$:

$$0 = \int \mathcal{D}\Phi \left[E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J\right] \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$$

...choose E (but not J) diagonal, use $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

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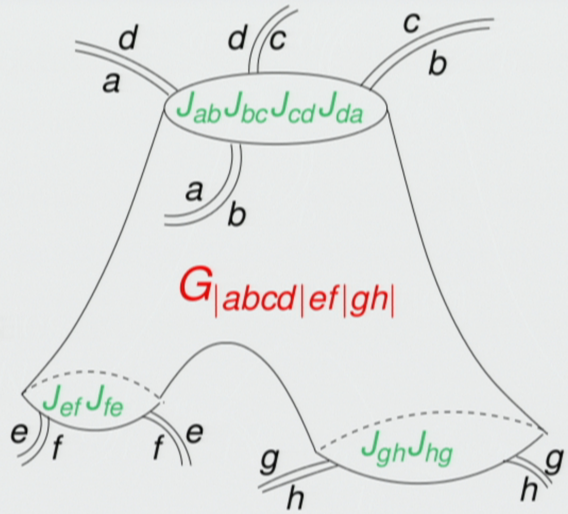
Ward identity [Disertori-Gurau-Magnen-Rivasseau, 2007]

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

Topological expansion

- Feynman graphs in matrix models are ribbon graphs.
- Encode genus-g Riemann surface with B boundary components.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



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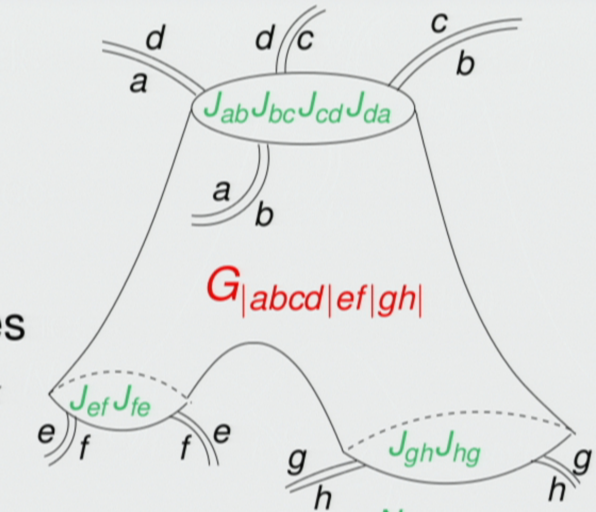

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- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure.
- The $G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$ become (smeared) Schwinger functions.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with (1) gluing and (2) symmetry.

For E of compact resolvent, the kernel of $E_p - E_a$ can be determined from the J-cycle structure in $\log \mathcal{Z}$:

Theorem (2012): Ward identity for E of compact resolvent

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$

$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)$$

$$+V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]$$

$$+\frac{V}{E_p - E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

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- J-derivatives of $\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E}$, where $\langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}$, lead to Schwinger-Dyson equations.
- The Theorem lets the usually infinite tower collape:

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

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2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- 2. uses reality $\mathcal{Z} = \overline{\mathcal{Z}}$
- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|}$ $\Rightarrow$ β-function zero!

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Digression: quantum gravity

We solved the quartic cousin of the Kontsevich model [1992]

$$\mathcal{Z}[E] = \frac{\int \mathcal{D}\Phi \, \exp\left(\mathrm{tr}(-\frac{1}{2}E\Phi^2 + \frac{\mathrm{i}}{6}\Phi^3)\right)}{\int \mathcal{D}\Phi \, \exp\left(\mathrm{tr}(-\frac{1}{2}E\Phi^2)\right)}$$

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1. Provides formulation of 2D quantum gravity:
   - proves Witten's conjecture about equivalence of two versions of 2D QG
   - generates ribbon graphs which are dual to triangulation of manifolds
   - quandrangulations (resulting from quartic model) should give the same result ...

