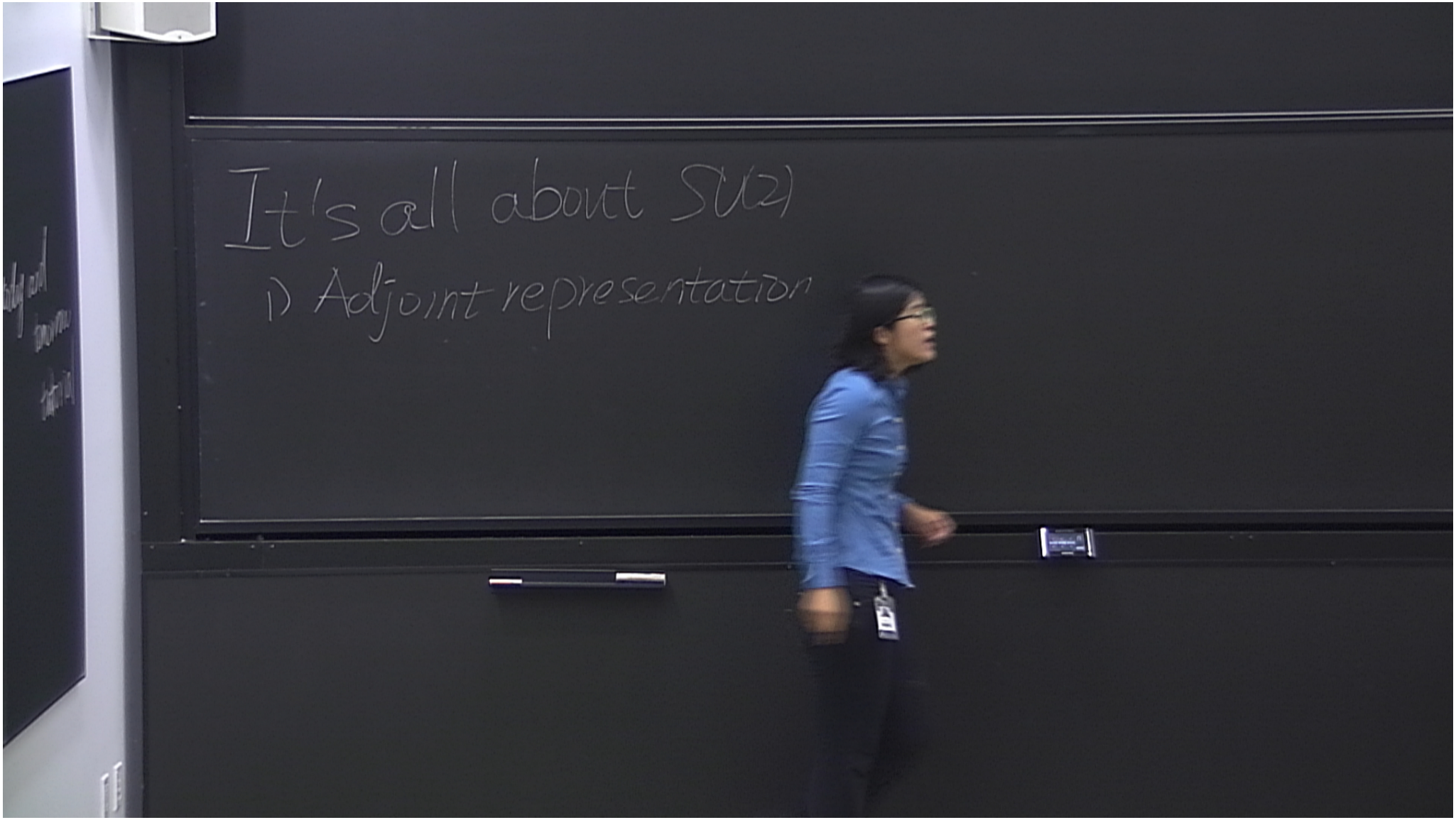
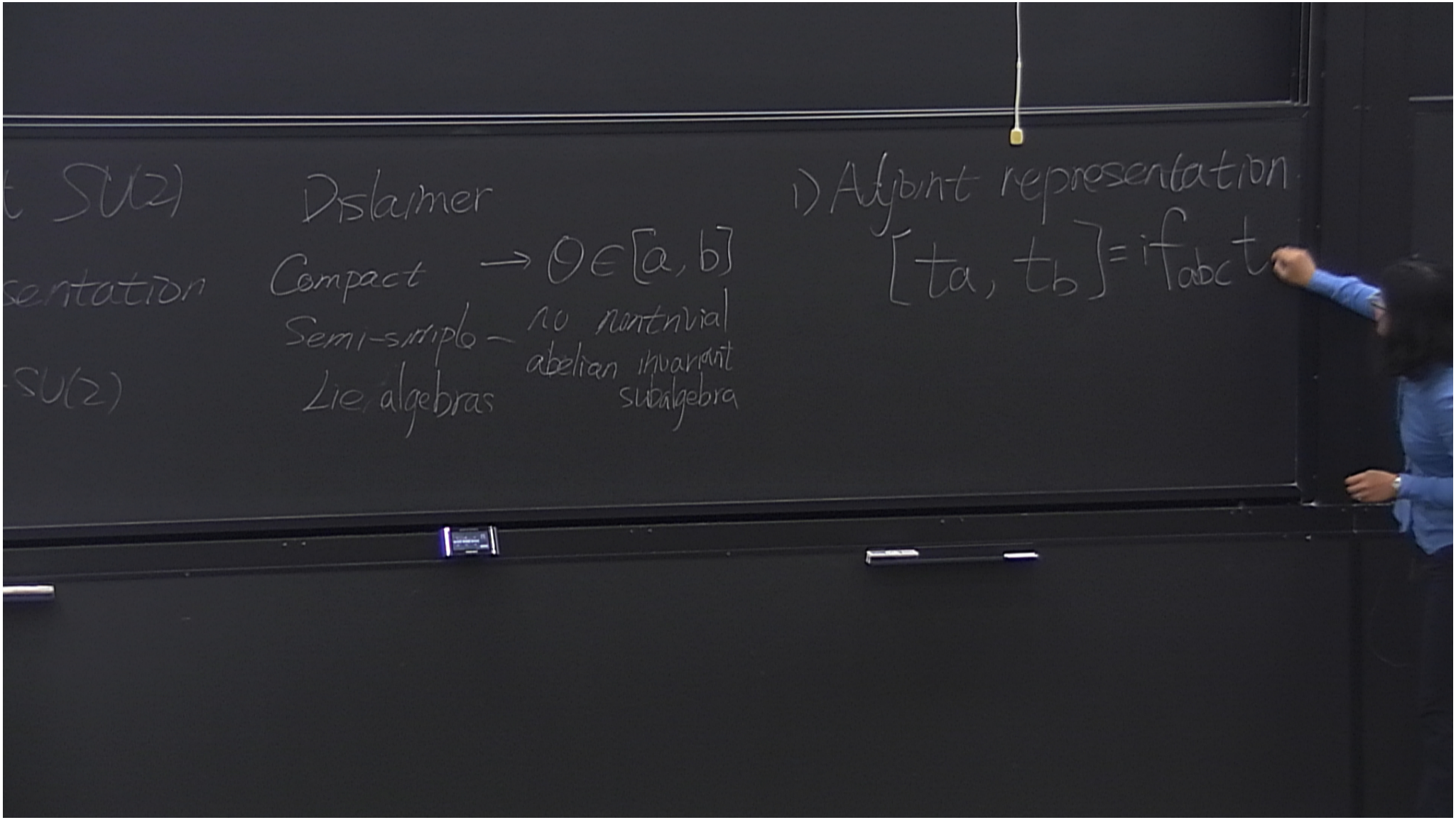
Title: 14/15 PSI - Lie Groups

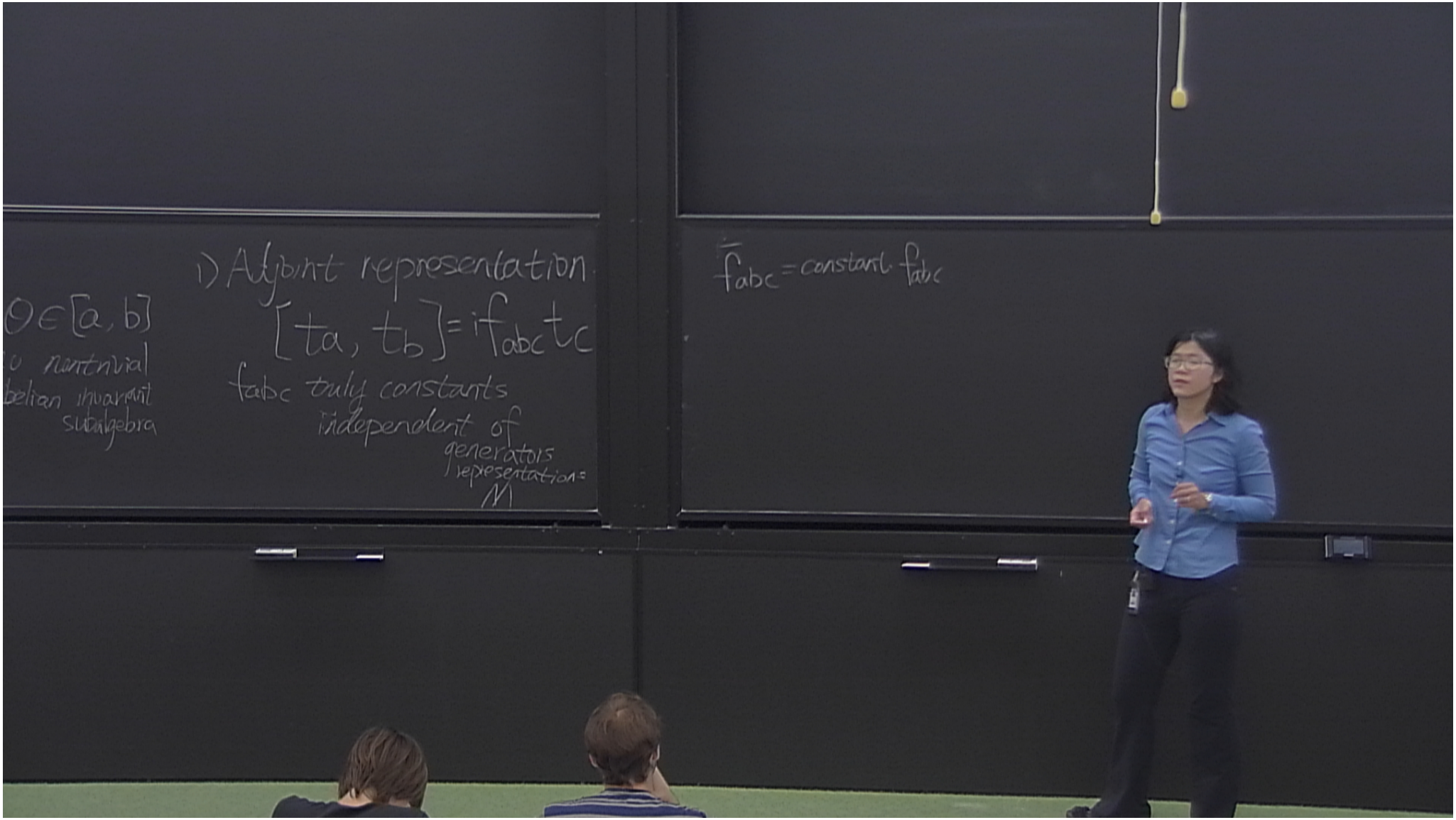
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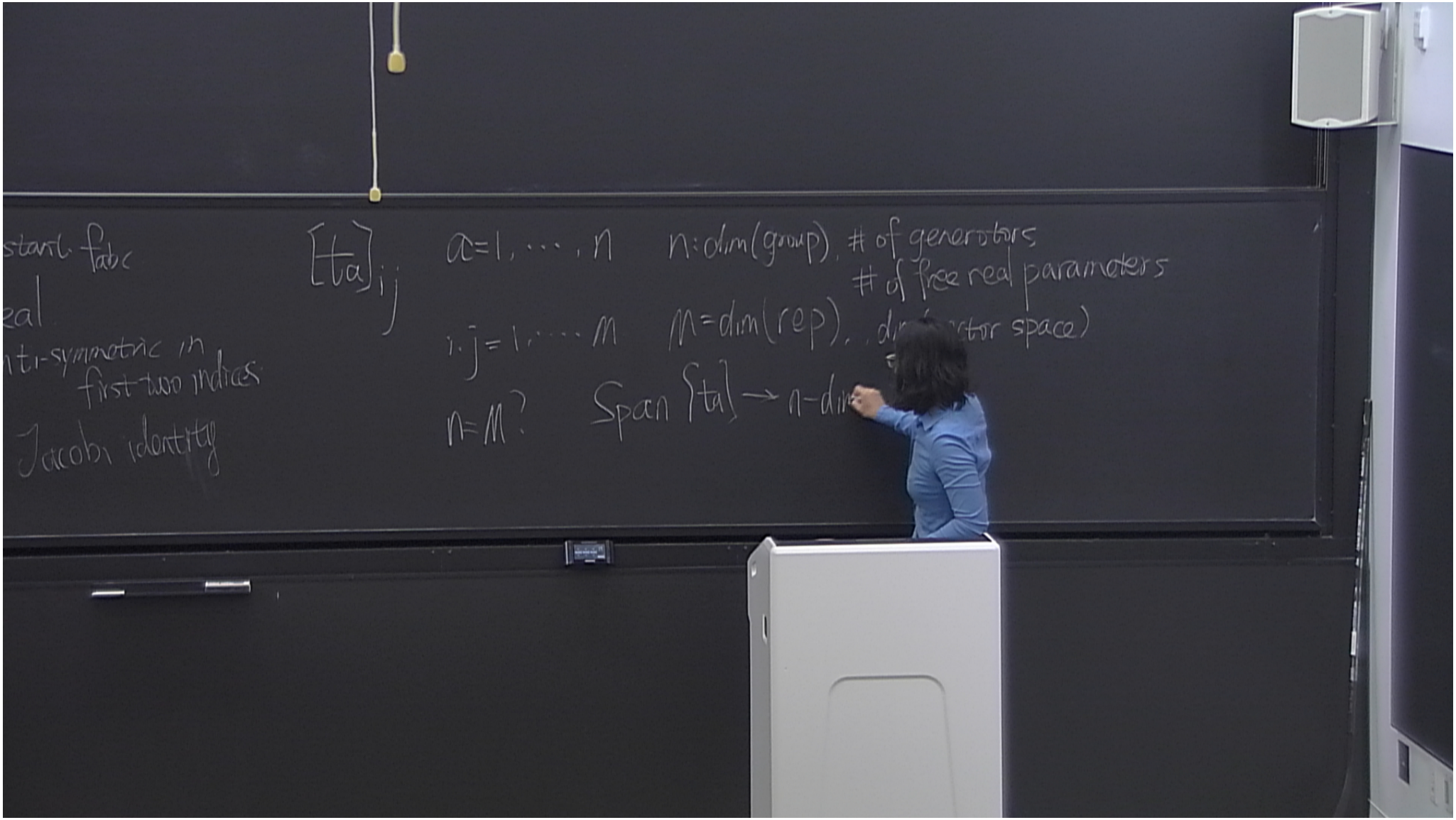
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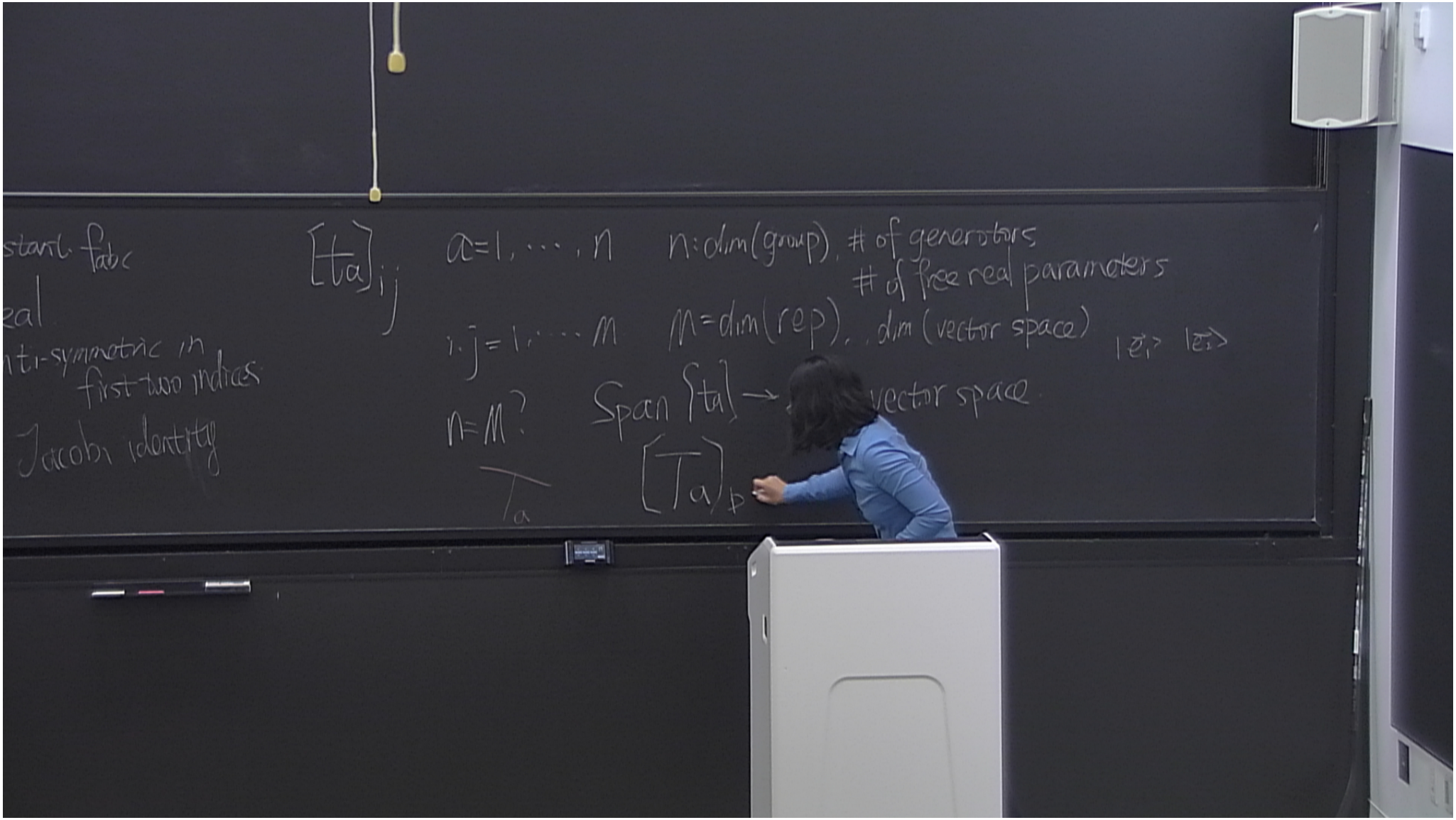
Abstract:

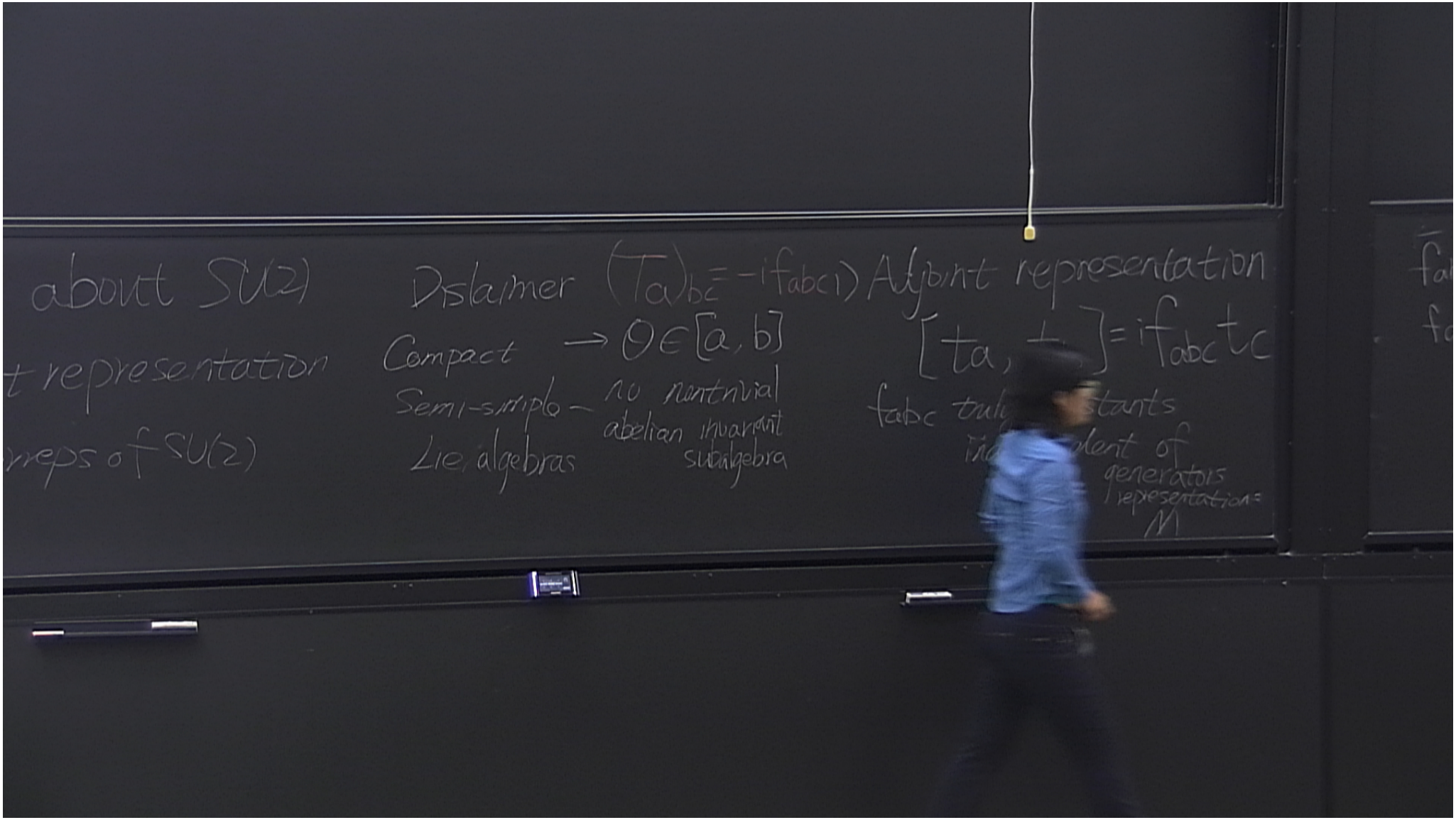

It's all about SU(2)
1) Adjoint representation

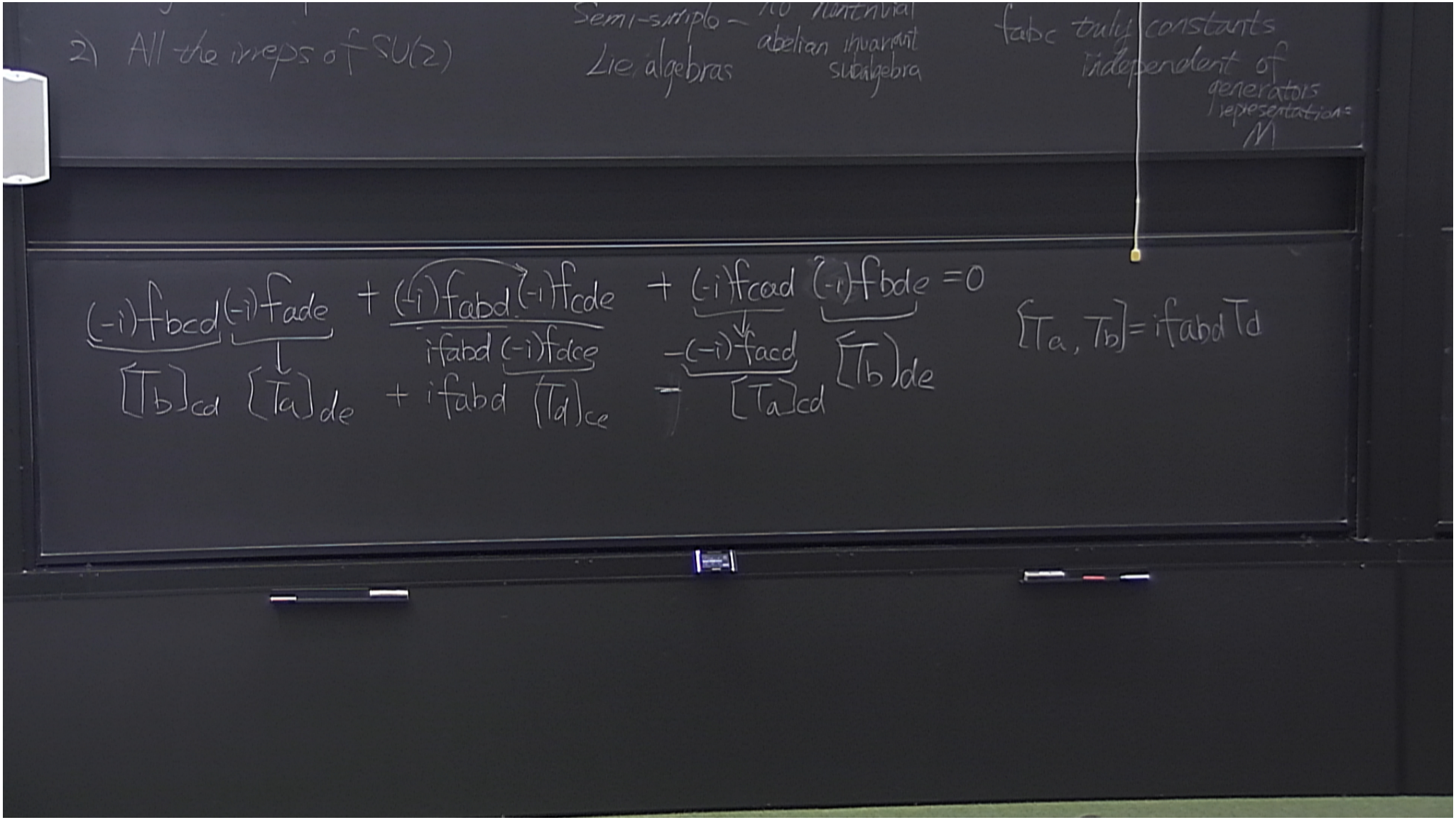

SU(2)
sentation
SU(2)
Dislaimer
Compact → θ ∈ [a, b]
Semi-simple – no nontrivial abelian invariant subalgebra
Lie algebras
1) Adjoint representation
[ta, tb] = if

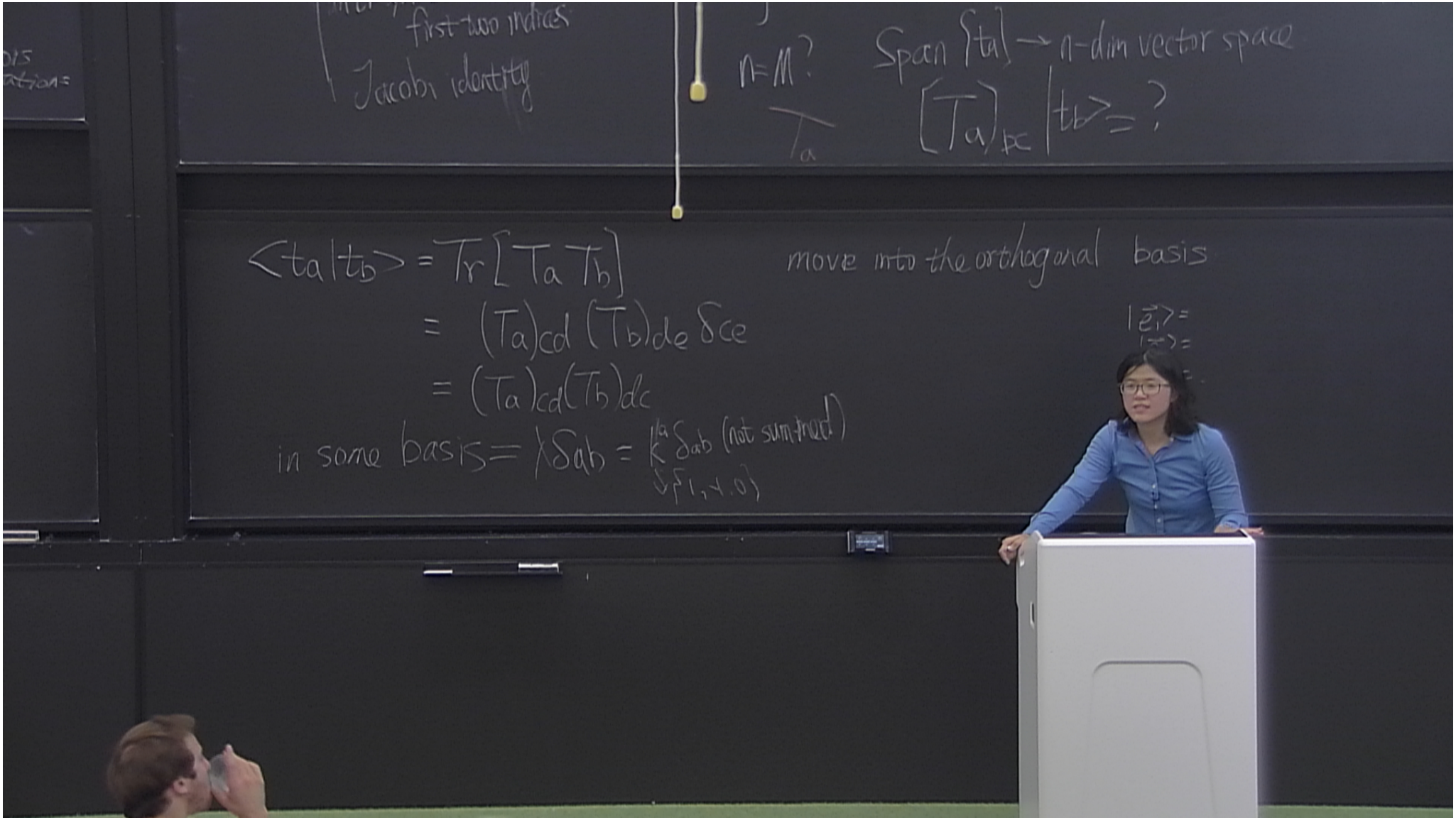

$\theta \in [a, b]$
non trivial
belian invariant
subalgebra
1) Adjoint representation
$[t_a, t_b] = i f_{abc} t_c$
f_{abc} truly constants
independent of
generators
representation =
M
$\bar{f}_{abc} = \text{constant} \cdot f_{abc}$

