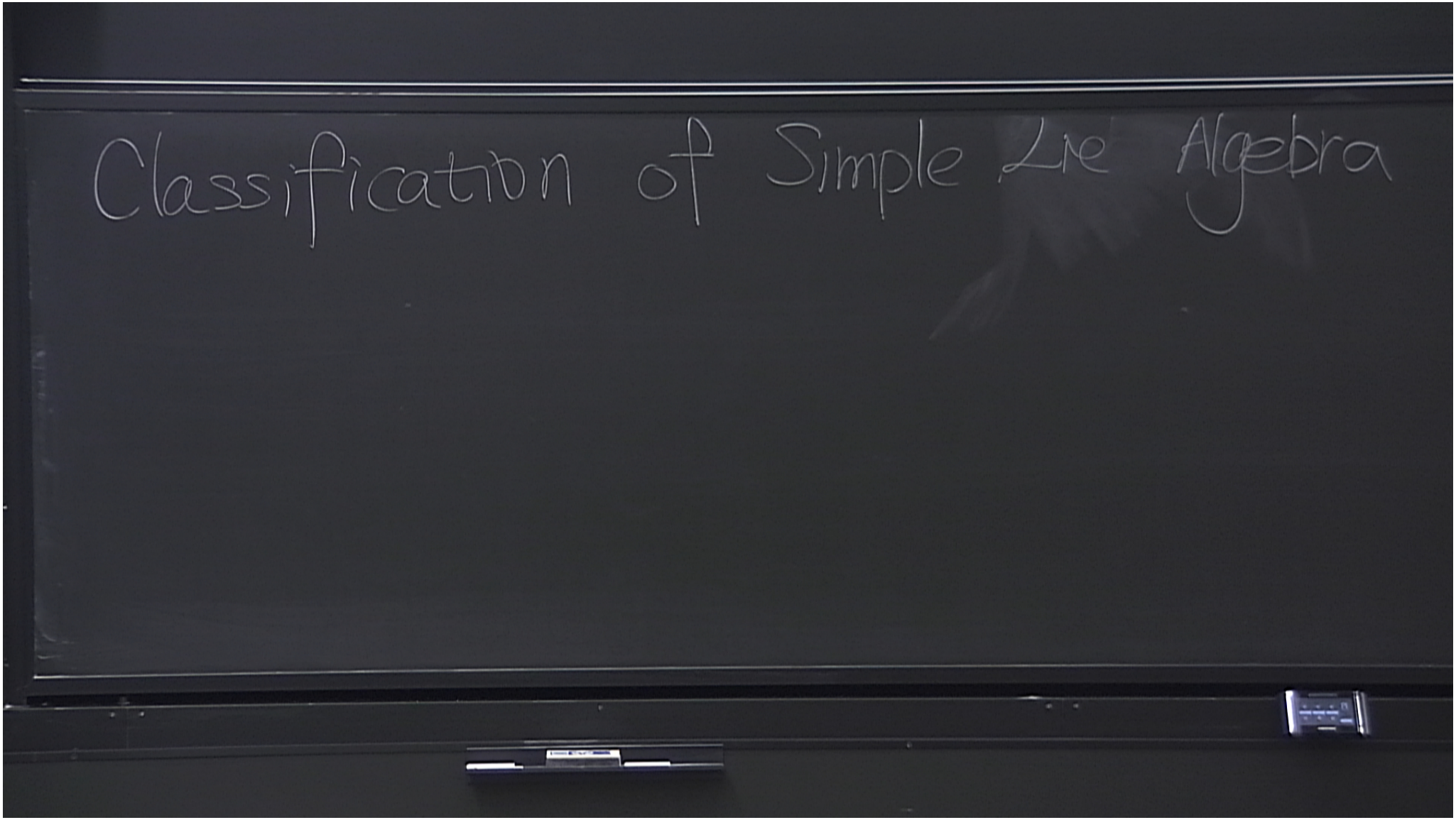
Title: 14/15 PSI - Lie Groups

Date: Aug 28, 2014  09:00 AM

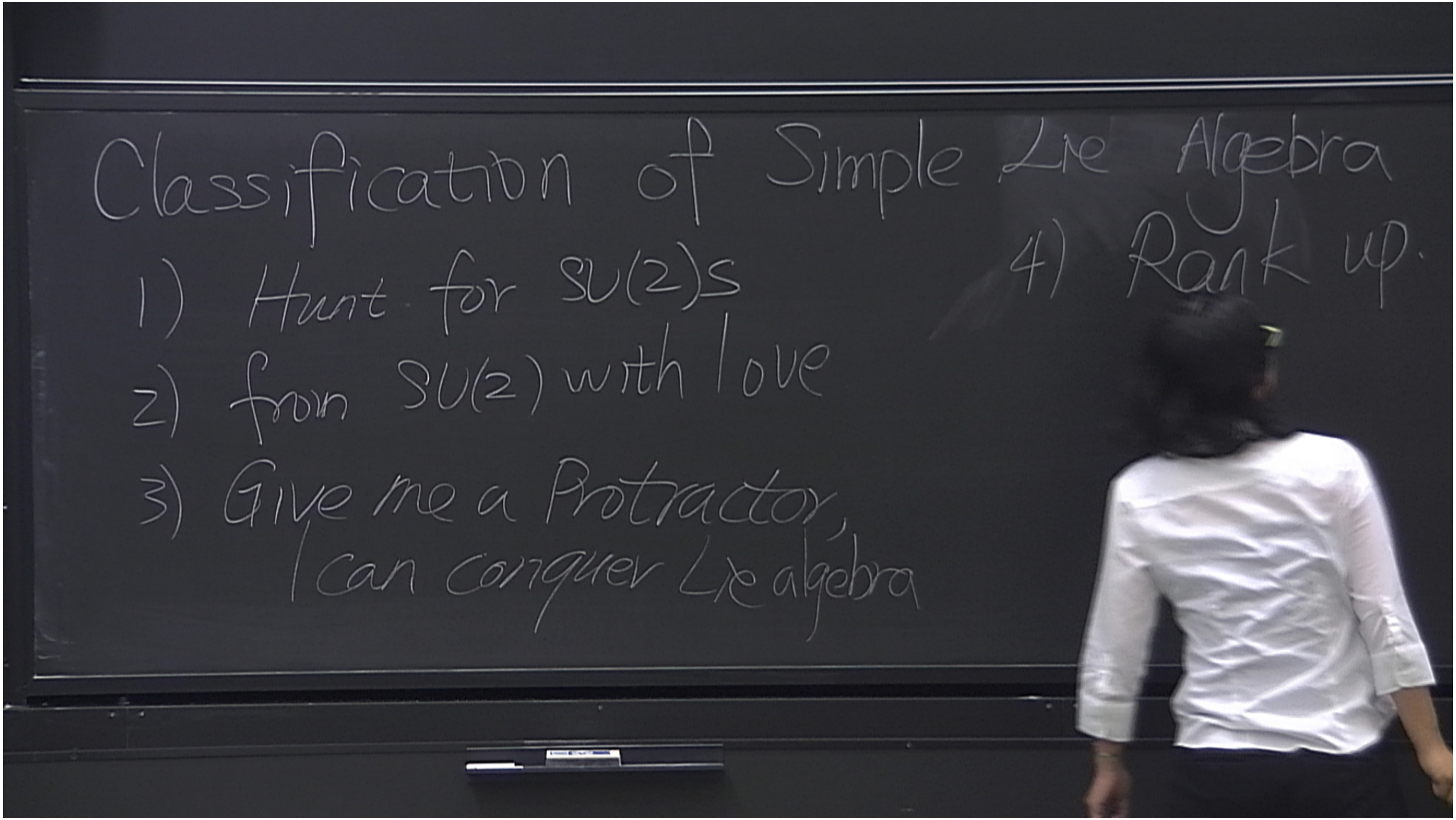
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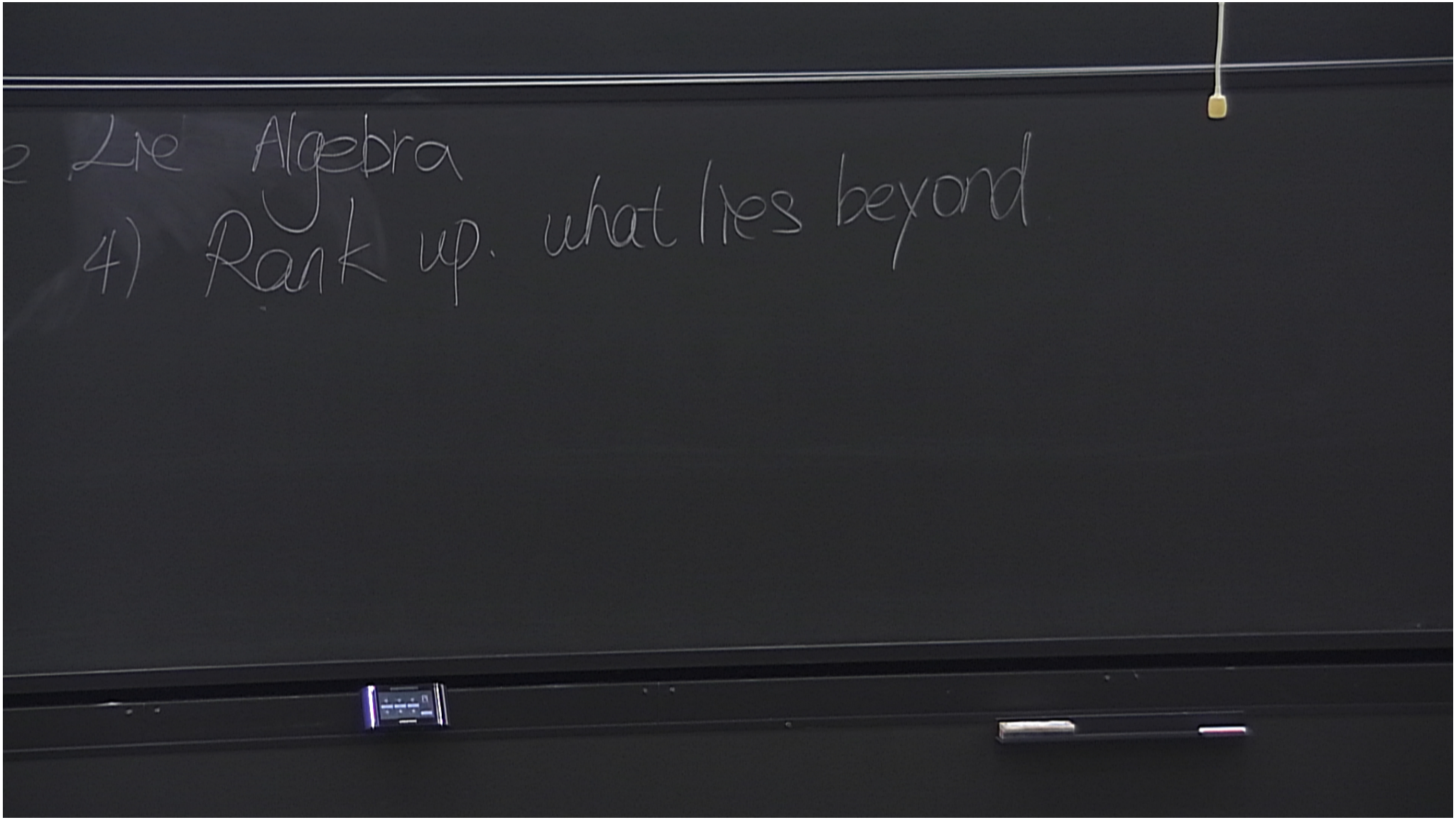
Abstract:

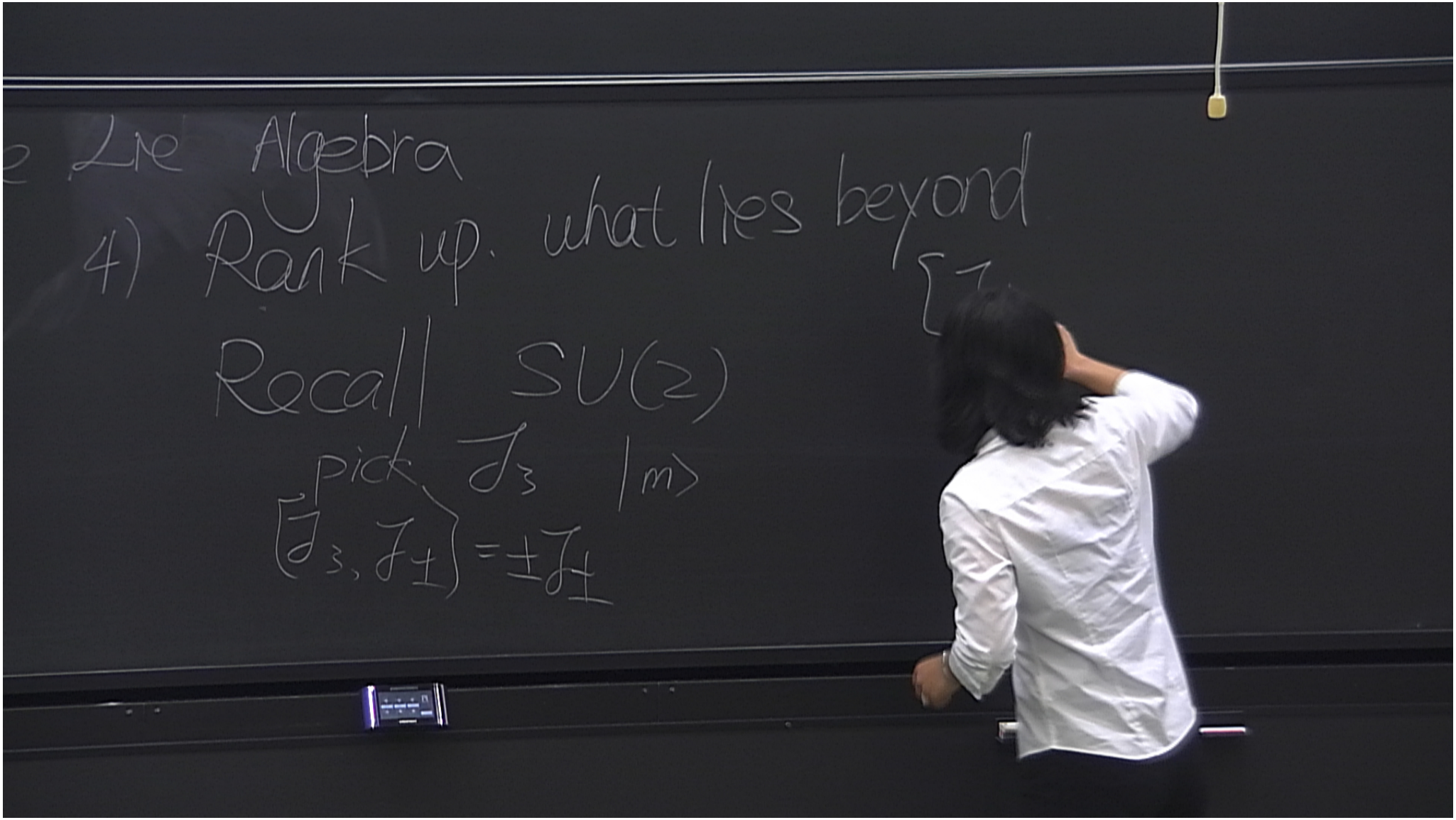

Classification of Simple Lie Algebra

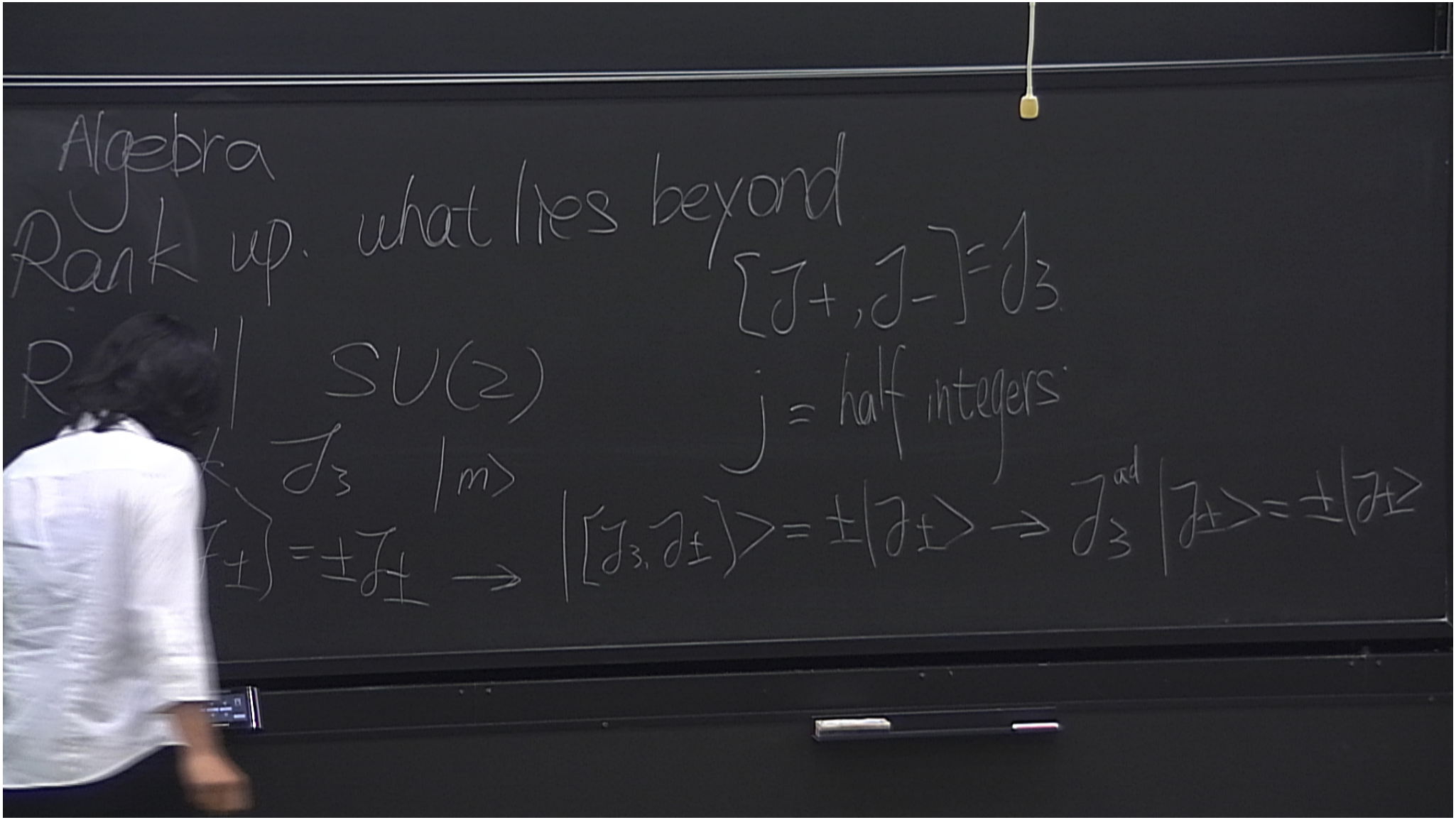
# Classification of Simple Lie Algebra

1) Hunt for SU(2)s
2) from SU(2) with love
3)


Classification of Simple Lie Algebra
1) Hunt for SU(2)s
2) from SU(2) with love
3) Give me a Protractor,
I can conquer Lie algebra
4) Rank up.


Lie Algebra
4) Rank up. what lies beyond


Lie Algebra
4) Rank up. what lies beyond
Recall SU(2)
pick $J_3$ $|m\rangle$
$[J_3, J_\pm] = \pm J_\pm$


Algebra
Rank up. what lies beyond
$[J_+, J_-] = J_3$
SU(2)
$J_3 \; |m\rangle$
$j$ = half integers
$\{J_\pm\} = \pm J_\pm \rightarrow |[J_3, J_\pm]\rangle = \pm |J_\pm\rangle \rightarrow J_3^{ad} |J_\pm\rangle = \pm |J_\pm\rangle$

rank 2 r=2

such that they are e

1) find Cartan generators

$\{H_i\}$ $i=1,\cdots,r$

finding $J_3$

$\forall i,j\ [H_i, H_j]=0$

2) Form linear combinations of other generators.

such that they are eigenvectors.