Digression: quantum gravity

2. Higher-dimensional simplicial manifolds captured by coloured tensor models [Gurau, 2009]

- have analogue of $\frac{1}{N}$-expansion [Gurau, 2010]
- simplified models are renormalisable [Ben Geloun & Rivasseau, 2011], [Ben Geloun & Samary, 2012]
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   - became very active research field
3. Do the solution techniques generalise from matrix models to coulored tensor models?
   - first success: equation for 2-point function [Samary, 2014]
   - expect much more . . .

Back to $\lambda\phi_4^4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \cdots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \cdots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$

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- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform.*

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Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\operatorname{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{0\bullet}]}{G_{0a}}}\right)$ and G_{0b} solution of

$$G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{0\bullet}]}{G_{0p}}\right)^2}\right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}, \dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi_4^4$ on extreme Moyal space in terms of

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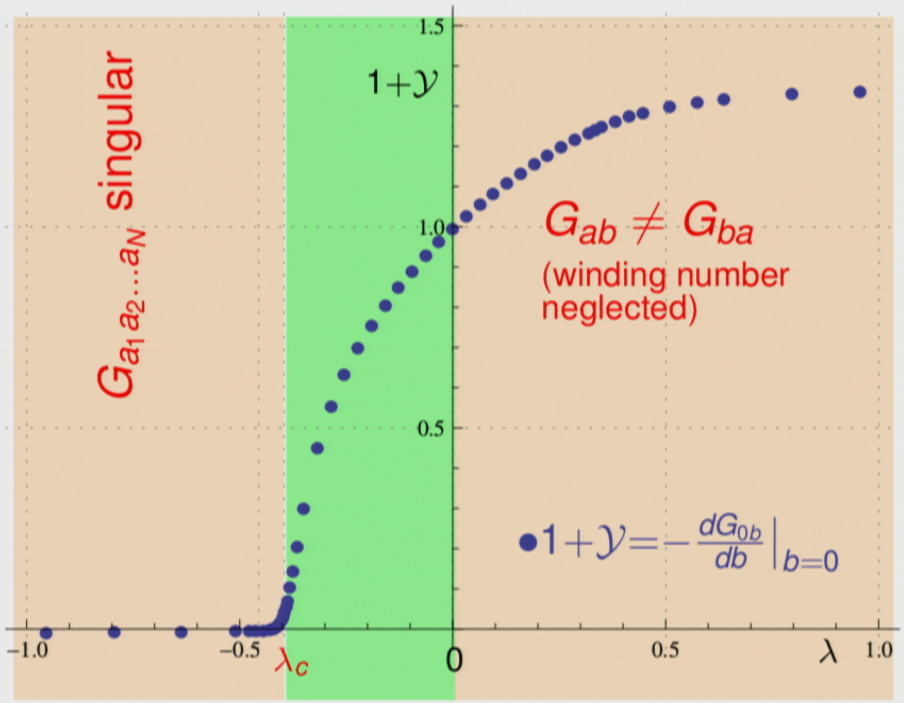
1. For $\lambda>0$ solution exists by Schauder fixed point theorem (but ambiguity due to winding number)
2. For $\lambda<0$ and $\Lambda^2\to\infty$ one exact solution is $G_{0b}=1$
3. Formula can be put on a computer and solved by iteration.
4. Shows that $G_{0b}=1$ is unstable, but attractive solution G_{0b} exists for all $\lambda\in\mathbb{R}$.

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2{=}10^7$ and 2000 sample points. Consider $1{+}\mathcal{Y}{:=}{-}\frac{dG_{0b}}{db}\big|_{b=0}$



- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda{=}\sup\{b \,:\, G_{0b}{=}1\}$ non-zero for $\lambda < \lambda_c$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation
- A key property for Schwinger functions is realised in subinterval of $[\lambda_c, 0]$, not outside!

