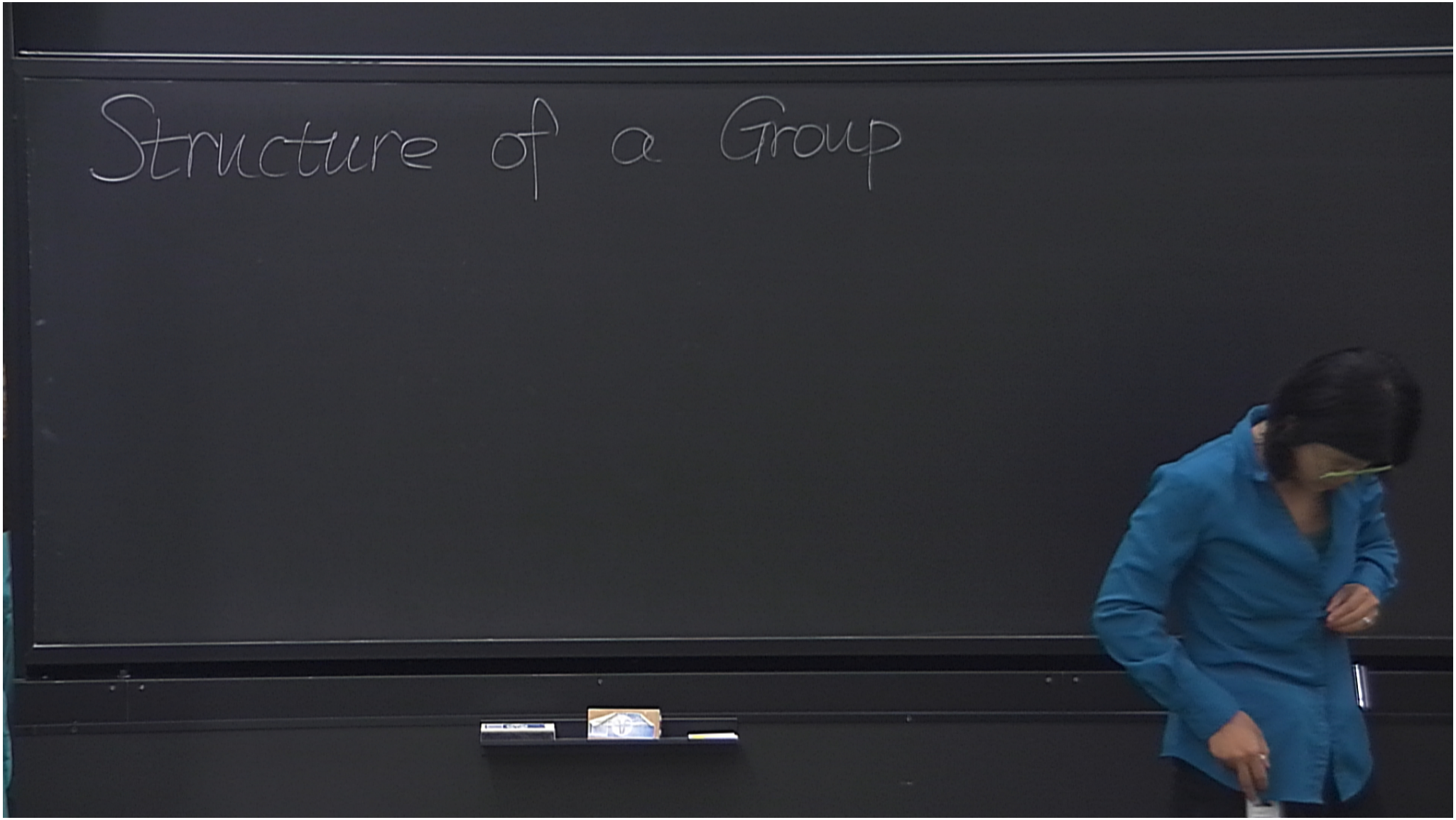
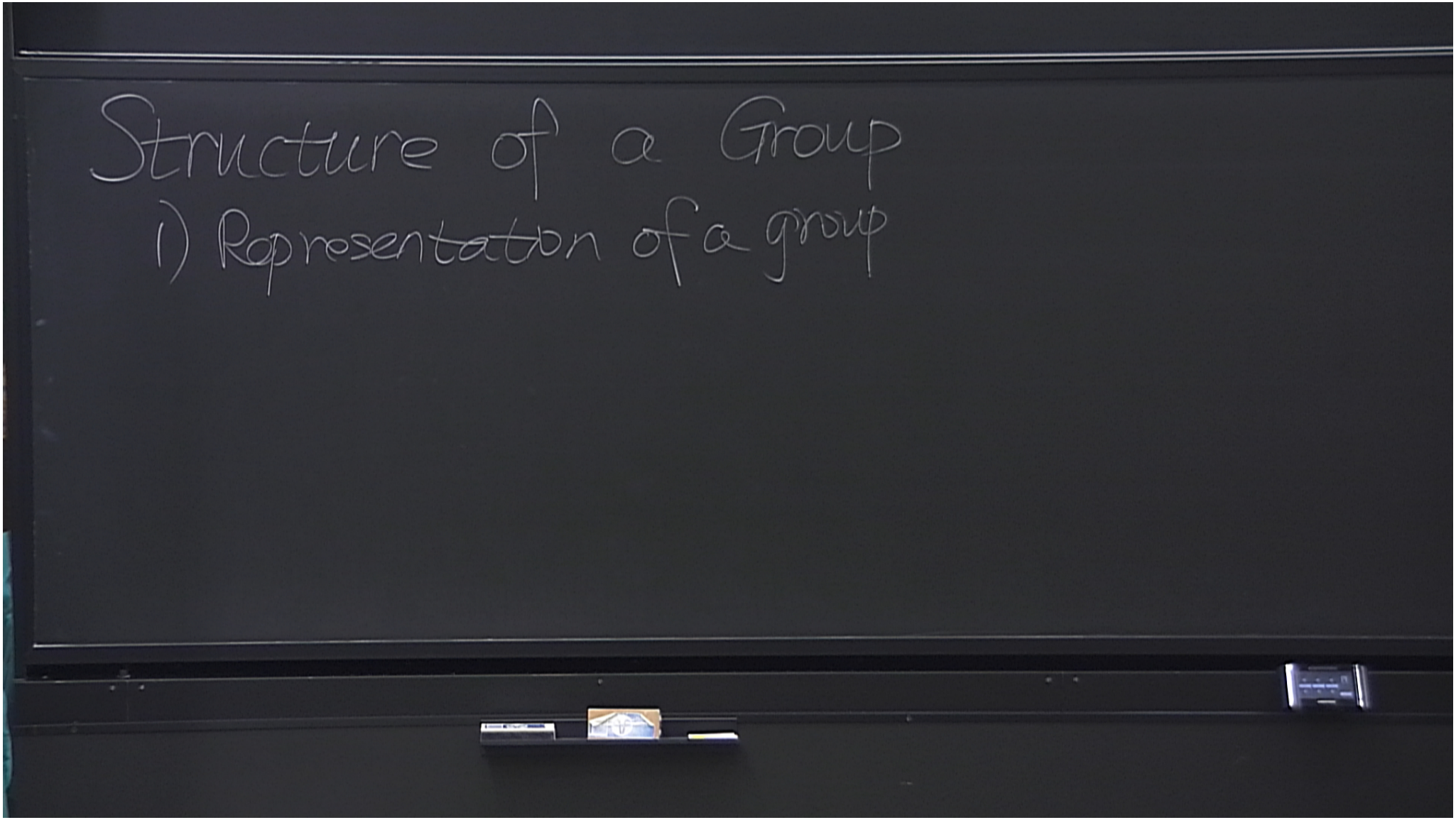
Title: 14/15 PSI - Lie Groups

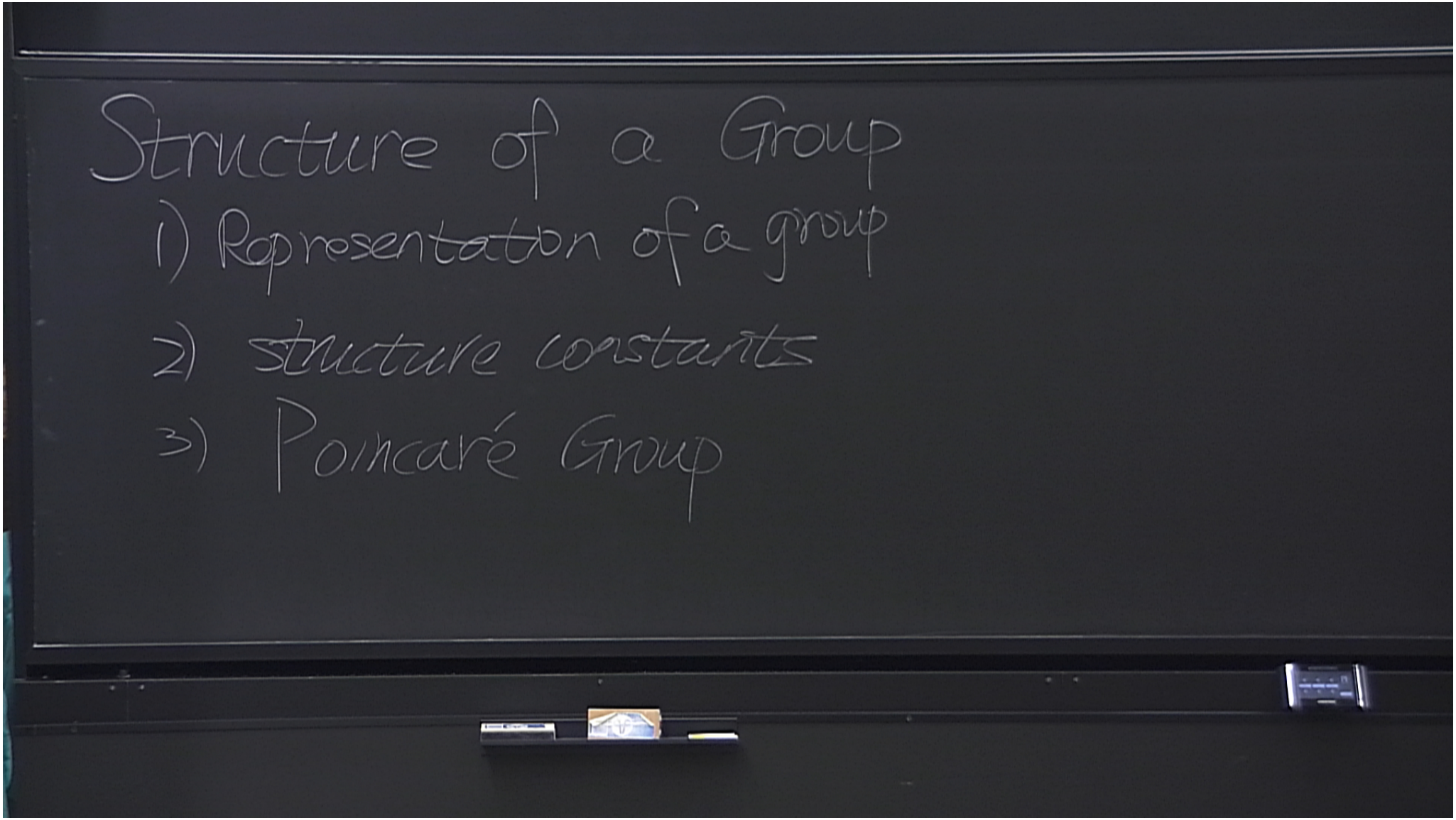
Date: Aug 26, 2014 09:00 AM

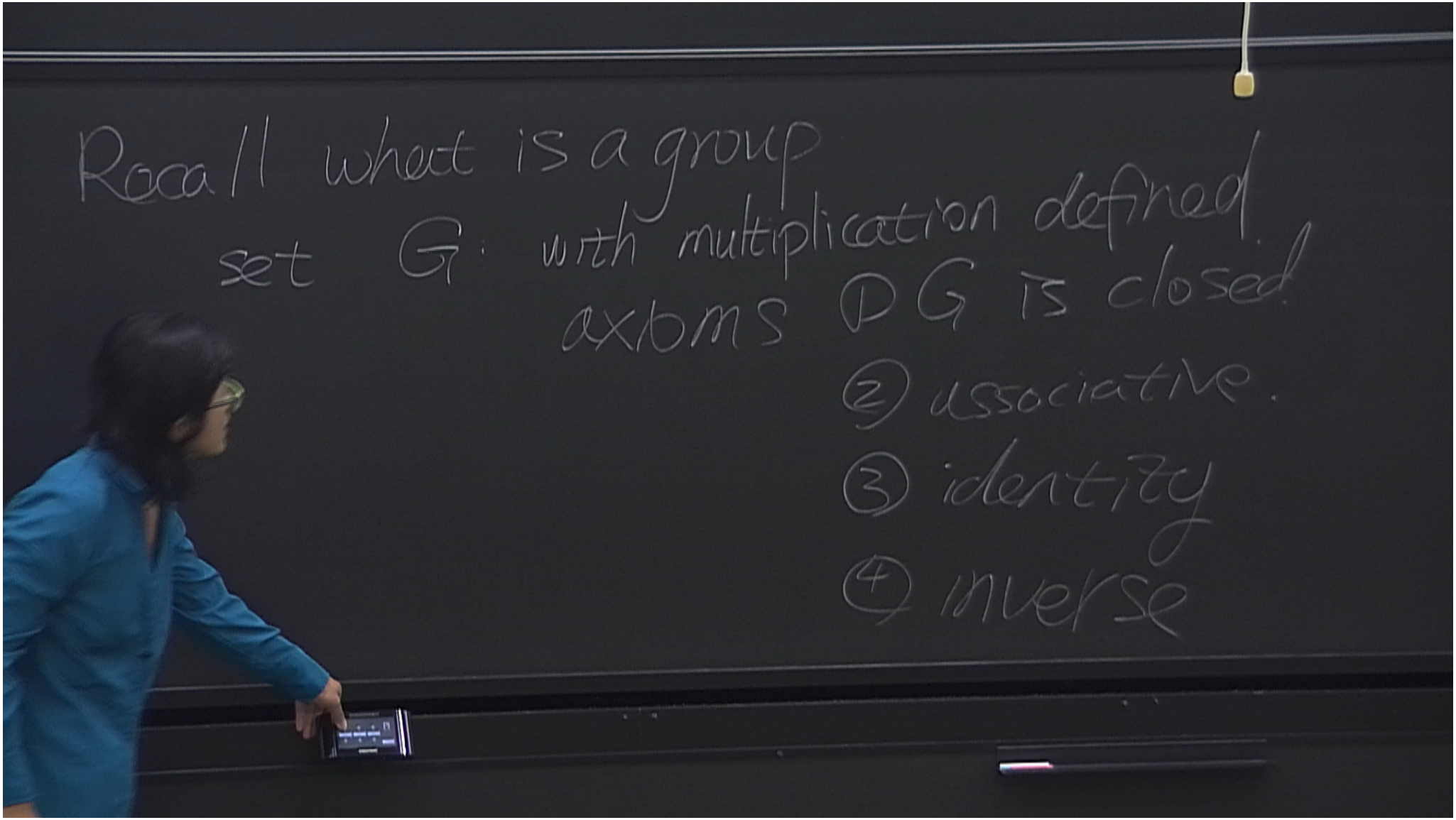
URL: http://pirsa.org/14080029

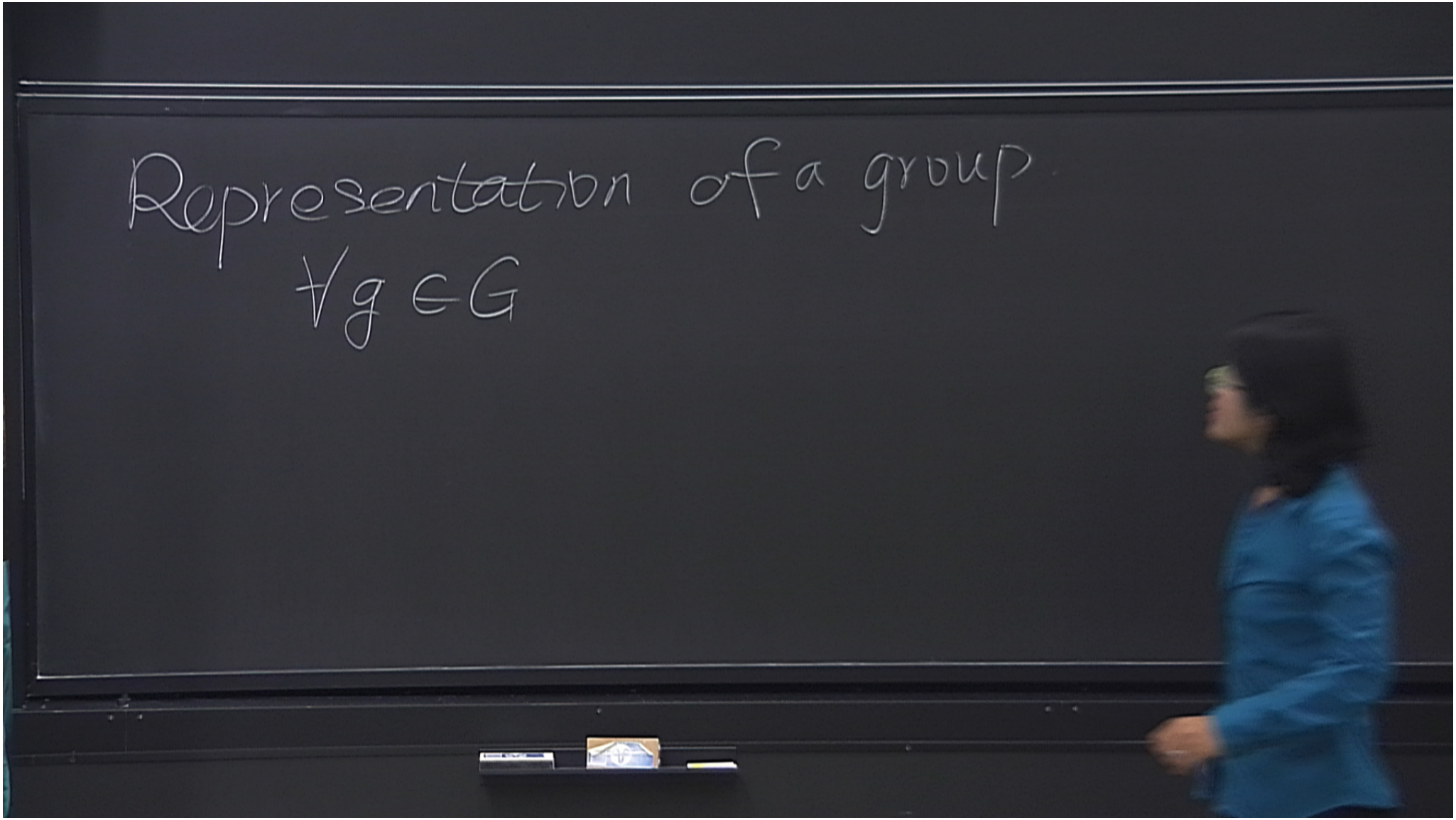
Abstract:

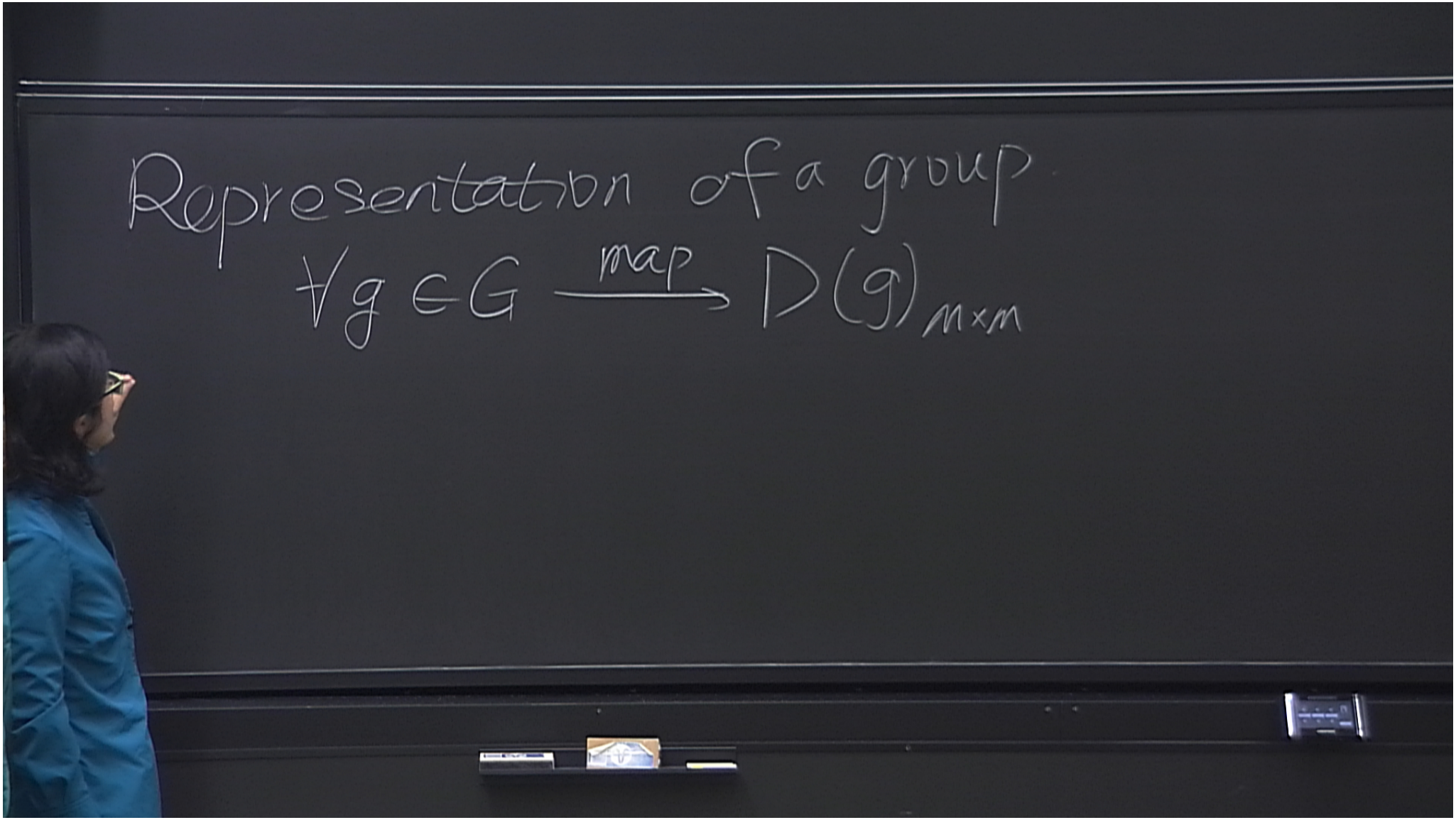

Structure of a Group

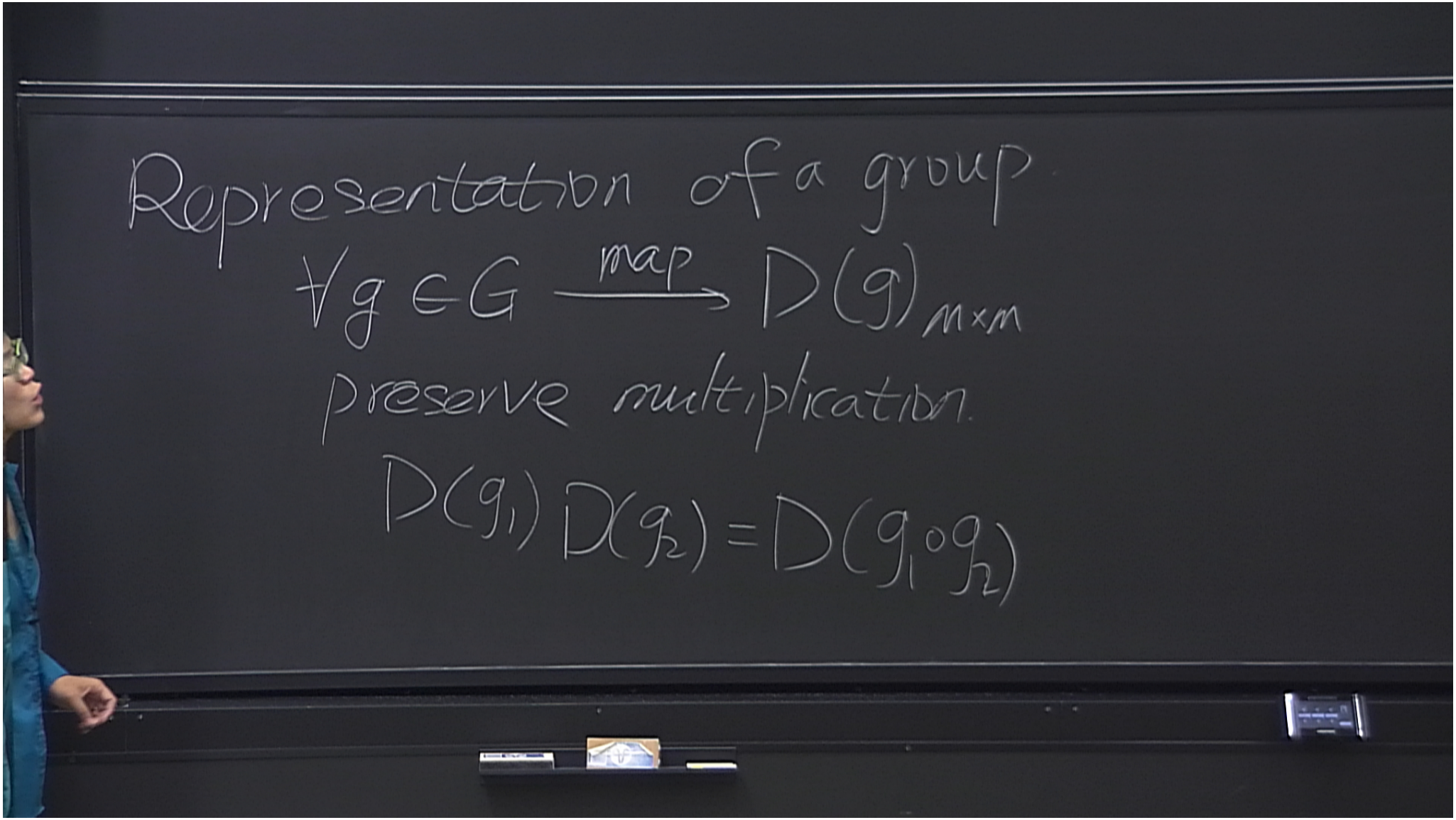

Structure of a Group
1) Representation of a group

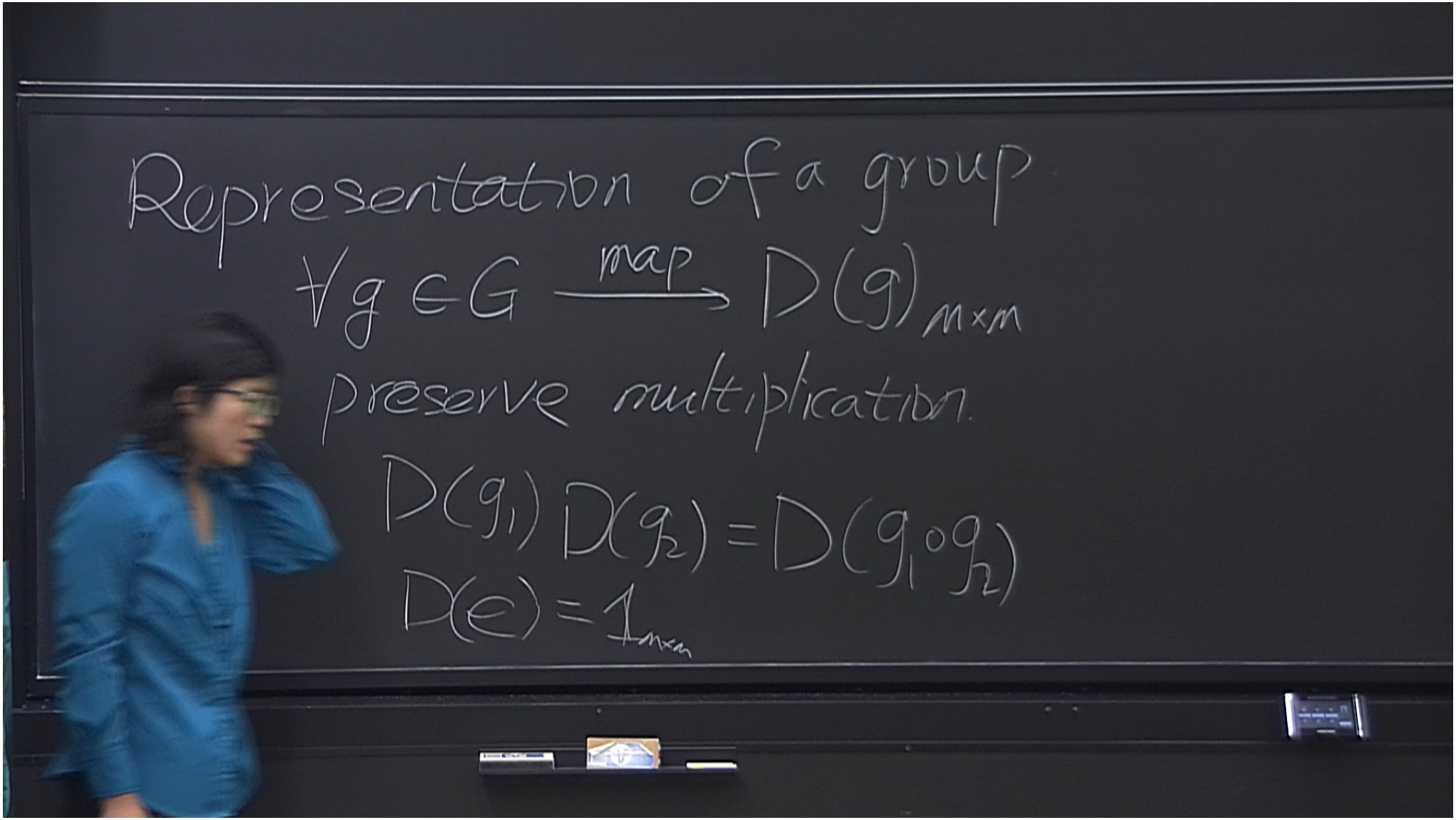

Structure of a Group
1) Representation of a group
2) structure constants
3) Poincaré Group

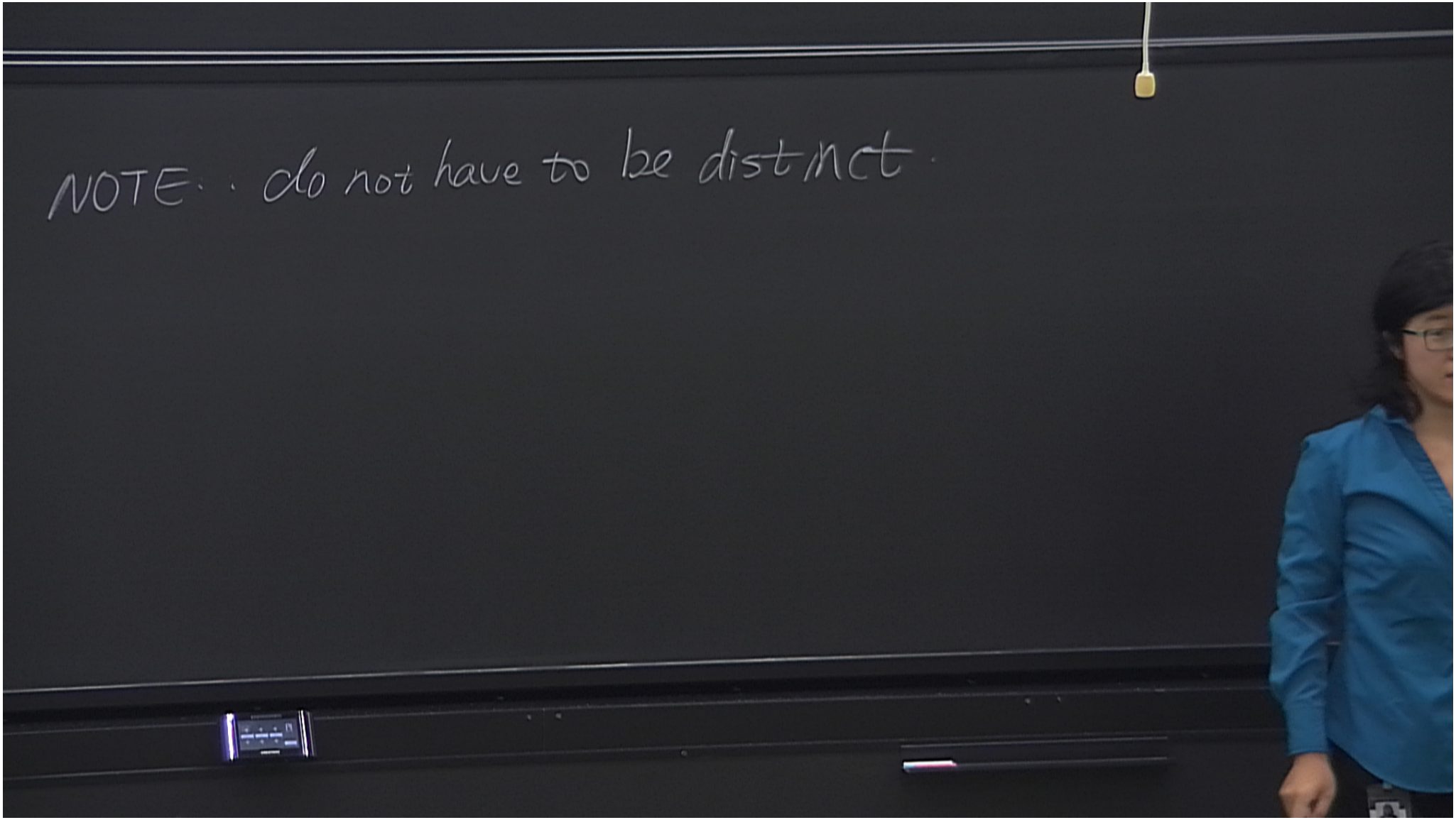

Recall what is a group
set G: with multiplication defined.
axioms ① G is closed
② associative.
③ identity
④ inverse

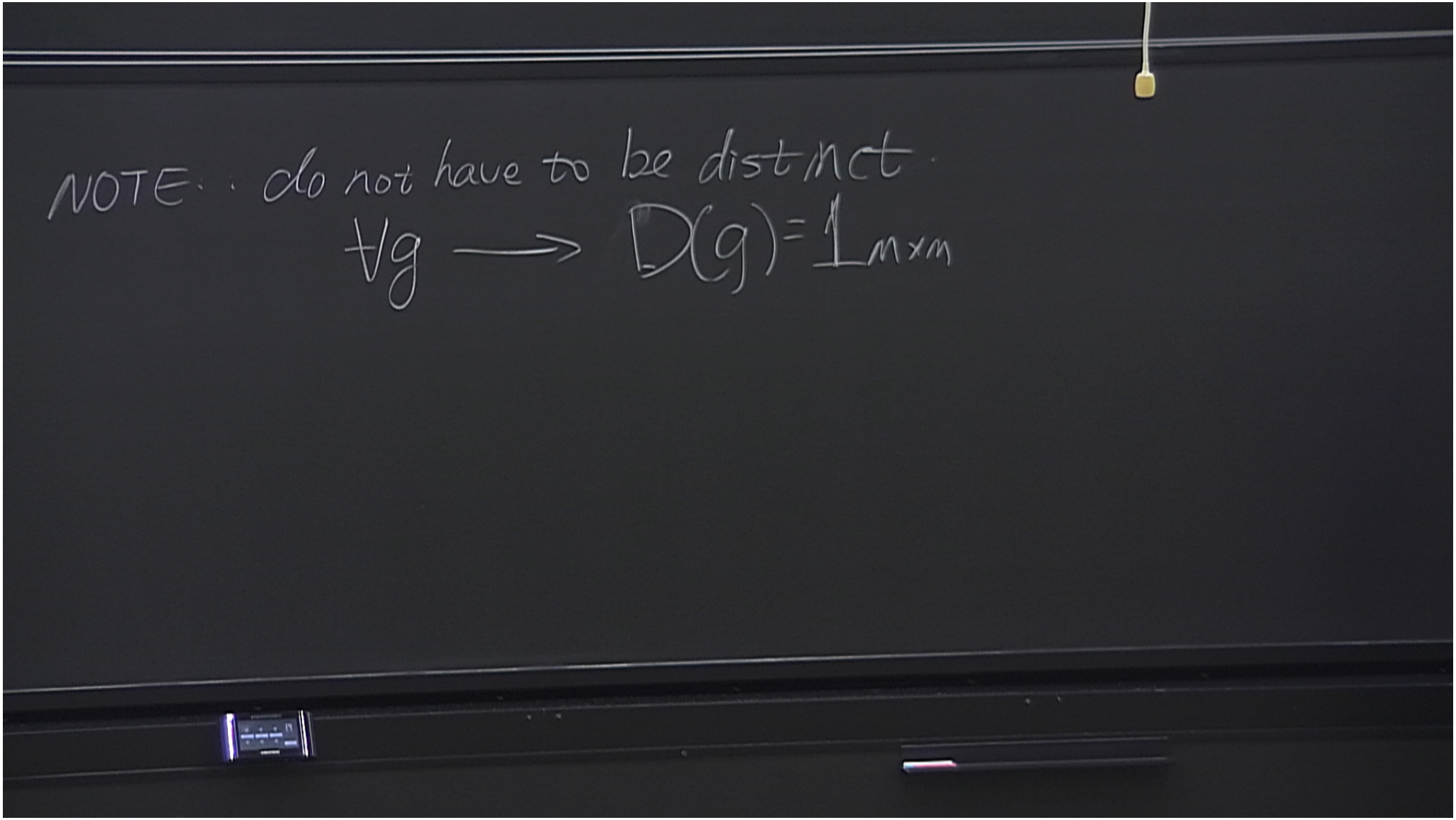

Representation of a group
$\forall g \in G$

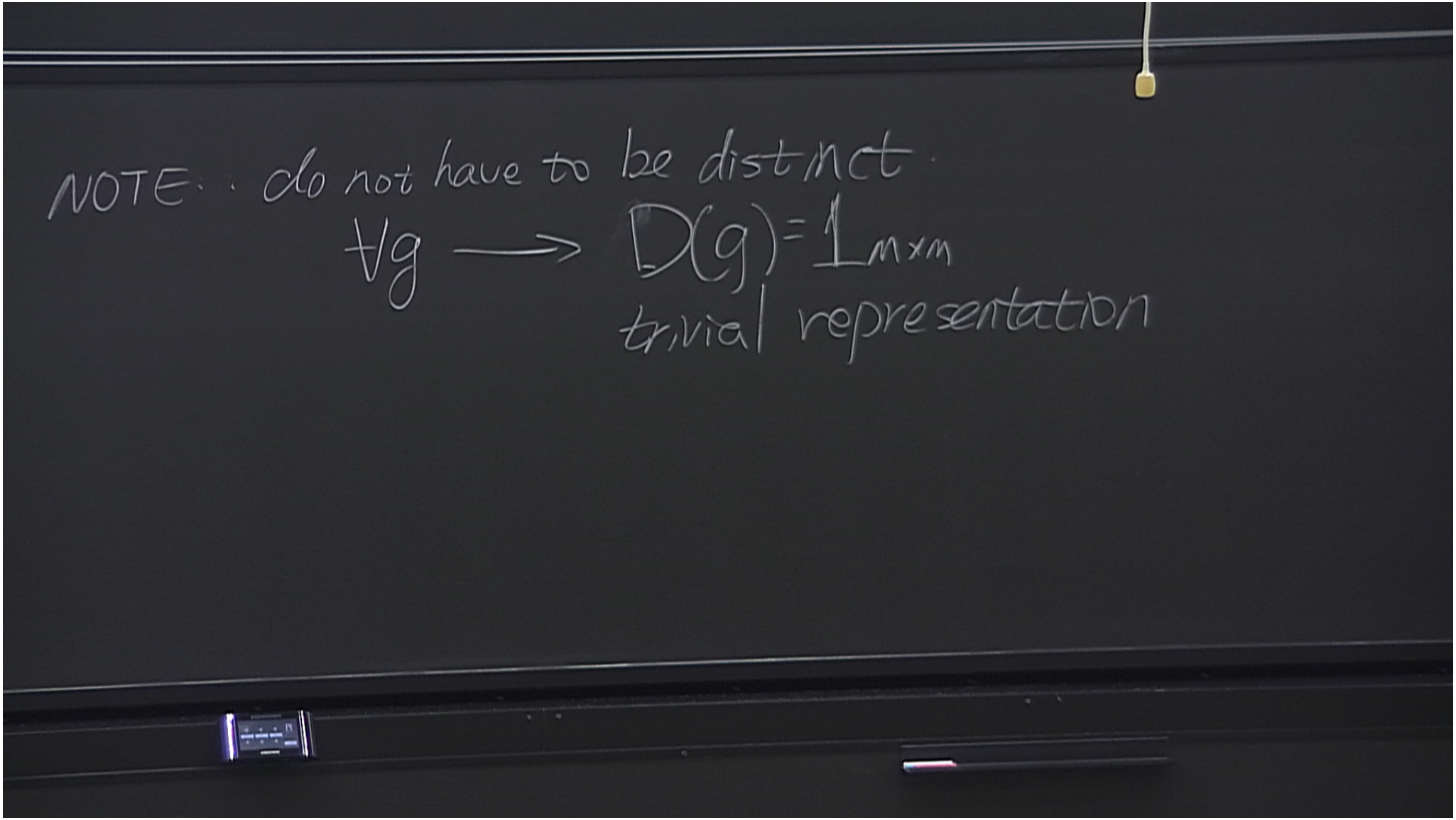

Representation of a group
$\forall g \in G \xrightarrow{\text{map}} D(g)_{n \times n}$