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$$

$$[\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha} E_{\vec{\alpha}}$$

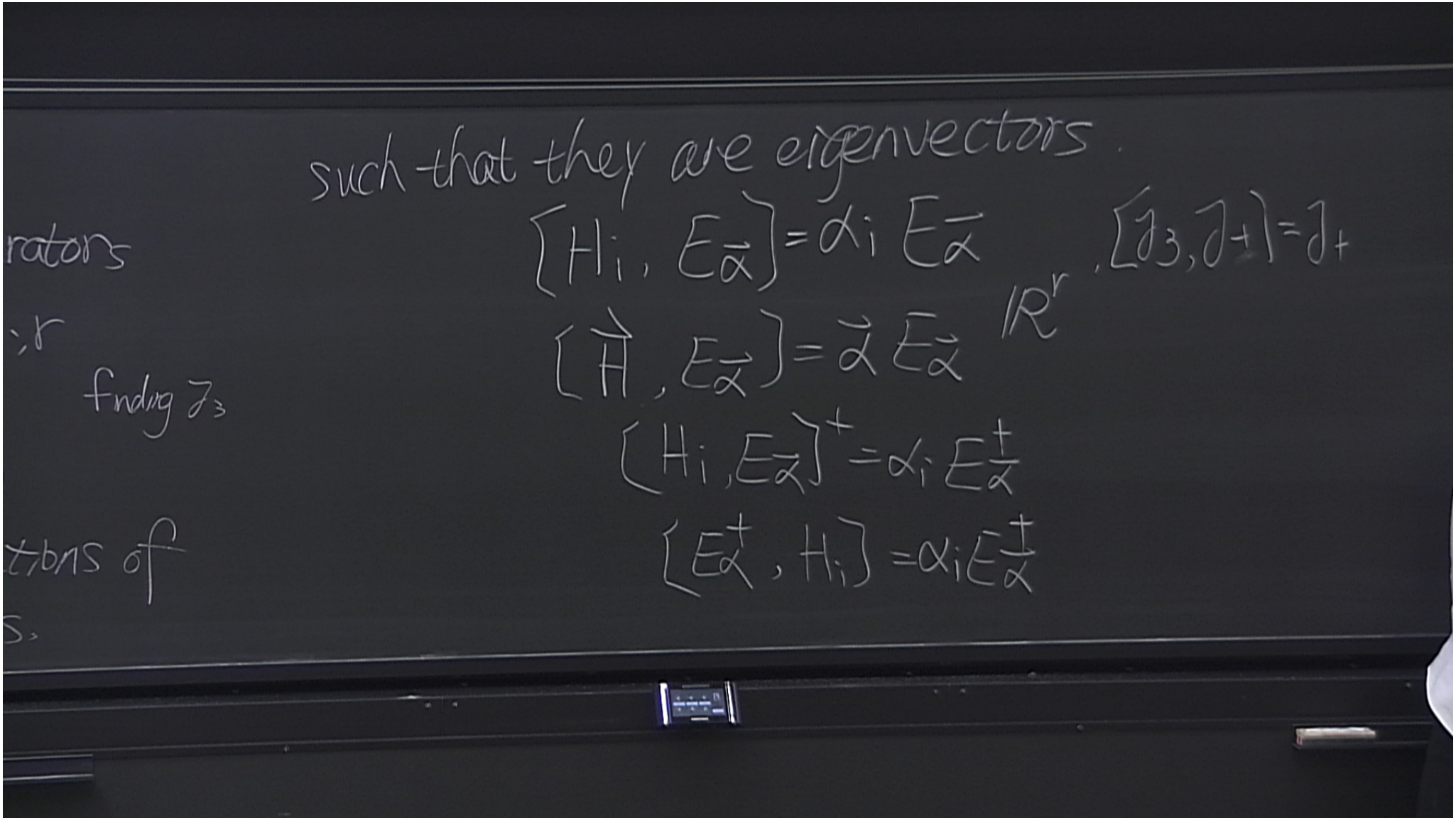
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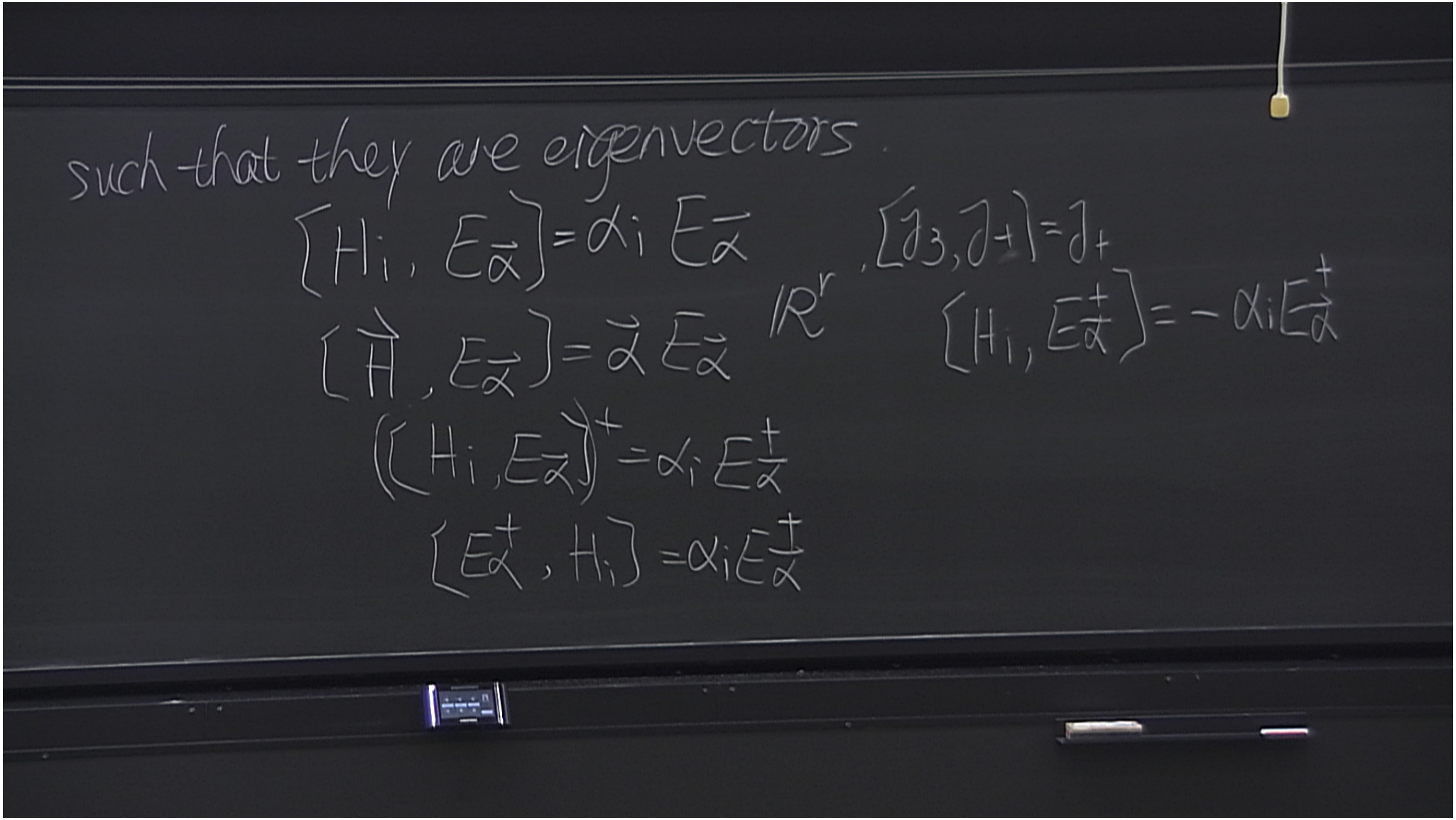
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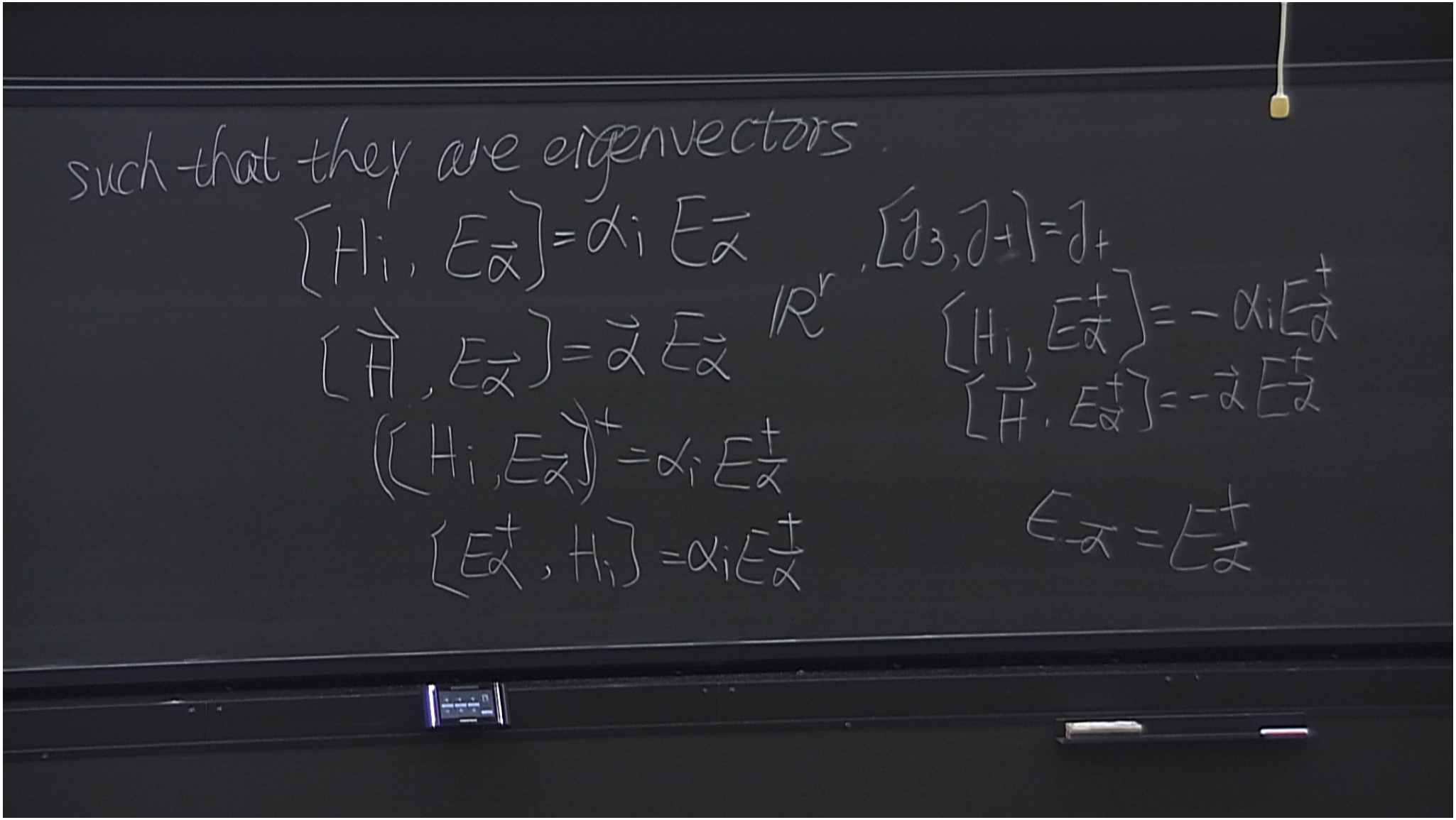
finding $\gamma_3$

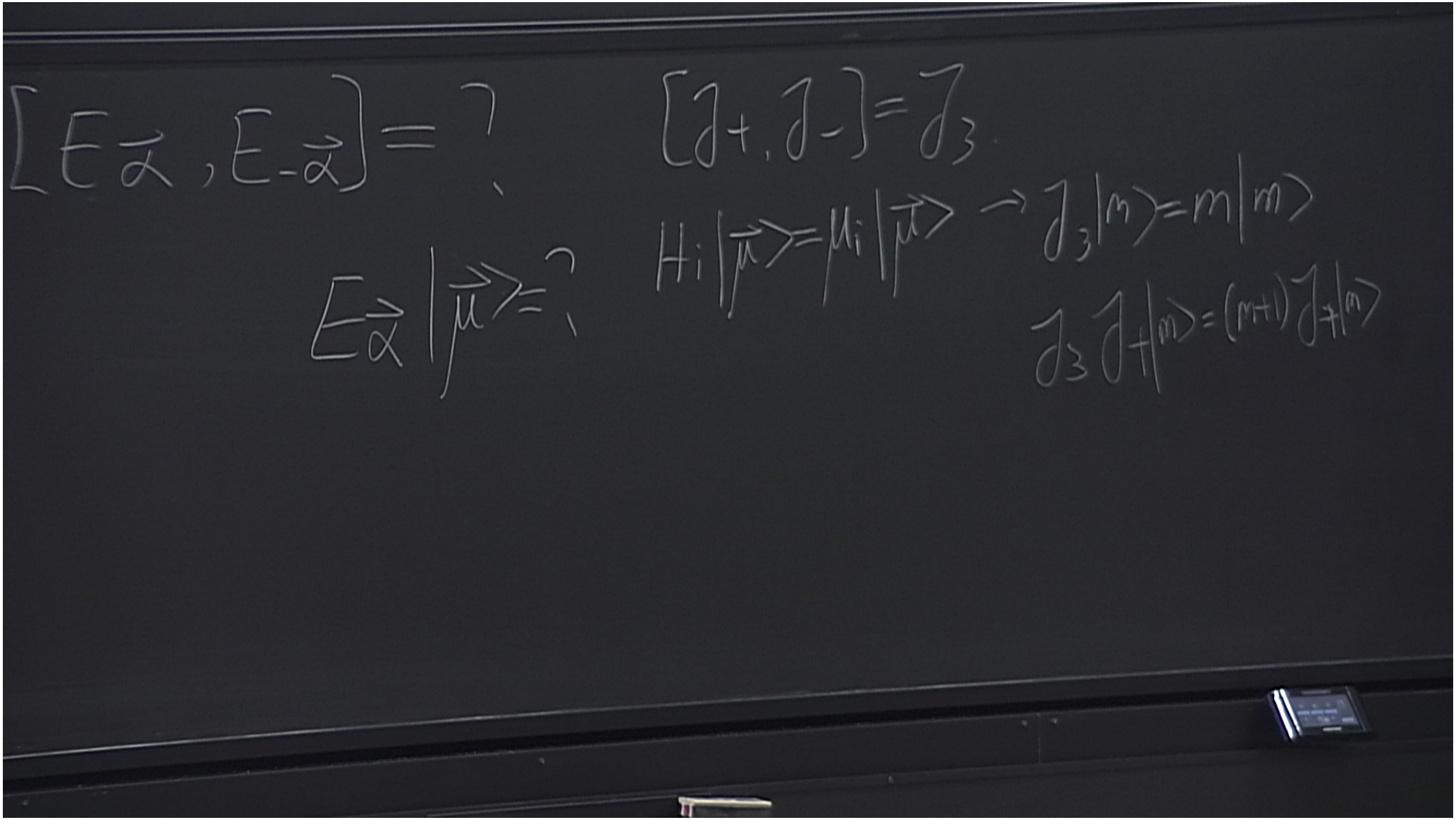
tions of

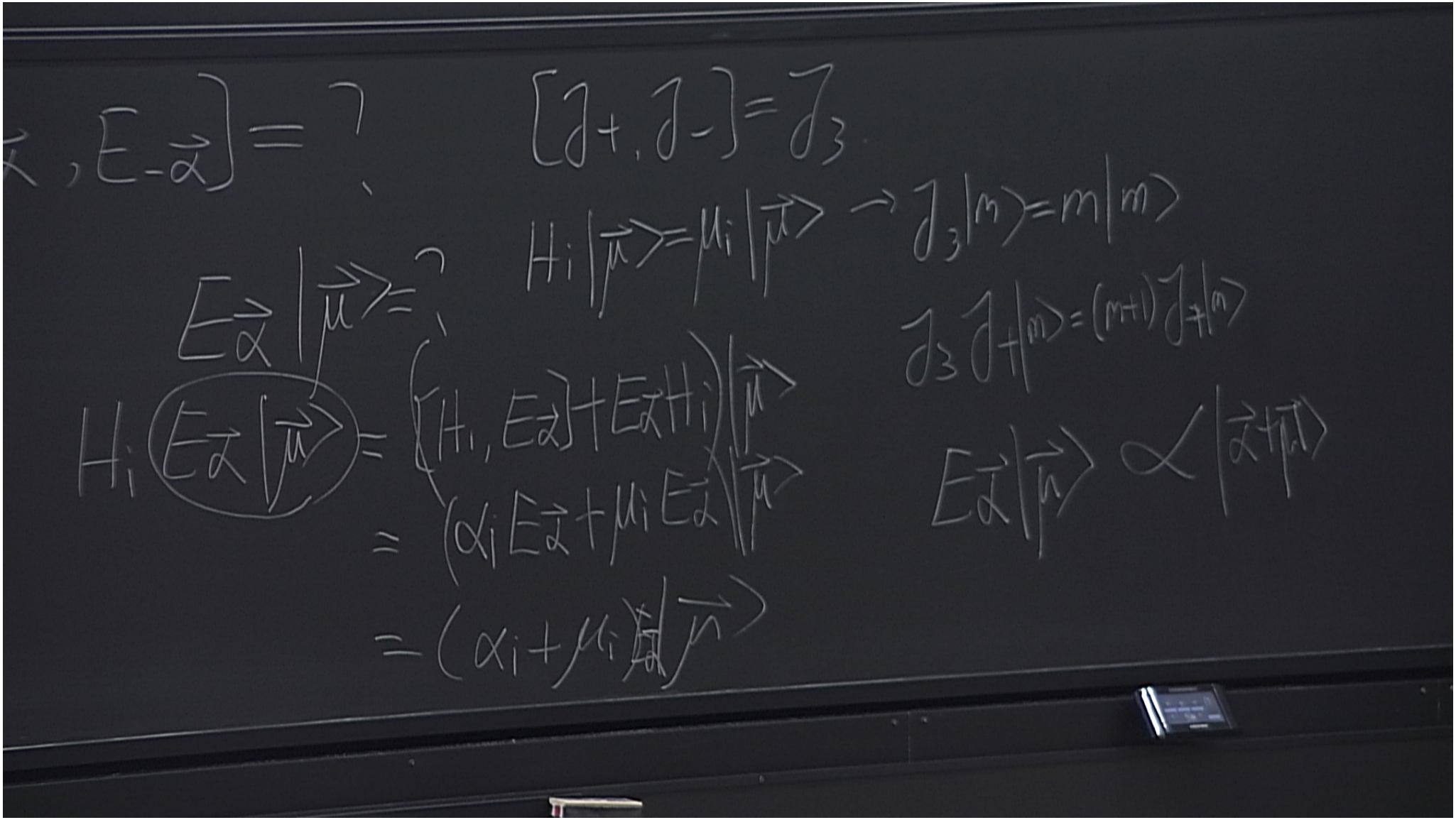
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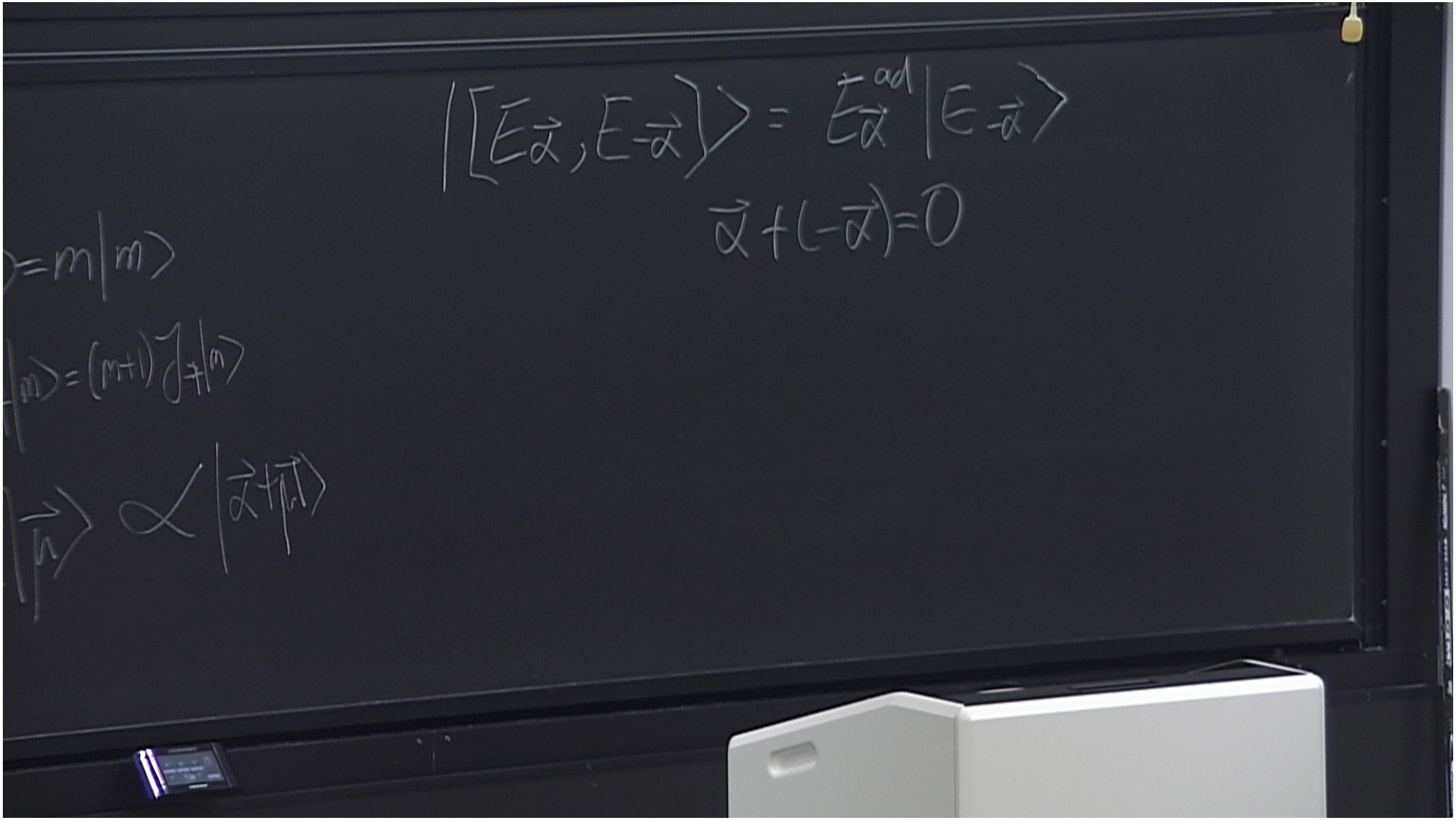

such that they are eigenvectors.
$[H_i, E_\alpha] = \alpha_i E_\alpha$
$[\vec{H}, E_\alpha] = \vec{\alpha} E_\alpha$
$[H_i, E_\alpha]^+ = \alpha_i E_\alpha^+$
$[E_\alpha^+, H_i] = \alpha_i E_\alpha^+$
$R^r$
$[J_3, J_+] = J_+$
finding $J_3$