Osterwalder-Schrader reconstruction theorem (1974)

Assume for Schwinger functions $S(x_1, ..., x_N)$:

- S0 growth rate: $\left|\int dx\, f(x_1, ..., x_N) S(x_1, ..., x_N)\right| \leq c_1 (N!)^{c_2} |f|_{Nc_3}$
- S1 Euclidean invariance: $S(x_1, ..., x_N) = S(Rx_1 + a, ..., Rx_N + a)$
- S2 reflection positivity: for each $(f_0, ..., f_K)$ with $f_N \in \mathcal{S}(\mathbb{R}^{ND})$,

$$\sum_{M,N=0}^{K} \int dxdy\; S(x_N, ..., x_1, y_1, ...y_M) \overline{f_N(^r x_1, ..., ^r x_N)} f_M(y_1, ..., y_M) \geq 0$$

where $^r(x^0, x^1, ...x^{D-1}) := (-x^0, x^1, ...x^{D-1})$

- S3 permutation symmetry: $S(x_1, ..., x_N) = S(x_{\sigma(1)}, ..., x_{\sigma(N)})$

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$$\sum_{M,N=0}^{K} \int dxdy\, S(x_N, ..., x_1, y_1, ...y_M) \overline{f_N(\theta x_1, ..., \theta x_N)} f_M(y_1, ..., y_M) \geq 0$$

where $\theta(x^0, x^1, ...x^{D-1}) := (-x^0, x^1, ...x^{D-1})$

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Then the $S(\xi_1, ..., \xi_{N-1})\big|_{\xi_i^0 > 0}$, with $\xi_i = x_i - x_{i+1}$, are inverse Laplace-Fourier transforms of FT $\hat{W}(q_1, ..., q_{N-1})$ of Wightman distributions in a relativistic QFT.

If in addition the $S(x_1, ..., x_N)$ satisfy

- S4 clustering

then the Wightman QFT has a unique vacuum state

From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0} \cdots$

Theorem (2013): *connected* Schwinger functions

$$S_c(\mu x_1, \dots, \mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \Big)$$

$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

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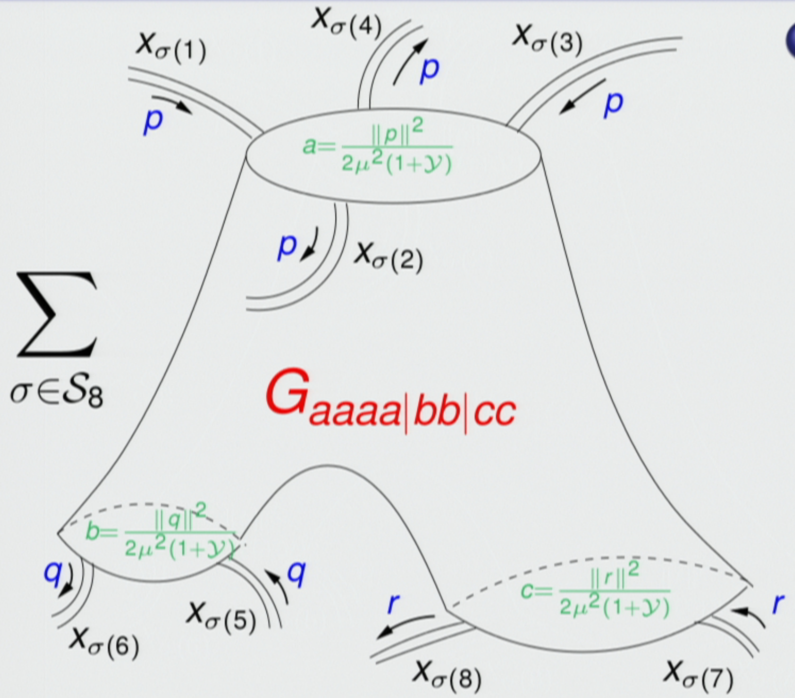
- Schwinger functions are symmetric S3 and invariant under full Euclidean group S1 (completely unexpected for NCQFT)
- growth conditions S0 established
- clustering S4 is violated: The $(N_1+\ldots+N_B)$-point functions are insensitive to the distance of different boundaries.
- remains: reflection positivity S2