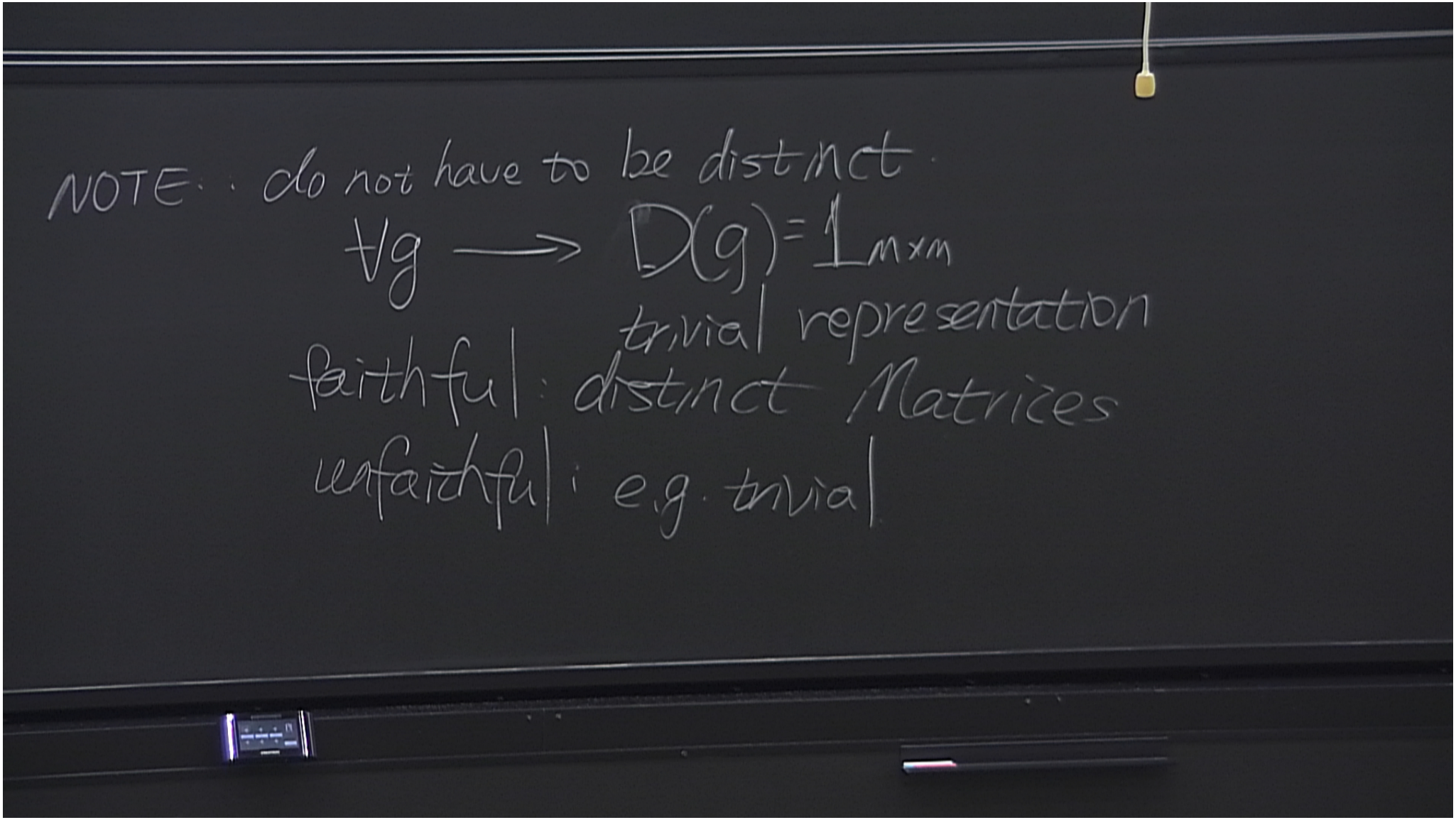

Representation of a group
$\forall g \in G \xrightarrow{\text{map}} D(g)_{m\times m}$
preserve multiplication.
$D(g_1)\,D(g_2) = D(g_1 \circ g_2)$

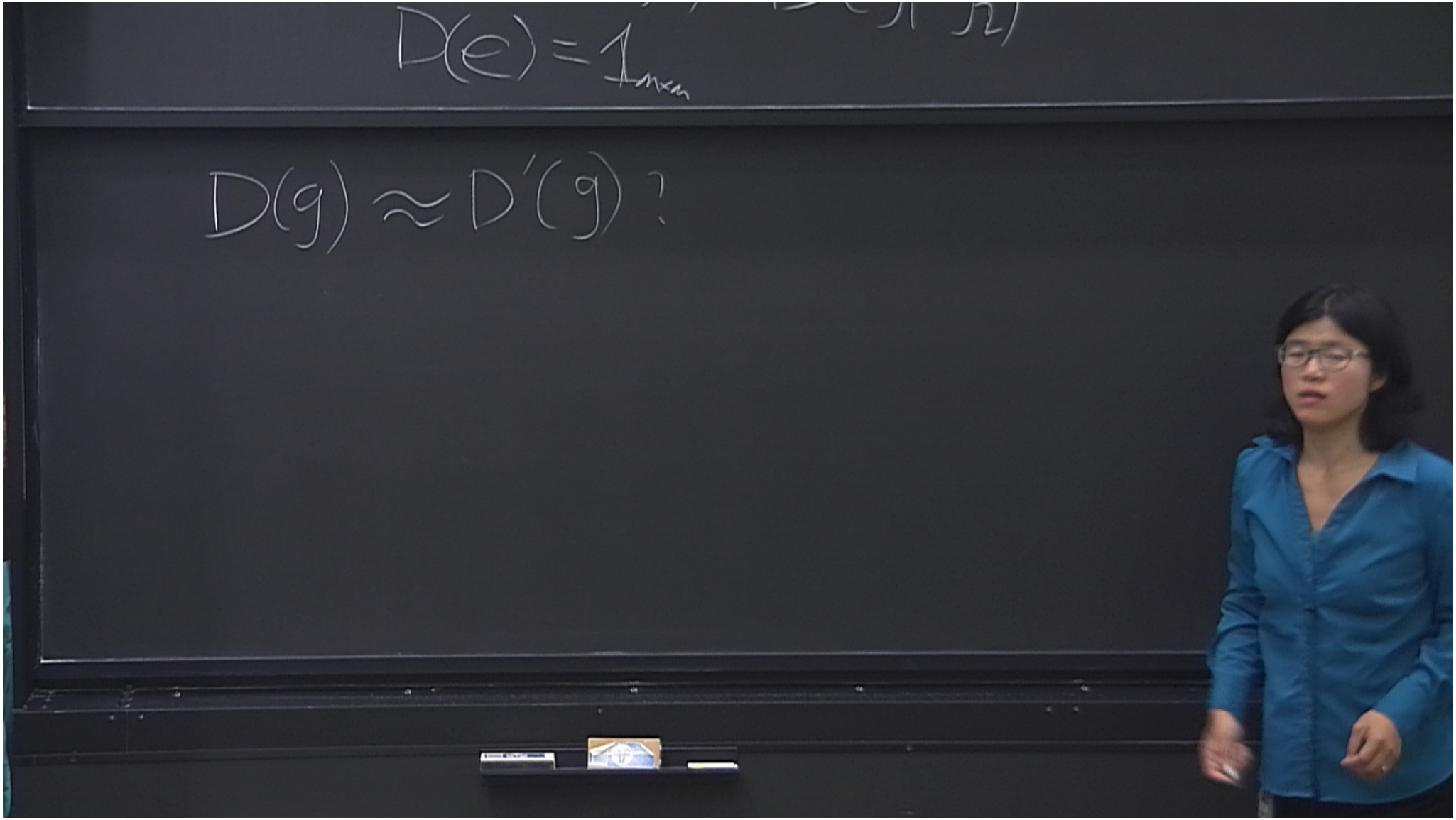

Representation of a group
$\forall g \in G \xrightarrow{\text{map}} D(g)_{m\times m}$
preserve multiplication.
$D(g_1)\,D(g_2) = D(g_1 \circ g_2)$
$D(e) = 1_{m\times m}$


NOTE: . do not have to be distinct.


NOTE: do not have to be distinct.
$\forall g \longrightarrow D(g) = 1_{m \times m}$


NOTE: do not have to be distinct.
$\forall g \longrightarrow D(g) = 1_{m \times m}$
trivial representation


NOTE: do not have to be distinct
$\forall g \longrightarrow D(g) = 1_{m \times m}$
trivial representation
faithful: distinct Matrices
unfaithful: e.g. trivial

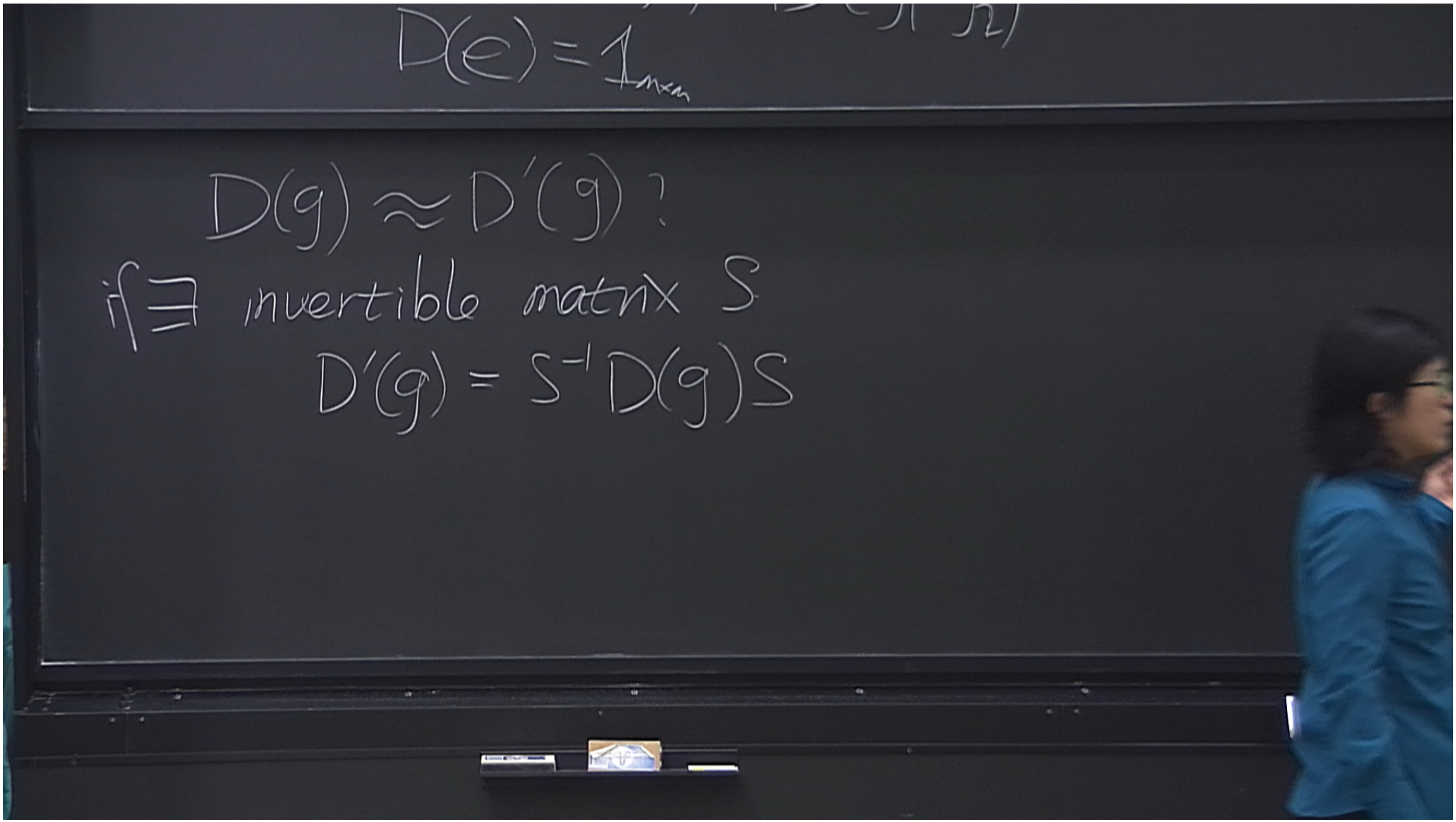

$D(e) = 1_{m \times m}$
$D(g) \approx D'(g)$?

$D(e) = 1_{n \times n}$

$D(g) \approx D'(g)$?

if $\exists$ invertible matrix S

$$D'(g) = S^{-1} D(g) S$$



$D(e) = 1_{n \times n}$

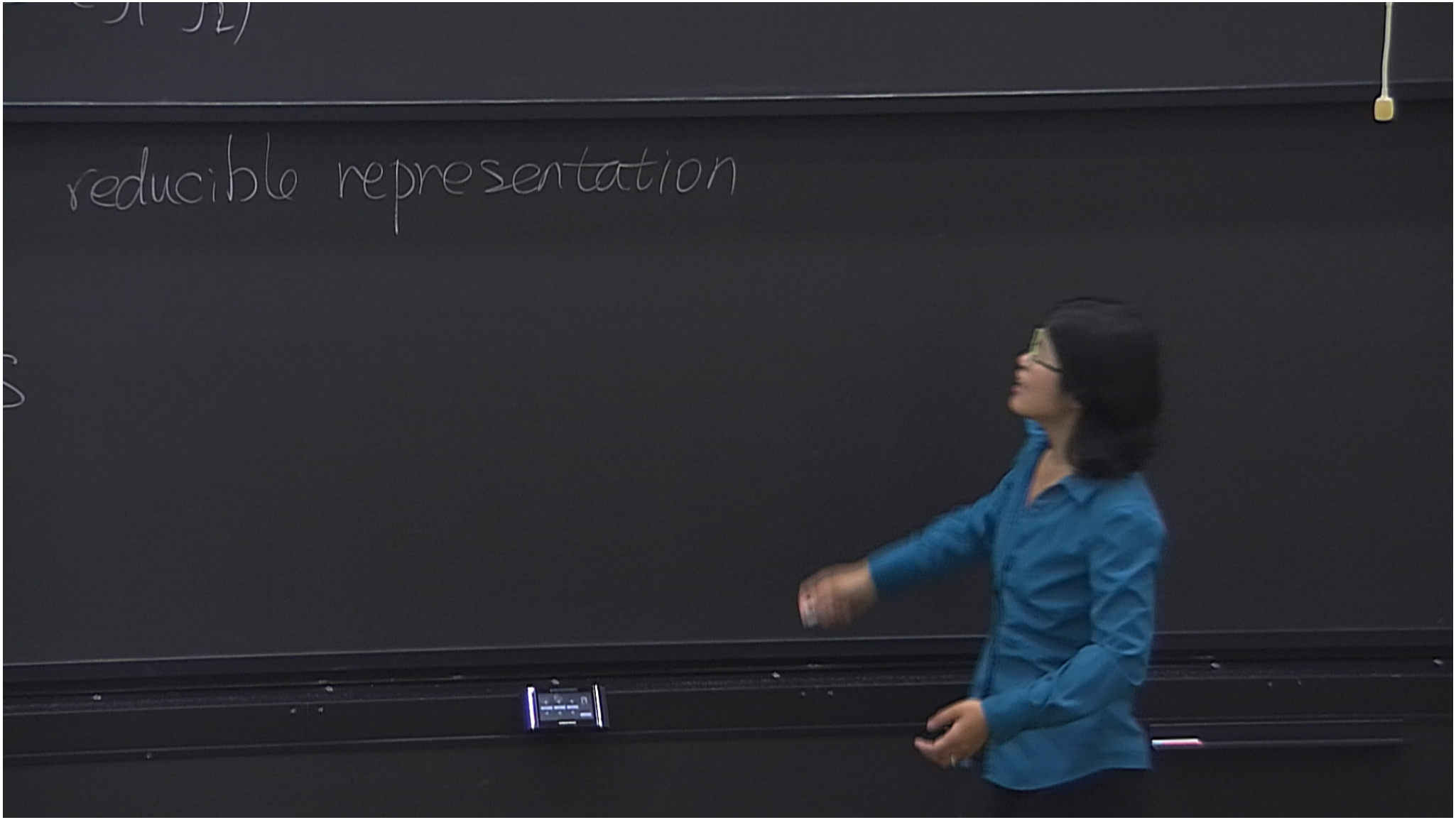
$D(g) \approx D'(g)$?

if $\exists$ invertible matrix S

$D'(g) = S^{-1} D(g) S$

equivalent!