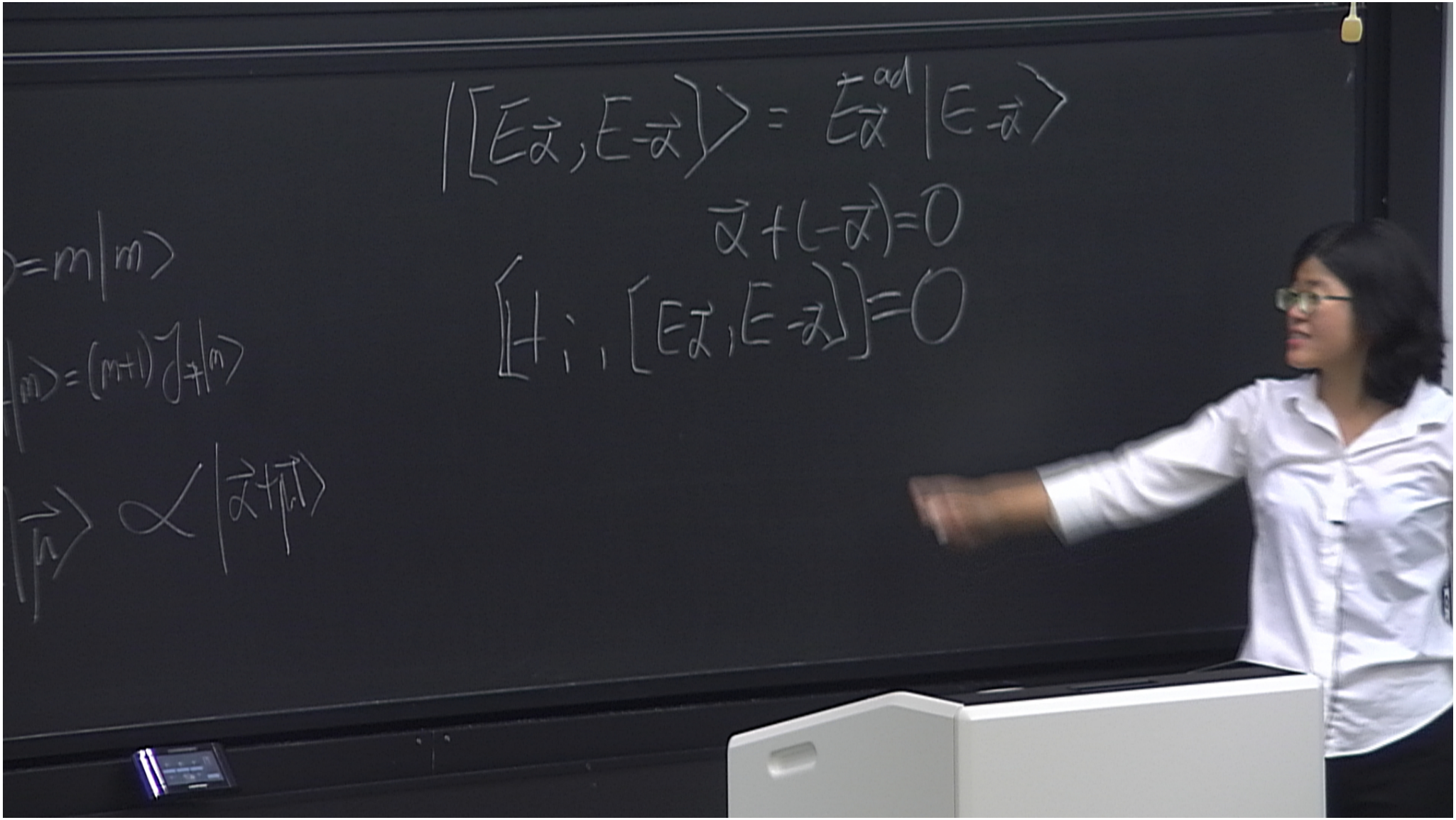

such that they are eigenvectors.
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$
$[\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha} E_{\vec{\alpha}}$
$\mathbb{R}^r$
$([H_i, E_{\vec{\alpha}}])^\dagger = \alpha_i E^\dagger_{\vec{\alpha}}$
$[E^\dagger_{\vec{\alpha}}, H_i] = \alpha_i E^\dagger_{\vec{\alpha}}$
$[J_3, J_+] = J_+$
$[H_i, E^\dagger_{\vec{\alpha}}] = -\alpha_i E^\dagger_{\vec{\alpha}}$

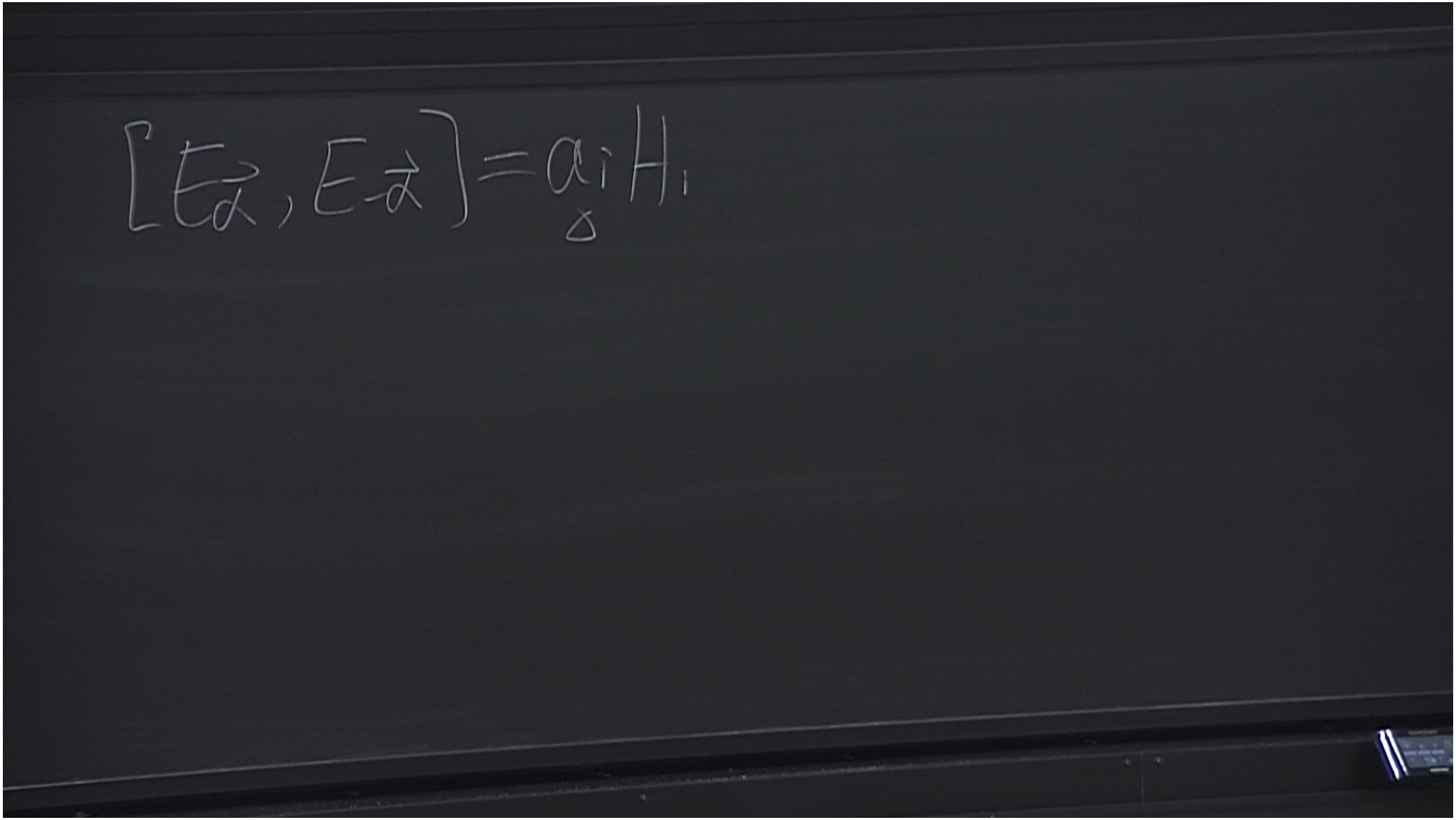

such that they are eigenvectors.
$[H_i, E_\alpha] = \alpha_i E_\alpha$
$[\vec{H}, E_\alpha] = \vec{\alpha} E_\alpha$
$([H_i, E_\alpha])^+ = \alpha_i E_\alpha^+$
$[E_\alpha^+, H_i] = \alpha_i E_\alpha^+$
$\mathbb{R}^r$
$[J_3, J_+] = J_+$
$[H_i, E_\alpha^+] = -\alpha_i E_\alpha^+$
$[\vec{H}, E_\alpha^+] = -\vec{\alpha} E_\alpha^+$
$E_{-\alpha} = E_\alpha^+$

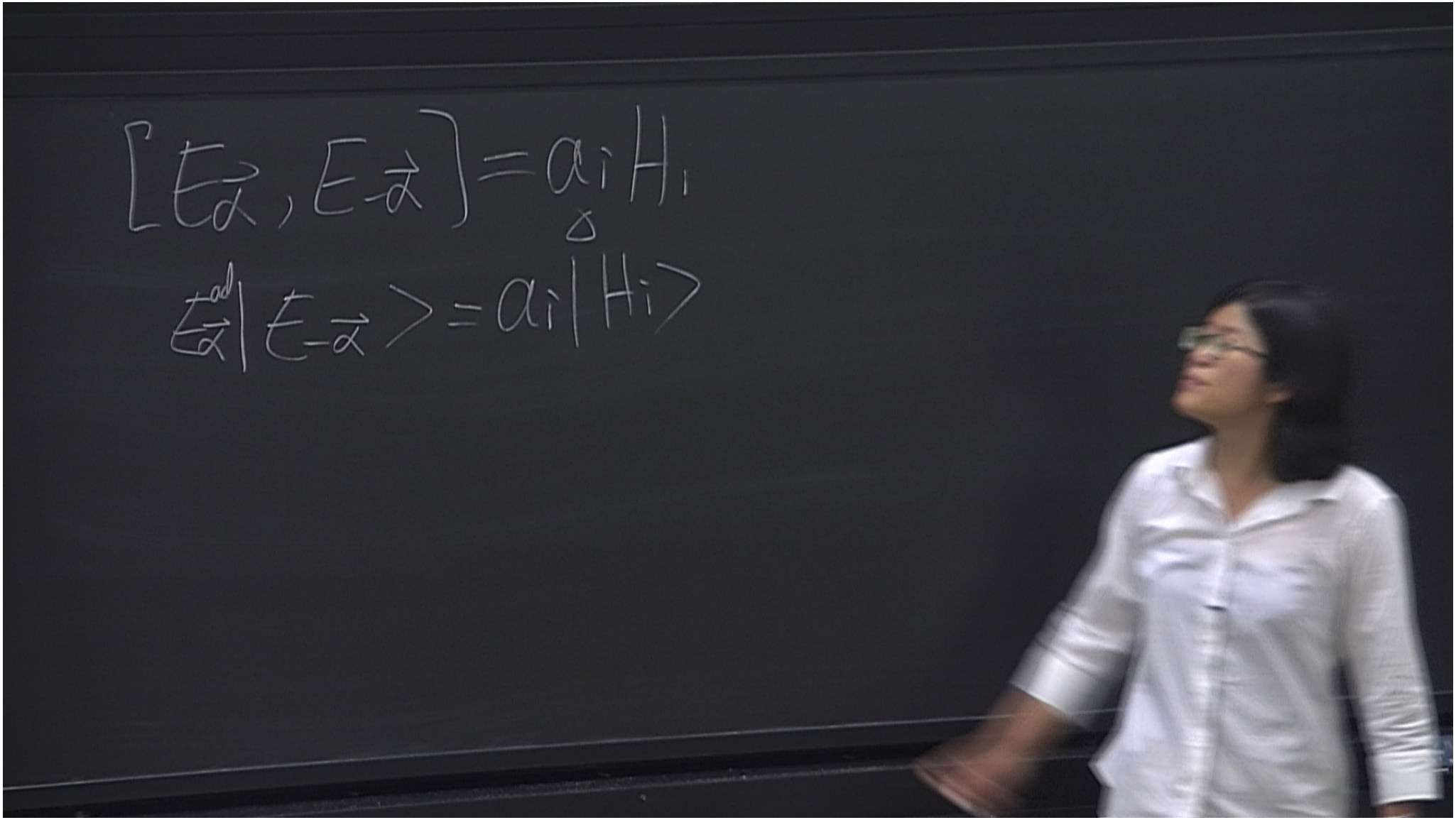

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = ?$
$[J_+, J_-] = J_3$
$E_{\vec{\alpha}} |\vec{\mu}\rangle = ?$
$H_i |\vec{\mu}\rangle = \mu_i |\vec{\mu}\rangle \rightarrow J_3 |m\rangle = m |m\rangle$
$J_3 J_+ |m\rangle = (m+1) J_+ |m\rangle$

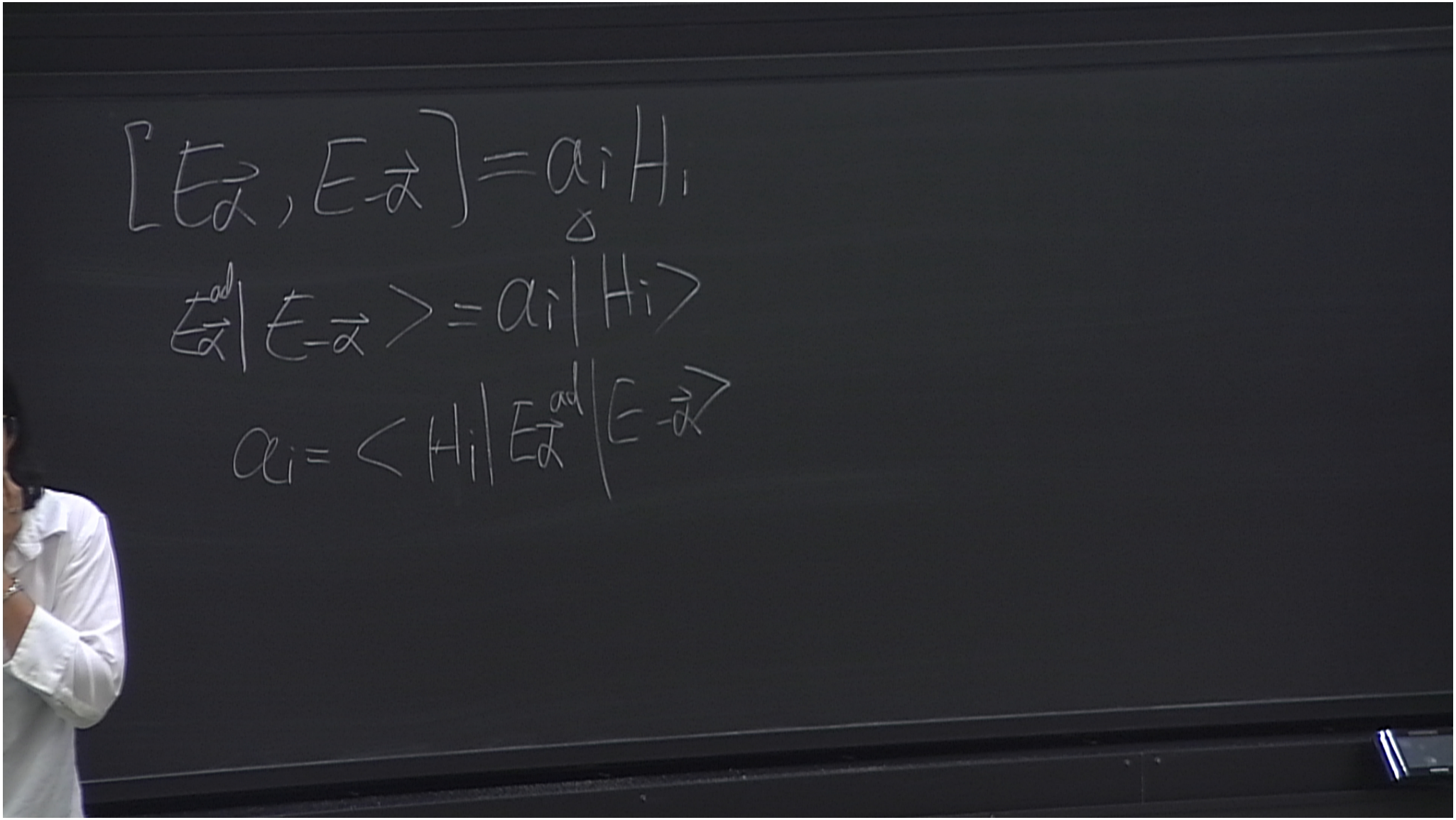

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = ?$
$[J_+, J_-] = J_3$
$H_i|\vec{\mu}\rangle = \mu_i|\vec{\mu}\rangle \rightarrow J_3|m\rangle = m|m\rangle$
$E_{\vec{\alpha}}|\vec{\mu}\rangle = ?$
$J_3 J_+|m\rangle = (m+1) J_+|m\rangle$
$H_i (E_{\vec{\alpha}}|\vec{\mu}\rangle) = ([H_i, E_{\vec{\alpha}}] + E_{\vec{\alpha}} H_i)|\vec{\mu}\rangle$
$= (\alpha_i E_{\vec{\alpha}} + \mu_i E_{\vec{\alpha}})|\vec{\mu}\rangle$
$= (\alpha_i + \mu_i) E_{\vec{\alpha}}|\vec{\mu}\rangle$
$E_{\vec{\alpha}}|\vec{\mu}\rangle \propto |\vec{\alpha}+\vec{\mu}\rangle$

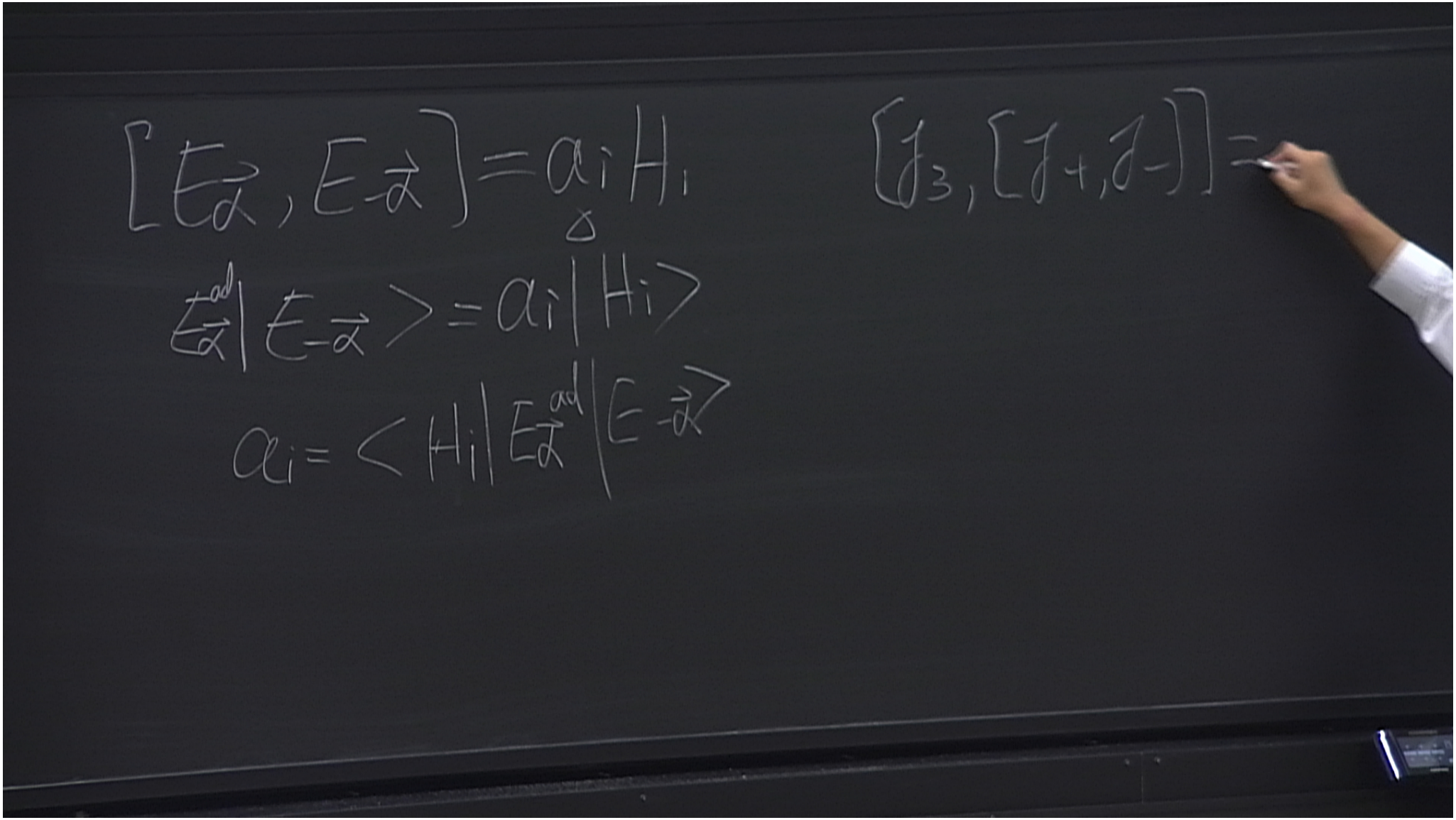

$|[E_{\vec{\alpha}}, E_{-\vec{\alpha}}]\rangle = \pm E^{ad}_{\vec{\alpha}} |E_{-\vec{\alpha}}\rangle$
$\vec{\alpha} + (-\vec{\alpha}) = 0$
$= m|m\rangle$
$|m\rangle = (m+1)\tilde{J}_{\pm}|m\rangle$
$|\vec{\mu}\rangle \propto |\vec{\alpha} + \vec{\mu}\rangle$