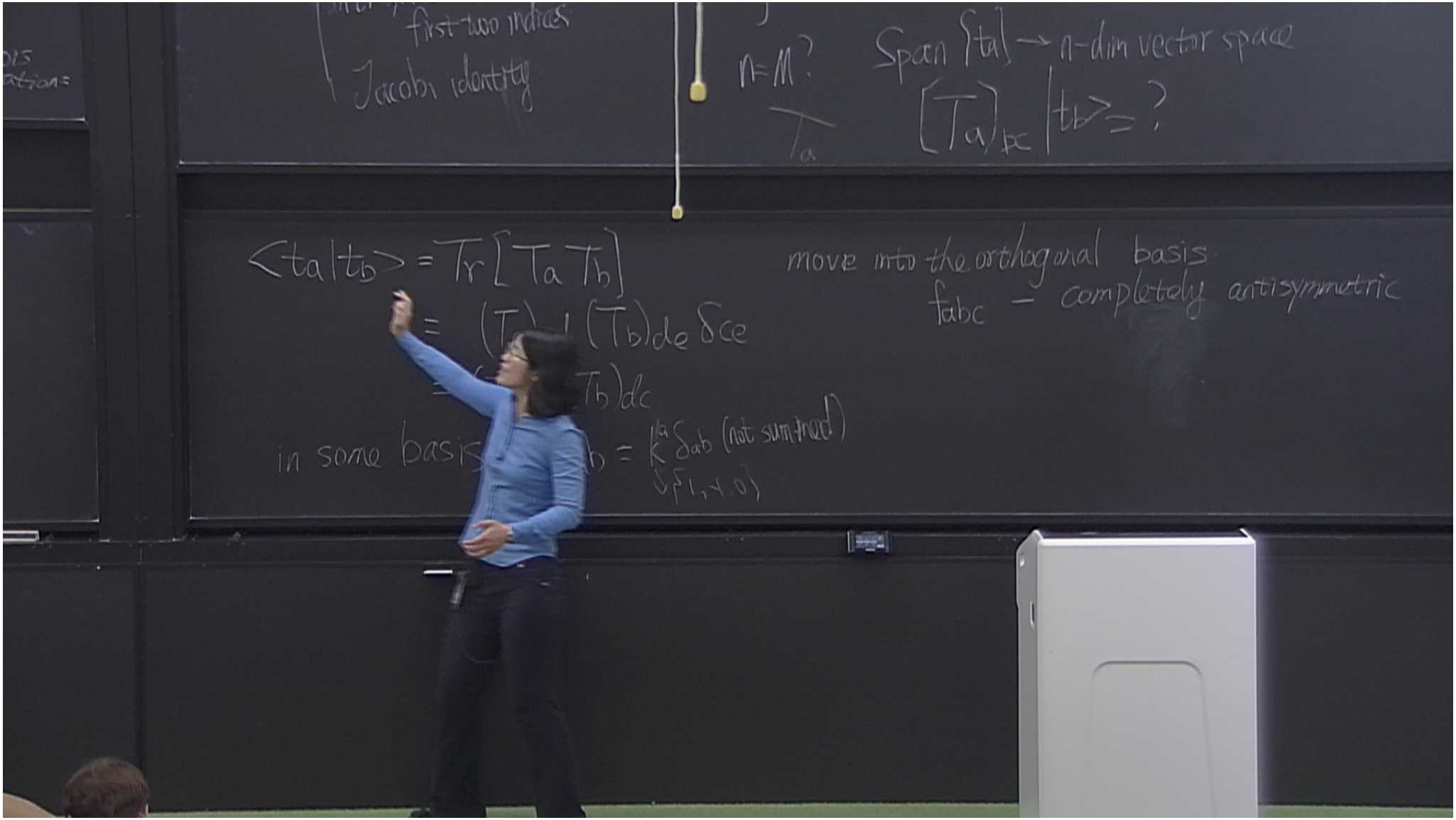

[ta]ij
a=1,...,n
n: dim(group). # of generators
# of free real parameters
i,j=1,...,M
M=dim(rep). dim(vector space)
n=M? Span{ta} → n-dim

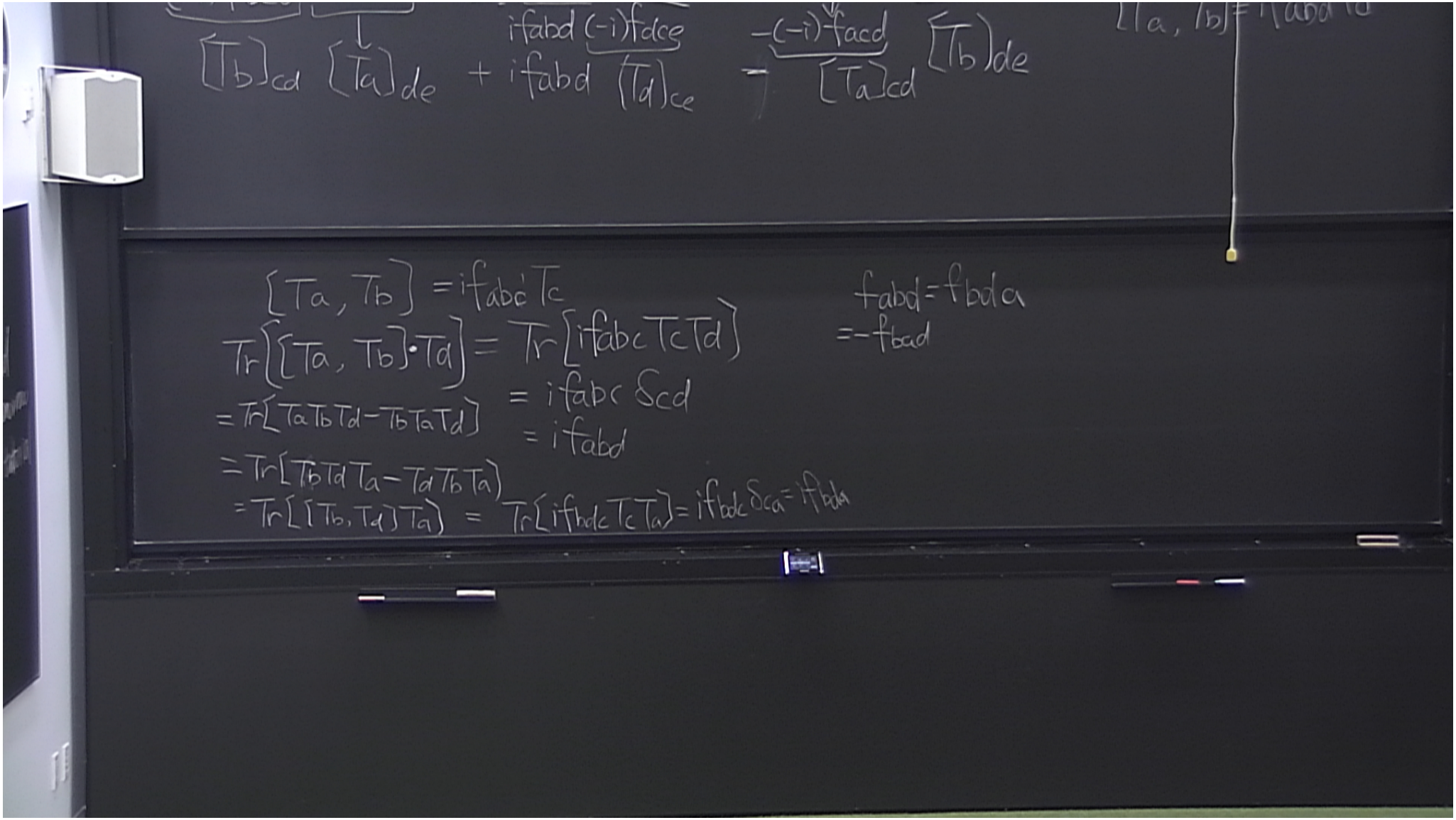

[ta]ij
a=1,...,n
n: dim(group). # of generators
# of free real parameters
i,j=1,...,M
M=dim(rep). dim(vector space)
|ei> |ei>
n=M?
Span {ta} → vector space
[Ta]b
Ta

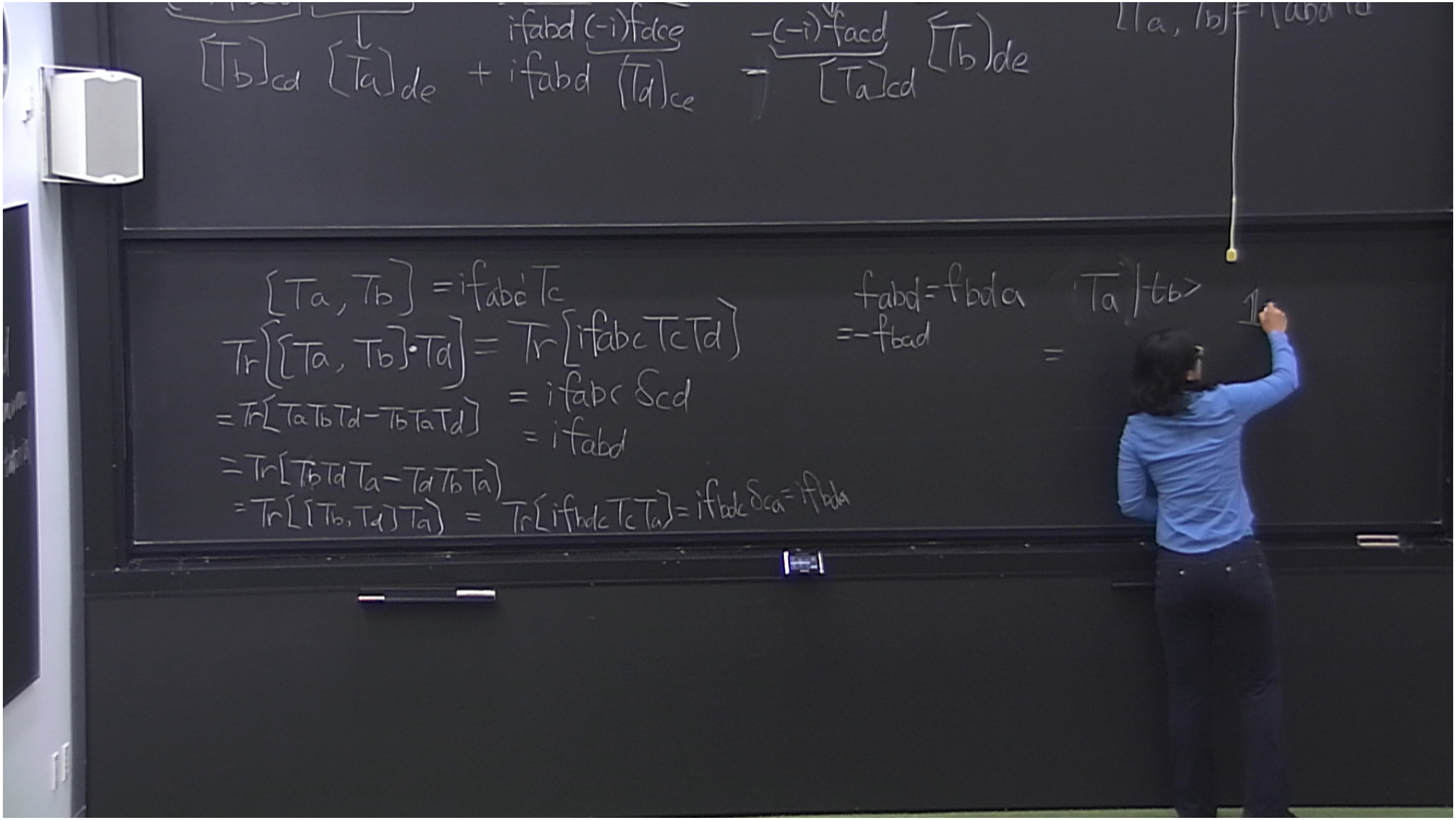

about SU(2)
t representation
reps of SU(2)
Dislaimer
Compact → θ ∈ [a, b]
Semi-simple Lie algebras
no nontrivial abelian invariant subalgebra
Adjoint representation
[ta,] = if
fabc
generators
representation =
M

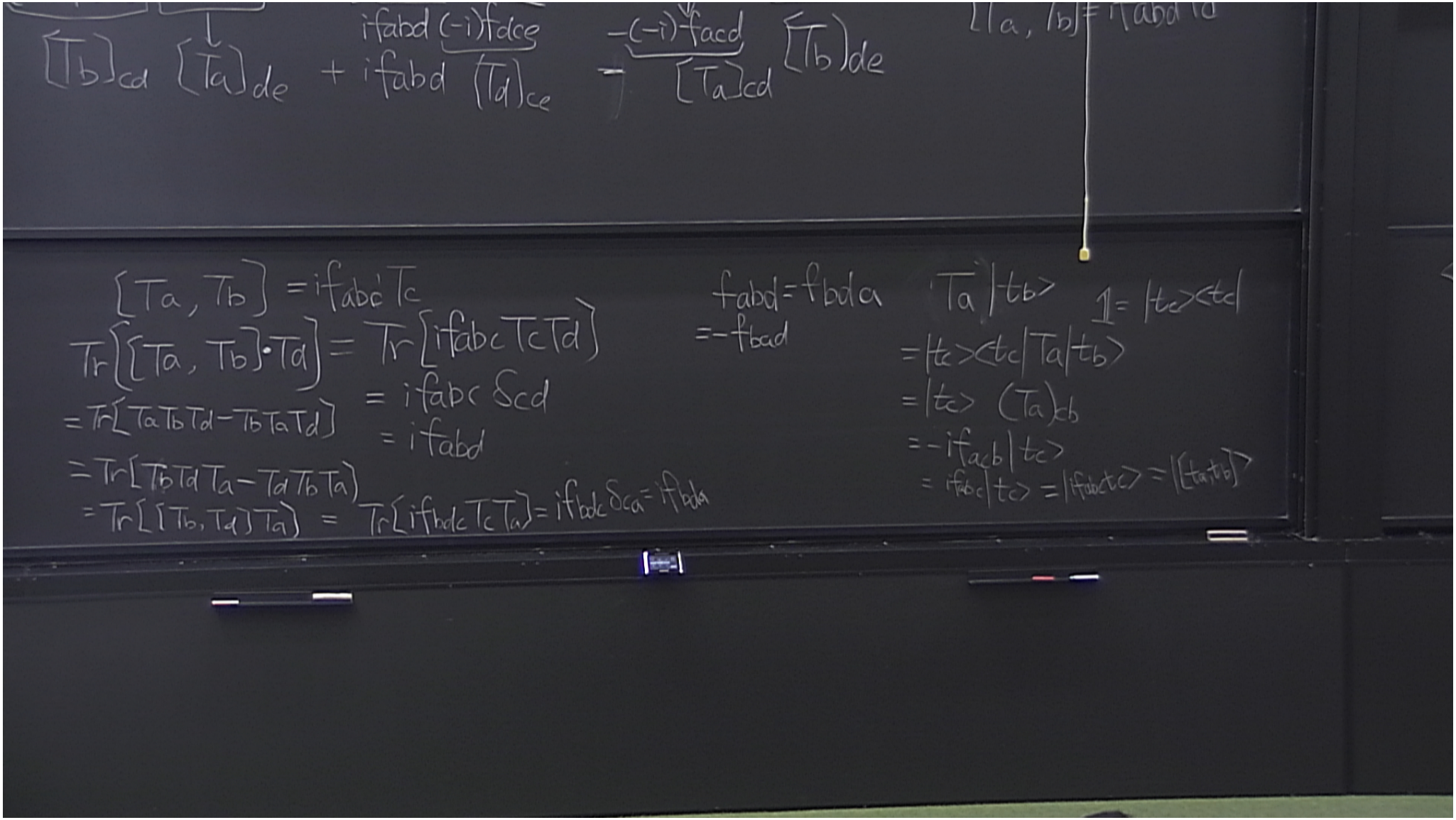

2) All the irreps of SU(2)
Semi-simple Lie algebras
abelian invariant subalgebra
f^{abc} fully constants independent of generators representation: M
$(-i)f^{bcd}(-i)f^{ade} + (-i)f^{abd}(-i)f^{cde} + (-i)f^{cad}(-i)f^{bde} = 0$
$[T_b]_{cd}[T_a]_{de} + if^{abd}(-i)f^{dce} - (-i)f^{acd}[T_b]_{de}$
$+ if^{abd}[T_d]_{ce} = -(-i)f^{acd}[T_a]_{cd}$
$[T_a, T_b] = if^{abd}T_d$