Connected (4+2+2)-point function



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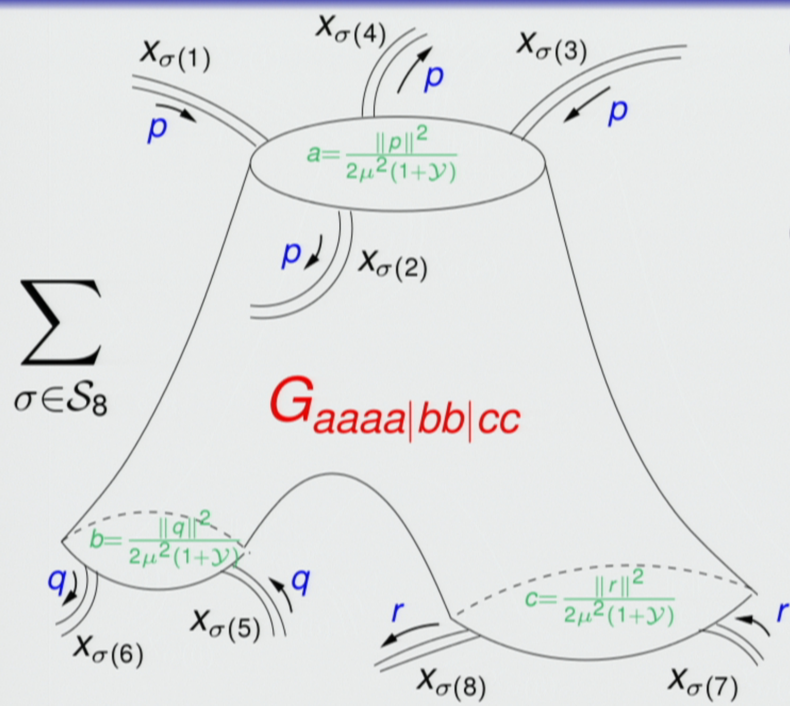

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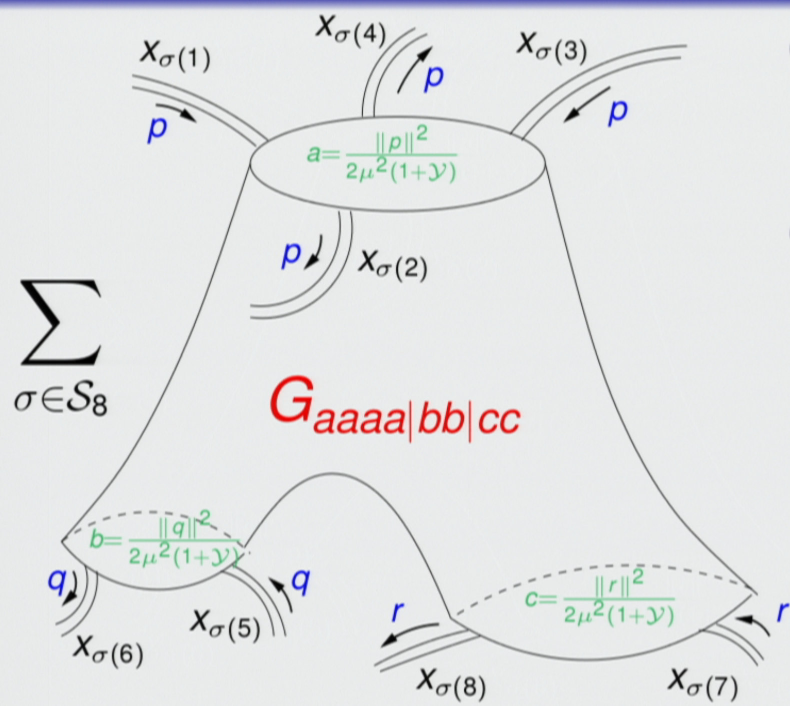