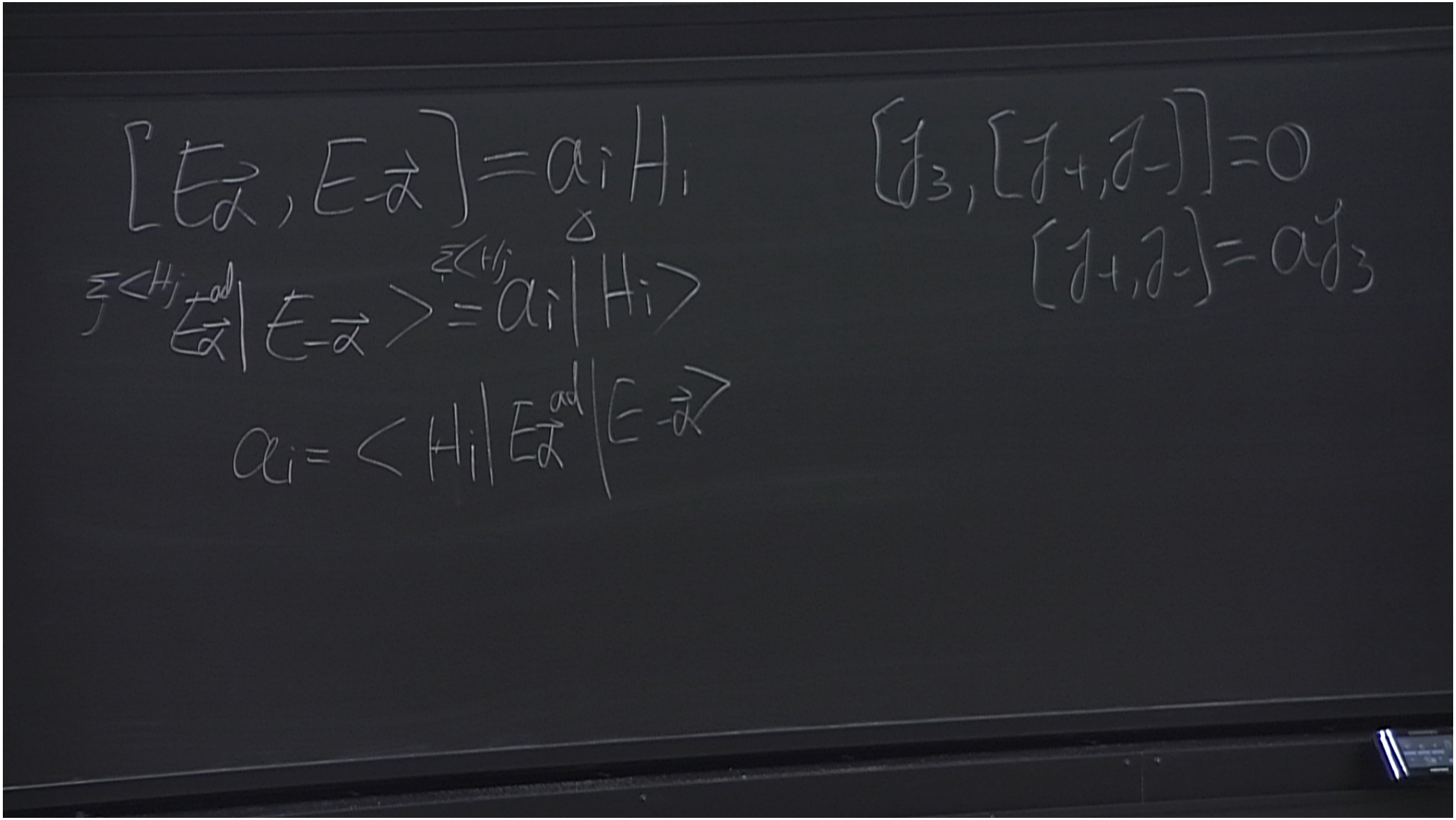

$|[E_{\vec{\alpha}}, E_{-\vec{\alpha}}]\rangle = \hat{E}^{ad}_{\vec{\alpha}} |E_{-\vec{\alpha}}\rangle$
$\vec{\alpha} + (-\vec{\alpha}) = 0$
$[H_i, [E_{\vec{\alpha}}, E_{-\vec{\alpha}}]] = 0$
$= m|m\rangle$
$|m\rangle = (m+1)|m\rangle$
$|\vec{\alpha}\rangle \propto |\vec{\alpha}+\vec{\beta}\rangle$


$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i$

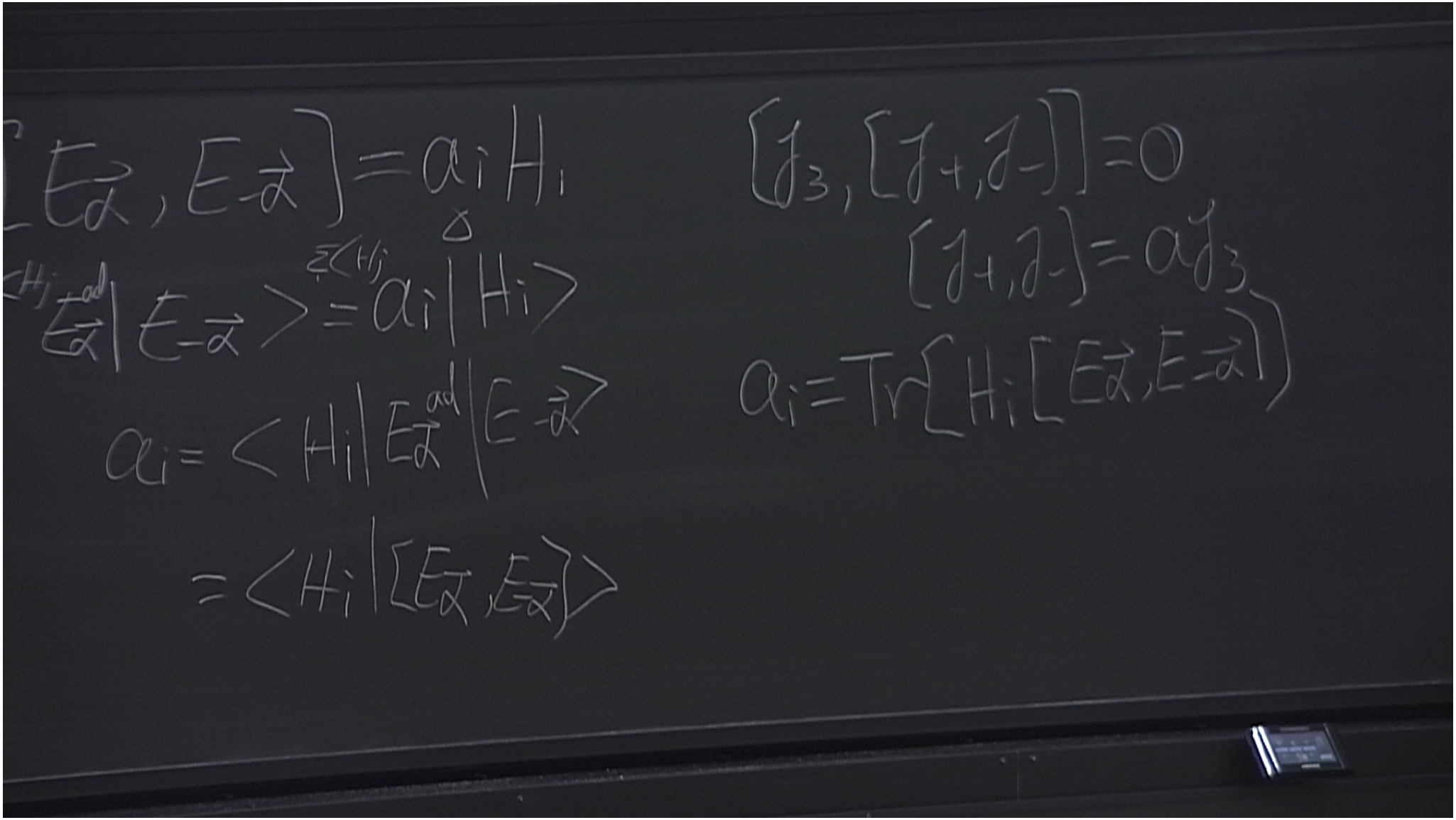

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i$
$\Delta$
$\text{ad}_{E_{\vec{\alpha}}} |E_{-\vec{\alpha}}\rangle = \alpha_i |H_i\rangle$

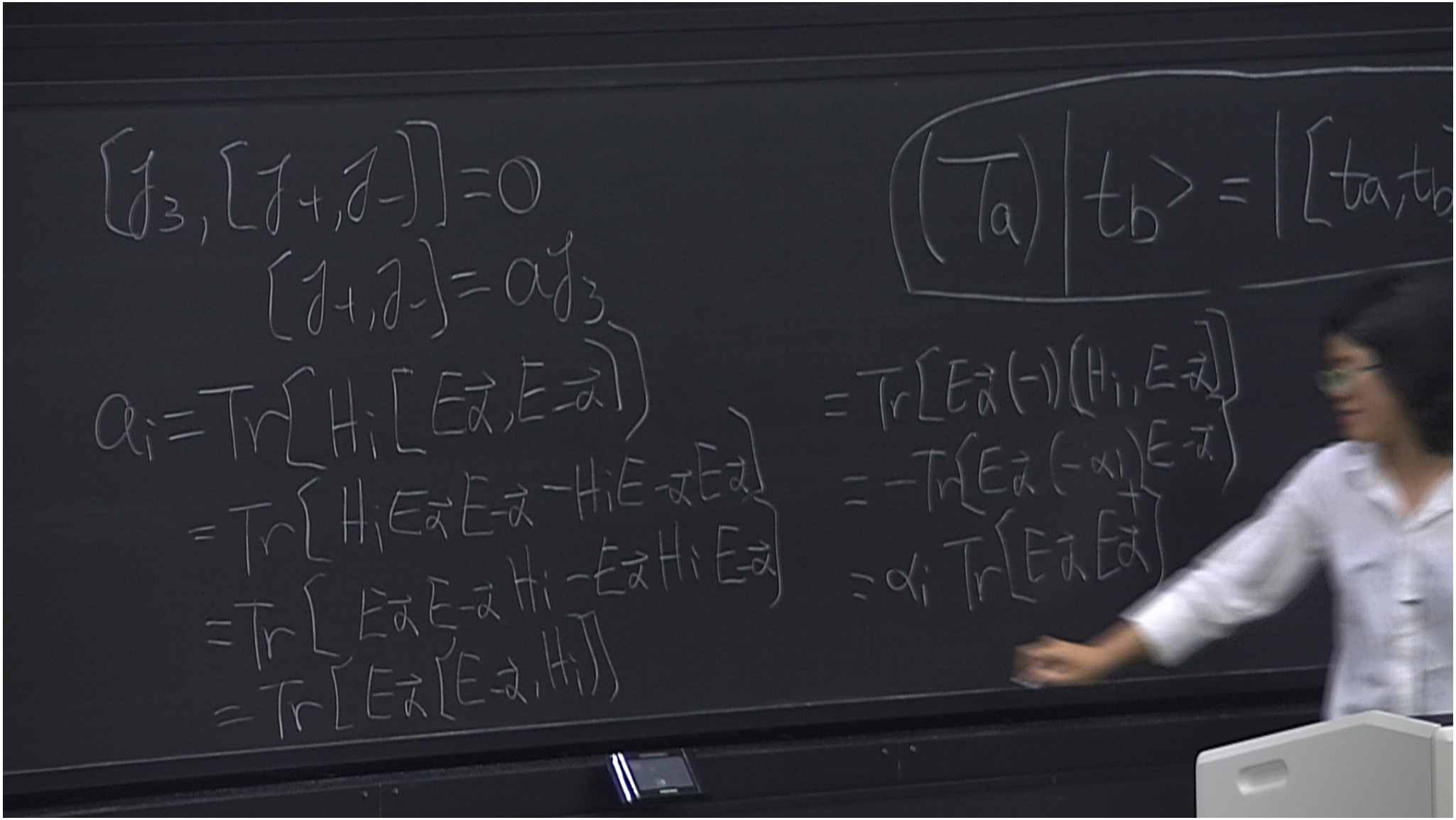

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i$
$\sum_{\alpha}$
$E_{\vec{\alpha}}^{ad} |E_{-\vec{\alpha}}\rangle = \alpha_i |H_i\rangle$
$\alpha_i = \langle H_i | E_{\vec{\alpha}}^{ad} | E_{-\vec{\alpha}} \rangle$

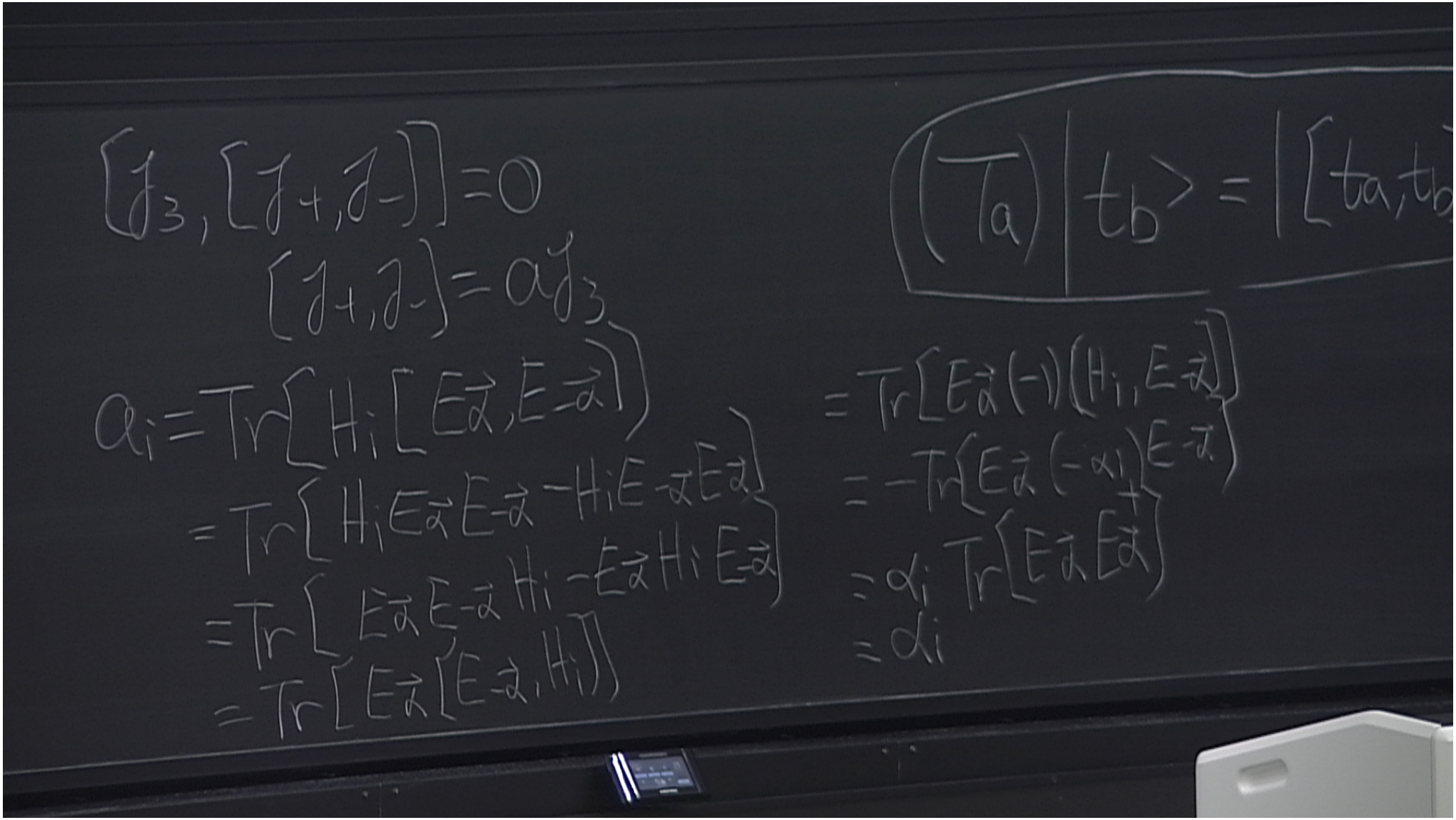

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i$
$[J_3, [J_+, J_-]] =$
$\Delta$
$E_{\vec{\alpha}}^{ad} |E_{-\vec{\alpha}}\rangle = \alpha_i |H_i\rangle$
$\alpha_i = \langle H_i | E_{\vec{\alpha}}^{ad} | E_{-\vec{\alpha}} \rangle$

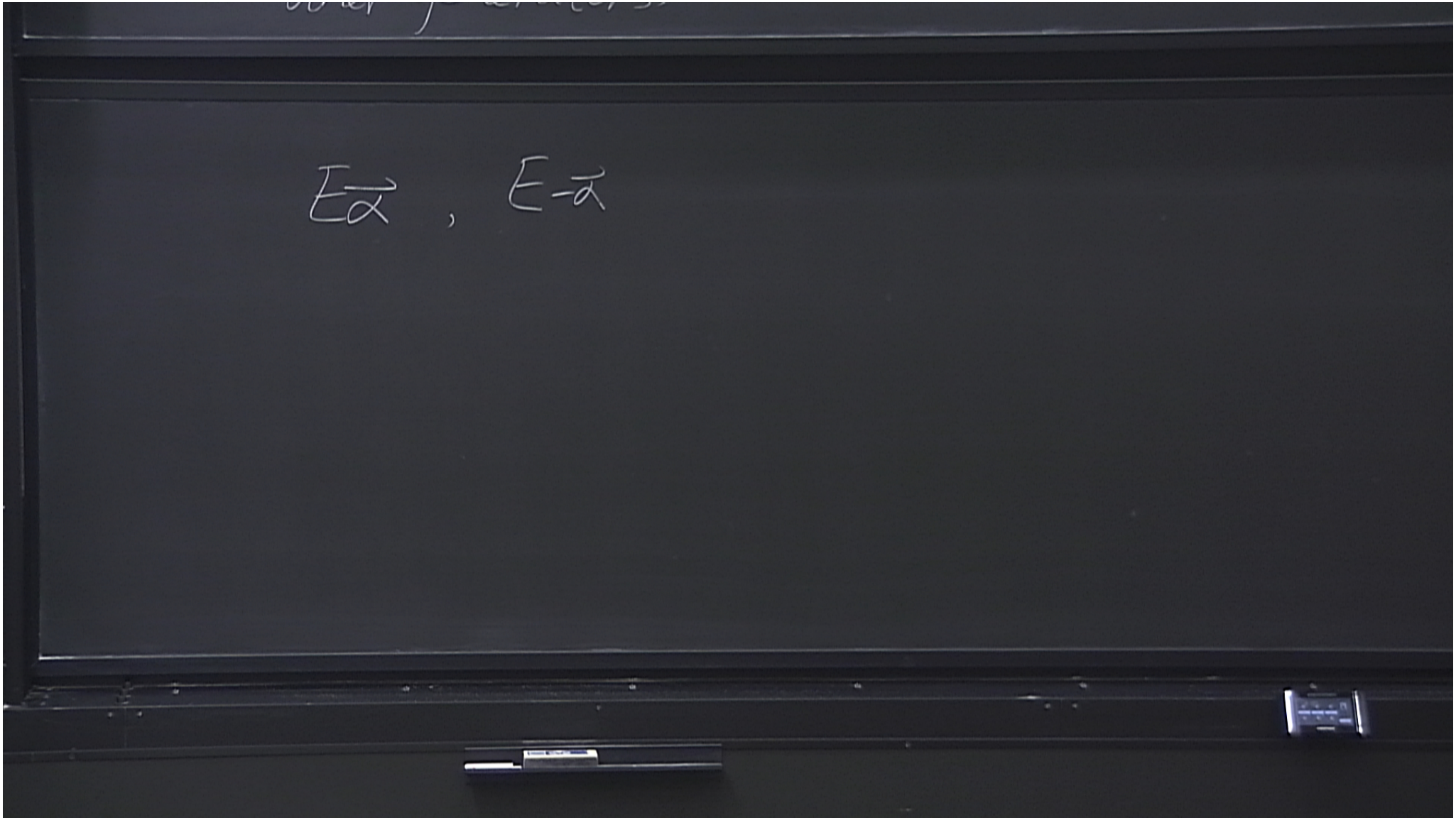

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = a_i H_i$
$\xi < H_j | E_{\vec{\alpha}}^{ad} | E_{-\vec{\alpha}} > = a_i | H_i >$
$a_i = < H_i | E_{\vec{\alpha}}^{ad} | E_{-\vec{\alpha}} >$
$[J_3, [J_+, J_-]] = 0$
$[J_+, J_-] = a J_3$