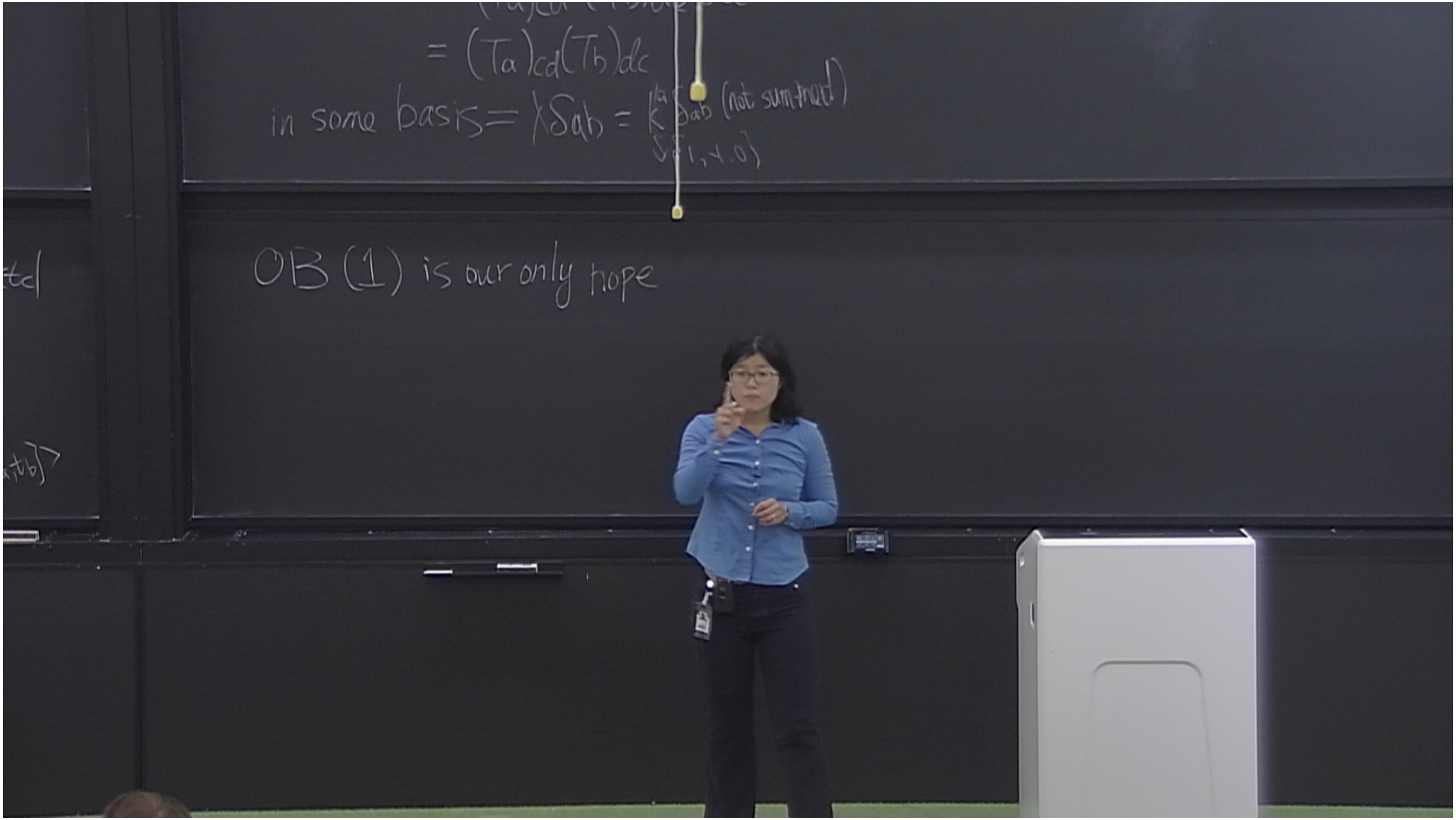

first two indices
Jacobi identity
n=M?
Span {|ta⟩} → n-dim vector space
[Ta]bc |tb⟩ = ?
⟨ta|tb⟩ = Tr[Ta Tb]
= (Ta)cd (Tb)de δce
= (Ta)cd (Tb)dc
move into the orthogonal basis
in some basis = λab = k^(ta) δab (not summed)
{1, 4, 0}

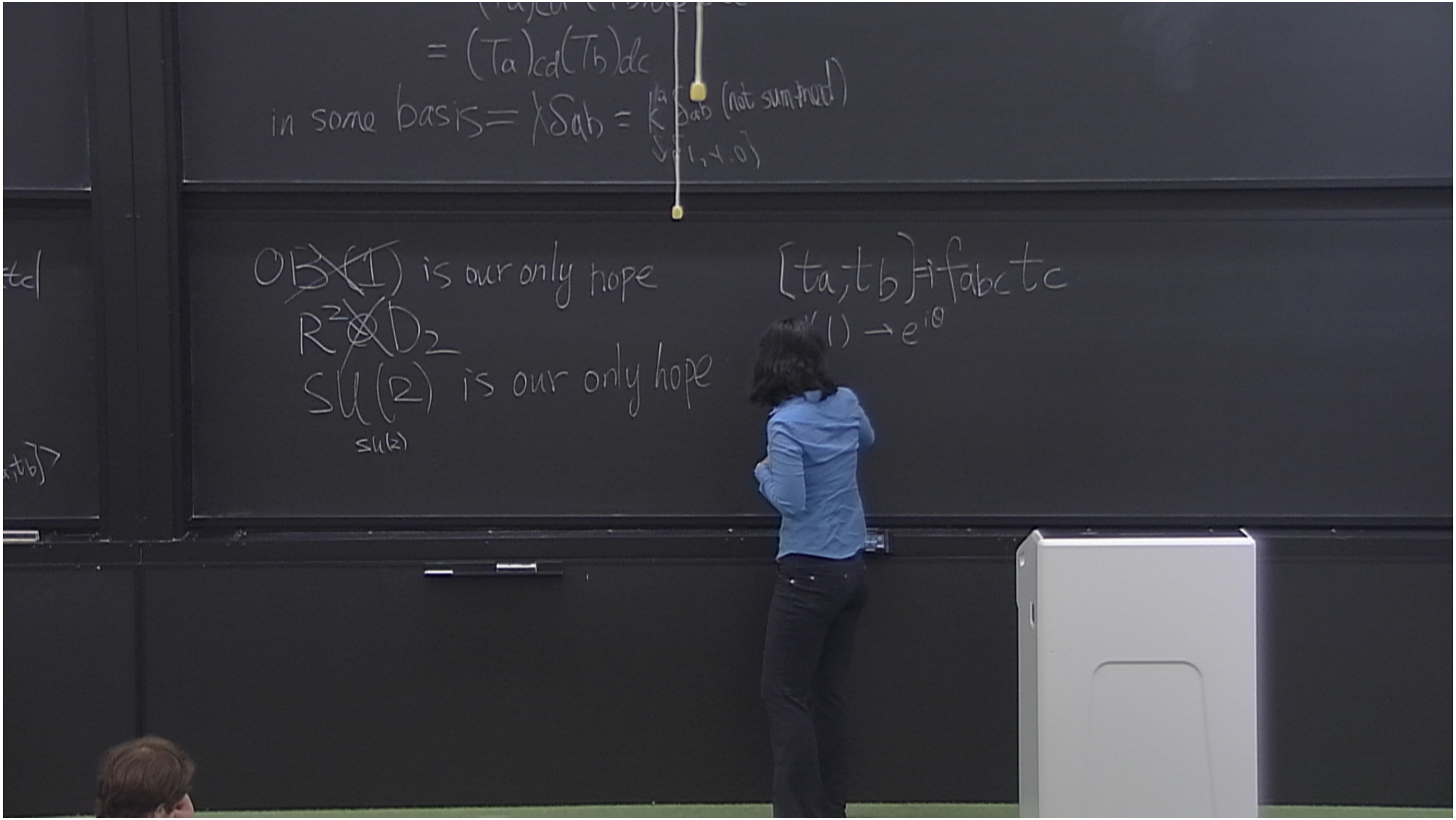

first two indices
Jacobi identity
n=M?
Span {ta} → n-dim vector space
[Ta]bc |tb> = ?
<ta|tb> = Tr[Ta Tb]
move into the orthogonal basis
fabc - completely antisymmetric
in some basis
= Kᵃ δab (not summed)

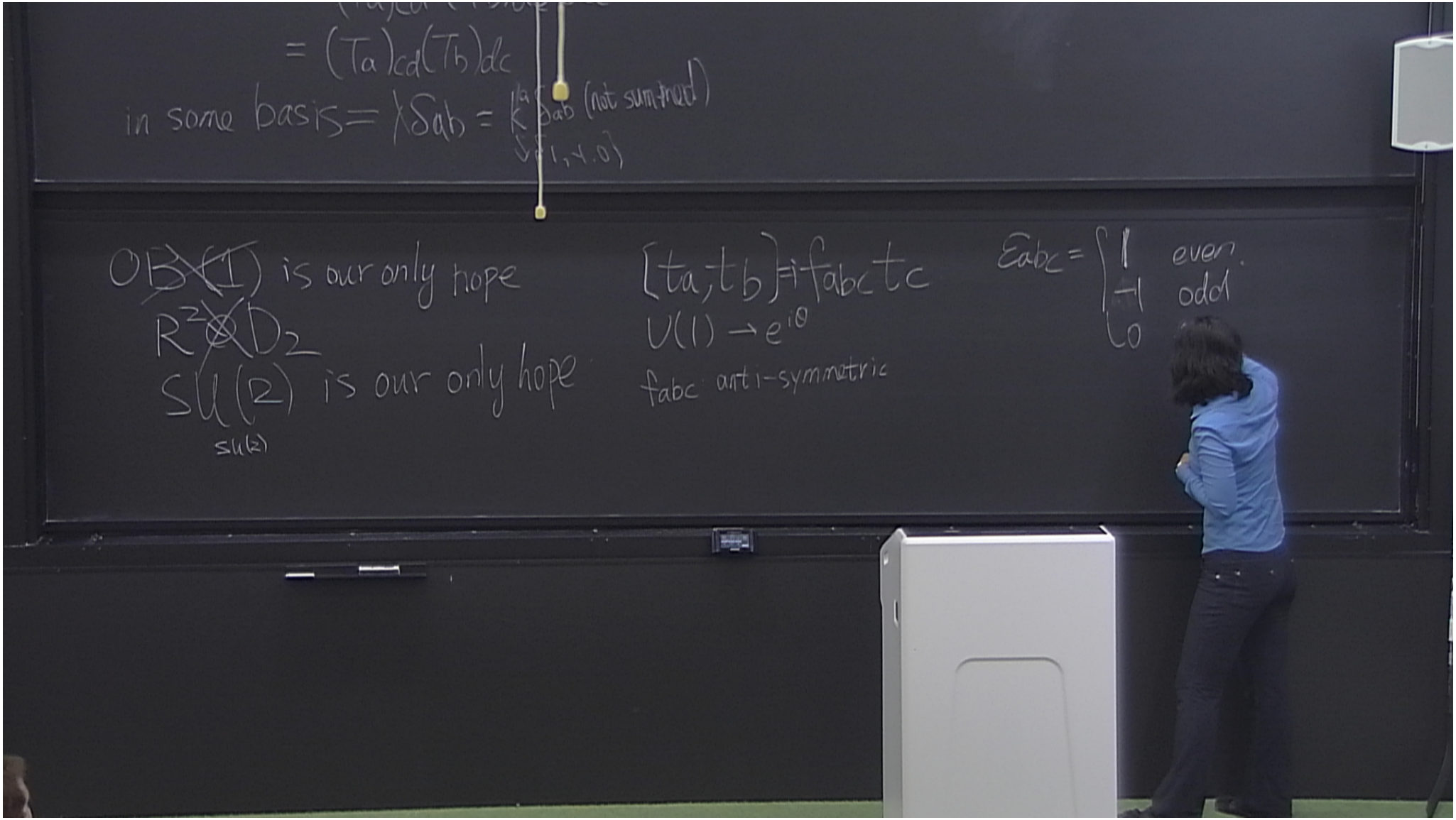

$[T_b]_{cd}\,[T_a]_{de} + i f_{abd}\,[T_d]_{ce} = -(-i) f_{acd}\,[T_b]_{de}$
$[T_a, T_b] = i f_{abc} T_c$
$f_{abd} = f_{bdc}$
$= -f_{bad}$
$\mathrm{Tr}\{[T_a, T_b] \cdot T_d\} = \mathrm{Tr}\{i f_{abc} T_c T_d\}$
$= i f_{abc} \delta_{cd}$
$= i f_{abd}$
$= \mathrm{Tr}\{T_a T_b T_d - T_b T_a T_d\}$
$= \mathrm{Tr}\{T_b T_d T_a - T_d T_b T_a\}$
$= \mathrm{Tr}\{[T_b, T_d] T_a\} = \mathrm{Tr}\{i f_{bdc} T_c T_a\} = i f_{bdc} \delta_{ca} = i f_{bda}$