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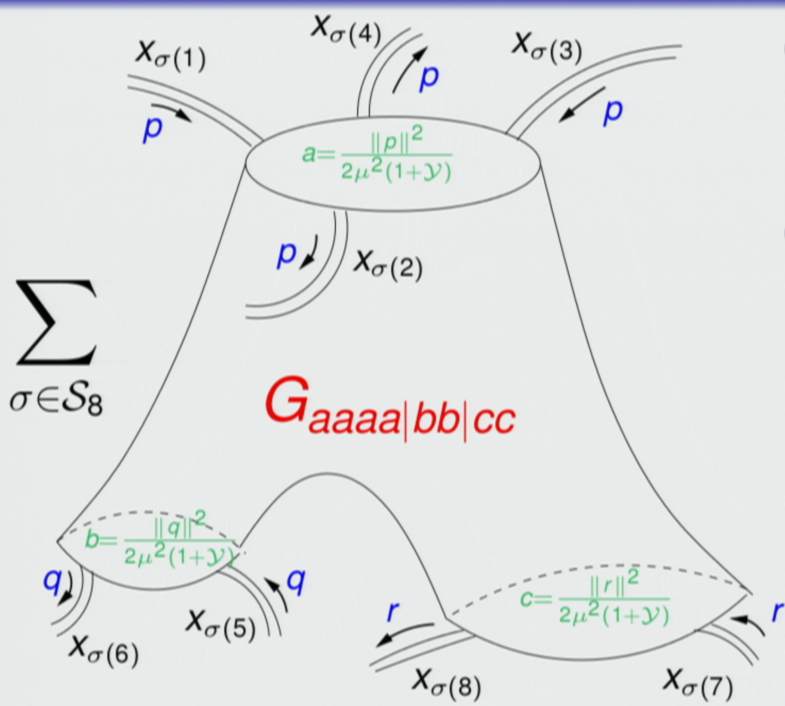

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Is there a link between the solution of our 4D model and traditional integrability in 2D? What about Yang-Baxter?

Osterwalder-Schrader reflection positivity

- Reflection positivity S2 gives spectrum condition which guarantees representation as Laplace transform in ξ^0, hence analyticity in $\mathrm{Re}(\xi^0) > 0$.

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}$$

with ρ positive and non-decreasing. Proof: Källén-Lehmann

- Excluded for any $\lambda > 0$ (due to renormalisation)!
- The Stieltjes property is a particularly strong positivity in mathematics.

Classes of positive definite functions

$f : \mathbb{R}_+ \to \mathbb{R}$ is positive definite if for any $x_1, \ldots, x_n \in \mathbb{R}_+$ the matrix $F = \big(f(x_i + x_j)\big)_{ij}$ is positive (semi-)definite. These are:

1. $\mathcal{C}$ = completely monotonic functions: $(-1)^n f^{(n)} \geq 0$
   - implies rep'n as Laplace transform $f(z) = \int_0^\infty d\mu(t)\, e^{-tz}$
   - related to Bernstein and Pick/Nevanlinna functions and Hausdorff moment problem
2. $\mathcal{L} \subset \mathcal{C}$ = logarithmically completely monotonic functions: $(-1)^n (\log f)^{(n)} \geq 0$

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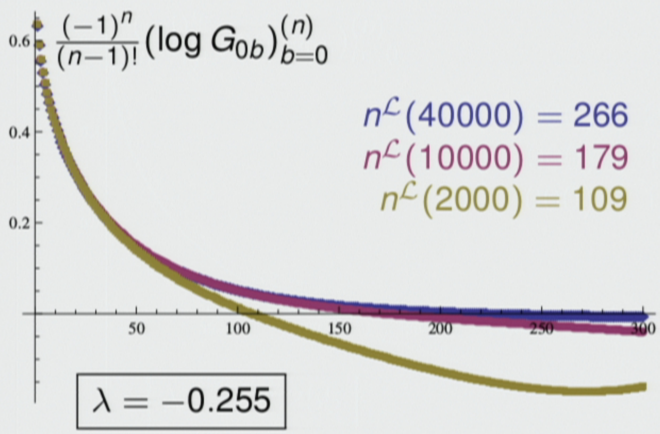
2. $\mathcal{L} \subset \mathcal{C}$ = logarithmically completely monotonic functions: $(-1)^n (\log f)^{(n)} \geq 0$
3. $\mathcal{S} \subset \mathcal{L} \subset \mathcal{C}$ Stieltjes functions: $L_{k,t}[f(\bullet)] \geq 0$ where[Widder, 1938]