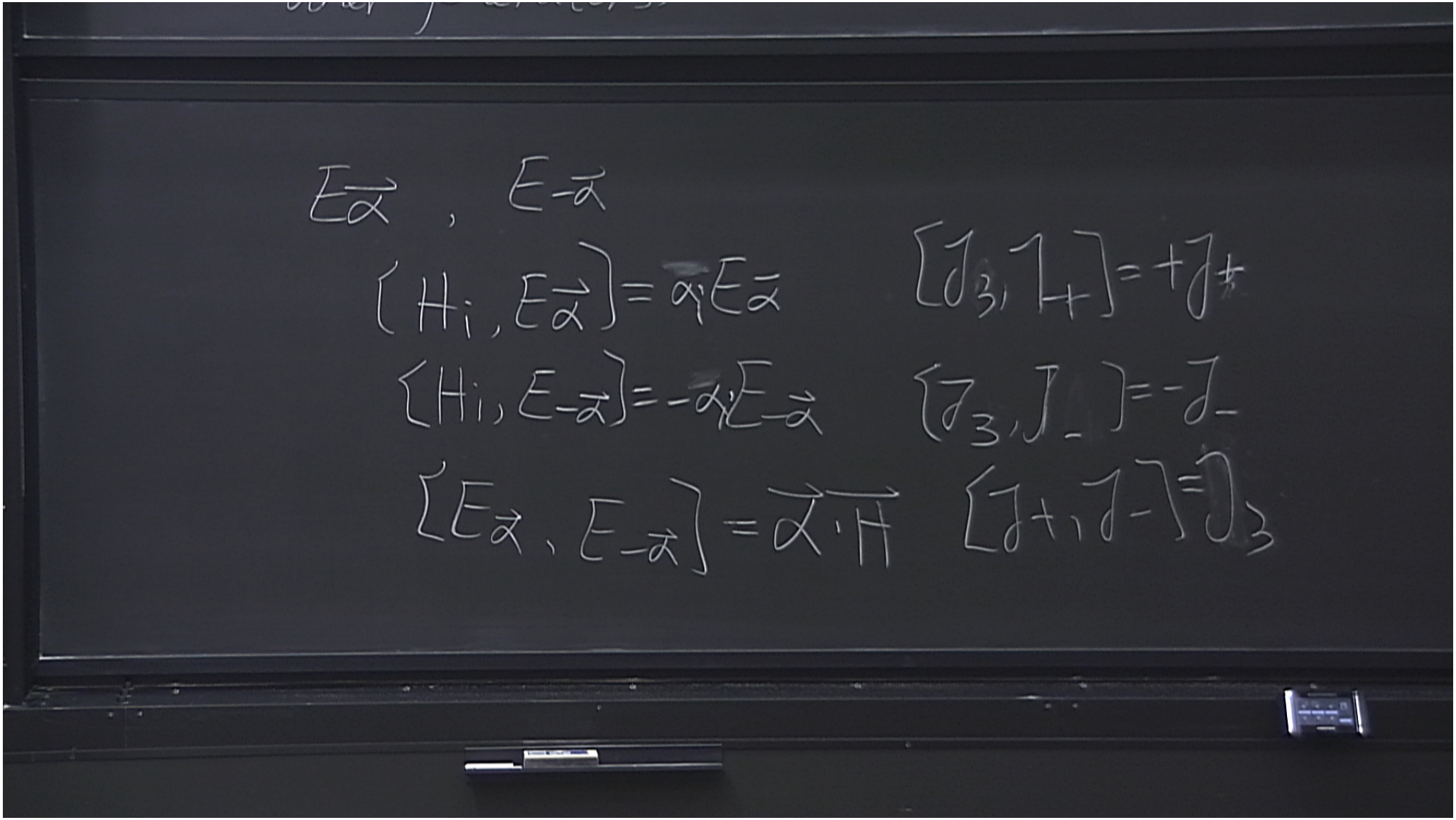

$(T_a)|t_b\rangle = |[t_a, t_b]\rangle$

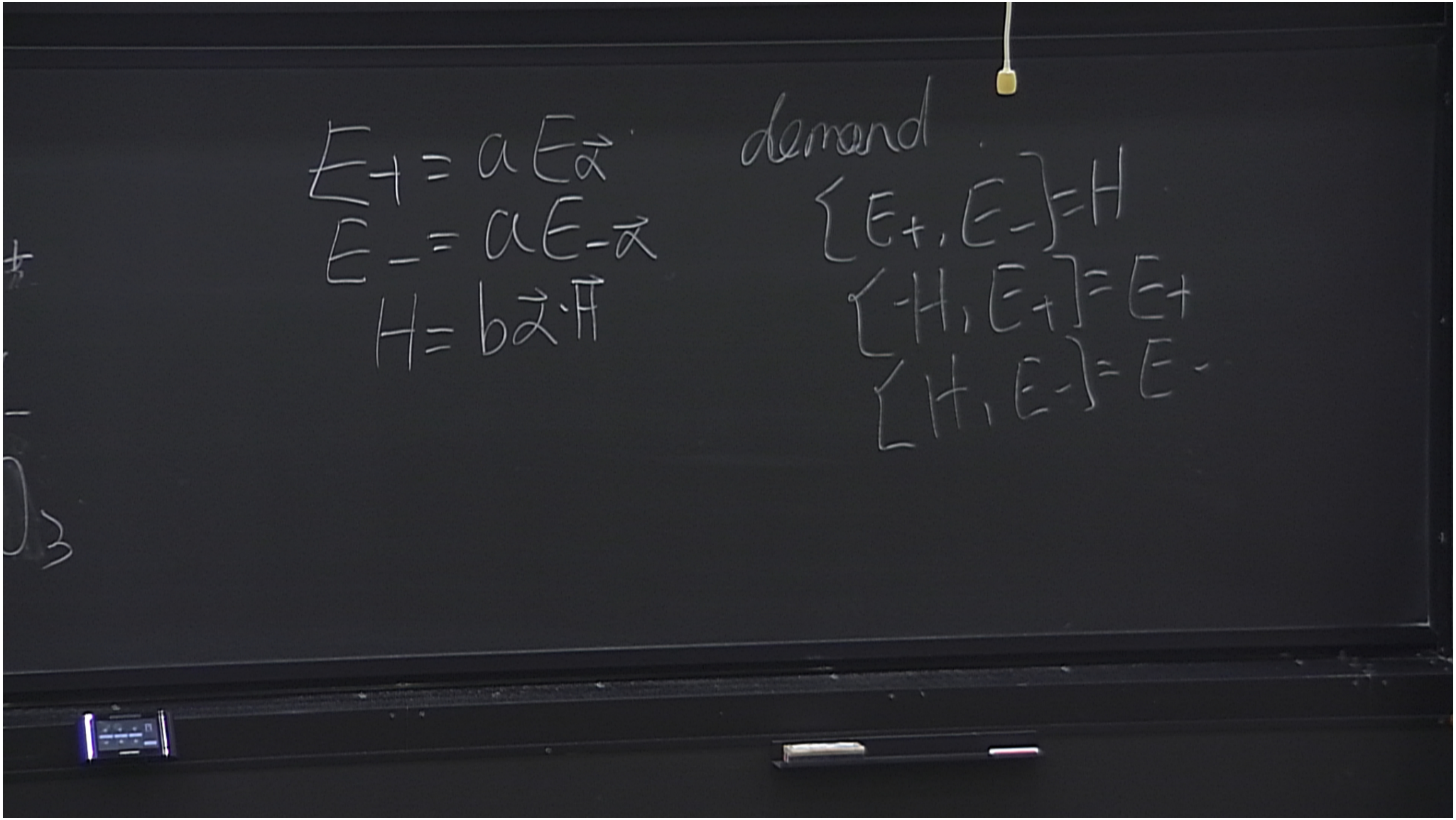

$[E_\alpha, E_{-\alpha}] = a_i H_i$
$\langle H_i | E_\alpha^{ad} | E_{-\alpha} \rangle = a_i \langle H_i | H_i \rangle$
$a_i = \langle H_i | E_\alpha^{ad} | E_{-\alpha} \rangle$
$= \langle H_i | [E_\alpha, E_{-\alpha}] \rangle$
$[J_3, [J_+, J_-]] = 0$
$[J_+, J_-] = a J_3$
$a_i = Tr[H_i [E_\alpha, E_{-\alpha}]]$

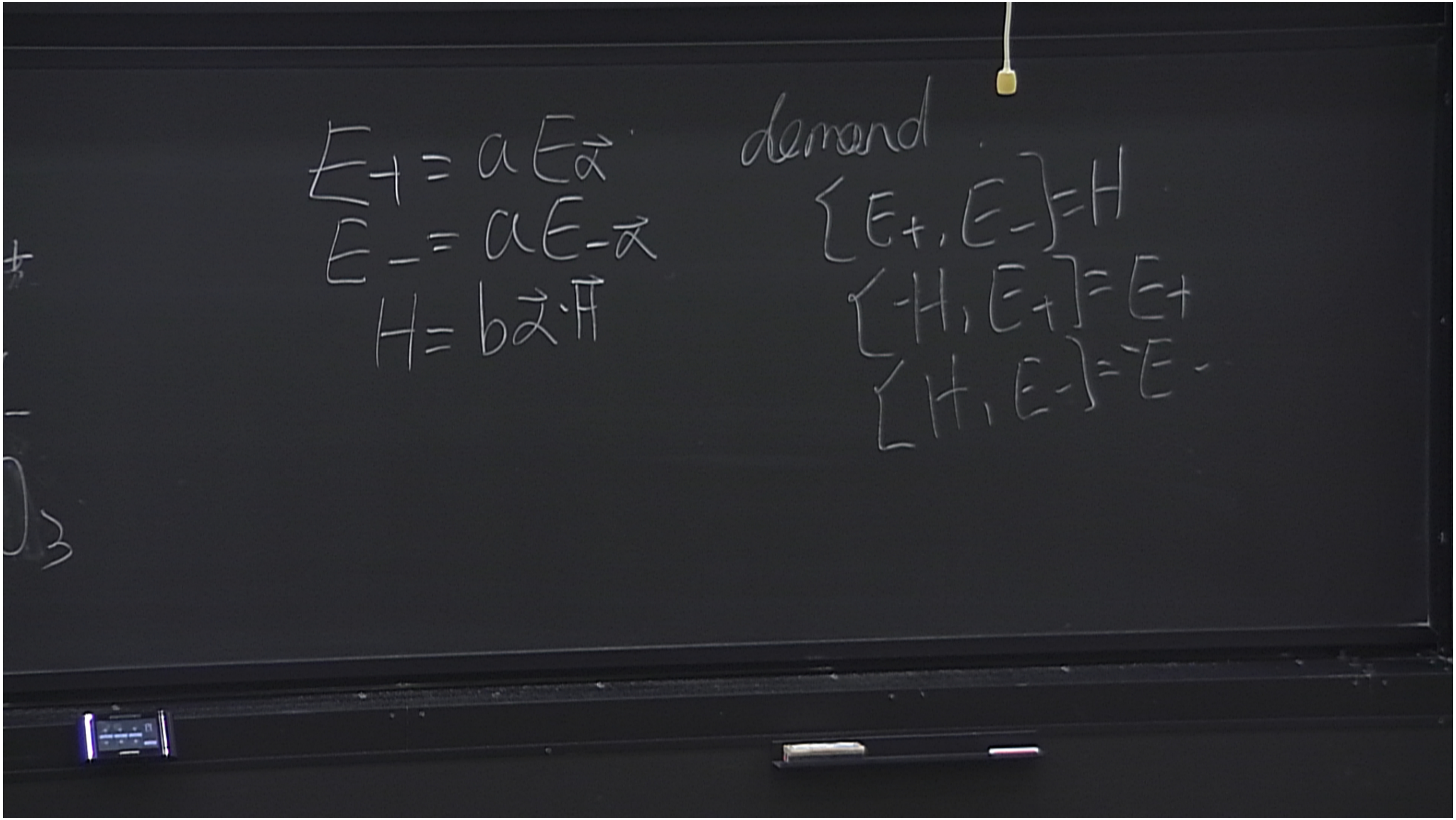

$[J_3,[J_+,J_-]]=0$
$[J_+,J_-]=\alpha J_3$
$\alpha_i = \mathrm{Tr}[H_i[E_{\vec{\alpha}},E_{-\vec{\alpha}}]]$
$= \mathrm{Tr}[H_i E_{\vec{\alpha}} E_{-\vec{\alpha}} - H_i E_{-\vec{\alpha}} E_{\vec{\alpha}}]$
$= \mathrm{Tr}[E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{\vec{\alpha}} H_i E_{-\vec{\alpha}}]$
$= \mathrm{Tr}[E_{\vec{\alpha}}[E_{-\vec{\alpha}},H_i]]$
$= \mathrm{Tr}[E_{\vec{\alpha}}(-1)[H_i,E_{-\vec{\alpha}}]]$
$= -\mathrm{Tr}[E_{\vec{\alpha}}(-\alpha_i)E_{-\vec{\alpha}}]$
$= \alpha_i \mathrm{Tr}[E_{\vec{\alpha}} E_{\vec{\alpha}}]$
$|T_a|t_b\rangle = |[t_a,t_b$

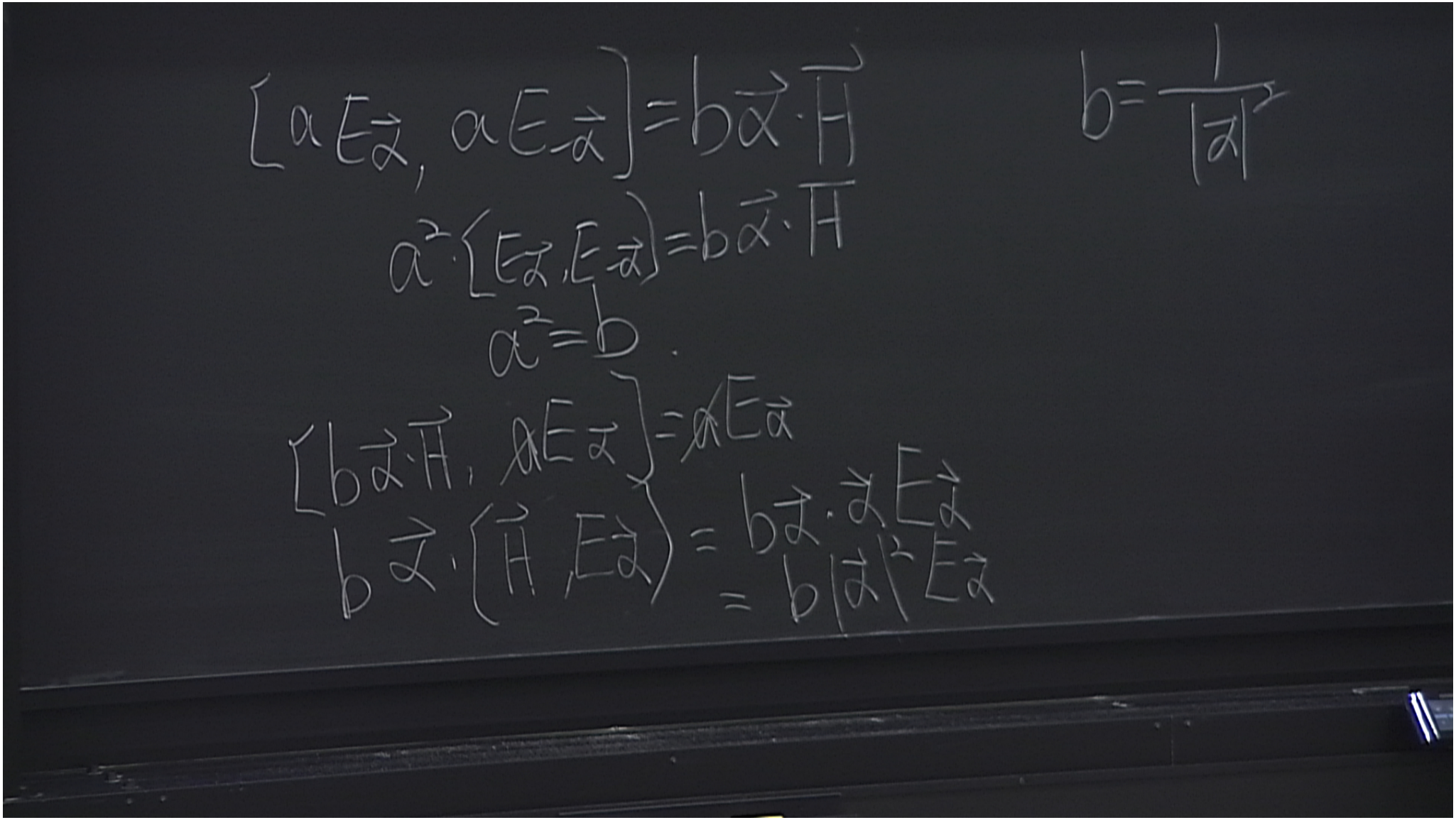

$[J_3,[J_+,J_-]]=0$
$[J_+,J_-]=\alpha J_3$
$\alpha_i = Tr[H_i[E_{\vec{\alpha}},E_{-\vec{\alpha}}]]$
$= Tr[H_i E_{\vec{\alpha}} E_{-\vec{\alpha}} - H_i E_{-\vec{\alpha}} E_{\vec{\alpha}}]$
$= Tr[E_{\vec{\alpha}} E_{-\vec{\alpha}} H_i - E_{\vec{\alpha}} H_i E_{-\vec{\alpha}}]$
$= Tr[E_{\vec{\alpha}}[E_{-\vec{\alpha}}, H_i]]$
$|T_a|t_b\rangle = |[t_a, t_b$
$= Tr[E_{\vec{\alpha}}(-1)(H_i, E_{-\vec{\alpha}}]$
$= -Tr[E_{\vec{\alpha}}(-\alpha_i)E_{-\vec{\alpha}}]$
$= \alpha_i Tr[E_{\vec{\alpha}} E_{\vec{\alpha}}]$
$= \alpha_i$