reducible representation

reducible representation

$\exists S$ such that $S^{-1} D(g) S$

$$= \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

reducible representation

$\exists S$ such that $S^{-1} D(g) S$

$$= \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

$(A_1)_{d_1 \times d_1}$

reducible representation

$\exists S$ such that $S^{-1} D(g) S$

$$= \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

$(A_i)_{d_i \times d_i}$

$\sum_{i=1}^{\ell} d_i = M$

$g <$

reducible representation

$\exists S$ such that $S^{-1} D(g) S$

$$= \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

$(A_i)_{d_i \times d_i}$

$\sum_{i=1}^{\ell} d_i = M$

$\ell > 1$

reducible representation

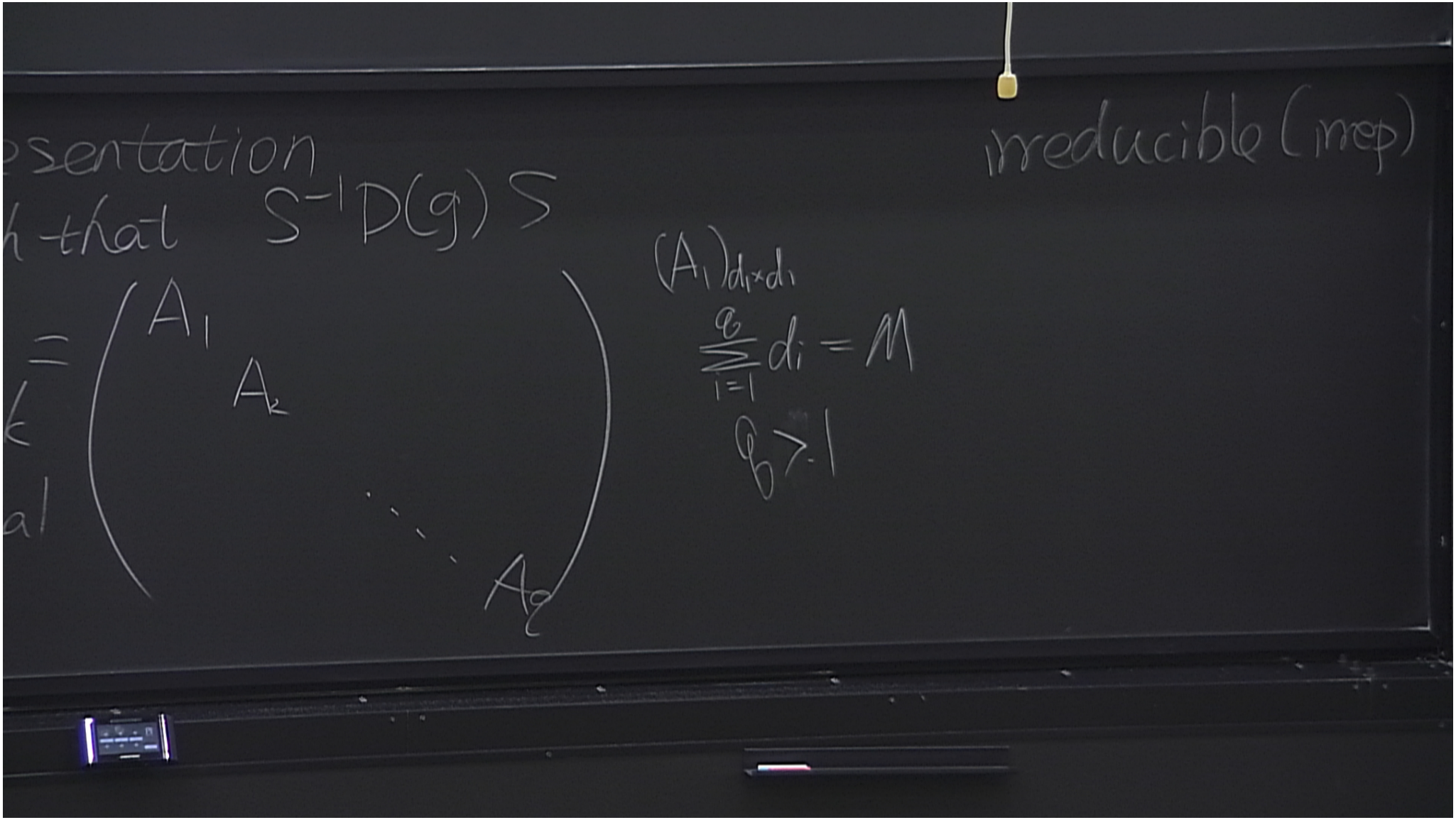
$\exists S$ such that $S^{-1} D(g) S$

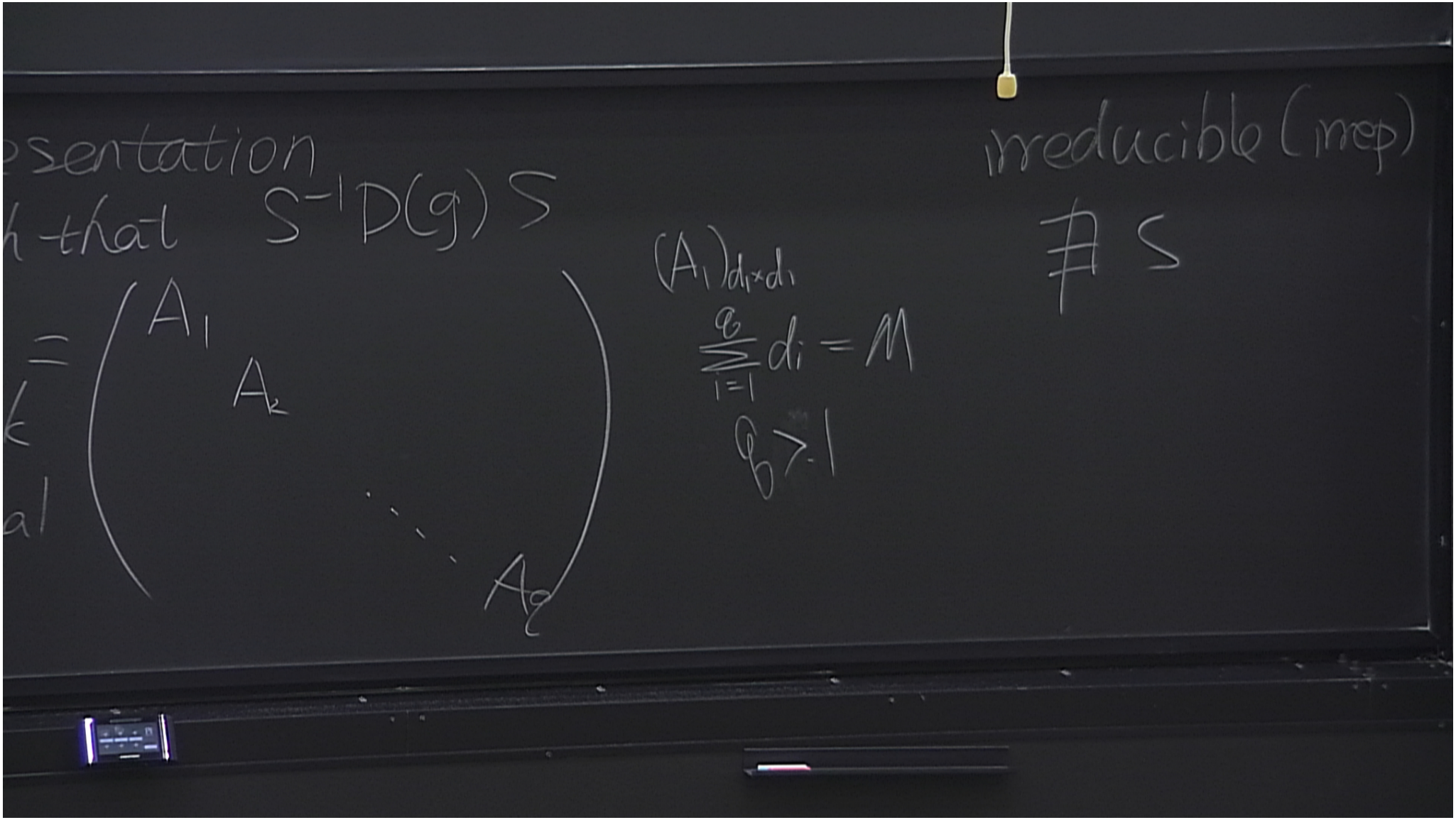
$= \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_q \end{pmatrix}$ block diagonal

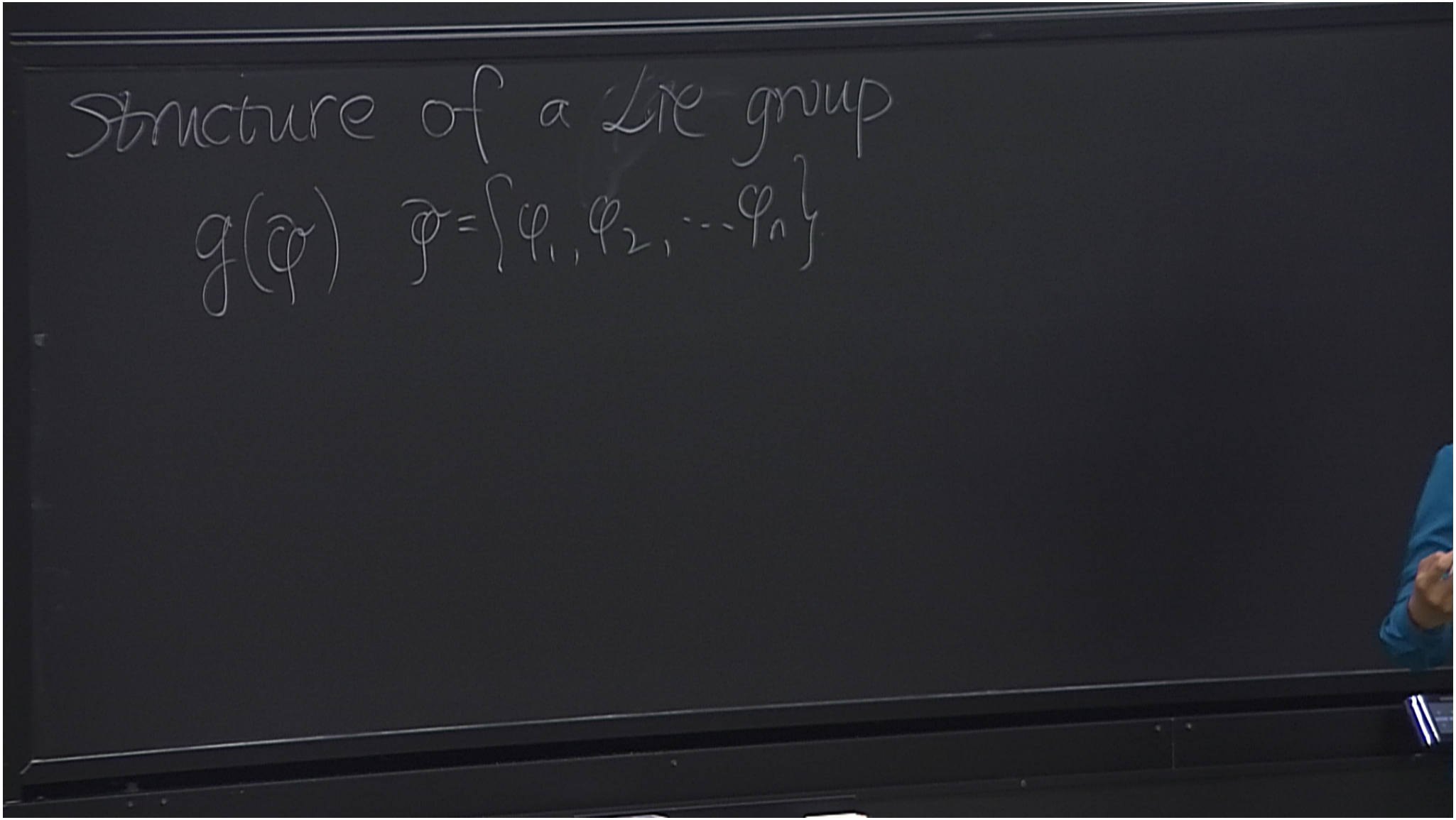
$(A_i)_{d_i \times d_i}$

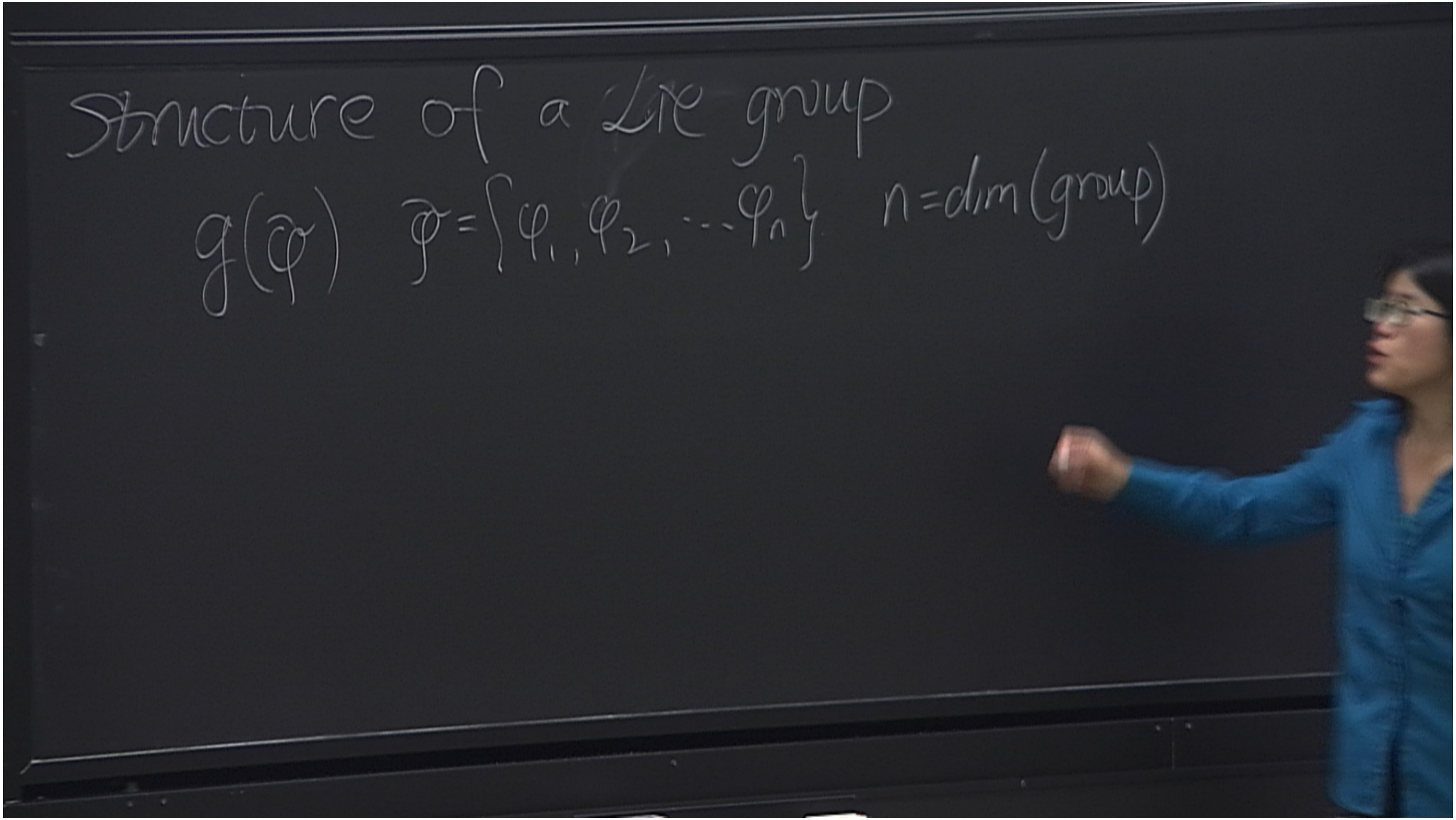
$\sum_{i=1}^{q} d_i = M$

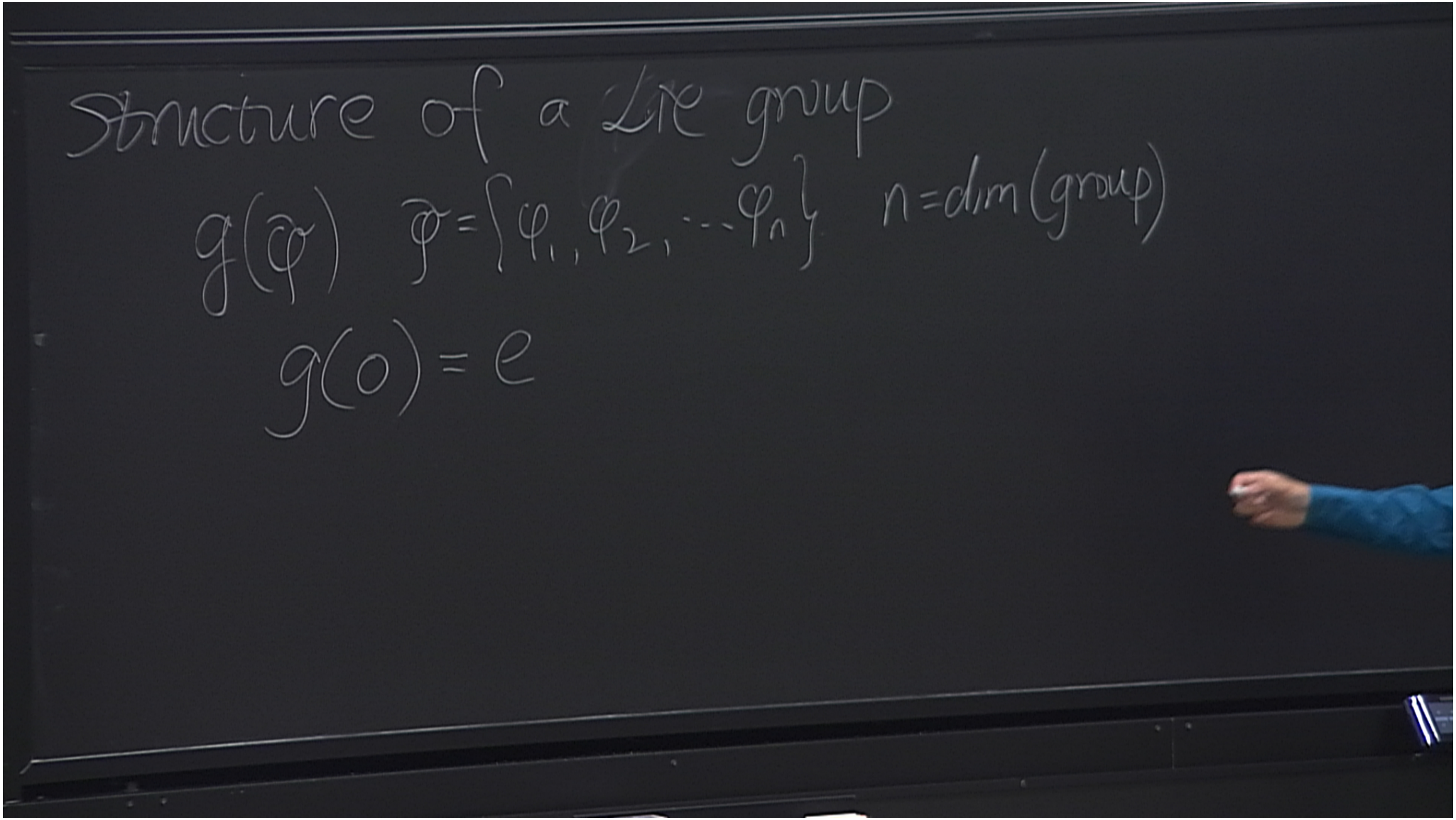
$q > 1$


esentation
h that $S^{-1}D(g)S$
irreducible (irrep)
$(A_1)_{d_1 \times d_1}$
$\sum_{i=1}^{q} d_i = M$
$q \geq 1$


esentation
h that $S^{-1}D(g)S$
irreducible (rep)
$\nexists$ S
$(A_1)_{d_1 \times d_1}$
$\sum_{i=1}^{q} d_i = M$
$q > 1$
A_1
A_2
A_q


Structure of a Lie group
$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$


Structure of a Lie group
$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$ $n = \dim(\text{group})$

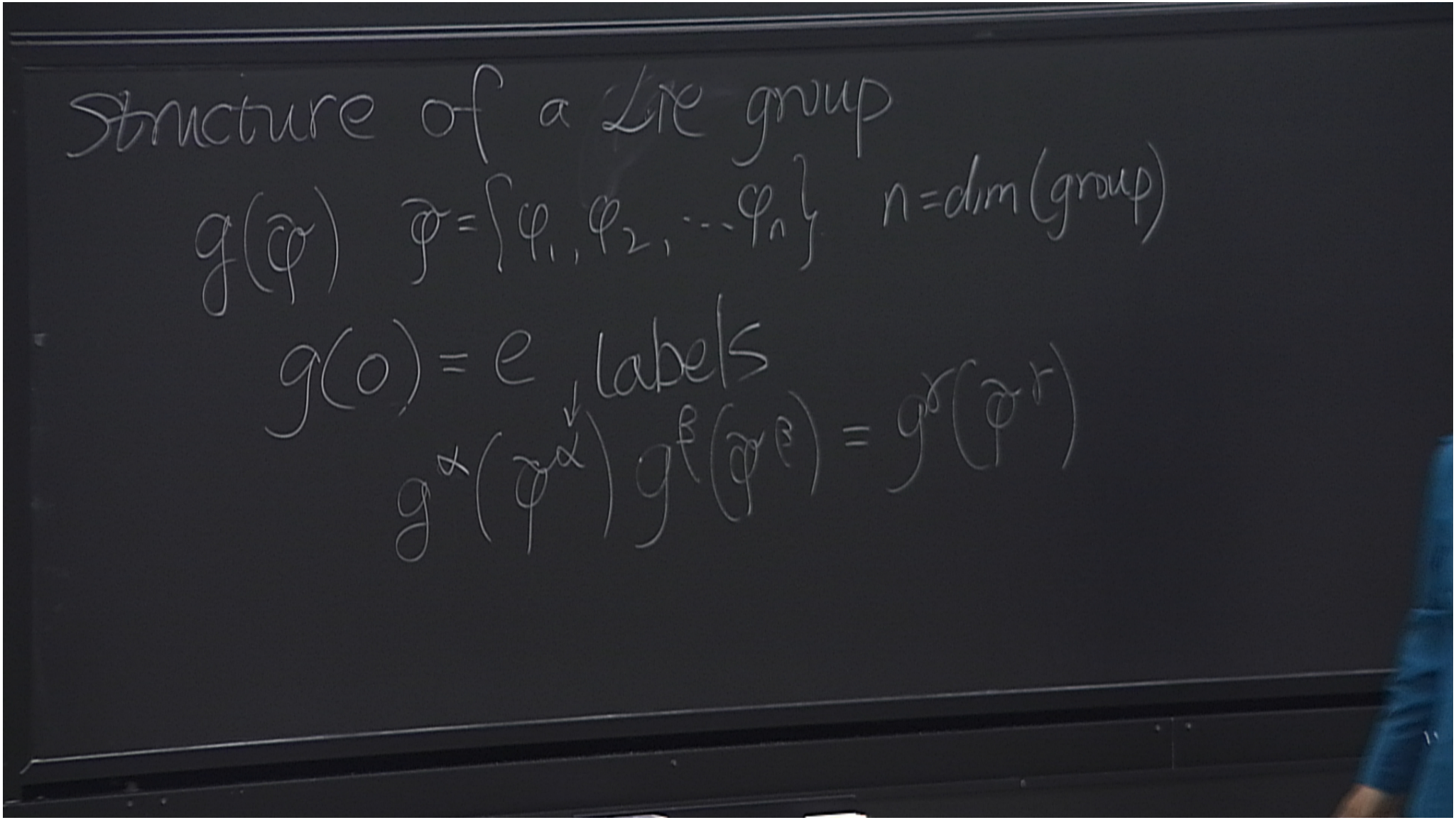

Structure of a Lie group
g(φ) φ = {φ1, φ2, ... φn} n = dim(group)
g(0) = e

Structure of a Lie group

$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$ $n = \dim(\text{group})$

$g(0) = e$

$g^\alpha(\varphi^\alpha)\, g^\beta(\varphi^\beta) = g^\gamma(\varphi^\gamma)$


Structure of a Lie group
$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$ $n = \dim(\text{group})$
$g(0) = e$ labels
$g^\alpha(\varphi^\alpha)\, g^\beta(\varphi^\beta) = g^\gamma(\varphi^\gamma)$