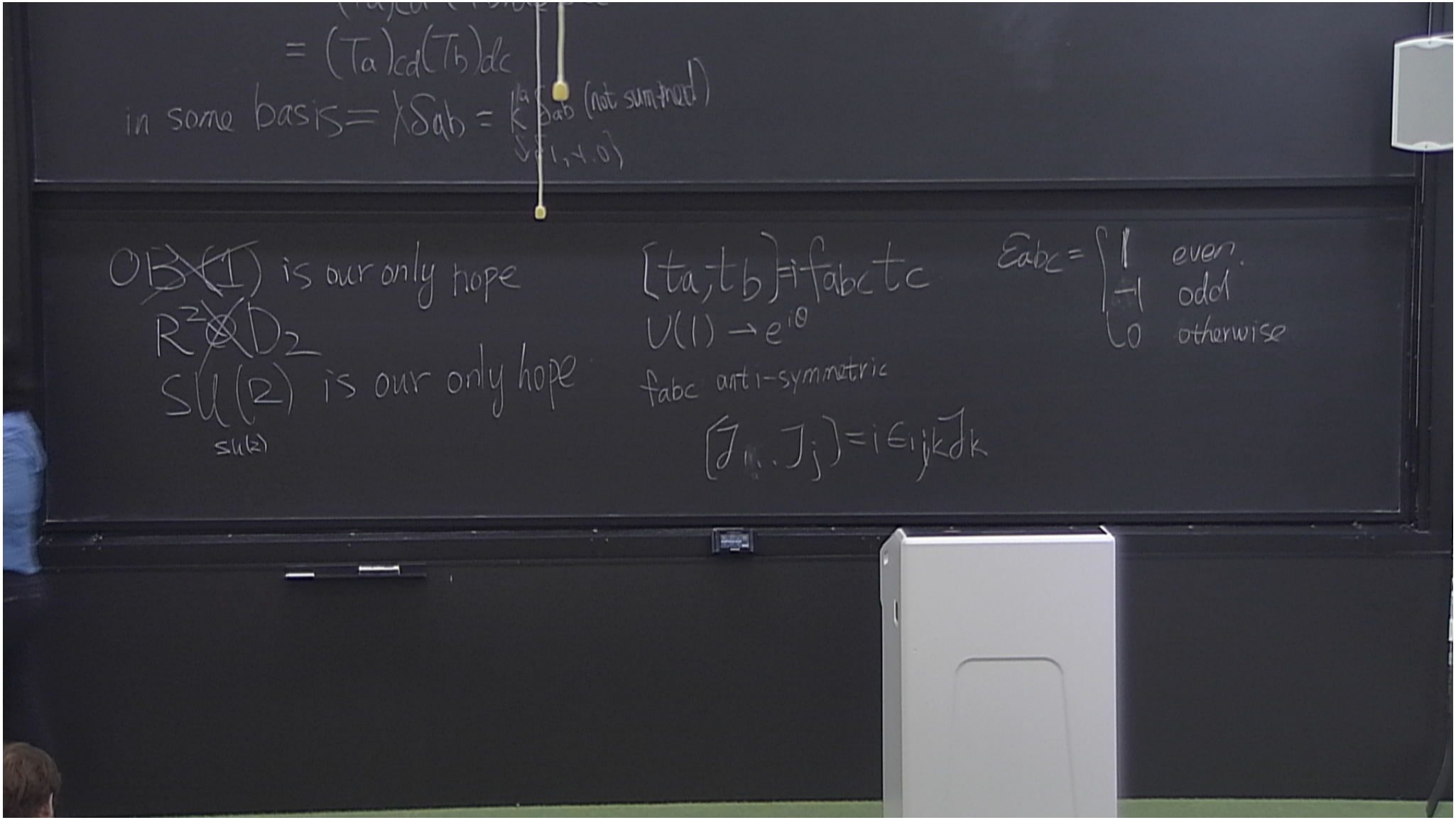

[Ta, Tb] = i fabc Tc
Tr{[[Ta, Tb]·Td] = Tr[i fabc Tc Td]
= Tr[Ta Tb Td − Tb Ta Td]
= Tr[Tb Td Ta − Td Tb Ta]
= Tr[[Tb, Td] Ta] = Tr[i fbdc Tc Ta] = i fbdc δca = i fbda
fabd = fbda
= − fbad
= i fabc δcd
= i fabd

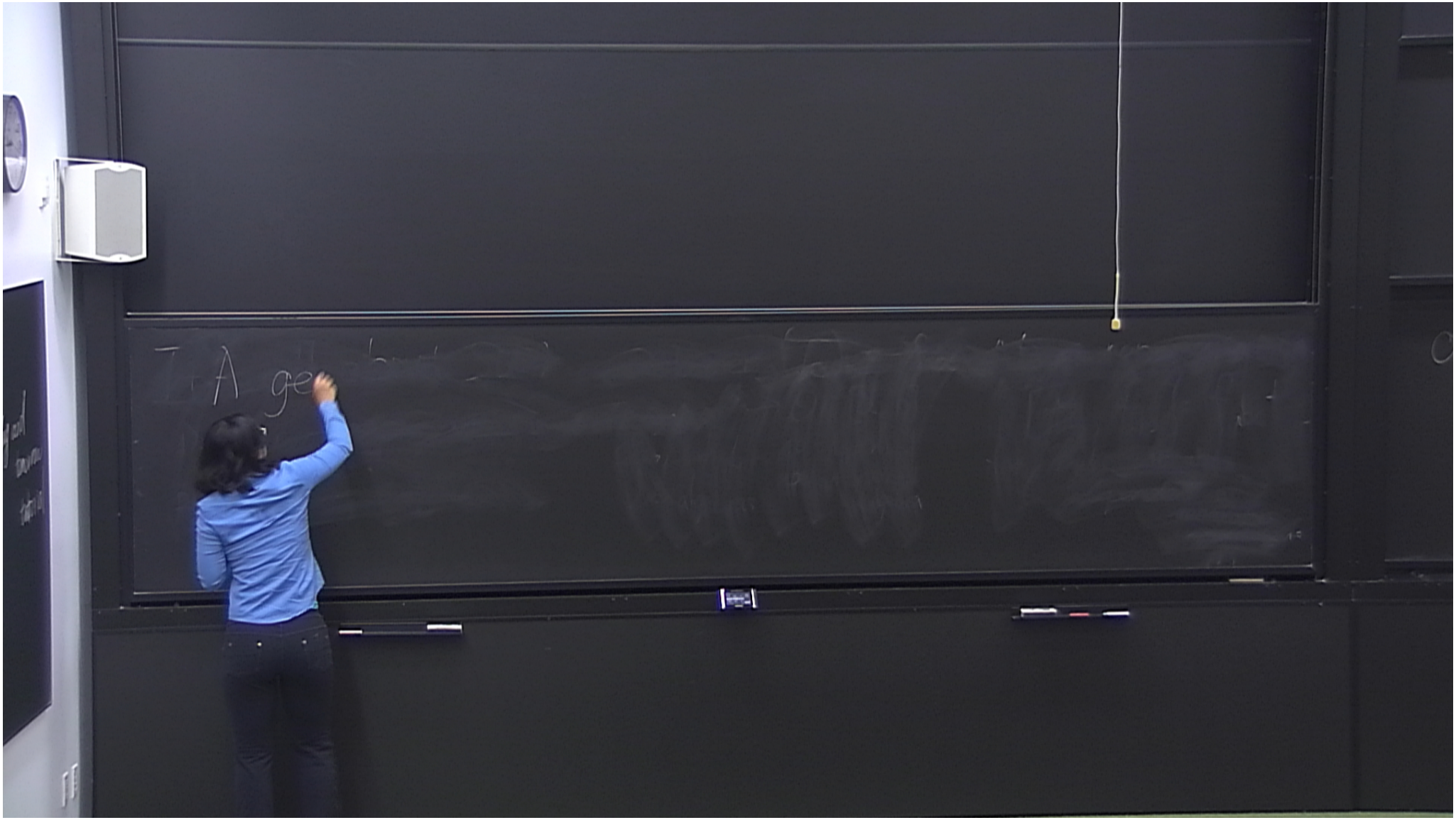

$[T_b]_{cd}\,[T_a]_{de} + i f_{abd}\,[T_d]_{ce} = i f_{abd}(-i) f_{dce}\,[T_b]_{de} - (-i) f_{acd}\,[T_a]_{cd}$
$[T_a, T_b] = i f_{abd} T_d$
$[T_a, T_b] = i f_{abc} T_c$
$\mathrm{Tr}\{[T_a, T_b] T_d\} = \mathrm{Tr}\{i f_{abc} T_c T_d\}$
$= \mathrm{Tr}[T_a T_b T_d - T_b T_a T_d]$
$= i f_{abc} \delta_{cd}$
$= i f_{abd}$
$= \mathrm{Tr}[T_b T_d T_a - T_d T_b T_a]$
$= \mathrm{Tr}[[T_b, T_d] T_a] = \mathrm{Tr}\{i f_{bdc} T_c T_a\} = i f_{bdc} \delta_{ca} = i f_{bda}$
$f_{abd} = f_{bda}$
$= -f_{bad}$
$T_a |t_b\rangle$
$\mathbb{1} = |t_c\rangle\langle t_c|$
$= |t_c\rangle\langle t_c| T_a |t_b\rangle$
$= |t_c\rangle (T_a)_{cb}$
$= -i f_{acb} |t_c\rangle$
$= i f_{abc} |t_c\rangle = |i f_{abc} t_c\rangle = |[t_a, t_b]\rangle$

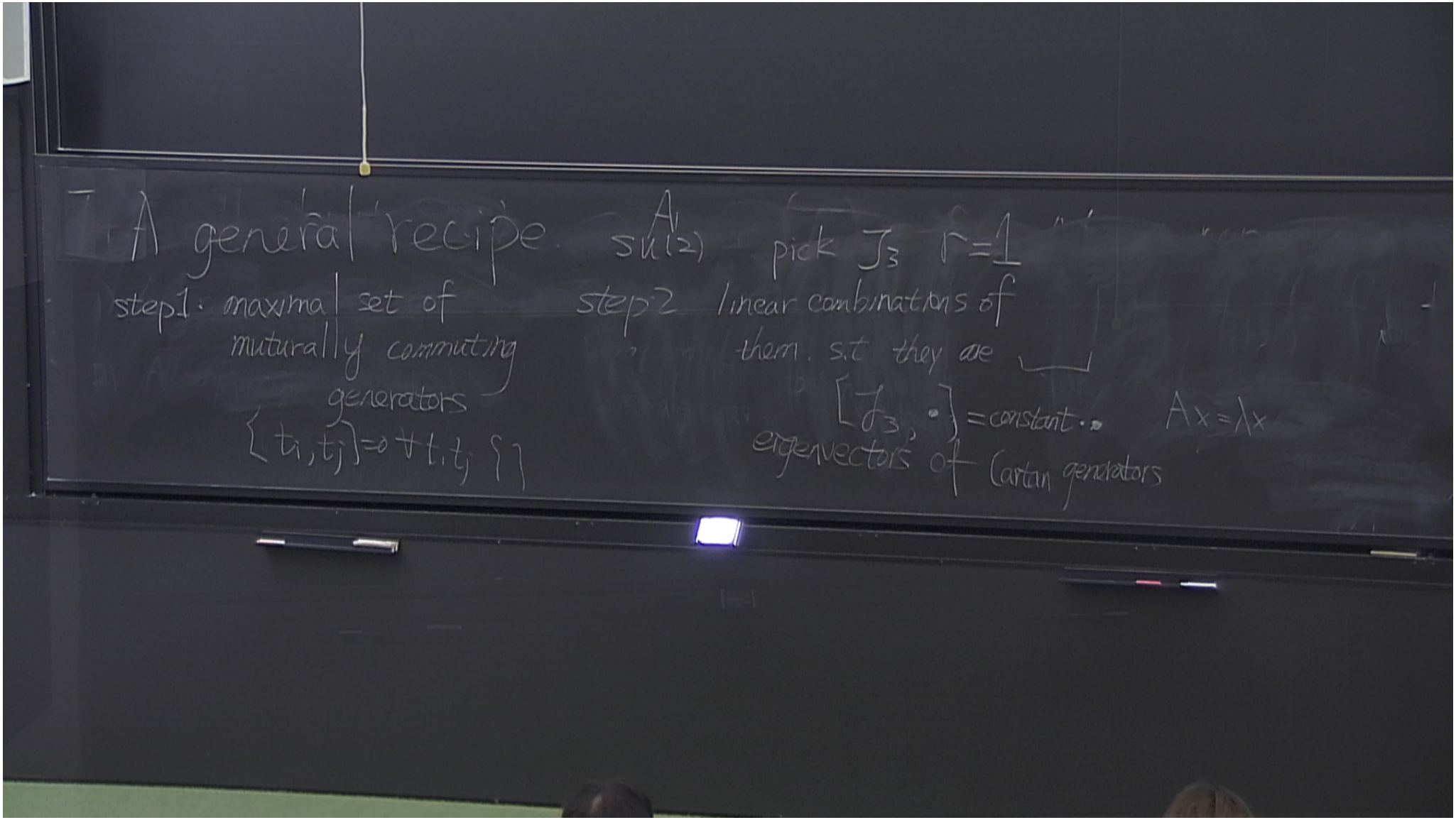

= (Ta)cd(Tb)dc
in some basis = λSab = k^a Sab (not summed)
OB(1) is our only hope

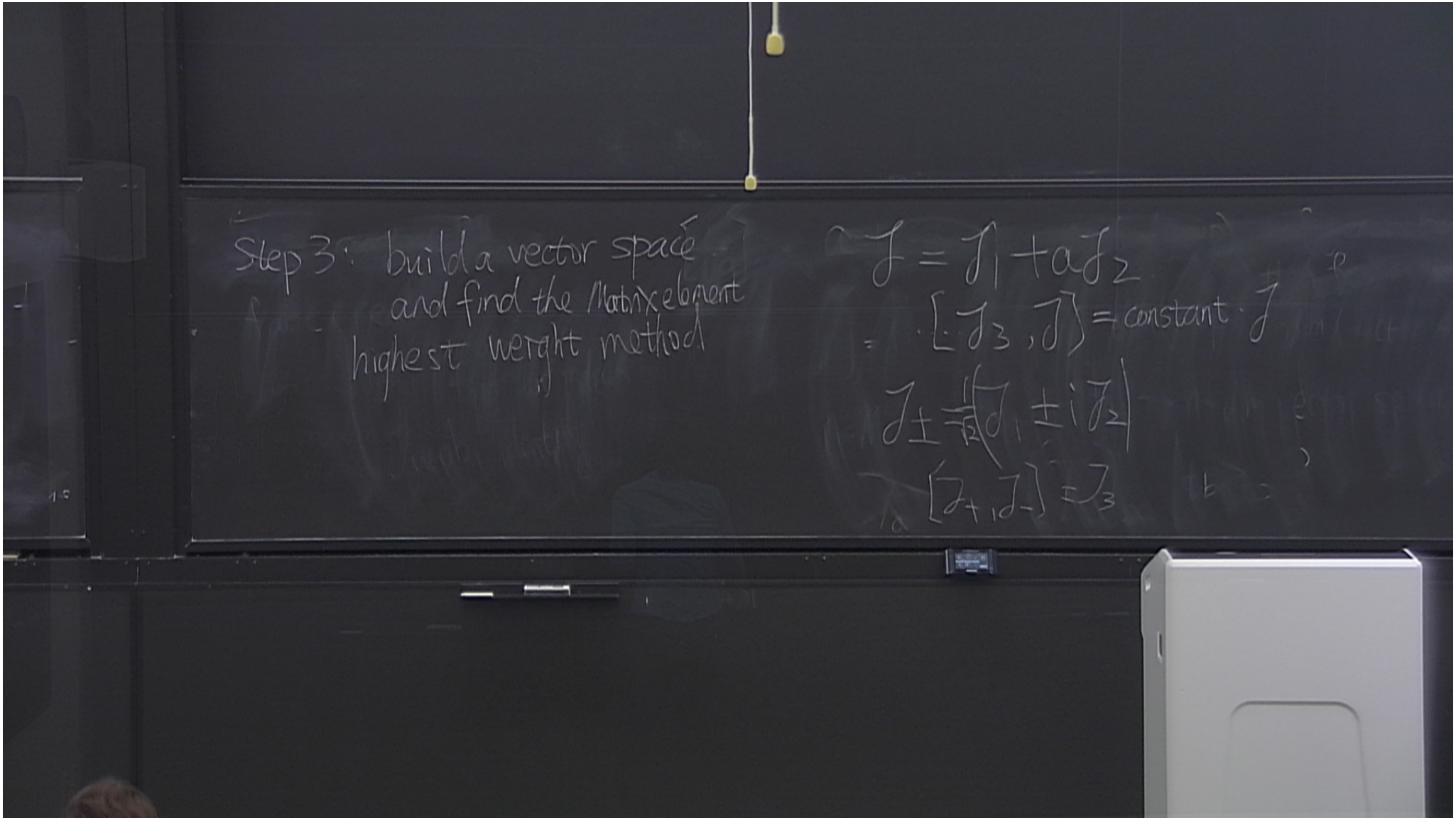

= (Ta)cd(Tb)dc
in some basis = λδab = k^a δab (not summed)
OB (I) is our only hope
R³ ⊗ D2
SU(2) is our only hope
SU(2)
[ta, tb] = i fabc tc
→ e^iθ