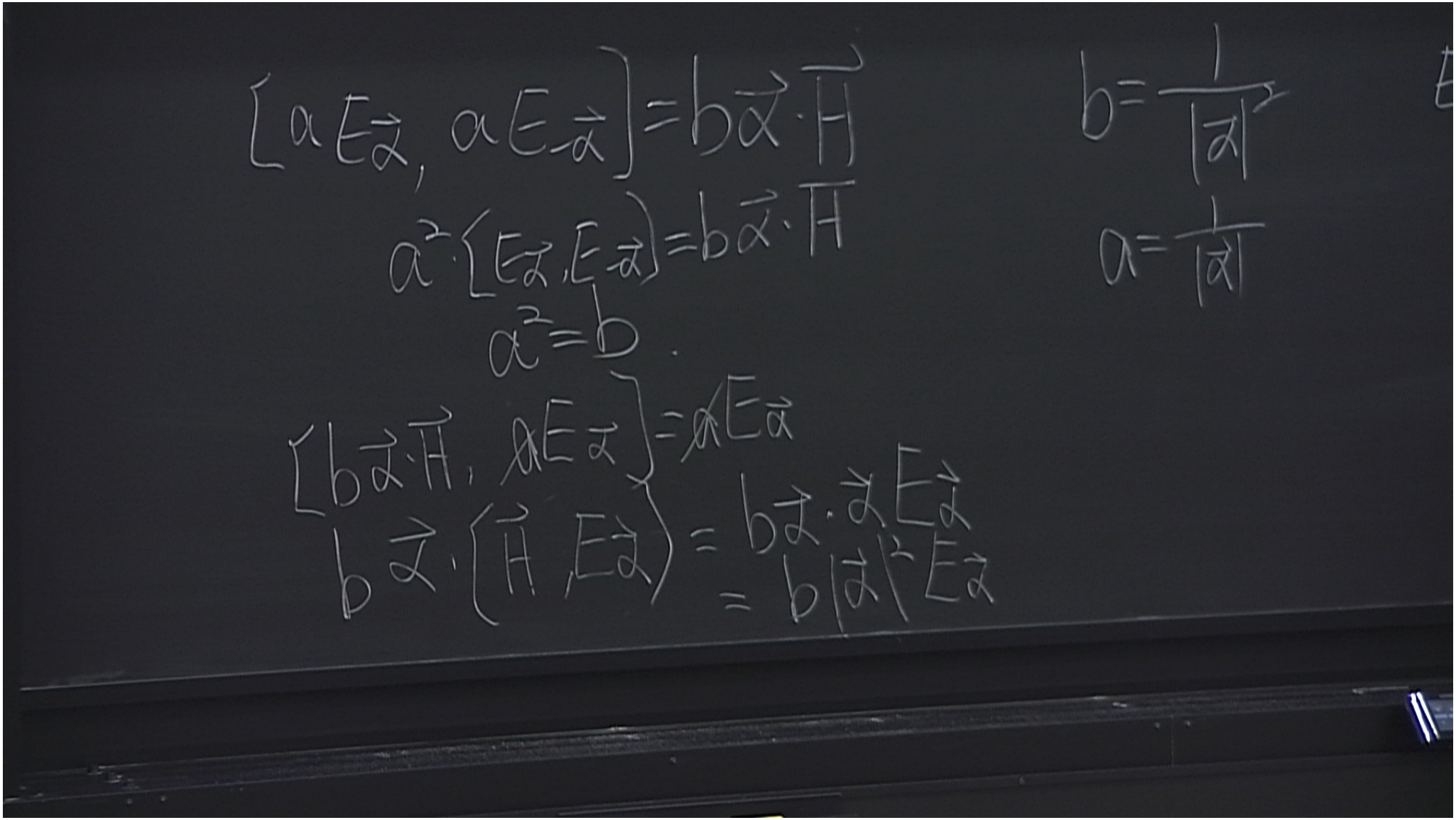

$E_{\vec{\alpha}}$ , $E_{-\vec{\alpha}}$

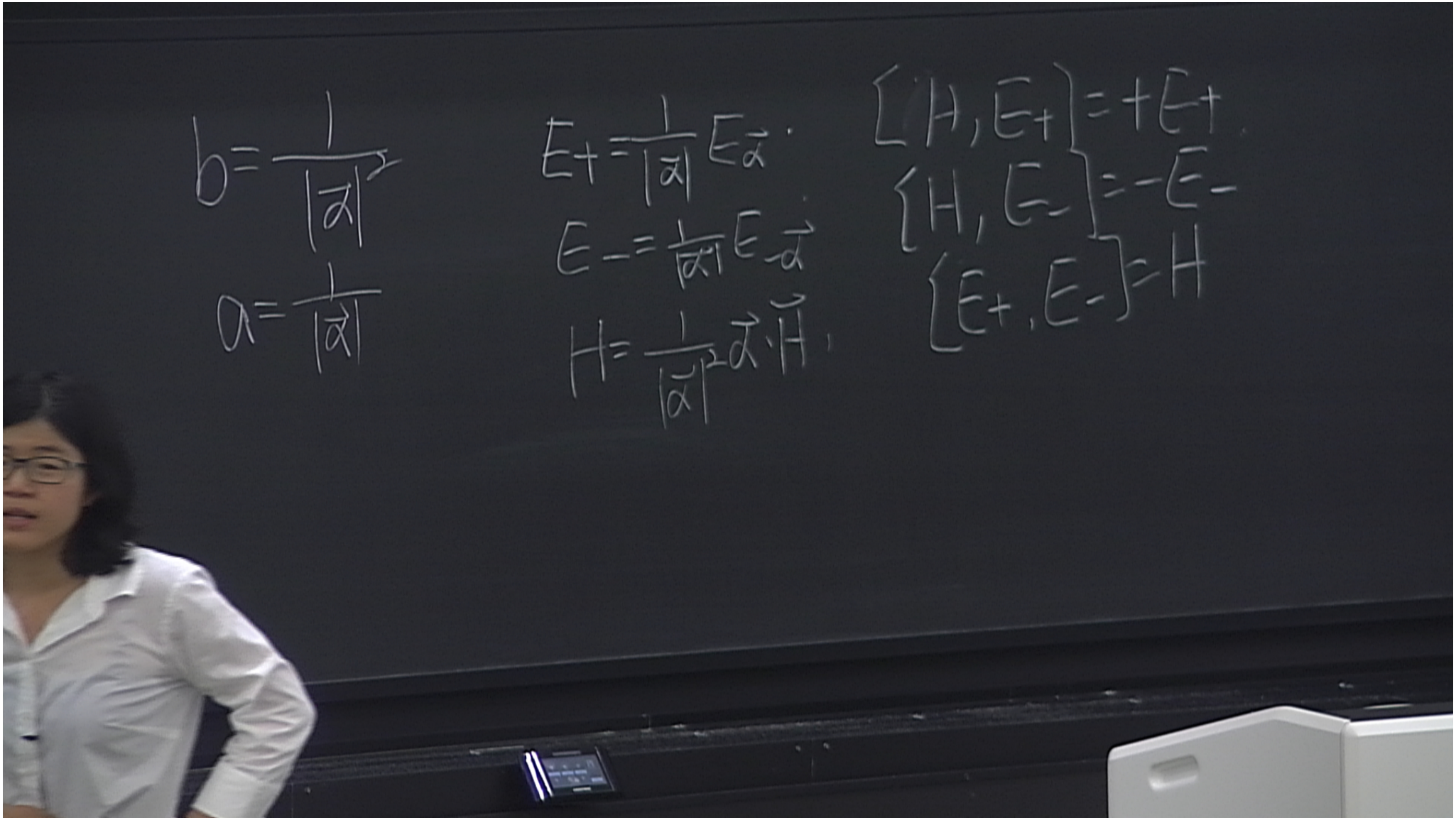

$E_{\vec{\alpha}}$ , $E_{-\vec{\alpha}}$
$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$
$[H_i, E_{-\vec{\alpha}}] = -\alpha_i E_{-\vec{\alpha}}$
$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha} \cdot \vec{H}$
$[J_3, J_+] = +J_+$
$[J_3, J_-] = -J_-$
$[J_+, J_-] = J_3$

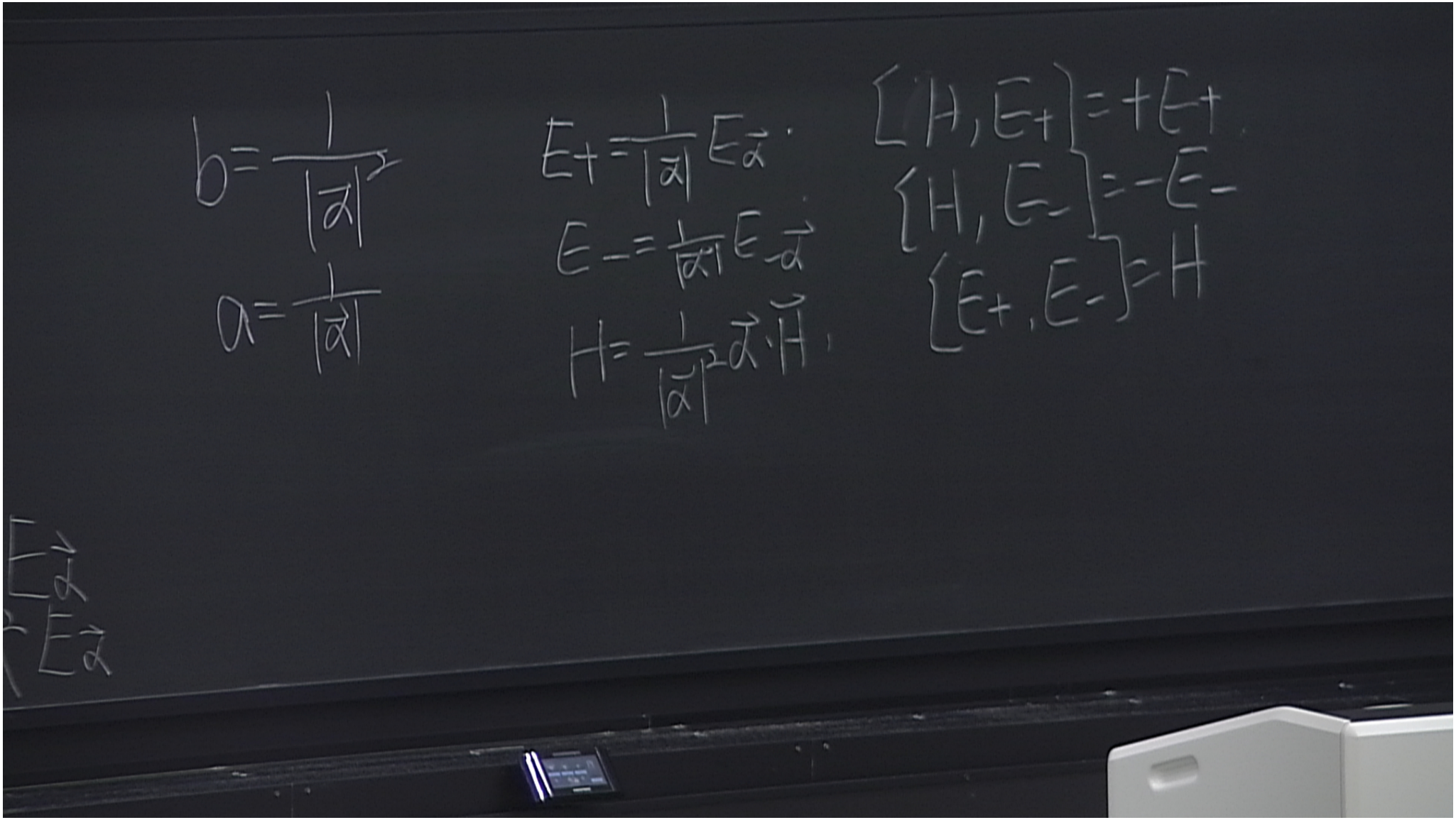

$E_+ = a E_{\vec{\alpha}}$
$E_- = a E_{-\vec{\alpha}}$
$H = b \vec{\alpha} \cdot \vec{H}$
demand
$[E_+, E_-] = H$
$[H, E_+] = E_+$
$[H, E_-] = E_-$

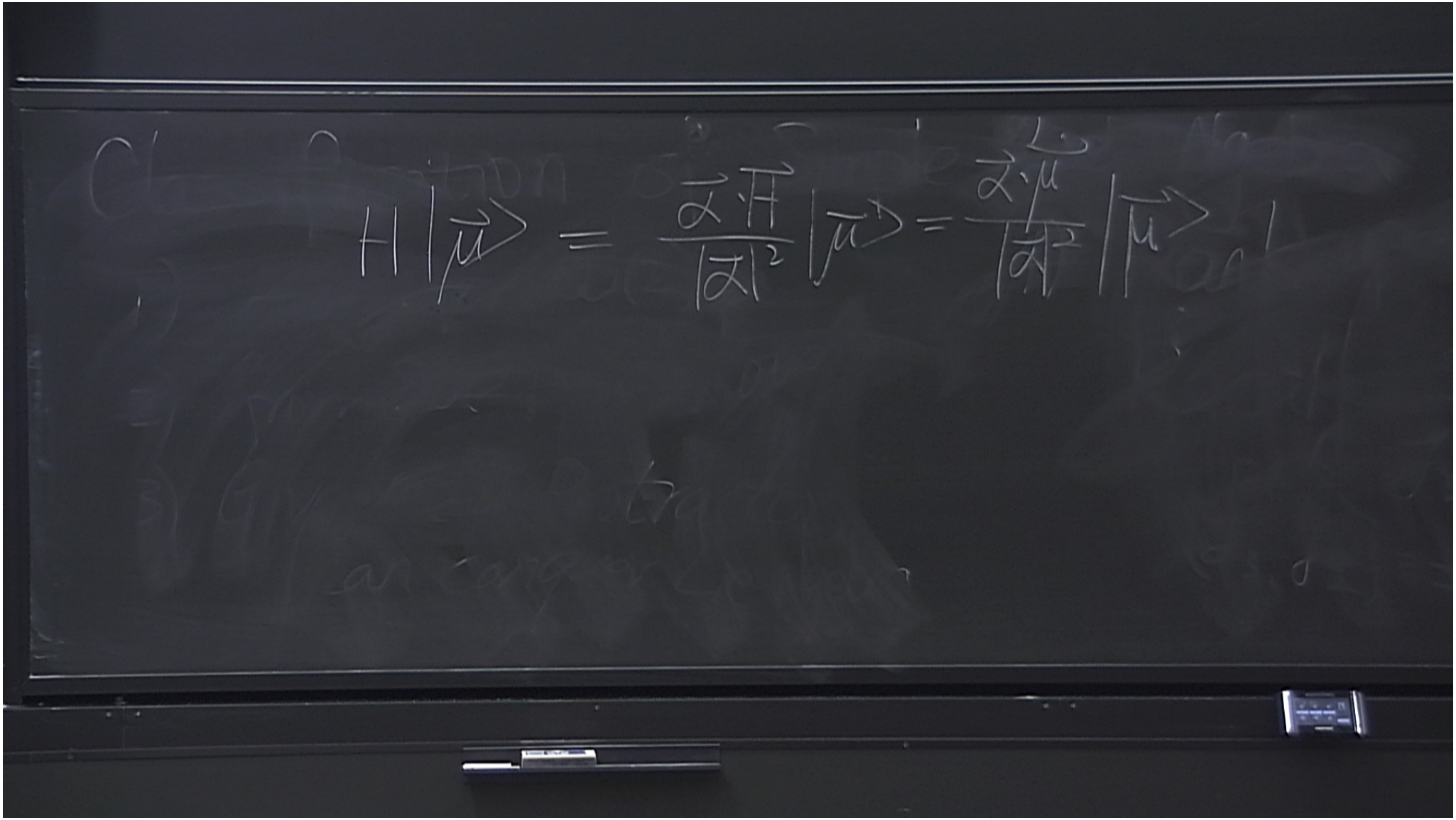

$E_+ = a E_{\vec{\alpha}}$
$E_- = a E_{-\vec{\alpha}}$
$H = b \vec{\alpha} \cdot \vec{H}$
demand
$[E_+, E_-] = H$
$[H, E_+] = E_+$
$[H, E_-] = -E_-$

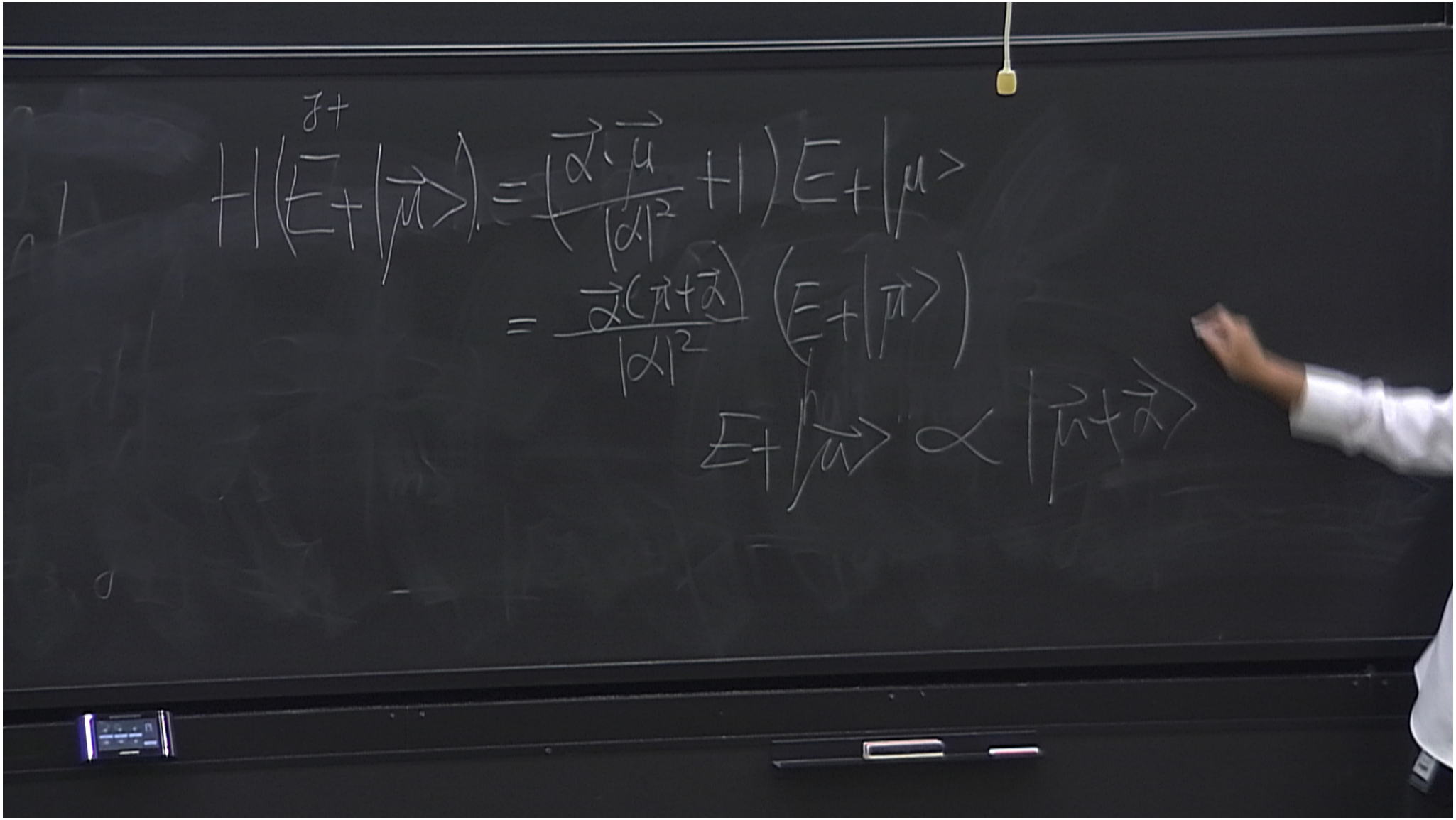

$[aE_{\vec{\alpha}}, aE_{-\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{H}$
$b = \frac{1}{|\vec{\alpha}|^2}$
$a^2\cdot[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{H}$
$a^2 = b$
$[b\vec{\alpha}\cdot\vec{H}, aE_{\vec{\alpha}}] = \alpha E_{\vec{\alpha}}$
$b\vec{\alpha}\cdot[\vec{H}, E_{\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{\alpha}E_{\vec{\alpha}}$
$= b|\vec{\alpha}|^2 E_{\vec{\alpha}}$

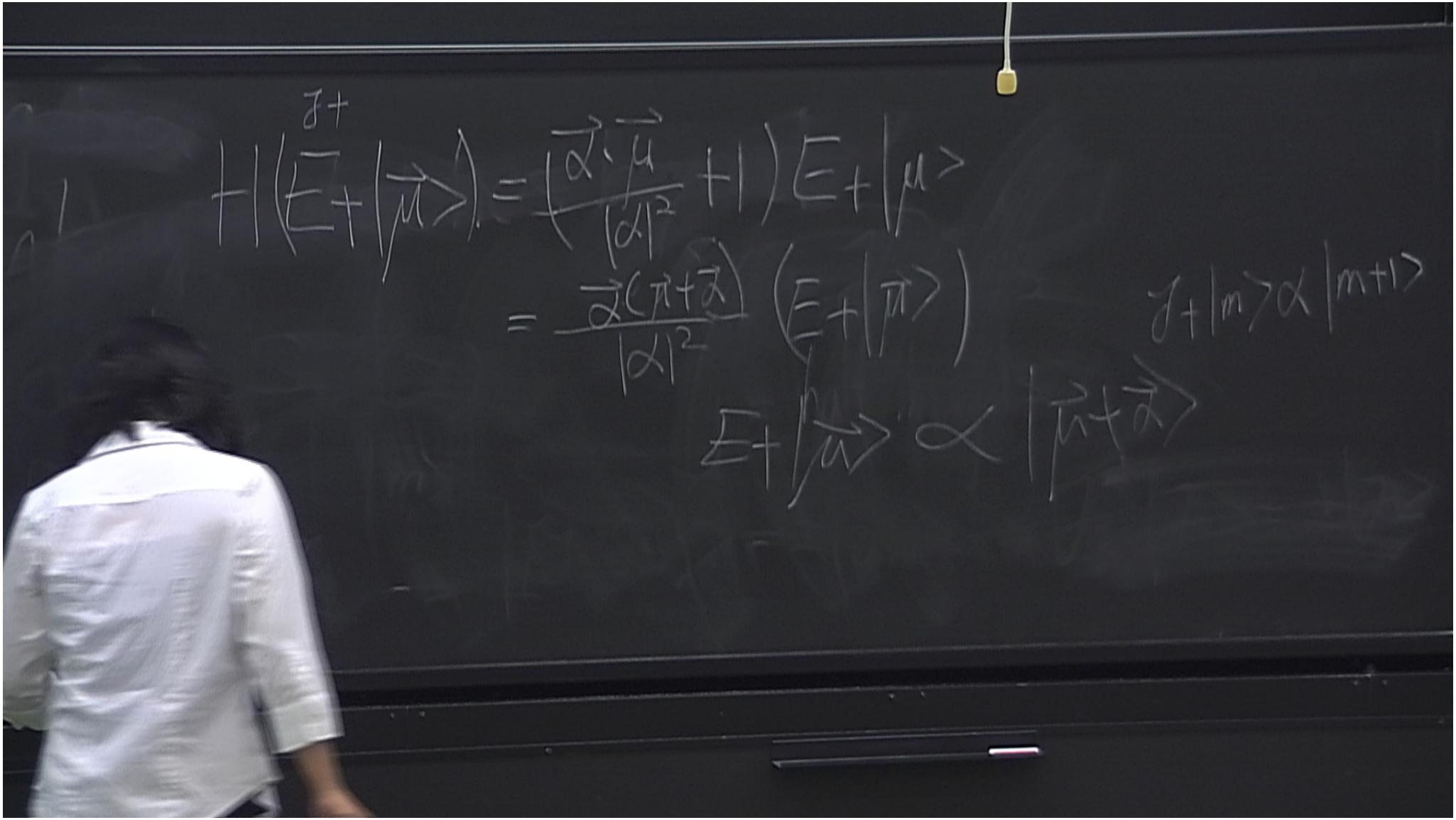

$[aE_{\vec{\alpha}}, aE_{-\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{H}$
$a^2\cdot[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{H}$
$a^2 = b$
$[b\vec{\alpha}\cdot\vec{H}, aE_{\vec{\alpha}}] = \alpha E_{\vec{\alpha}}$
$b\vec{\alpha}\cdot[\vec{H}, E_{\vec{\alpha}}] = b\vec{\alpha}\cdot\vec{\alpha}E_{\vec{\alpha}}$
$= b|\vec{\alpha}|^2 E_{\vec{\alpha}}$
$b = \frac{1}{|\vec{\alpha}|^2}$
$a = \frac{1}{|\vec{\alpha}|}$