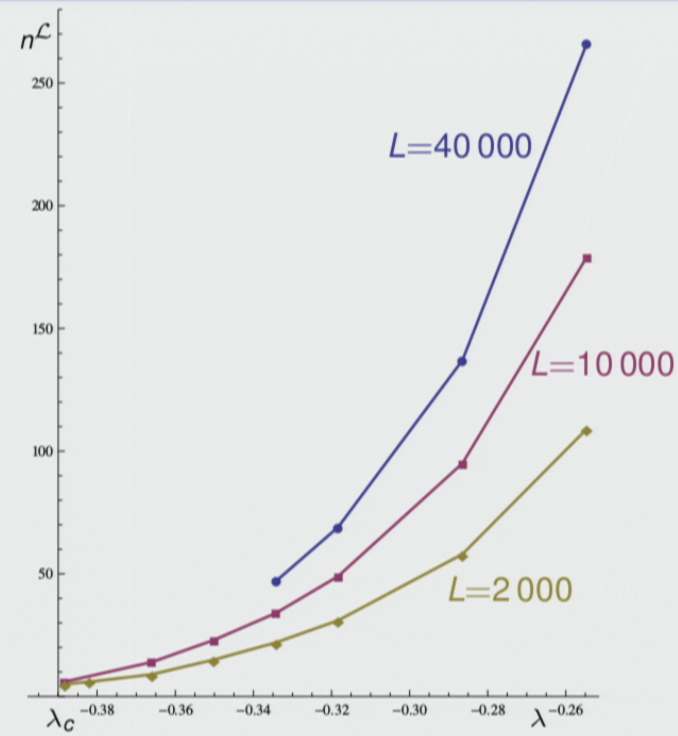
$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big)\,, \qquad c_1 = 1,\ c_{k>1} = k!(k-2)!$$

- imply analyticity in cut plane $\mathbb{C} \setminus]-\infty, 0]$ with $\mathrm{Im}(f(z)) < 0$ for $\mathrm{Im}(z) > 0$ (anti-Herglotz function)
- measure recoved from $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$

Positivity of approximated boundary function G_{0b}

λ	L	$n^{\mathcal{L}}$	$n^{\mathcal{C}}$	$n^{\mathcal{S}}$
−0.255	2000	109		
−0.255	10000	179		
−0.255	40000	266		
−0.318	2000	31	35	37
−0.318	10000	49	55	
−0.350	2000	15	17	18
−0.350	10000	23	25	26
−0.388	2000	5	5	6
−0.388	10000	6	7	8