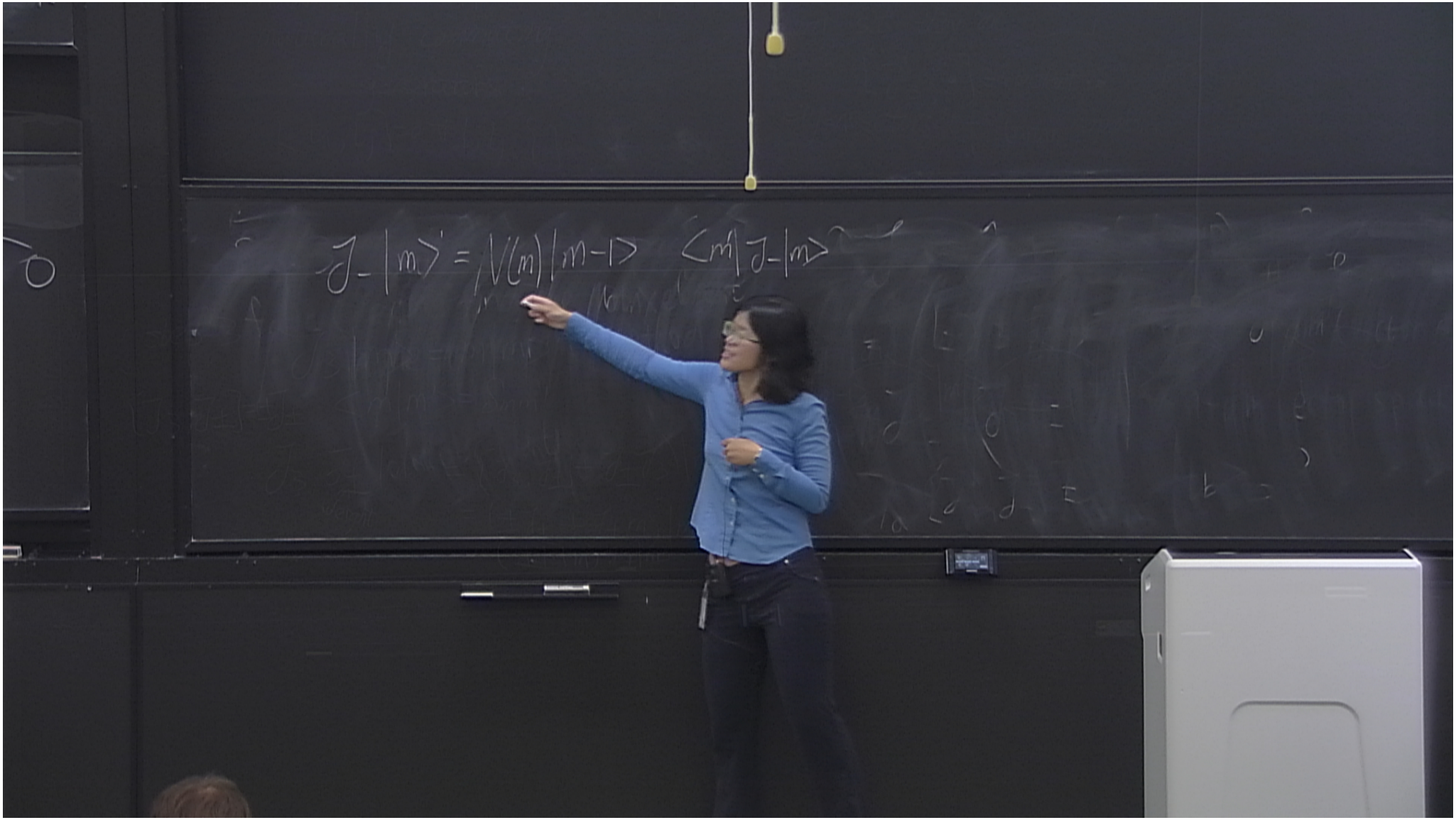

= (Ta)cd(Tb)dc
in some basis = λδab = k^a δab (not summed)
v_a (1, 4, 0)
O(1) is our only hope
R² ⊗ D2
SU(2) is our only hope
SU(2)
[ta, tb] = i fabc tc
U(1) → e^iθ
fabc anti-symmetric
εabc = 1 even
-1 odd
0

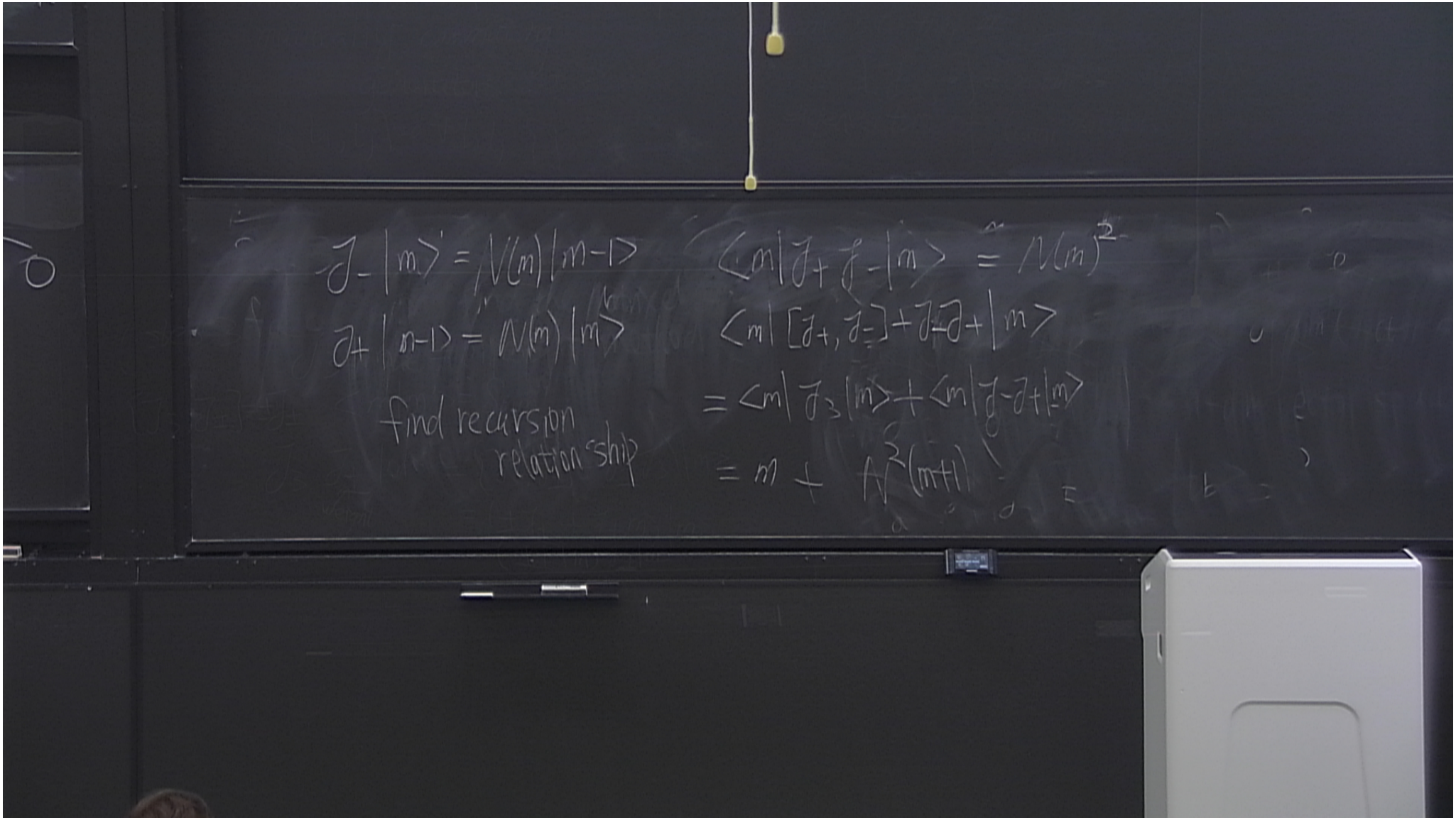

$= (T_a)_{cd}(T_b)_{dc}$
in some basis $= \lambda\delta_{ab} = k^{(a)}\delta_{ab}$ (not summed)
$\vec{v} = (1, 1, 0)$
$O(3) \times O(1)$ is our only hope
$R^2 \otimes D_2$
$SU(2)$ is our only hope
$SU(2)$
$[t_a, t_b] = i f_{abc} t_c$
$U(1) \to e^{i\theta}$
f_{abc} anti-symmetric
$[J_i, J_j] = i\epsilon_{ijk} J_k$
$\epsilon_{abc} = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \\ 0 & \text{otherwise} \end{cases}$

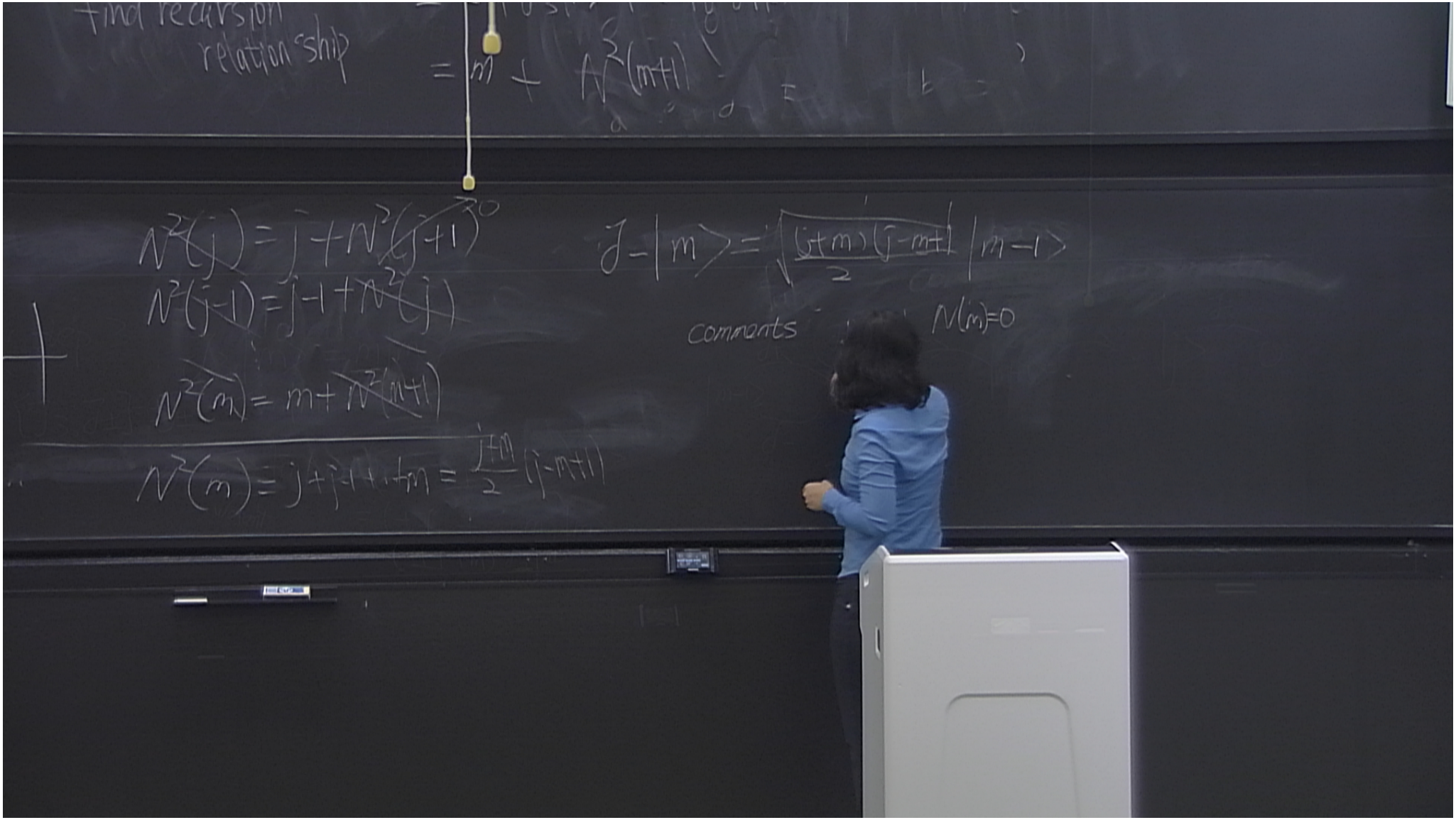


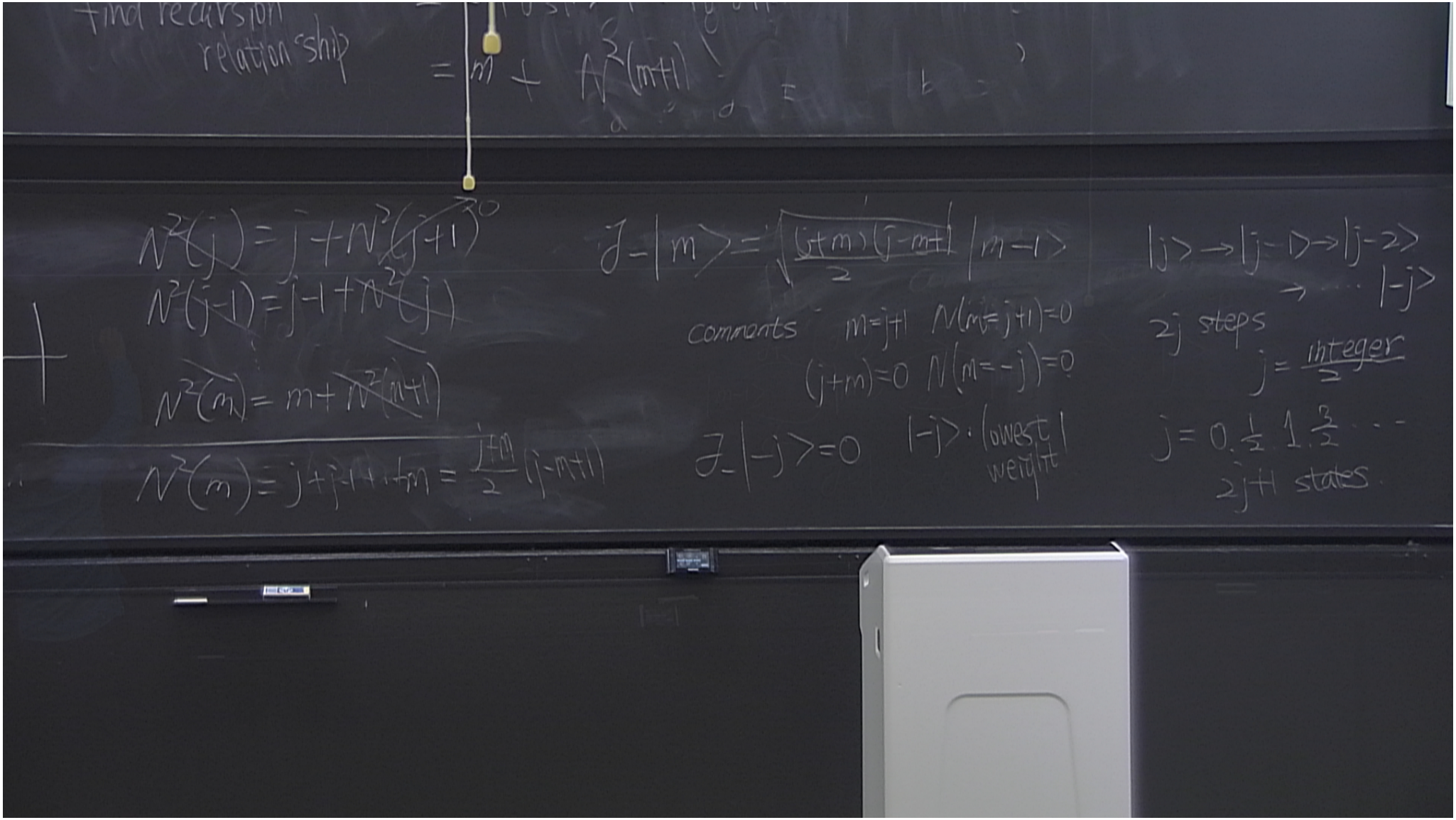

A general recipe
step 1. maximal set of mutually commuting generators
$\{t_i, t_j\} = 0 \quad \forall\, t_i, t_j$
A
$su(2)$
pick J_3 $r = 1$
step 2 linear combinations of them s.t they are
$[J_3, \cdot\,] = \text{constant} \cdot \cdot$
eigenvectors of Cartan generators
$Ax = \lambda x$