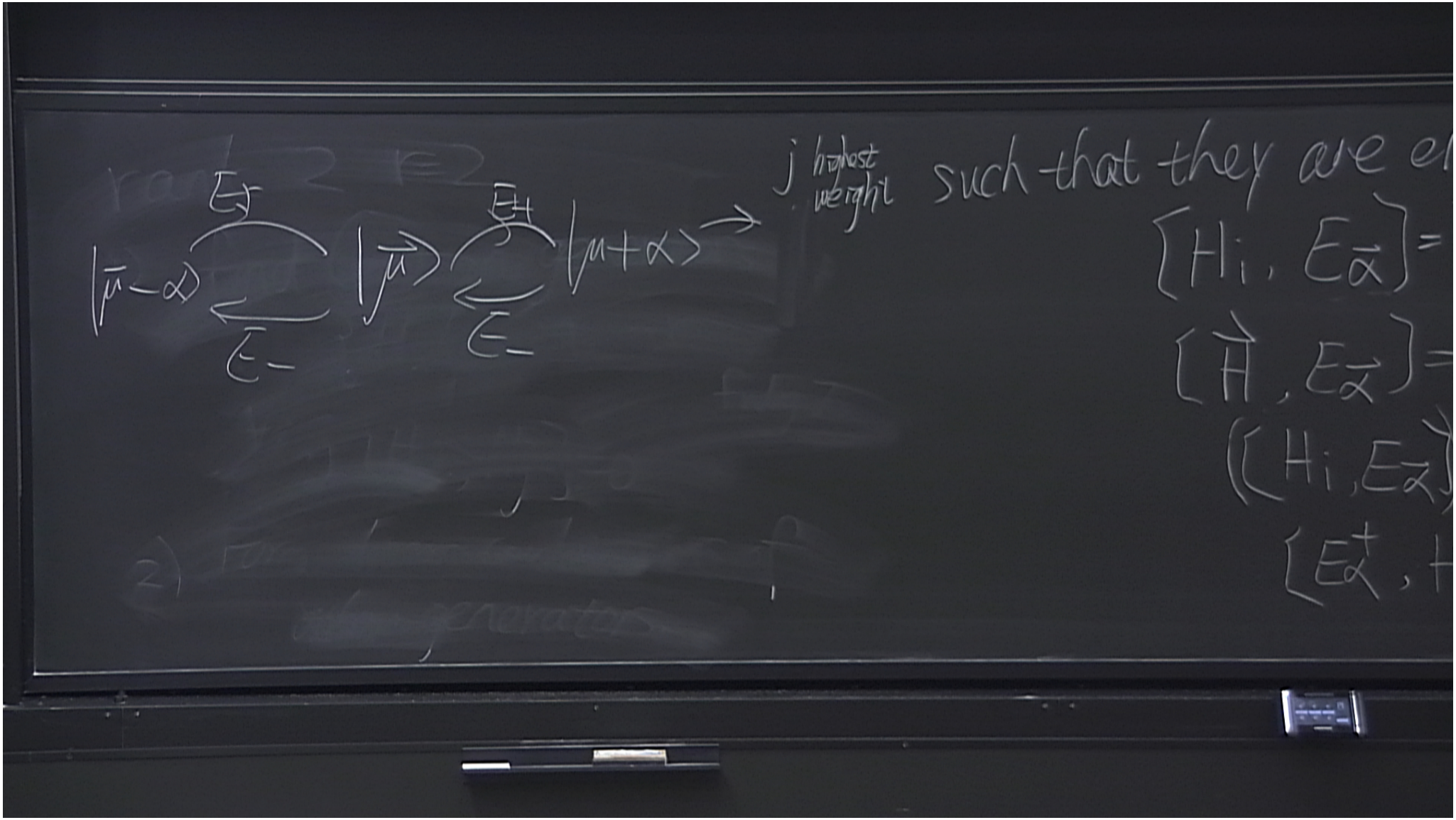

$b = \frac{1}{|\vec{\alpha}|^2}$
$a = \frac{1}{|\vec{\alpha}|}$
$E_+ = \frac{1}{|\vec{\alpha}|} E_{\vec{\alpha}}$
$E_- = \frac{1}{|\vec{\alpha}|} E_{-\vec{\alpha}}$
$H = \frac{1}{|\vec{\alpha}|^2} \vec{\alpha} \cdot \vec{H}$
$[H, E_+] = +E_+$
$[H, E_-] = -E_-$
$[E_+, E_-] = H$

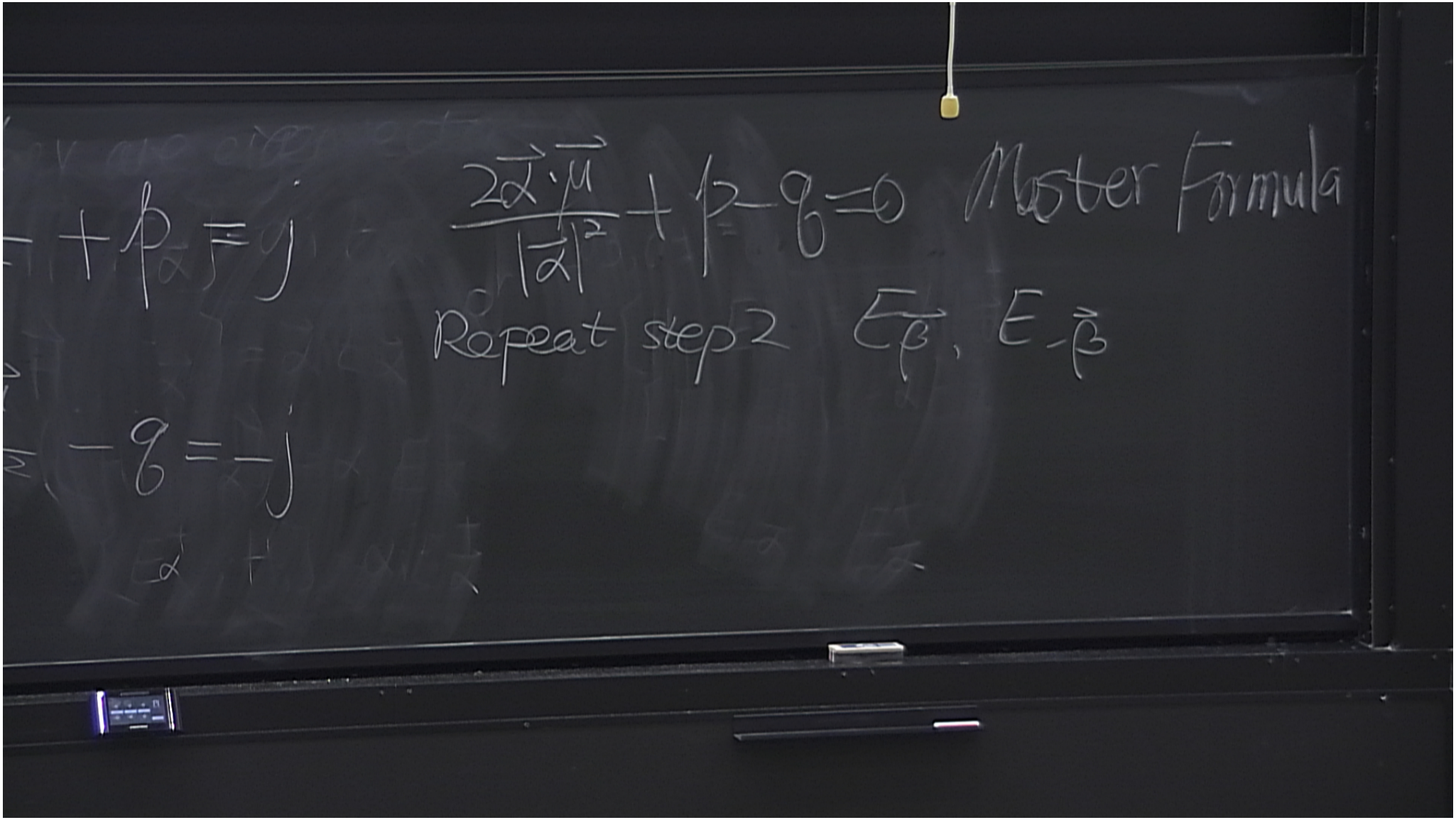

$b = \frac{1}{|\vec{\alpha}|^2}$
$a = \frac{1}{|\vec{\alpha}|}$
$E_+ = \frac{1}{|\vec{\alpha}|} E_{\vec{\alpha}}$
$E_- = \frac{1}{|\vec{\alpha}|} E_{-\vec{\alpha}}$
$H = \frac{1}{|\vec{\alpha}|^2} \vec{\alpha} \cdot \vec{H}$
$[H, E_+] = +E_+$
$[H, E_-] = -E_-$
$[E_+, E_-] = H$
$E_{\vec{\alpha}}$
$E_{\vec{\alpha}}$

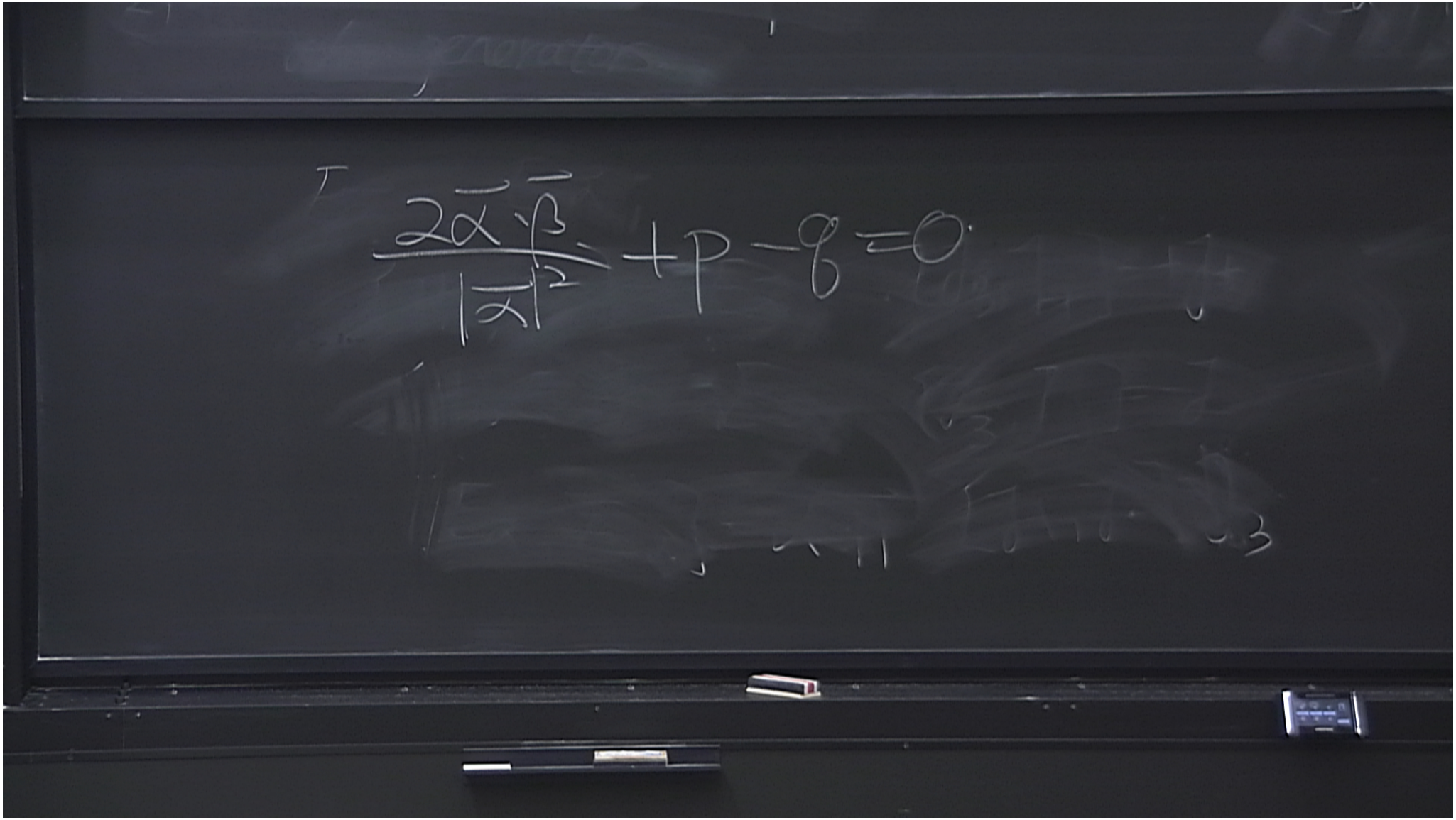

$H|\vec{\mu}\rangle = \frac{\alpha \cdot \vec{\Pi}}{|\alpha|^2}|\vec{\mu}\rangle = \frac{\alpha \cdot \vec{\mu}}{|\alpha|^2}|\vec{\mu}\rangle$

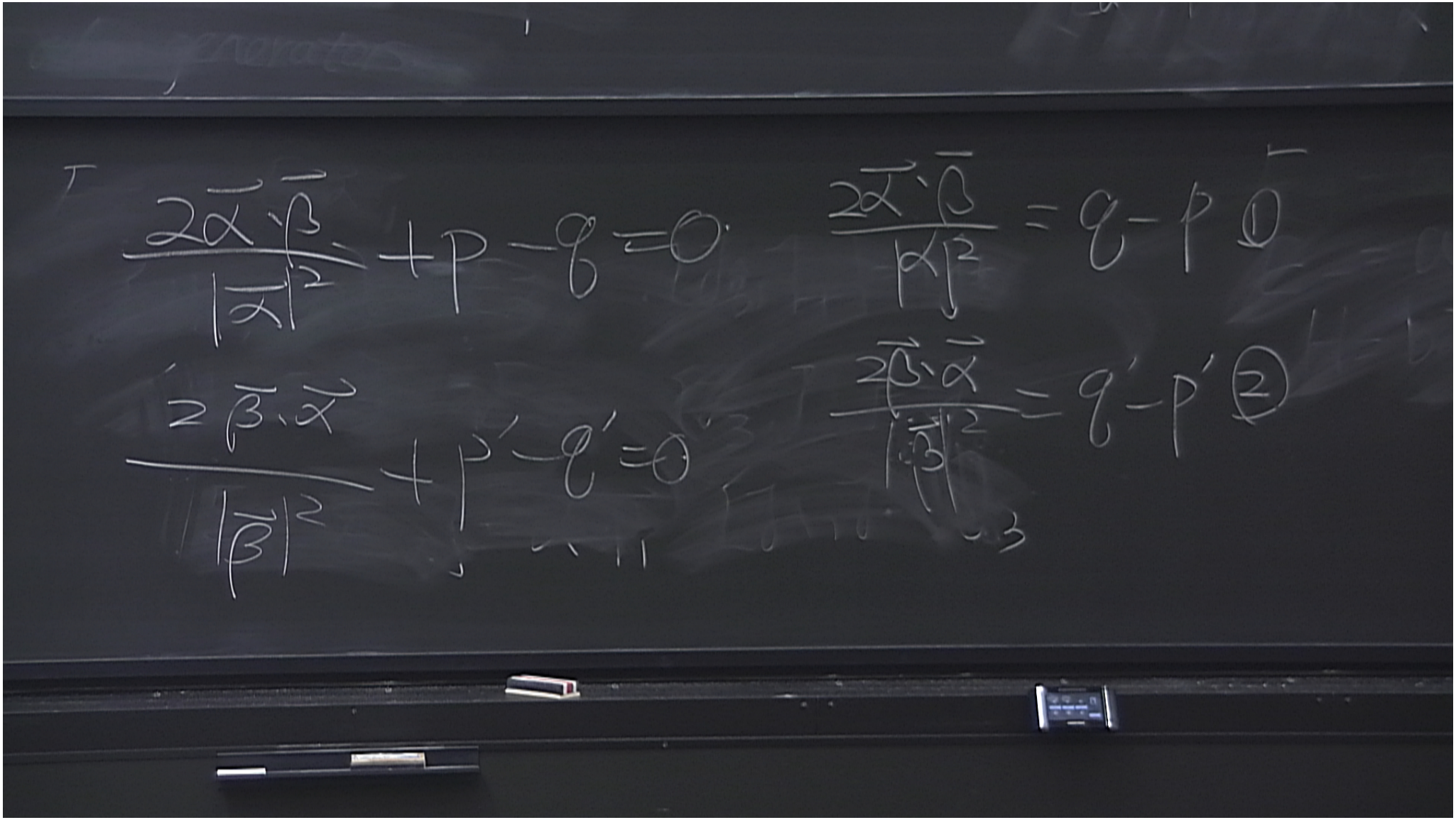

$H(E_+^{s+}|\vec{\mu}\rangle) = \left(\frac{\vec{\alpha}\cdot\vec{\mu}}{|\alpha|^2}+1\right)E_+|\mu\rangle$
$= \frac{\vec{\alpha}(\vec{\pi}+\vec{\alpha})}{|\alpha|^2}\left(E_+|\vec{\pi}\rangle\right)$
$E_+|\vec{\mu}\rangle \propto |\vec{\pi}+\vec{\alpha}\rangle$

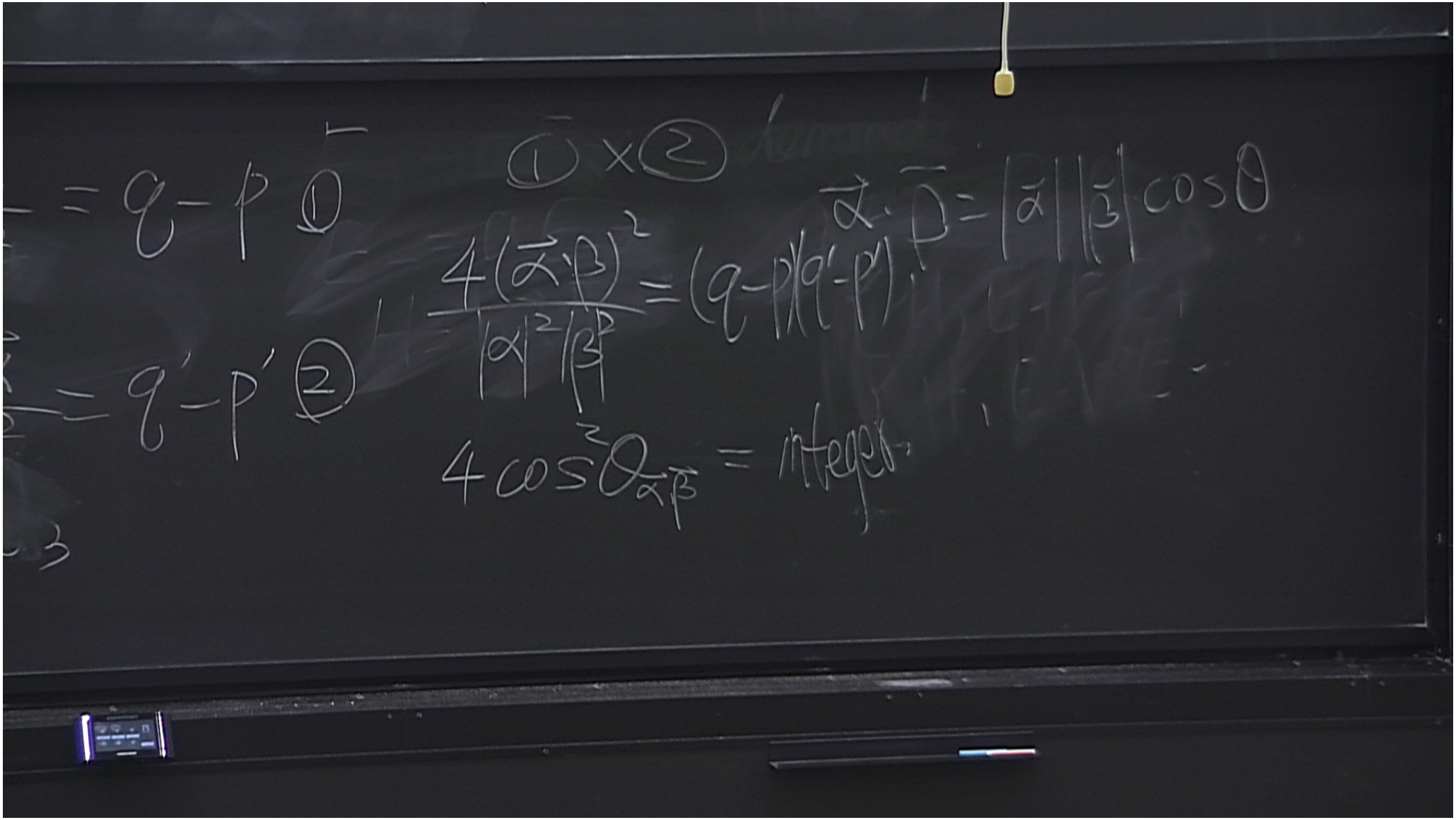

$H(E_+^{\vec{\pi}}|\vec{\mu}\rangle) = \left(\frac{\vec{\alpha}\cdot\vec{\mu}}{|\alpha|^2} + 1\right) E_+|\mu\rangle$
$= \frac{\vec{\alpha}(\vec{\pi}+\vec{\alpha})}{|\alpha|^2} \left(E_+|\vec{\pi}\rangle\right)$
$a_+|m\rangle \propto |m+1\rangle$
$E_+|\vec{\mu}\rangle \propto |\vec{\mu}+\vec{\alpha}\rangle$