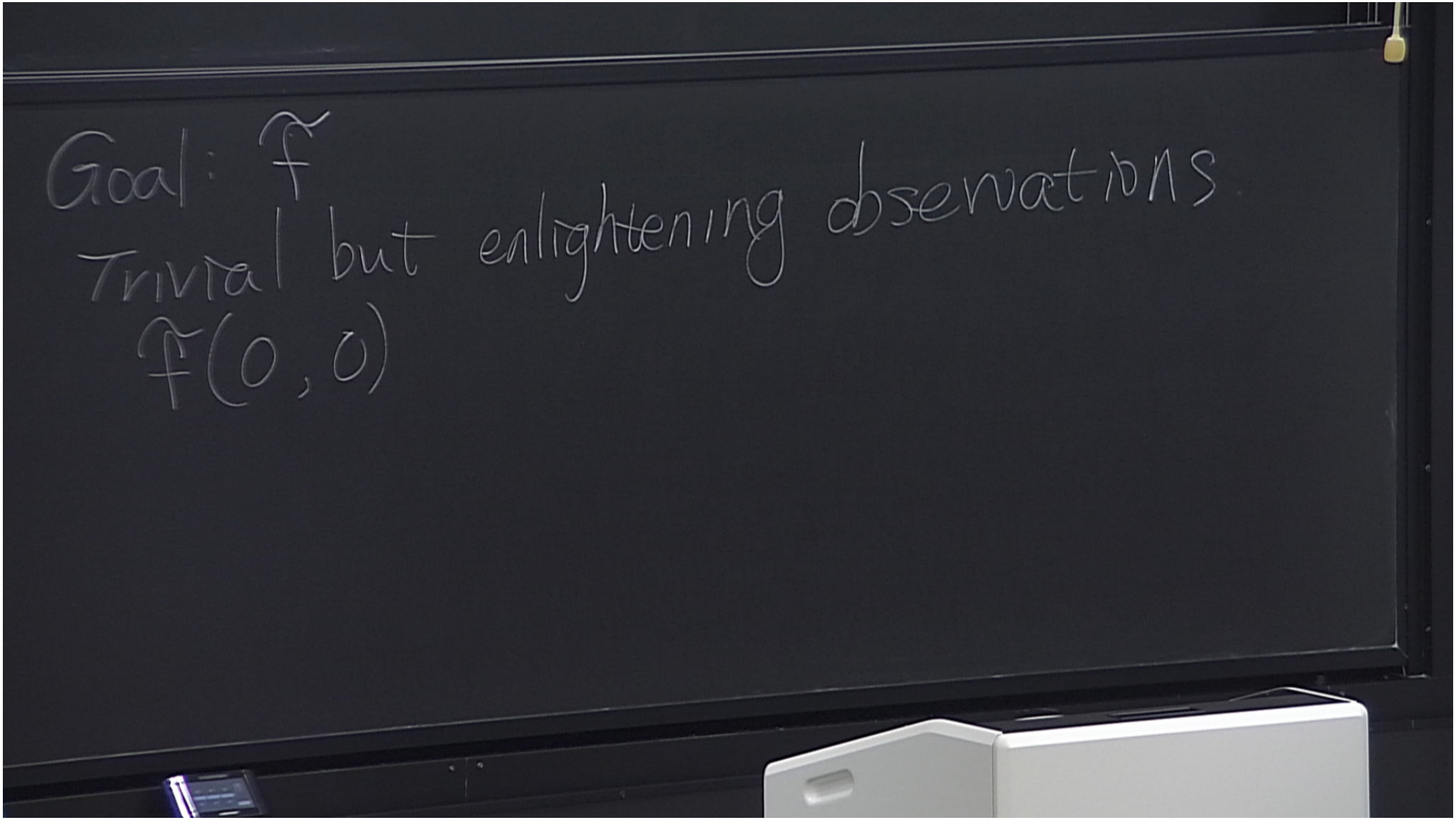
Structure of a Lie group

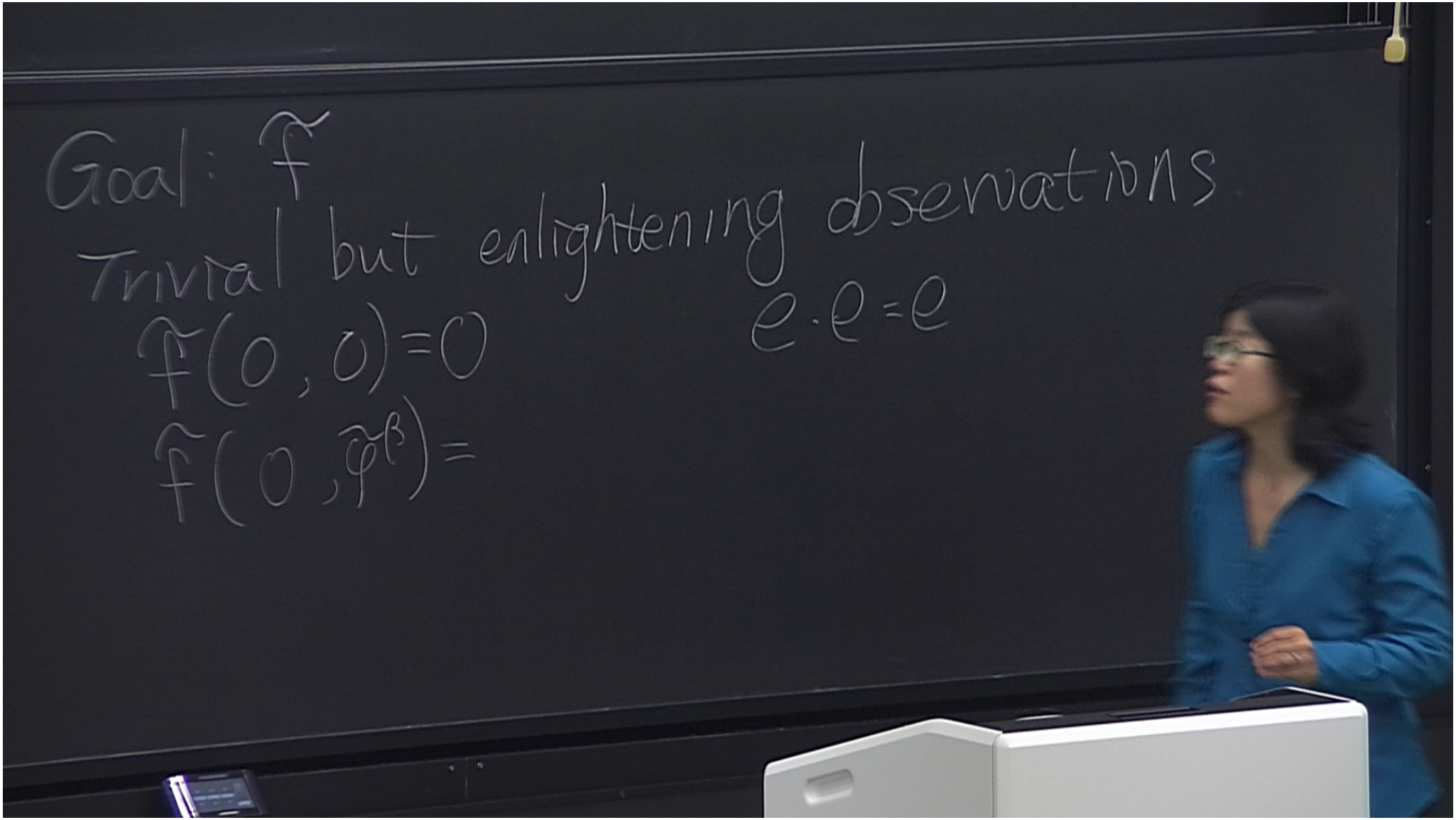
$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$ $n = \dim(\text{group})$

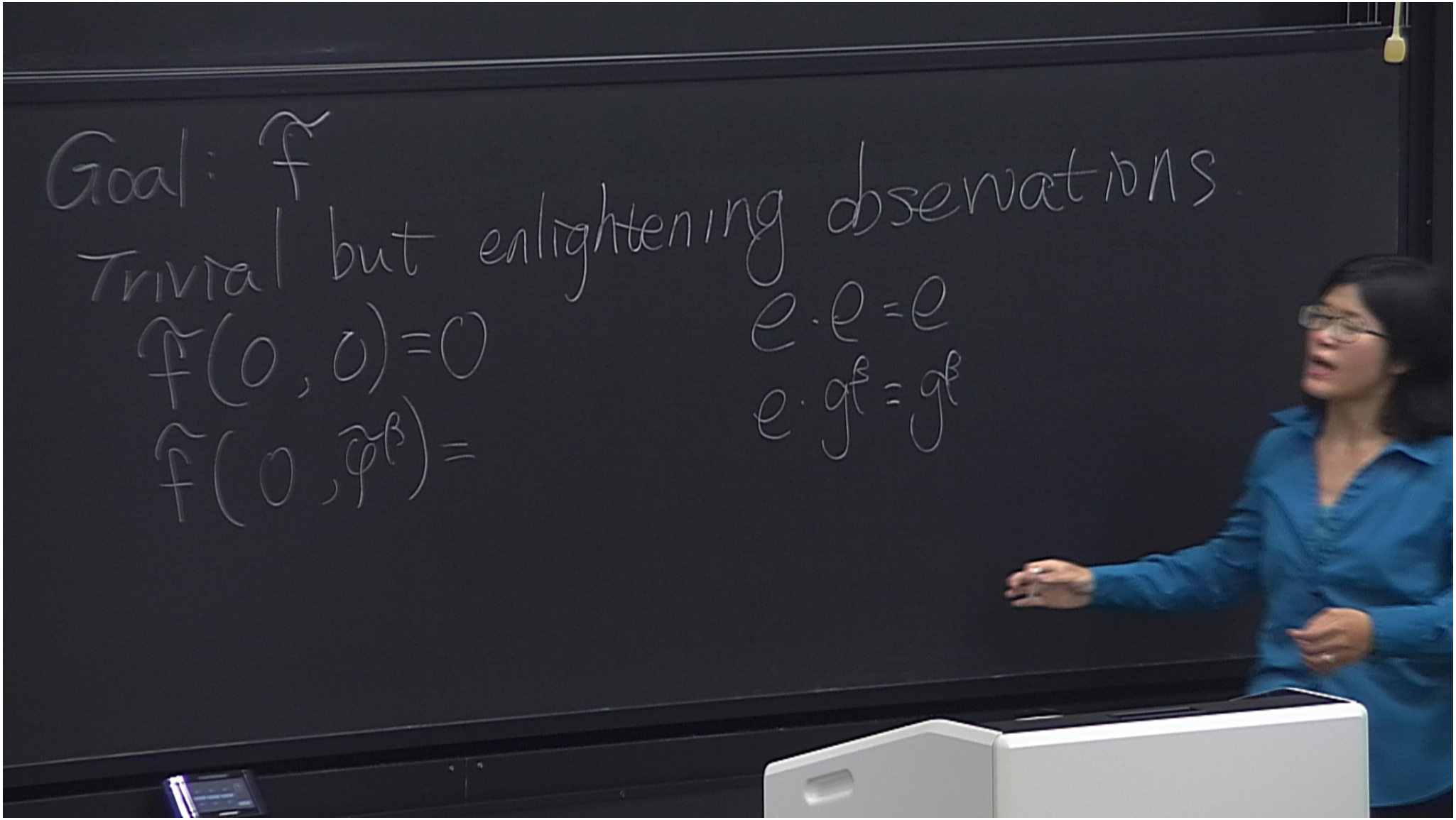
$g(0) = e$ labels

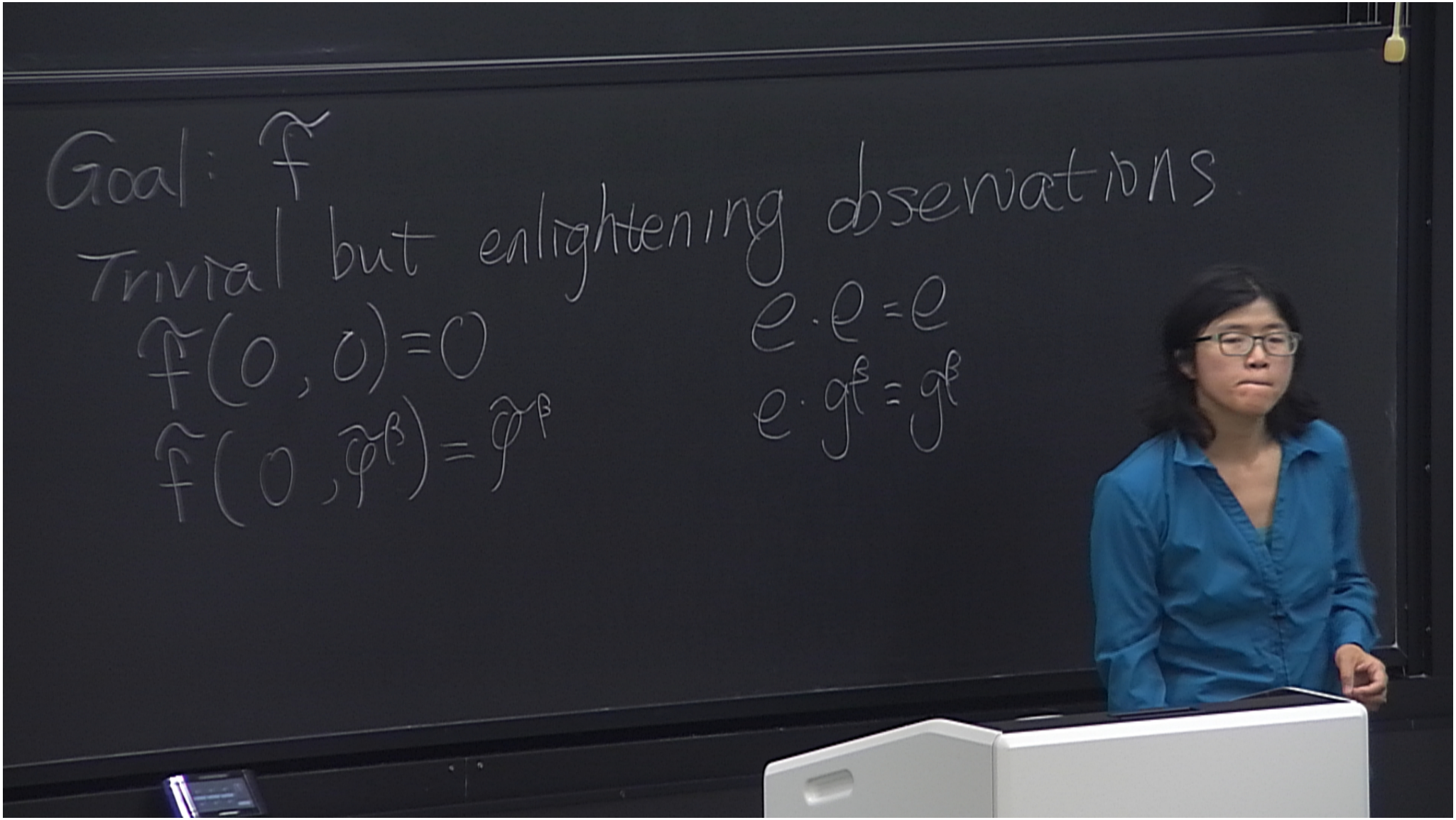
$g^\alpha(\varphi^\alpha)\, g^\beta(\varphi^\beta) = g^\gamma(\varphi^\gamma)$

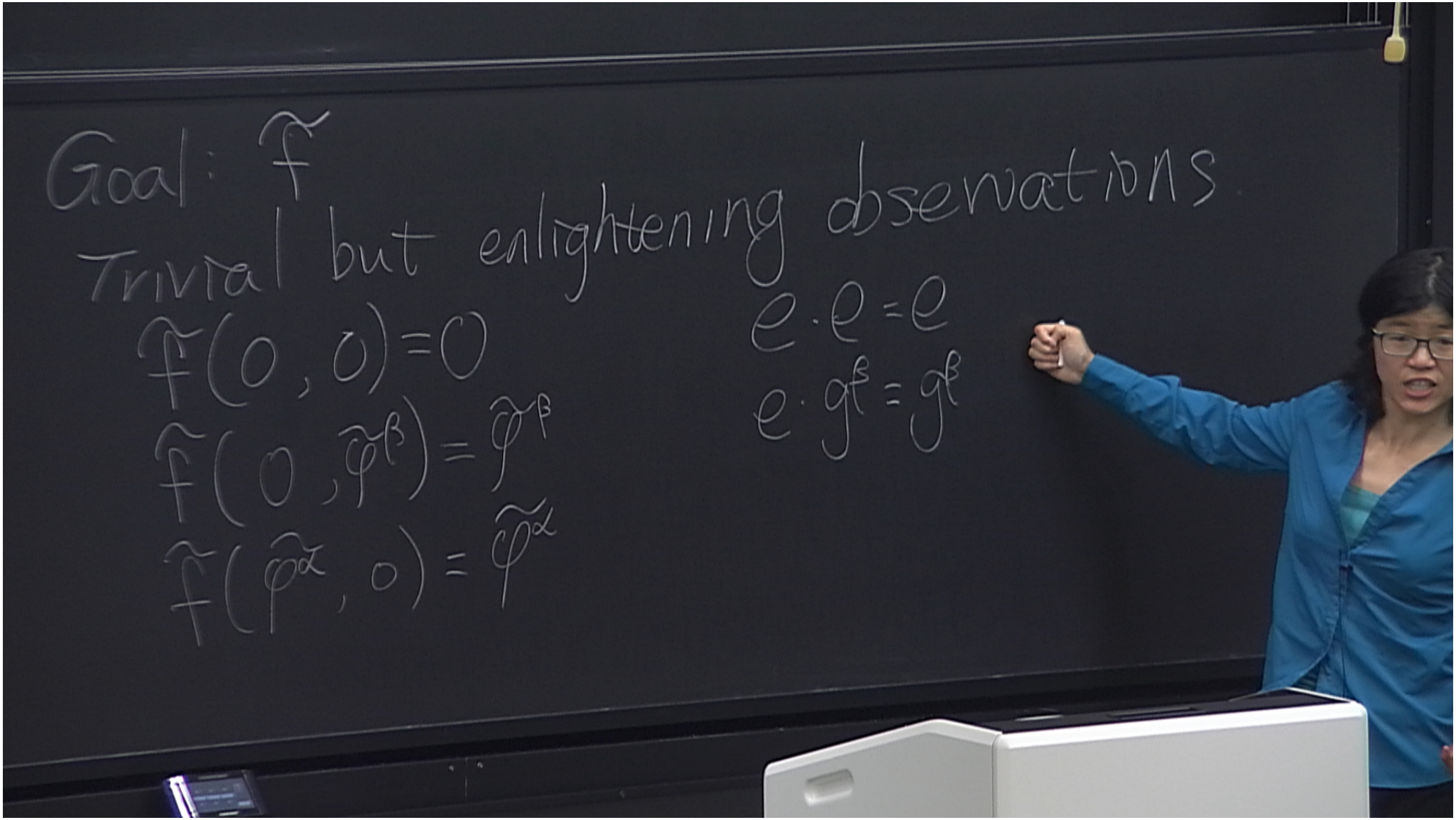
$\varphi^\gamma = f^\gamma(\varphi^\alpha, \varphi^\beta)$