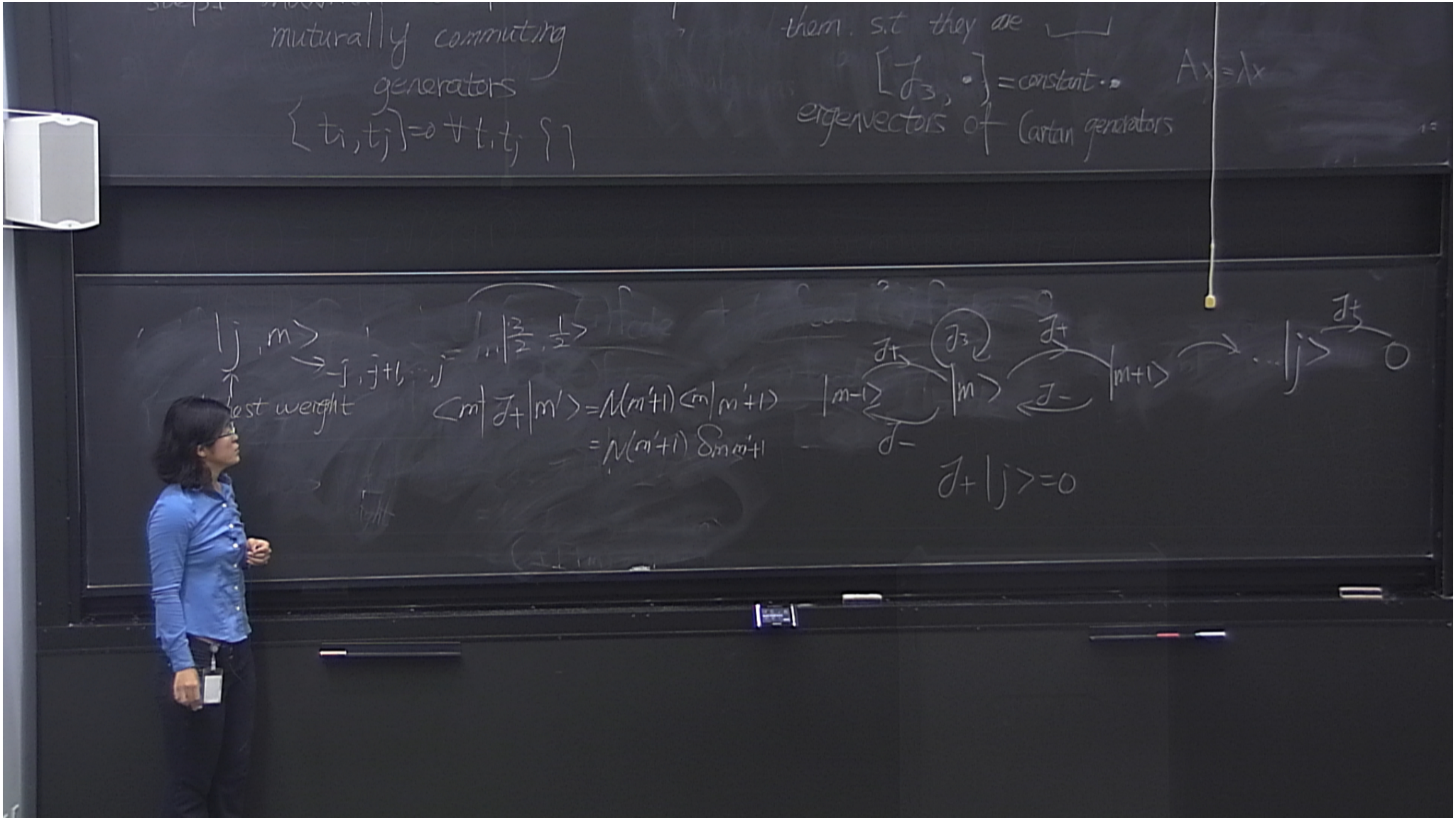

Step 3: build a vector space and find the Matrix element highest weight method
$J = J_1 + aJ_2$
$[J_3, J] = \text{constant} \cdot J$
$J_\pm = \frac{1}{2}(J_1 \pm iJ_2)$
$[J_+, J_-] = J_3$

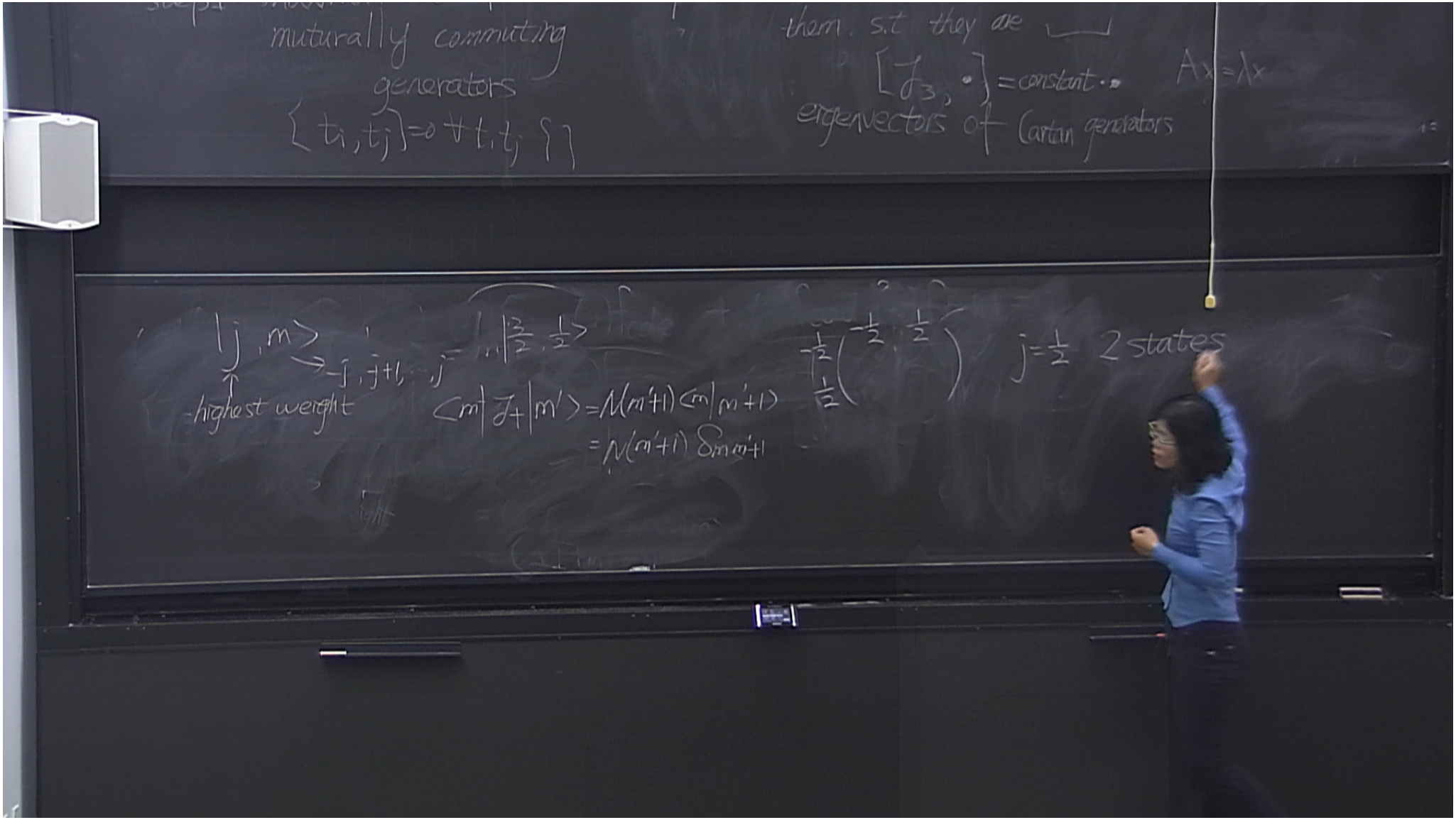

J_ |m> = V(m) |m-1>
<m| J_ |m>

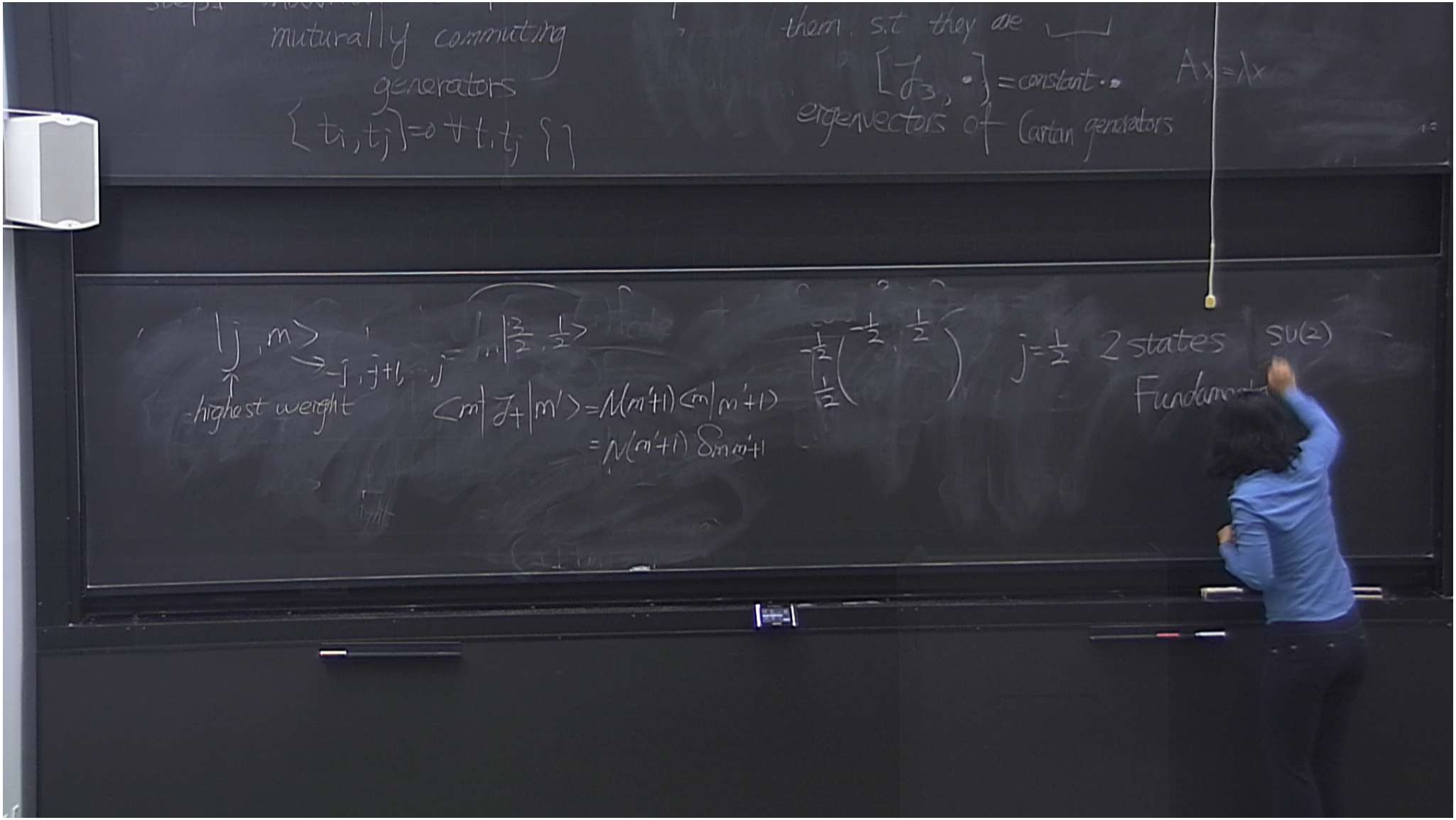

$J_- |m\rangle = N(m) |m-1\rangle$
$J_+ |m-1\rangle = N(m) |m\rangle$
find recursion relationship
$\langle m | J_+ J_- | m \rangle = N(m)^2$
$\langle m | [J_+, J_-] + J_- J_+ | m \rangle$
$= \langle m | J_3 | m \rangle + \langle m | J_- J_+ | m \rangle$
$= m + N^2(m+1)$