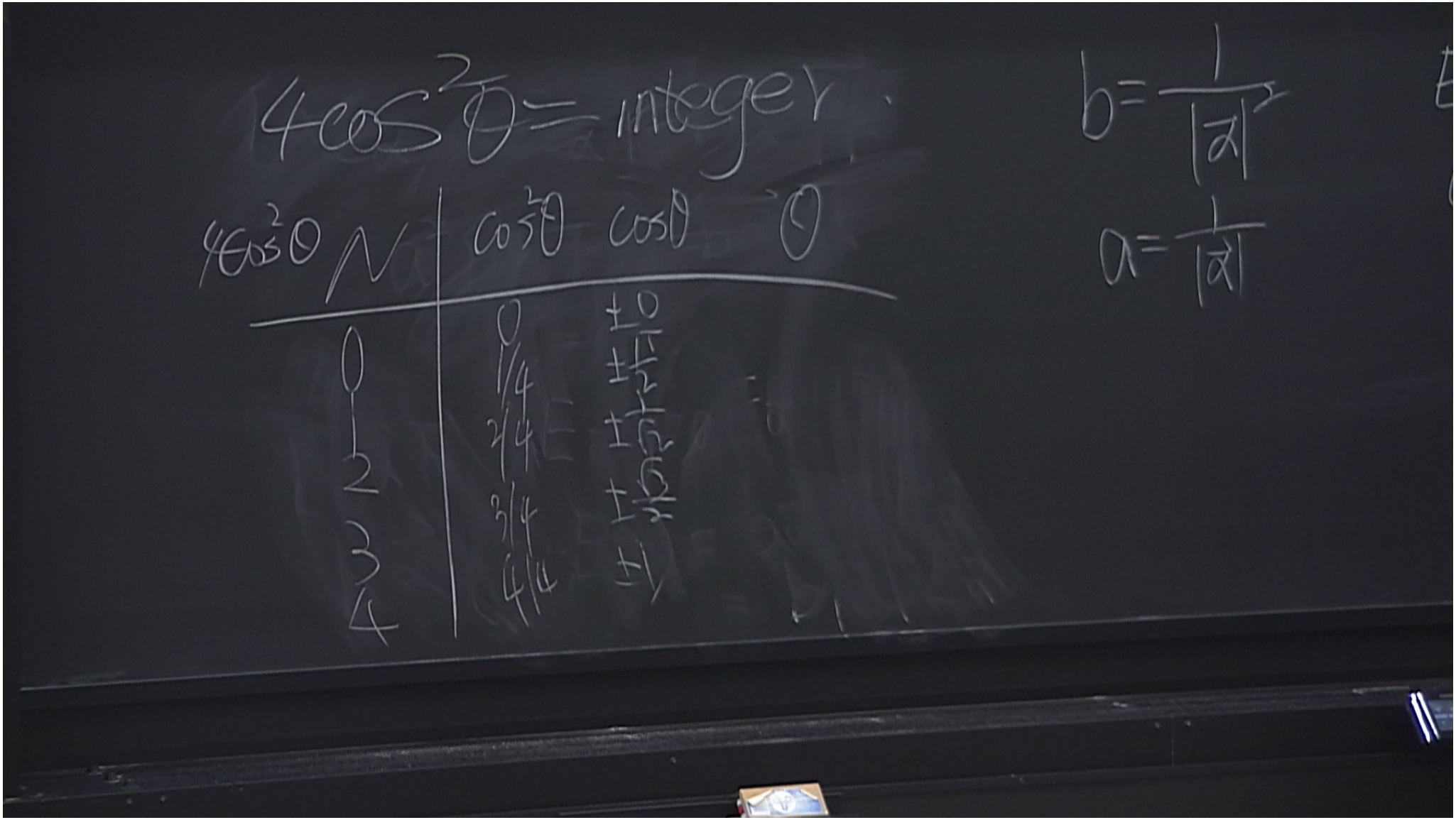

$|\mu-\alpha\rangle$
$E_+$
$E_-$
$|\mu\rangle$
$E_+$
$E_-$
$|\mu+\alpha\rangle \rightarrow$
j highest weight
such that they are e
$[H_i, E_\alpha] =$
$[\vec{H}, E_\alpha] =$
$([H_i, E_\alpha]$
$[E_\alpha^+, H$

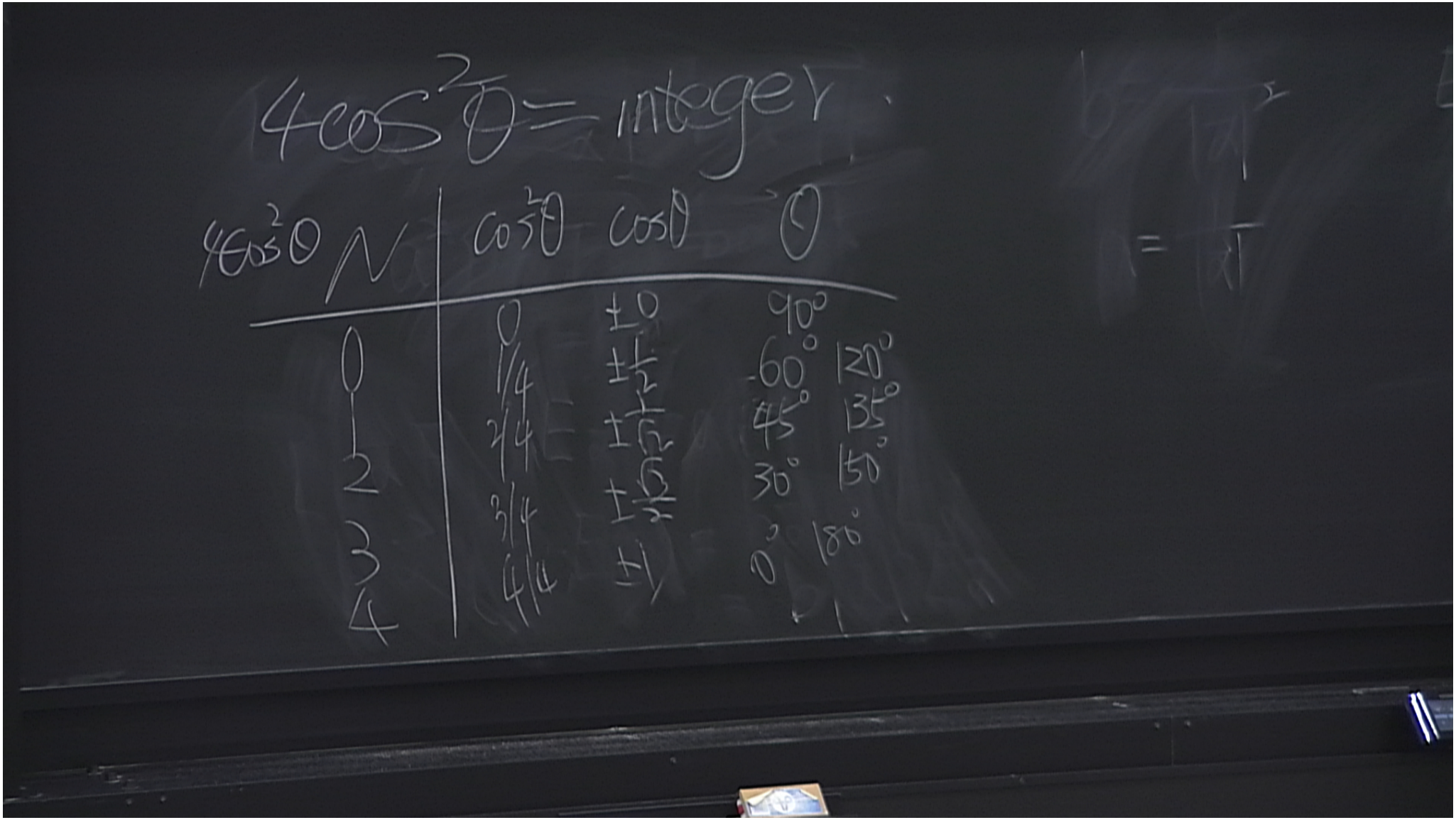

$+p=j$
$-q=-j$
$\frac{2\vec{\alpha}\cdot\vec{\mu}}{|\vec{\alpha}|^2}+p-q=0$ Master Formula
Repeat step 2 $E_{\vec{\beta}}$, $E_{-\vec{\beta}}$

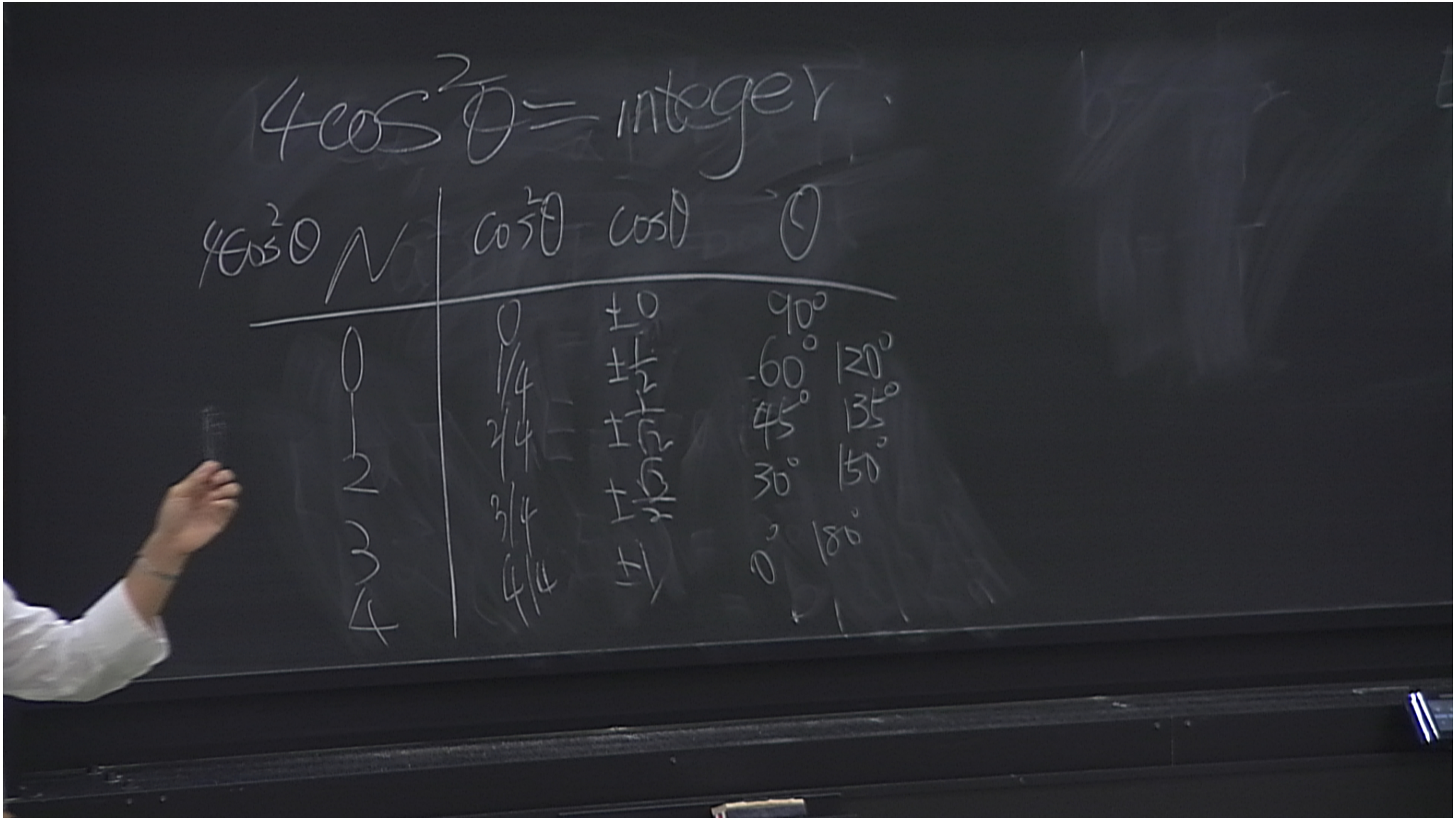

$\frac{2\bar{\alpha}\cdot\bar{\beta}}{|\bar{\alpha}|^2} + p - q = 0$

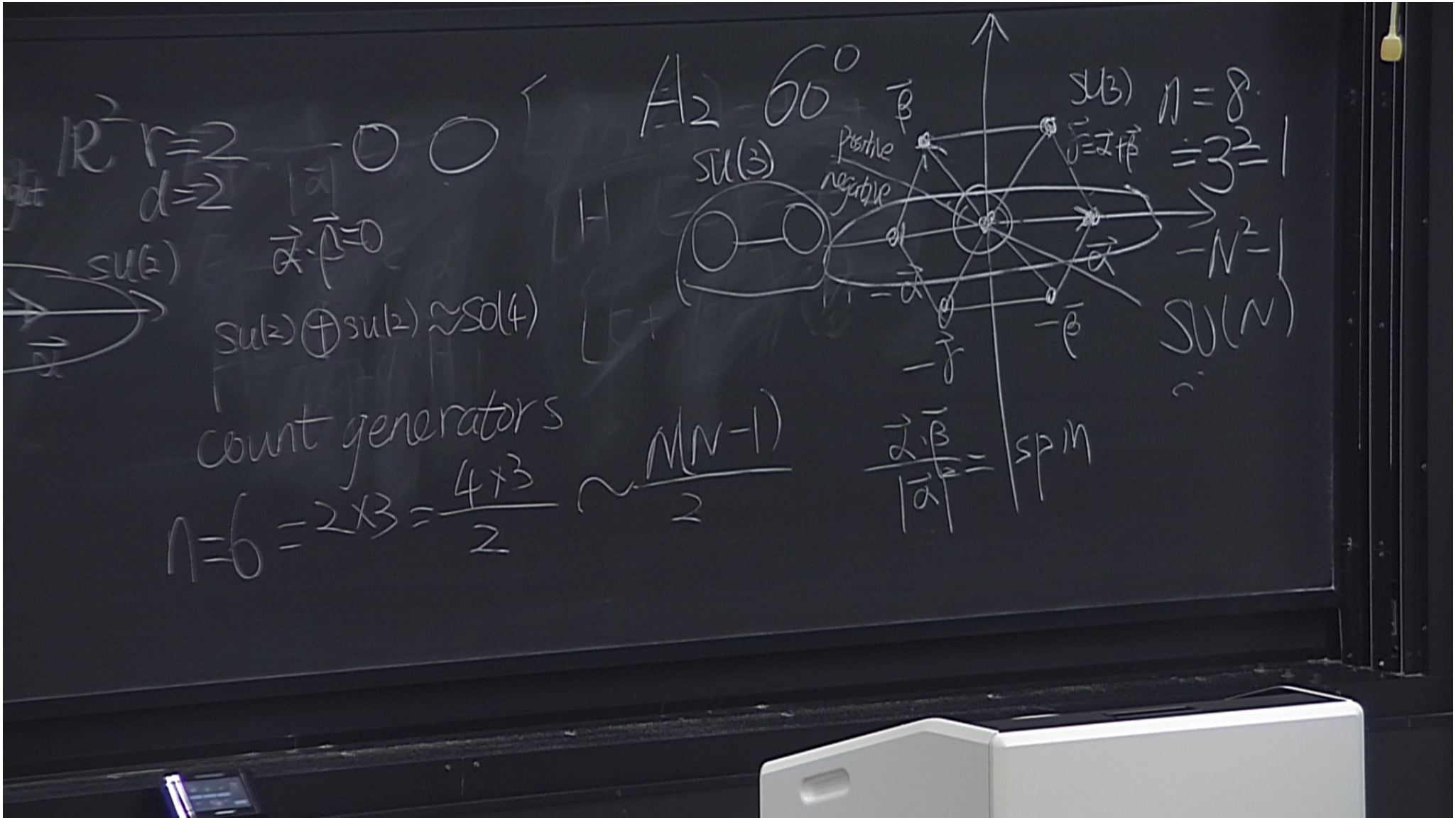

$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} + p - q = 0$
$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} = q - p$ ①
$\frac{2\vec{\beta}\cdot\vec{\alpha}}{|\vec{\beta}|^2} + p' - q' = 0$
$\frac{2\vec{\beta}\cdot\vec{\alpha}}{|\vec{\beta}|^2} = q' - p'$ ②

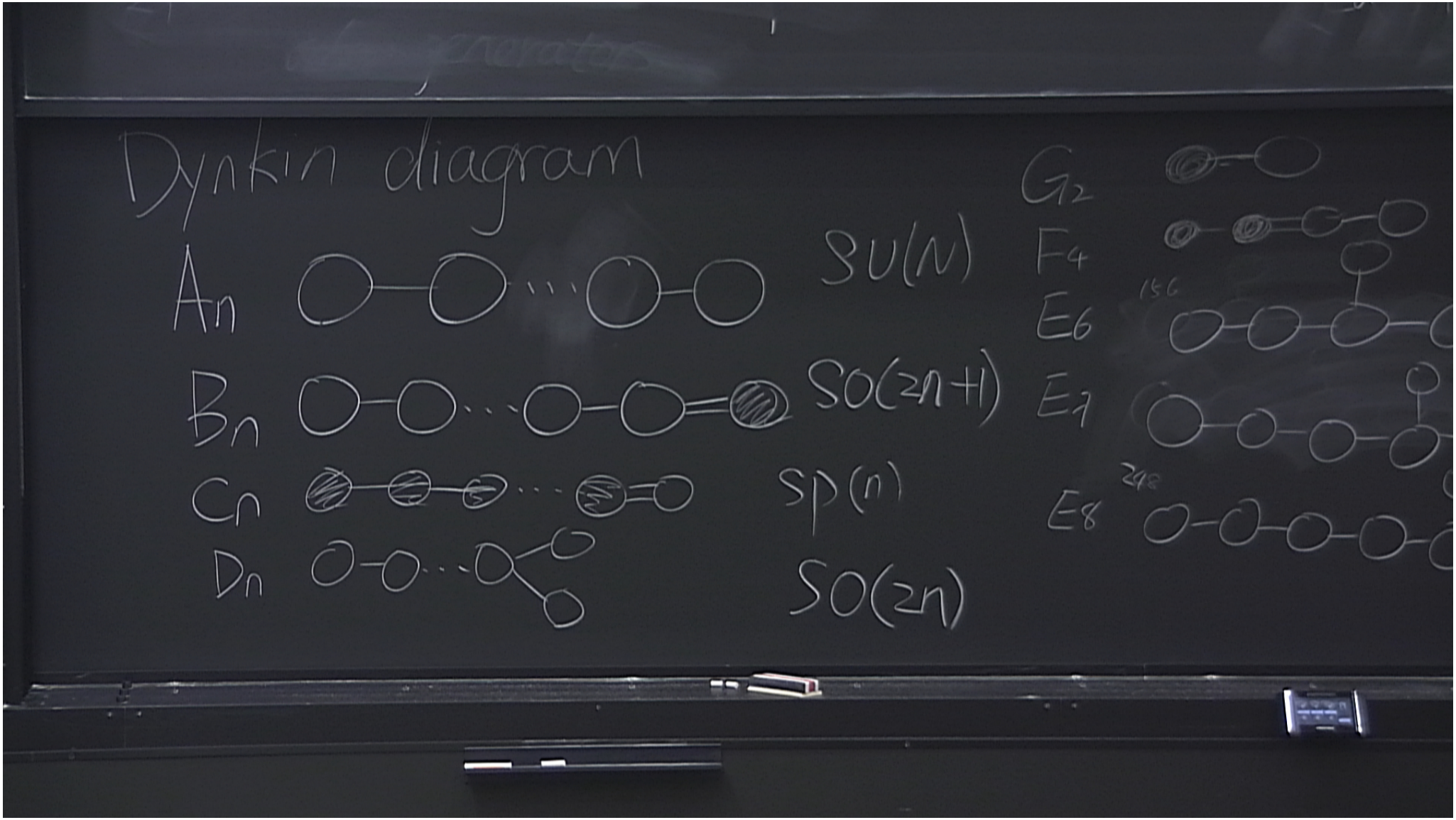

= q − p ①
= q′ − p′ ②
① × ②
$\frac{4(\vec{\alpha}\cdot\vec{\beta})^2}{|\alpha|^2|\beta|^2} = (q-p)(q'-p')$
$\vec{\alpha}\cdot\vec{\beta} = |\alpha||\beta|\cos\theta$
$4\cos^2\theta_{\alpha\beta}$ = integer.


$4\cos^2\theta = \text{integer}$
$b = \frac{1}{|\alpha|^2}$
$a = \frac{1}{|\alpha|}$
$4\cos^2\theta$ | $N$ | $\cos^2\theta$ | $\cos\theta$ | $\theta$
0 | 0 | $\pm 0$
1 | 1/4 | $\pm\frac{\pi}{3}$
2 | 2/4 | $\pm\frac{\pi}{2}$
3 | 3/4 | $\pm\frac{2\pi}{3}$
4 | 4/4 | $\pm 1$


$4\cos^2\theta = \text{integer}$
$4\cos^2\theta$ | $N$ | $\cos^2\theta$ | $\cos\theta$ | $\theta$
0 | 0/4 | 0 | ±0 | 90°
1 | 1/4 | 1/4 | ±1/2 | 60° 120°
2 | 2/4 | 2/4 | ±1/√2 | 45° 135°
3 | 3/4 | 3/4 | ±√3/2 | 30° 150°
4 | 4/4 | 1 | ±1 | 0° 180°


$4\cos^2\theta = \text{integer}$
$4\cos^2\theta$ | $N$ | $\cos^2\theta$ | $\cos\theta$ | $\theta$
0 | 0 | $\pm 0$ | $90°$
1 | 1/4 | $\pm\frac{1}{2}$ | $60°$ $120°$
2 | 2/4 | $\pm\frac{1}{\sqrt{2}}$ | $45°$ $135°$
3 | 3/4 | $\pm\frac{\sqrt{3}}{2}$ | $30°$ $150°$
4 | 4/4 | $\pm 1$ | $0°$ $180°$


$A_2$ 60°
SU(3) n = 8
positive
negative
$\vec{\alpha}\cdot\vec{\beta}=0$
SU(2) ⊕ SU(2) ≃ SO(4)
count generators
$n=6=2\times3=\frac{4\times3}{2}\sim\frac{N(N-1)}{2}$
$-N^2-1$
SU(N)
spin


Dynkin diagram
An
SU(N)
Bn
SO(2n+1)
Cn
Sp(n)
Dn
SO(2n)
G2
F4
E6
E7
E8