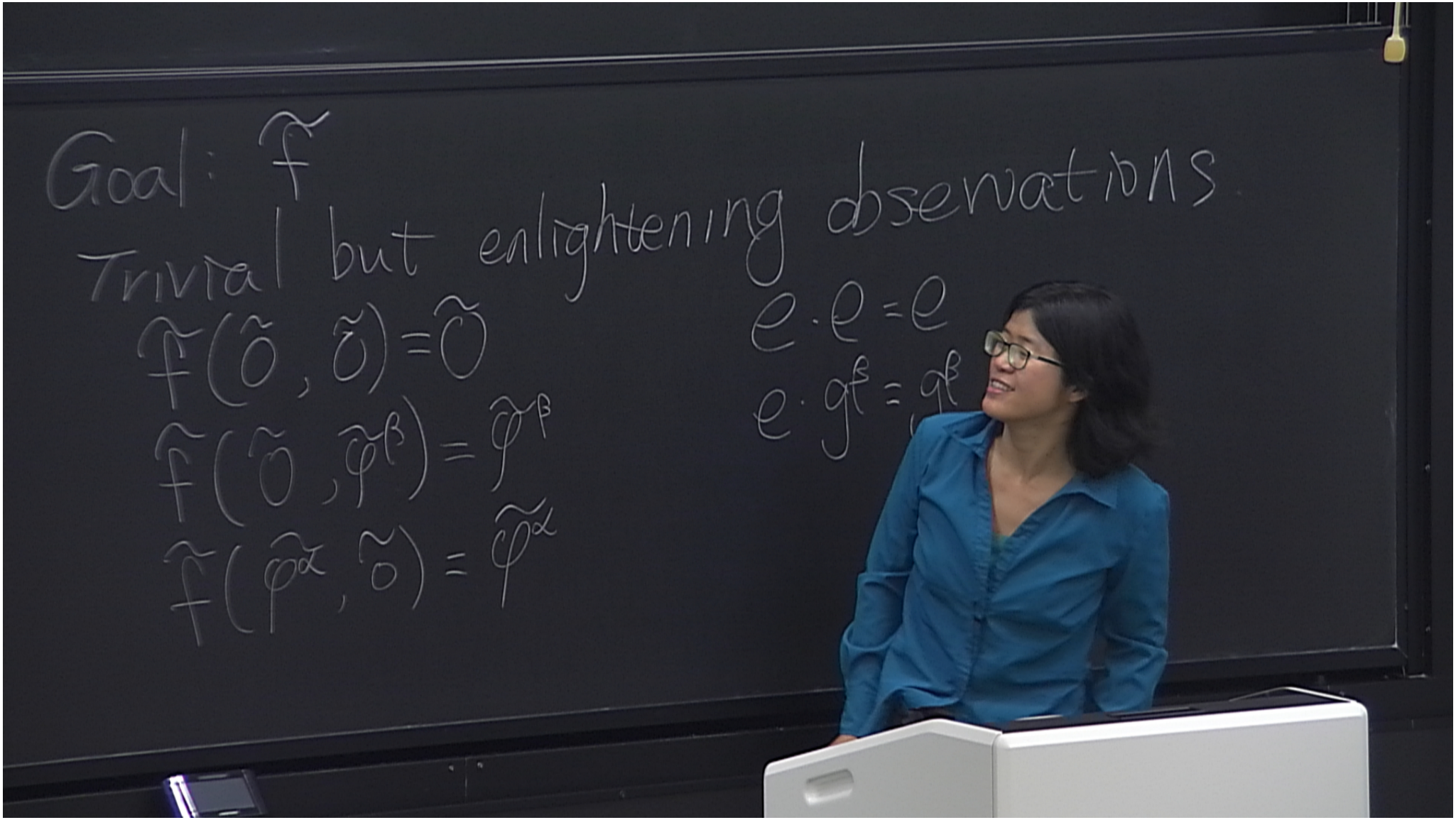

Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(0,0)$

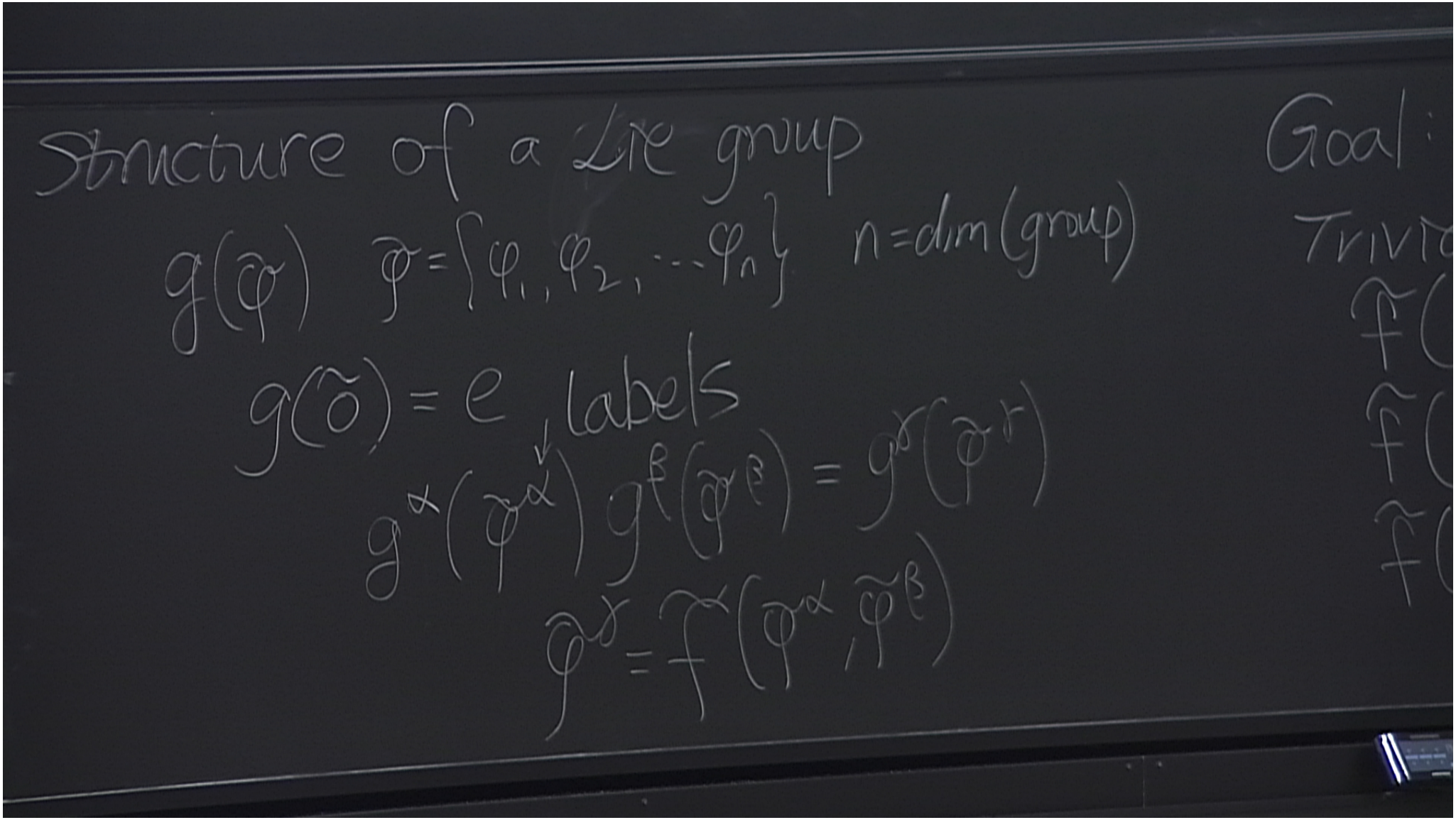

Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(0,0)=0$
$e \cdot e = e$
$\tilde{f}(0,\varphi^\beta)=$


Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(0,0)=0$
$\tilde{f}(0,\varphi^\beta)=$
$e \cdot e = e$
$e \cdot g^\beta = g^\beta$


Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(0,0)=0$
$\tilde{f}(0,\varphi^\beta)=\varphi^\beta$
$e\cdot e=e$
$e\cdot g^\beta=g^\beta$


Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(0,0)=0$
$\tilde{f}(0,\varphi^\beta)=\varphi^\beta$
$\tilde{f}(\varphi^\alpha,0)=\varphi^\alpha$
$e\cdot e=e$
$e\cdot g^\beta=g^\beta$


Goal: $\tilde{f}$
Trivial but enlightening observations
$\tilde{f}(\tilde{o}, \tilde{o}) = \tilde{o}$
$\tilde{f}(\tilde{o}, \tilde{\varphi}^{\beta}) = \tilde{\varphi}^{\beta}$
$\tilde{f}(\tilde{\varphi}^{\alpha}, \tilde{o}) = \tilde{\varphi}^{\alpha}$
$e \cdot e = e$
$e \cdot g^{\beta} = g^{\beta}$


Structure of a Lie group
Goal:
$g(\varphi)$ $\varphi = \{\varphi_1, \varphi_2, \ldots \varphi_n\}$ $n = \dim(\text{group})$
$g(\tilde{0}) = e$ labels
$g^\alpha(\varphi^\alpha)\, g^\beta(\varphi^\beta) = g^\gamma(\varphi^\gamma)$
$\varphi^\gamma = f(\varphi^\alpha, \varphi^\beta)$

Structure of a Lie group

$g(\vec{\varphi})$ $\vec{\varphi} = \{\varphi_1, \varphi_2, \ldots \varphi_n\}$ $n = \dim(\text{group})$

$g(\vec{0}) = e$ labels

$g(\vec{\varphi}^\alpha)\, g(\vec{\varphi}^\beta) = g(\vec{\varphi}^\gamma)$

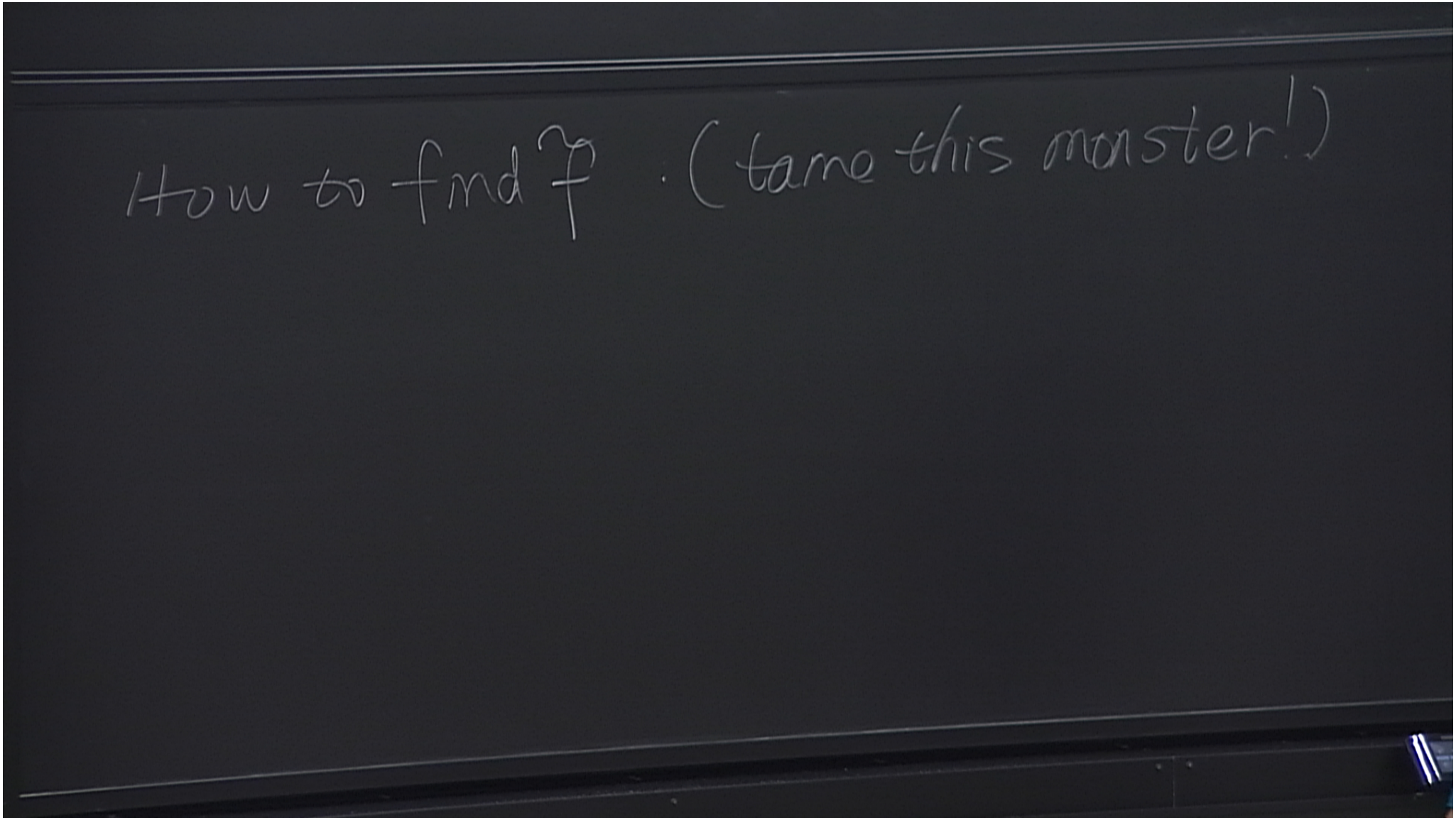
$\vec{\varphi}^\gamma = \vec{f}(\vec{\varphi}^\alpha, \vec{\varphi}^\beta)$