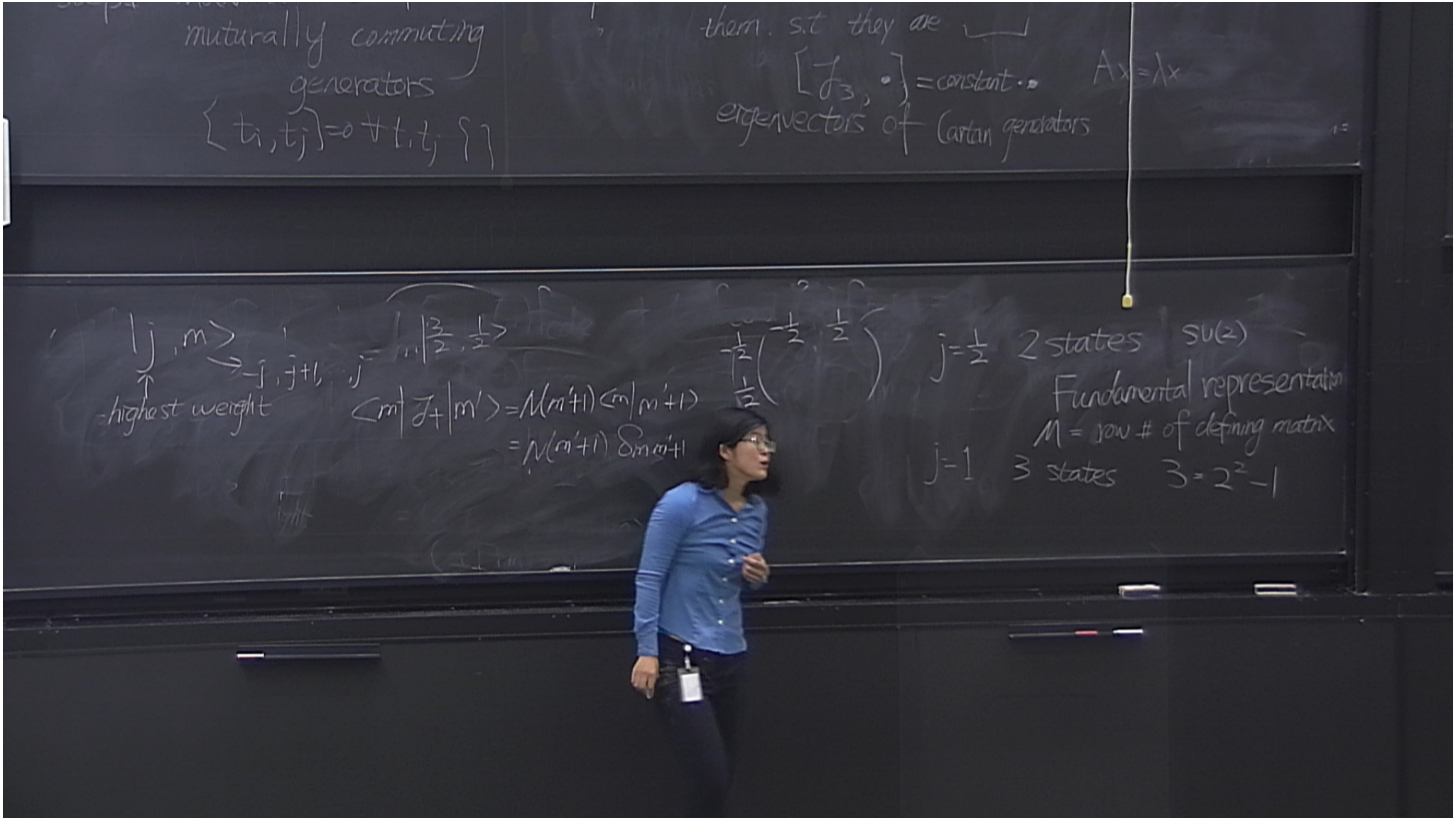

find recursion relationship
$N^2(j) = j + N^2(j+1)$
$N^2(j-1) = j-1 + N^2(j)$
$N^2(m) = m + N^2(m+1)$
$N^2(m) = j + j-1 + \dots + m = \frac{j+m}{2}(j-m+1)$
$J_-|m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}}|m-1\rangle$
comments
$N(m)=0$

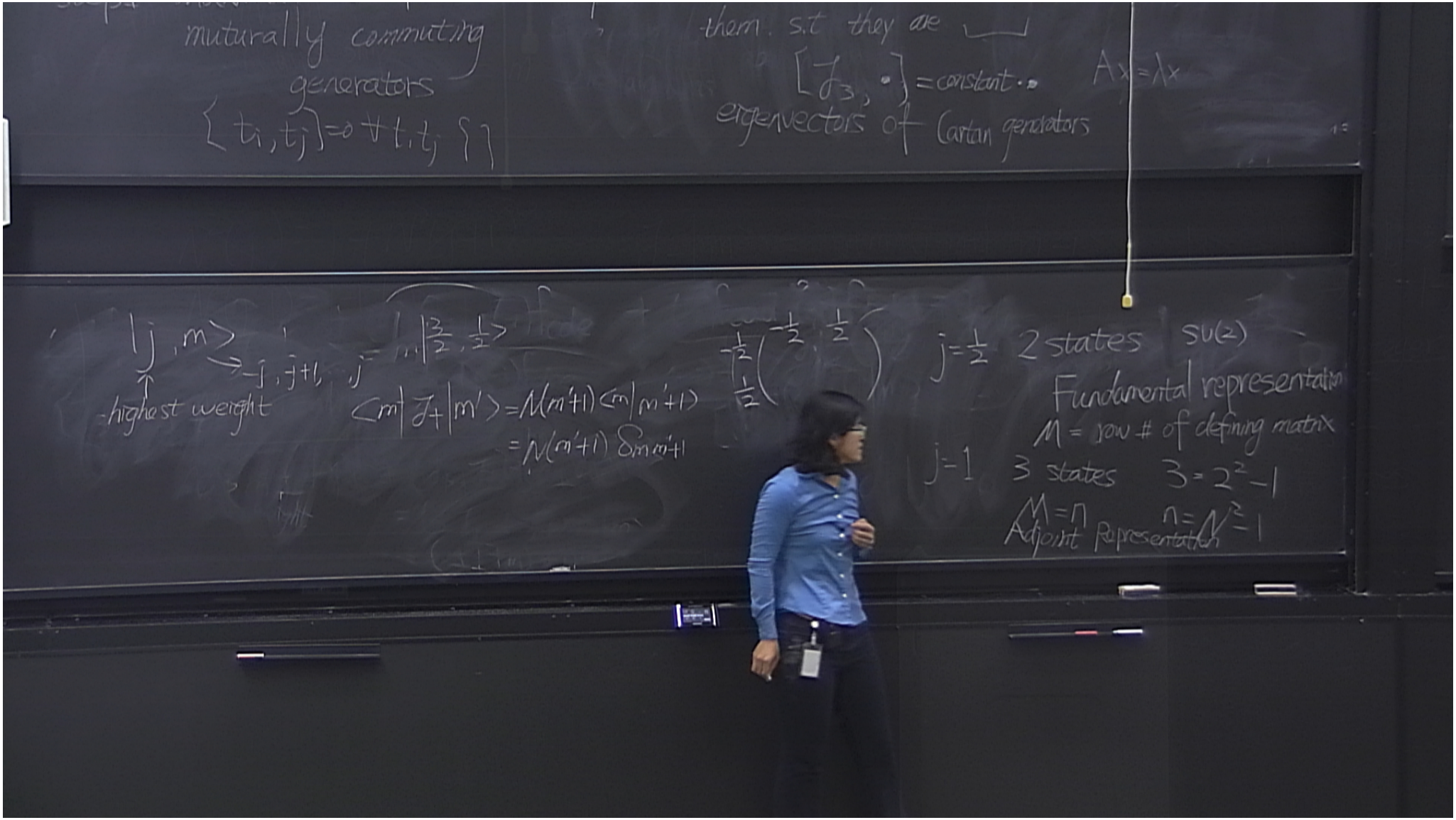

find recursion relationship
$N^2(j) = j + N^2(j-1)$
$N^2(j-1) = j-1 + N^2(j)$
$N^2(m) = m + N^2(m+1)$
$N^2(m) = j + j-1 + \cdots + m = \frac{j+m}{2}(j-m+1)$
$J_-|m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}}\,|m-1\rangle$
comments $m = j+1$ $N(m=j+1)=0$
$(j+m)=0$ $N(m=-j)=0$
$J_-|-j\rangle = 0$ $|-j\rangle$: lowest weight
$|j\rangle \to |j-1\rangle \to |j-2\rangle \to \cdots |-j\rangle$
2j steps
$j = \frac{\text{integer}}{2}$
$j = 0, \frac{1}{2}, 1, \frac{3}{2} \cdots$
2j+1 states

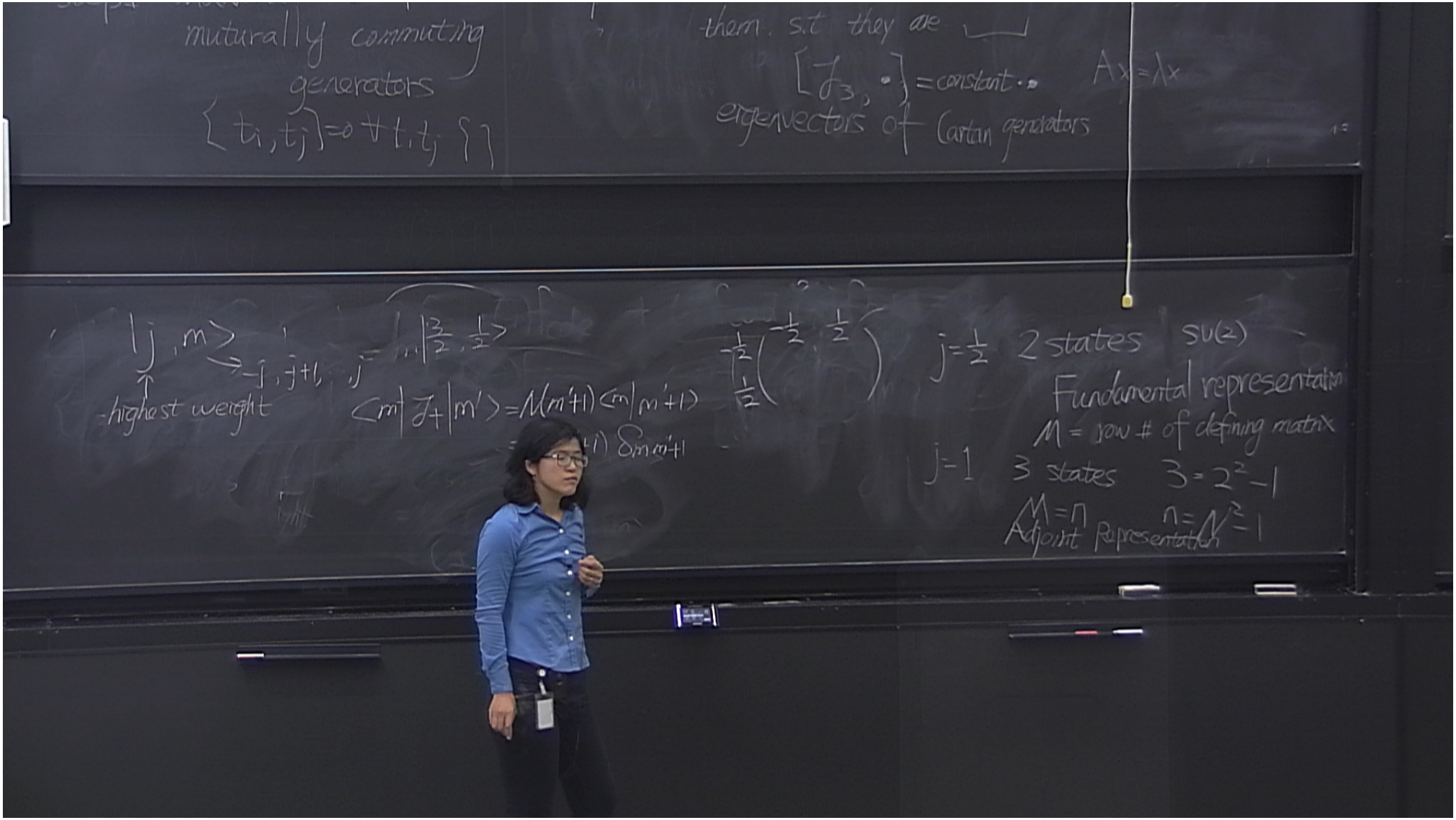

mutually commuting
generators
$\{t_i, t_j\}=0 \;\forall\; t_i, t_j$
them. s.t they are
$[J_3, \cdot\,]$ = constant
eigenvectors of Cartan generators
$A x = \lambda x$
$|j, m\rangle \to -j, -j+1, \ldots j$
$= \ldots \left|\frac{3}{2}, \frac{1}{2}\right\rangle$
lest weight
$\langle m'| J_+ |m\rangle = N(m+1)\langle m'|m+1\rangle$
$= N(m+1)\,\delta_{m'm+1}$
$|m-1\rangle$ J_+ J_3 J_+ $|m\rangle$ J_- $|m+1\rangle$ J_-
$J_+|j\rangle = 0$
$|j\rangle$ J_+ 0


mutually commuting
generators
$[t_i, t_j] = 0 \quad \forall\, t_i, t_j$
them. s.t they are
$[J_3, \cdot] =$ constant
$A x = \lambda x$
eigenvectors of Cartan generators
$|j, m\rangle \rightarrow |j, j+1, \ldots, j\rangle$
highest weight
$\left|\frac{3}{2}, \frac{1}{2}\right\rangle$
$\langle m'| J_+ |m\rangle = N(m+1)\langle m'|m+1\rangle$
$= N(m+1)\,\delta_{m',m+1}$
$j = \frac{1}{2}$ 2 states


mutually commuting
generators
$\{t_i, t_j\} = 0 \quad \forall\, t_i, t_j$
them s.t they are
$[J_3, \cdot] = \text{constant} \cdot \cdot$
$A x = \lambda x$
eigenvectors of Cartan generators
$|j, m\rangle \to -j, -j+1, \ldots j$
highest weight
$\langle m| J_+ |m'\rangle = N(m'+1)\langle m|m'+1\rangle$
$= N(m'+1)\,\delta_{m,m'+1}$
$j = \frac{1}{2}$ 2 states SU(2)
Fundam


mutually commuting
generators
$[t_i, t_j] = 0 \quad \forall\, i, j$
them. s.t they are
$[J_3, \cdot] =$ constant
eigenvectors of Cartan generators
$A\vec{x} = \lambda\vec{x}$
$|j, m\rangle$
highest weight
$\langle m'| J_+ |m'\rangle = N(m'+1)\langle m|m'+1\rangle$
$= N(m'+1)\,\delta_{m,m'+1}$
$j = \frac{1}{2}$ 2 states SU(2)
Fundamental representation
N = row # of defining matrix
$j = 1$ 3 states $3 = 2^2 - 1$


mutually commuting
generators
$\{t_i, t_j\} = 0 \quad \forall\, t_i, t_j$
them. s.t they are
$[J_3, \cdot] =$ constant
eigenvectors of Cartan generators
$A\chi = \lambda\chi$
$|j, m\rangle \to |j, j+1, \ldots, -j\rangle$
highest weight
$|\frac{3}{2}, \frac{1}{2}\rangle$
$\langle m'| J_+ |m\rangle = N(m'+1)\langle m|m'+1\rangle$
$= N(m'+1)\,\delta_{m,m'+1}$
$-\frac{1}{2}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}$
$j = \frac{1}{2}$ 2 states SU(2)
Fundamental representation
M = row # of defining matrix
$j = 1$ 3 states $3 = 2^2 - 1$
$M = n$ $n = N^2 - 1$
Adjoint Representation


mutually commuting
generators
$[t_i, t_j] = 0 \quad \forall\, t_i, t_j$
them. s.t they are
$[J_3, \cdot] =$ constant
eigenvectors of Cartan generators
$A\chi = \lambda\chi$
$|j, m\rangle$
$\rightarrow |j, j+1, \ldots\rangle$
highest weight
$|\frac{3}{2}, \frac{1}{2}\rangle$
$\langle m'| J_+ |m\rangle = N(m+1)\langle m'|m+1\rangle$
$= \ldots \delta_{m'm+1}$
$-\frac{1}{2}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}$
$j = \frac{1}{2}$ 2 states SU(2)
Fundamental representation
N = row # of defining matrix
$j = 1$ 3 states $3 = 2^2 - 1$
$N = n$ $n = N^2 - 1$
Adjoint Representation