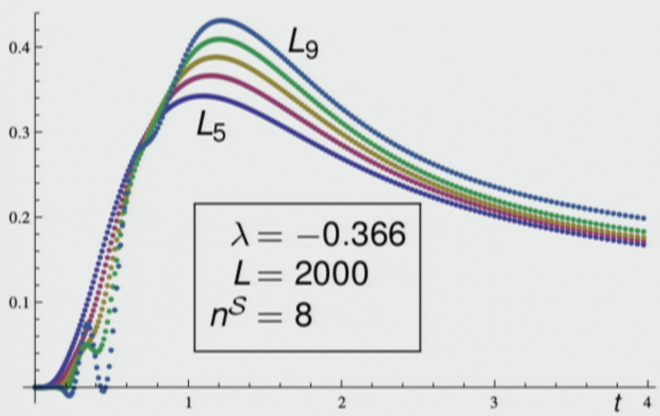


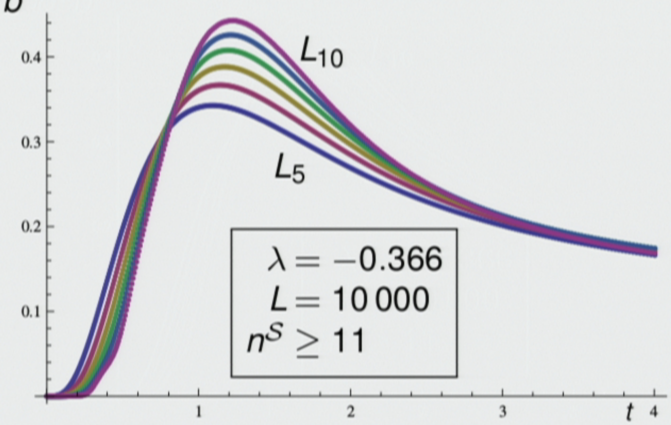


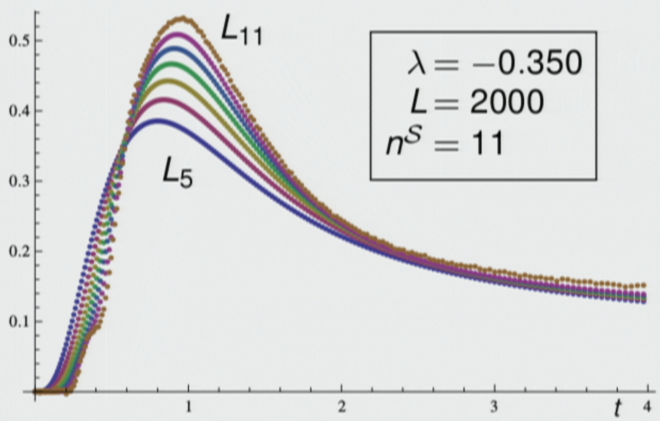
- improvement of $n^{\mathcal{L}}$ with $\uparrow L$ slows down precisely at λ_c!
- Stieltjes failure $n^{\mathcal{S}} > n^{\mathcal{L}}$!

Positivity of approximated G_{ab}: Widder's $L_{k,t}[G_{\bullet\bullet}]$

key step: integral formula for $\frac{\partial^{n+\ell} G_{ab}}{\partial^n a \partial^\ell b}$







- improvement of n^S with $\uparrow L$ and $\downarrow |\lambda|$
- convergence of $\int_0^{m^2} dt\, L_{k,t}[G_{\bullet\bullet}]$ to mass spectrum $\rho(m^2)$
- mass gap $\rho\big|_{[0,m_0^2]} = 0$, but no further gap!

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point solution
   - stable non-perturbative solution for $\lambda < 0$
     $\rightarrow$ planar wrong-sign $\lambda\phi^4_4$-model [t'Hooft; Rivasseau, 1983]
   - phase transitions and critical phenomena, hence interesting statistical physics model
   - non-trivial as a matrix model
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
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3. Reflection positivity S2 does not fail immediately. Why? Needs verification and extension to higher correlation functions

(Non)-triviality?

Projection to diagonal matrices brings the non-trivial intermediate matrix model close to triviality. This is more subtle:

- suppose we can prove S2, then reconstruct Hilbert space H, field operators $\varphi(f)$, unitaries $U(a, L)$ and some vacuum Ω
- uniqueness of Ω cannot be proved without clustering S4
- main problem: characterise set of Poincaré-invariant unit vectors of H, and find its extremal points Ω_e
- each restricted Hilbert space H_e, generated by its cyclic vector Ω_e, admits collision states (Haag-Ruelle theory) and (if asymptotically complete) an S-matrix
- involves new Wightman distributions

$$W_e(x_1, ..., x_N) = \langle \Omega_e, \varphi(x_1) \cdots \varphi(x_N)\Omega_e \rangle$$

expected to differ from $W(x_1, ..., x_N) = \langle \Omega, \varphi(x_1) \cdots \varphi(x_N)\Omega \rangle$

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Consequently, a non-trivial $S \neq \mathbf{1}$ is not impossible.