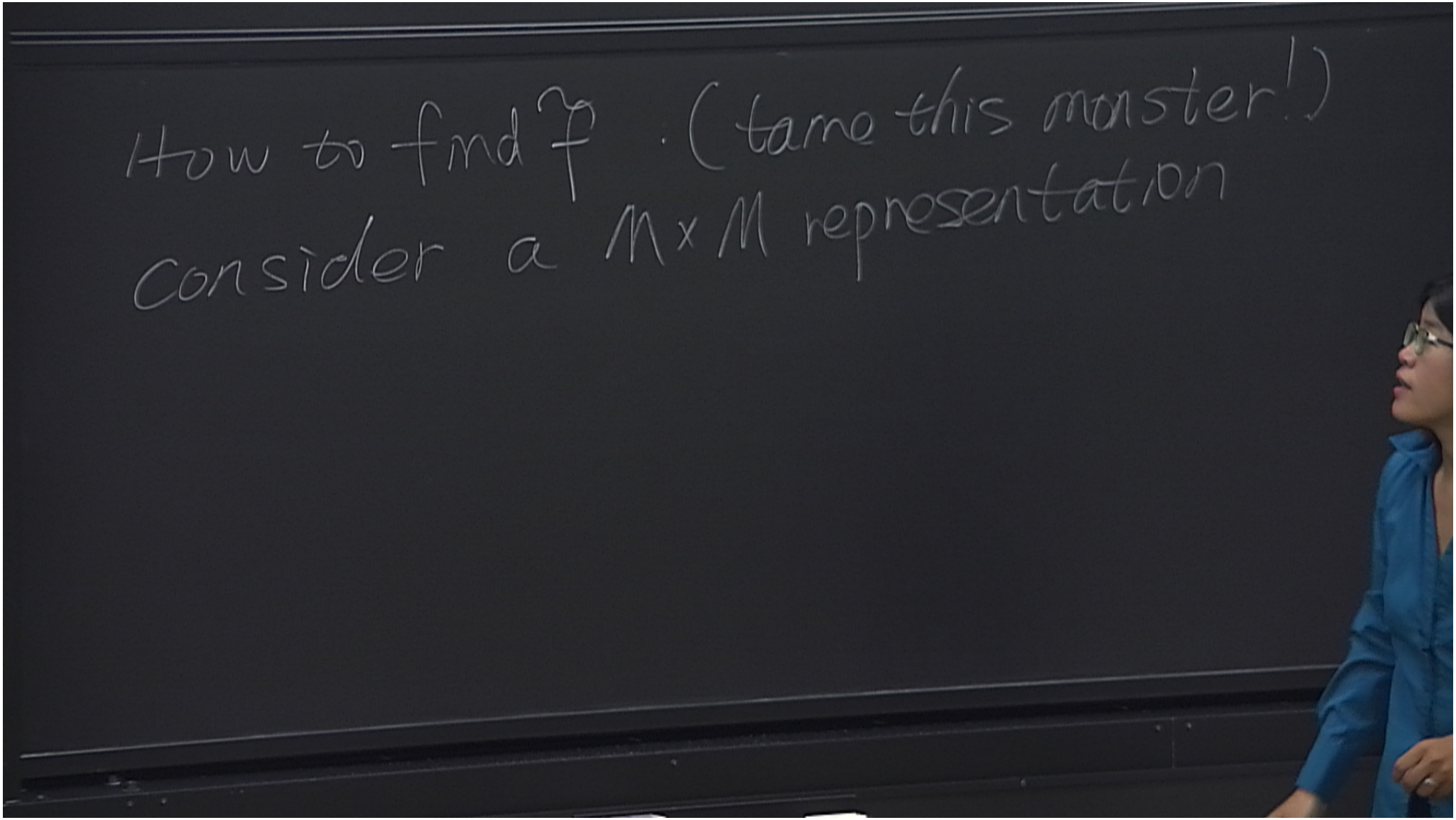
Goal: Trivia

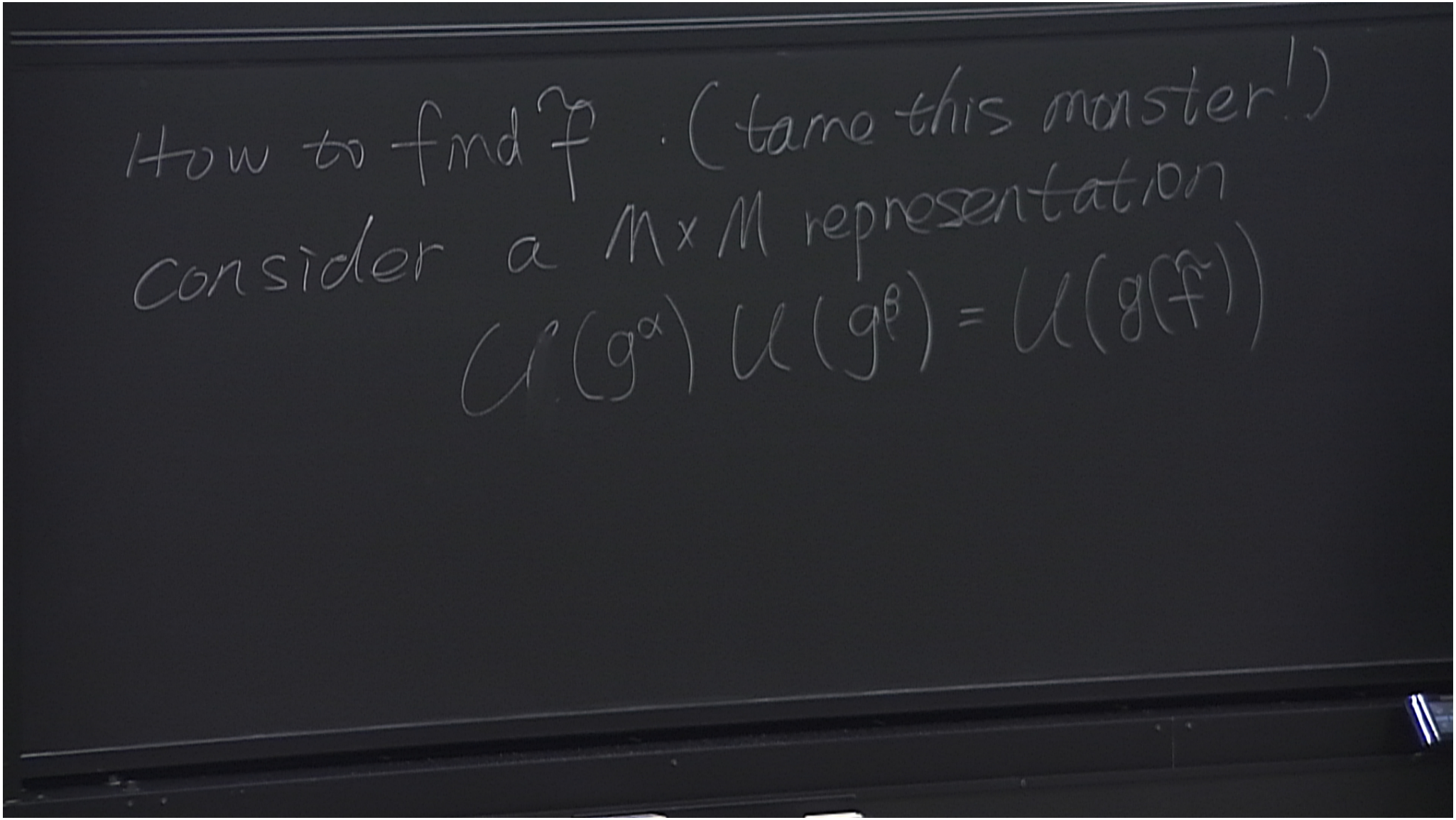
$\vec{f}($

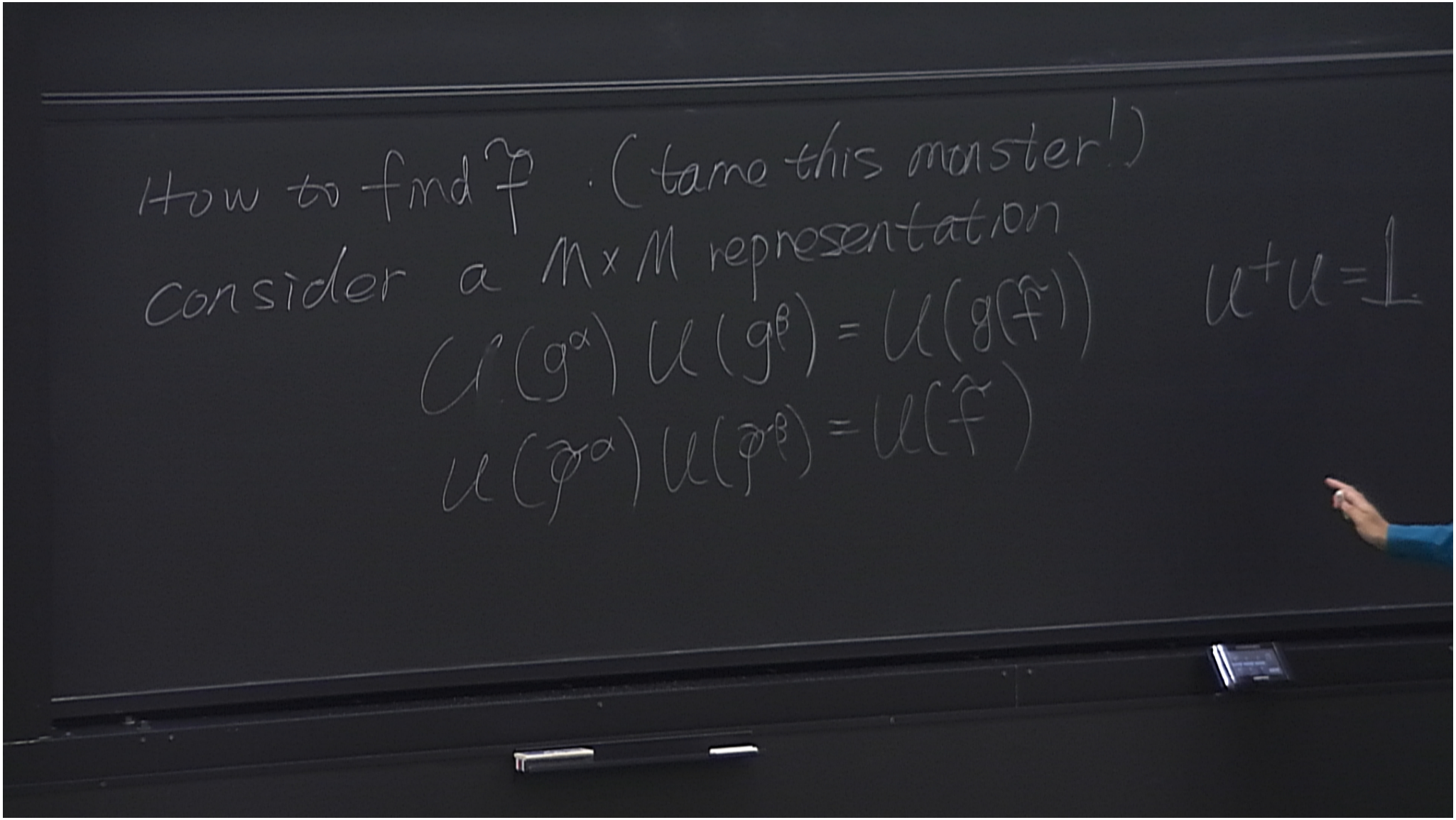
$\vec{f}($

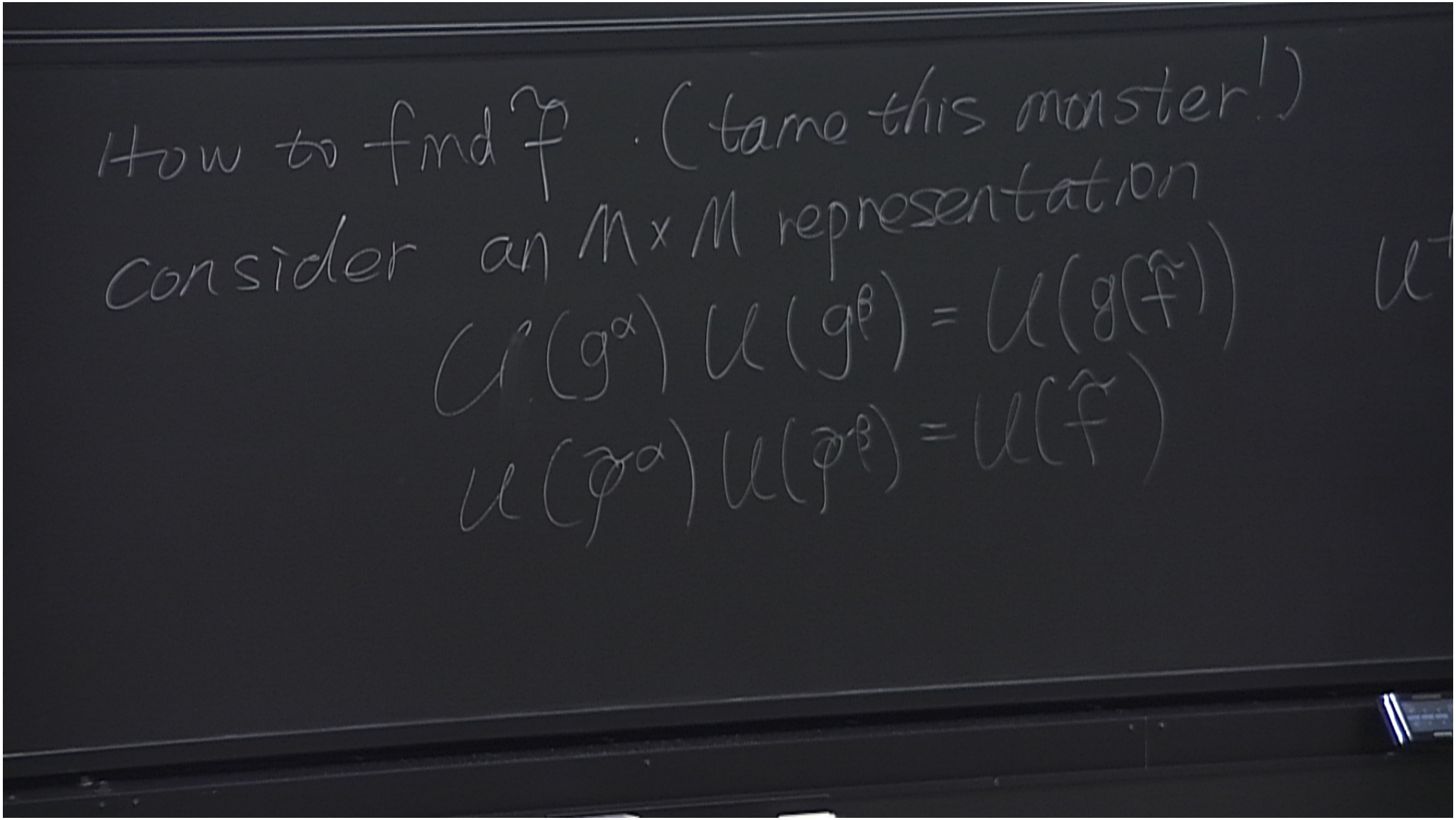
$\vec{f}($

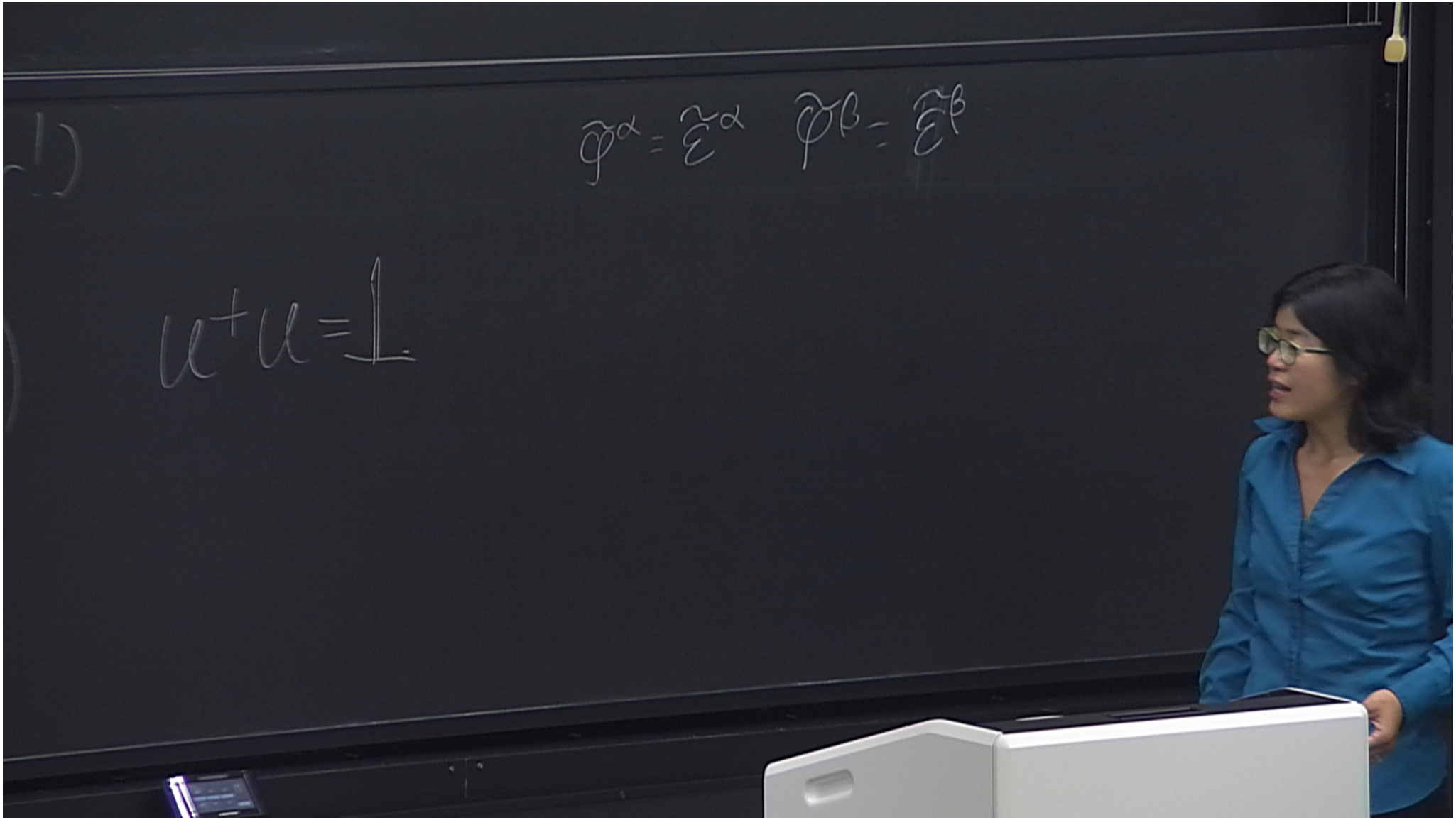

How to find ? . (tame this monster!)

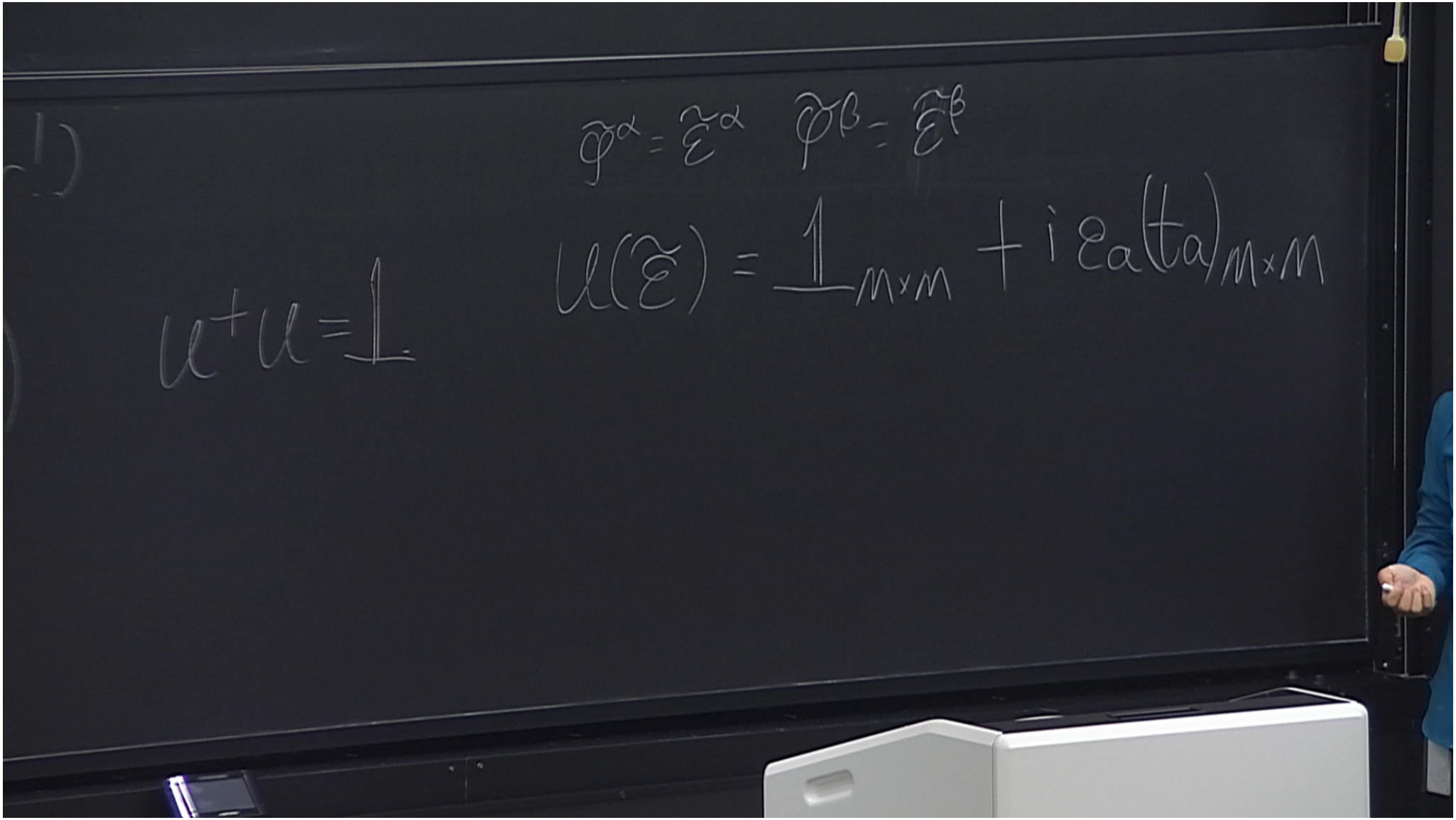

How to find ? . (tame this monster!)
Consider a M x M representation

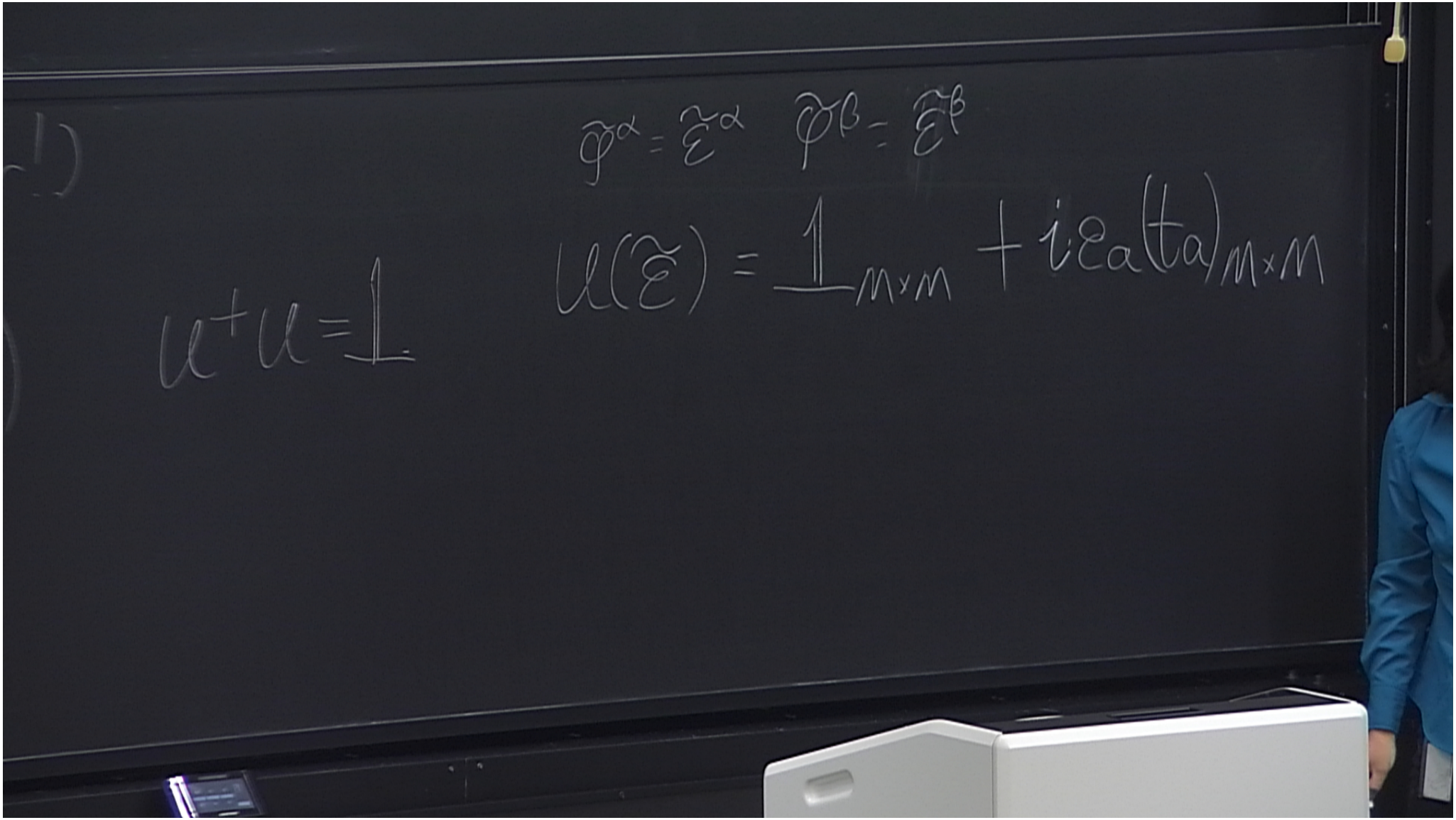

How to find ? . (tame this monster!)
Consider a $N \times M$ representation
$U(g^\alpha)\, U(g^\beta) = U(g(f))$

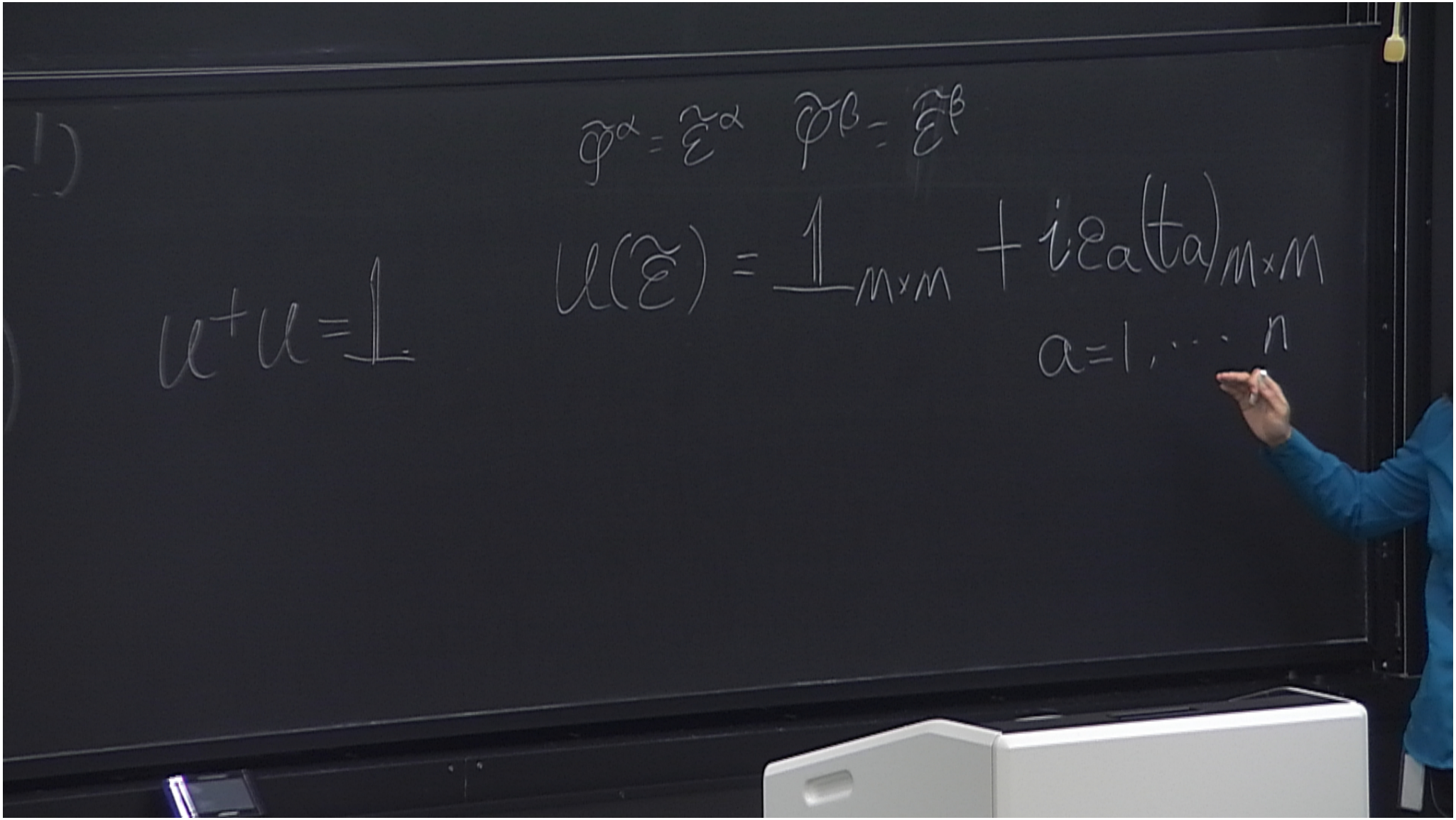

How to find ? . (tame this monster!)
Consider a M × M representation
U(g^α) U(g^β) = U(g(f̃))
U(φ^α) U(φ^β) = U(f̃)
U⁺U = 1

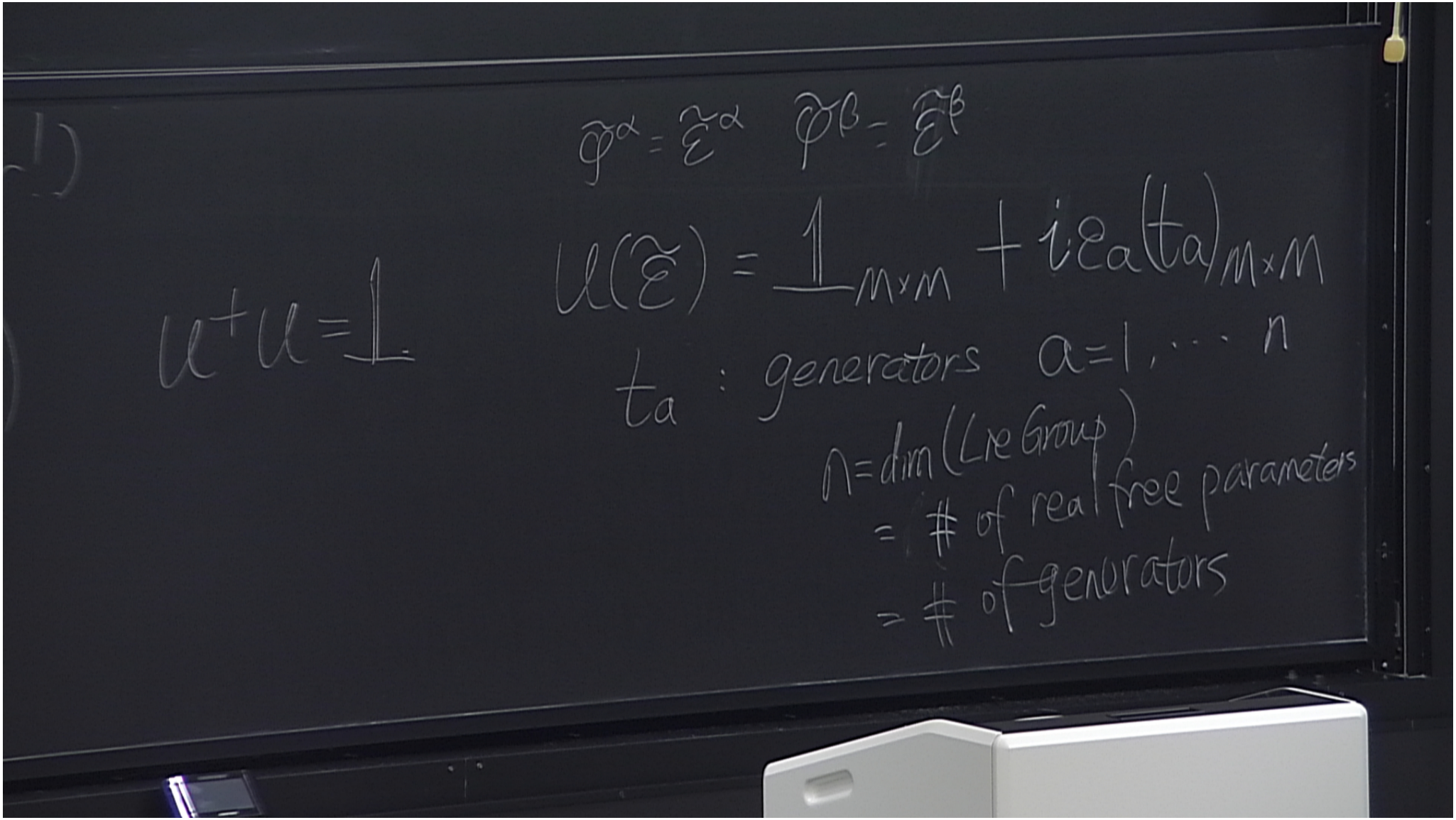

How to find $\tilde{\gamma}$. (tame this monster!)
Consider an $M \times M$ representation
$U(g^\alpha)\,U(g^\beta) = U(g(f))$
$U(\gamma^\alpha)\,U(\gamma^\beta) = U(\tilde{f})$

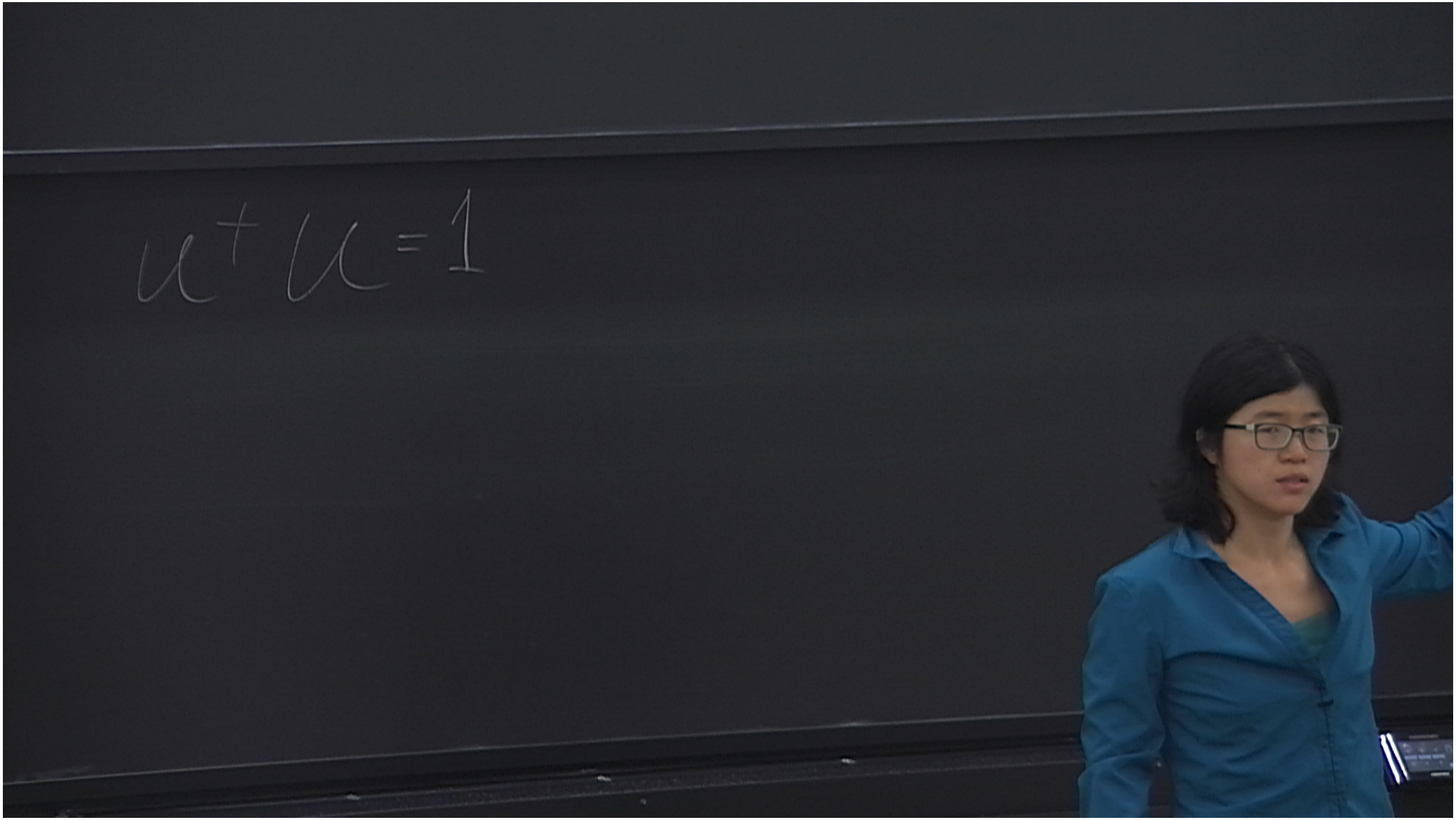

$\varphi^\alpha = \tilde{\varepsilon}^\alpha \quad \varphi^\beta = \varepsilon^\beta$
$U^+ U = \mathbb{1}$

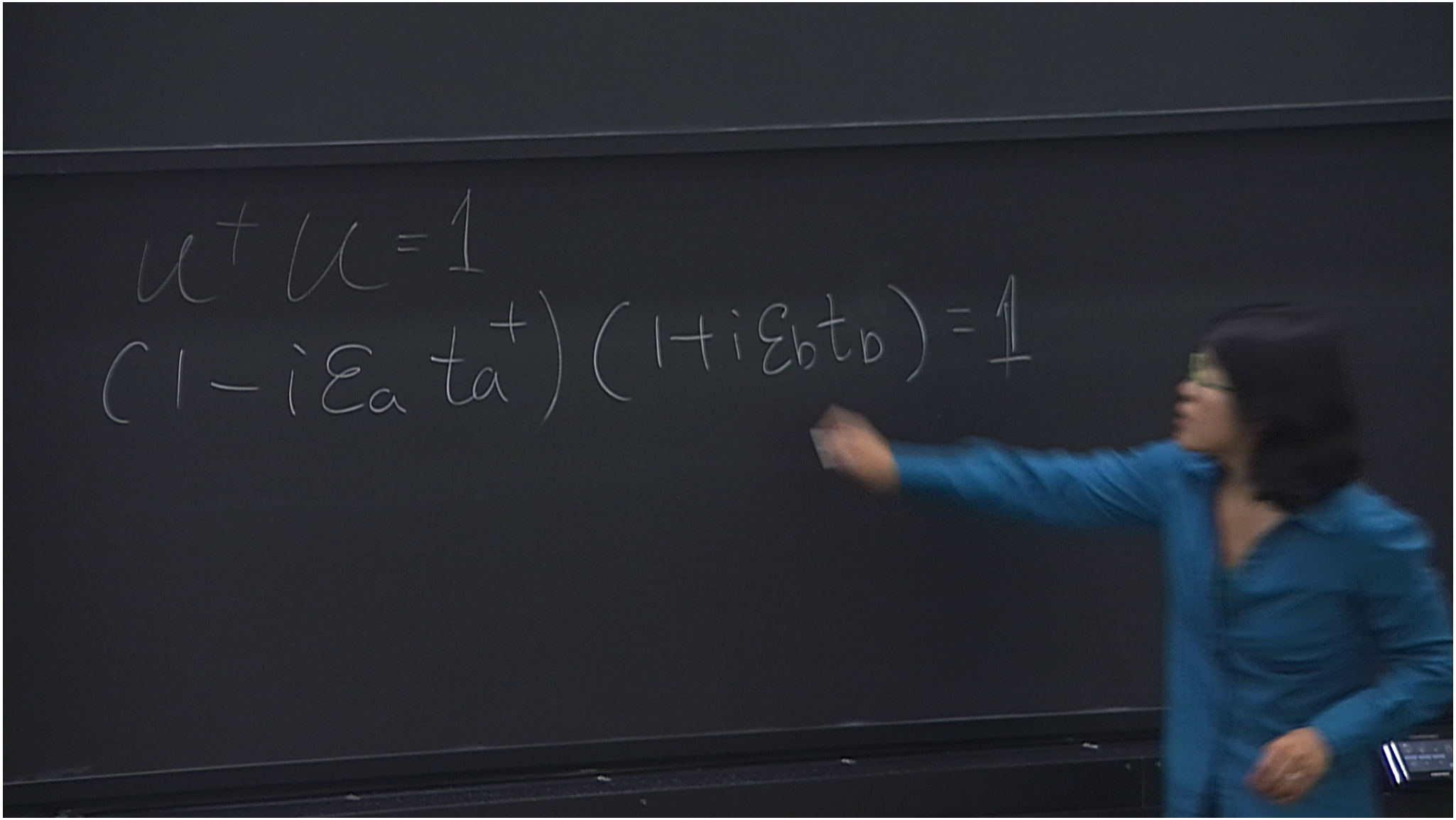

$\varphi^\alpha = \tilde{\varepsilon}^\alpha \quad \varphi^\beta = \tilde{\varepsilon}^\beta$
$U^+ U = \mathbb{1}$
$U(\tilde{\varepsilon}) = \mathbb{1}_{n\times n} + i\varepsilon_a (t_a)_{n\times n}$

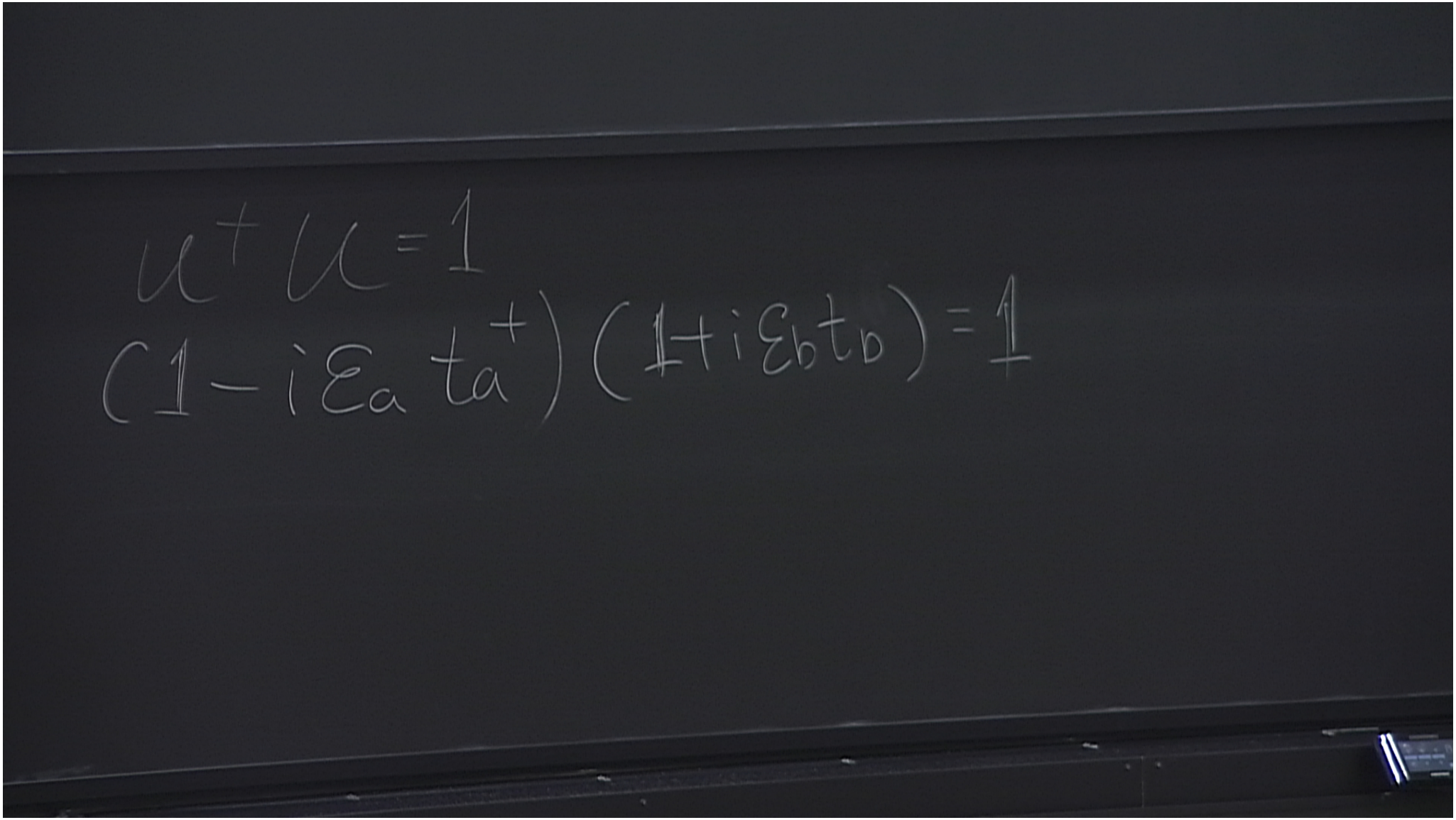

$\varphi^\alpha = \tilde{\varepsilon}^\alpha \quad \varphi^\beta = \tilde{\varepsilon}^\beta$
$U^+ U = \mathbb{1}$
$U(\tilde{\varepsilon}) = \mathbb{1}_{n\times n} + i\varepsilon_a (t_a)_{n\times n}$

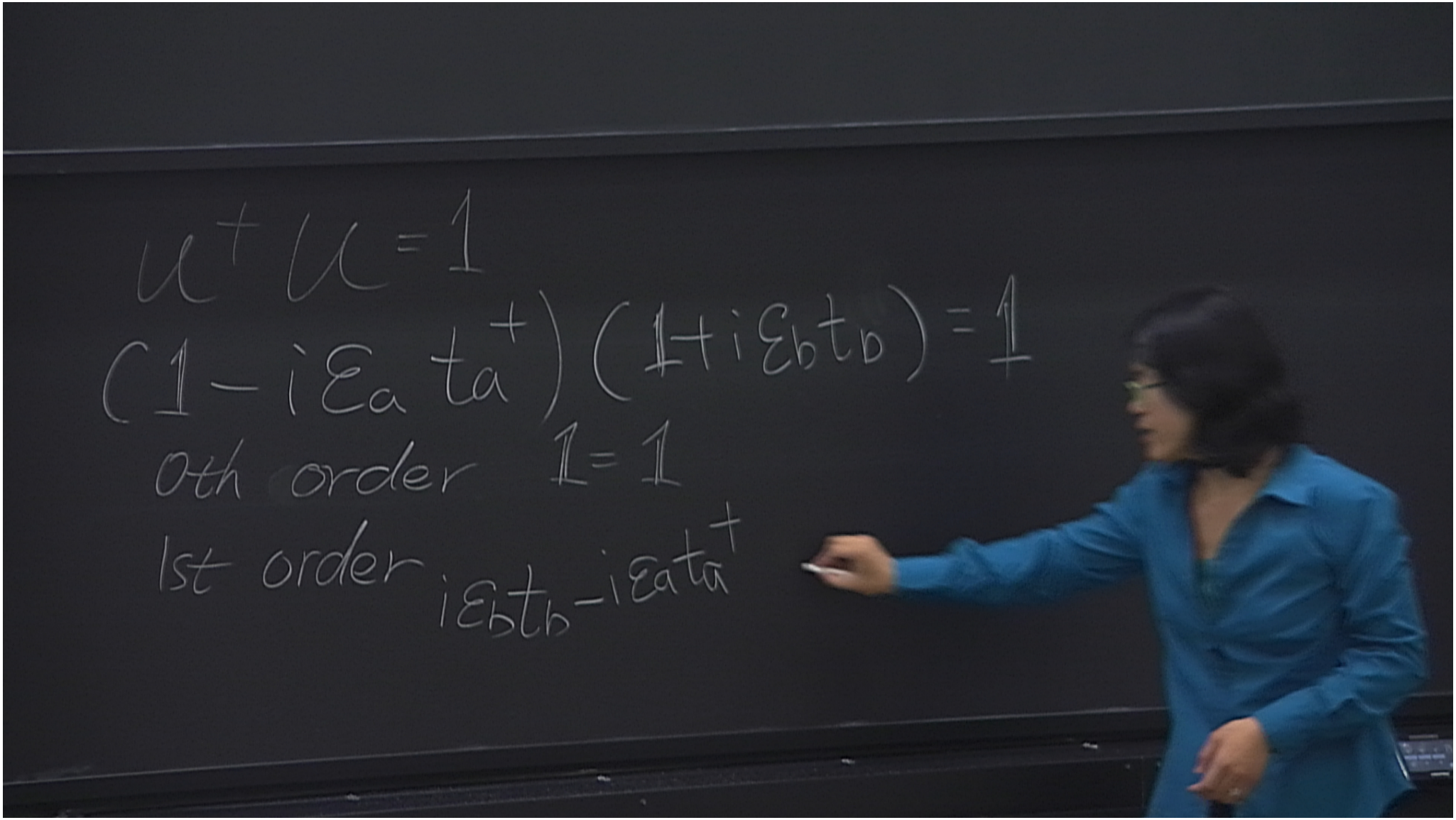

$\varphi^\alpha = \tilde{\varepsilon}^\alpha \quad \varphi^\beta = \tilde{\varepsilon}^\beta$
$U^+ U = \mathbb{1}$
$U(\tilde{\varepsilon}) = \mathbb{1}_{n\times n} + i\varepsilon_a (t_a)_{n\times n}$
$a = 1, \cdots n$


$\varphi^\alpha = \tilde{\varepsilon}^\alpha \quad \varphi^\beta = \tilde{\varepsilon}^\beta$
$U^+ U = \mathbb{1}$
$U(\tilde{\varepsilon}) = \mathbb{1}_{n\times n} + i\varepsilon_a (t_a)_{n\times n}$
t_a : generators $a = 1, \cdots n$
n = dim (Lie Group)
= # of real free parameters
= # of generators


u+u=1


$u^\dagger u = 1$
$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$


$u^\dagger u = 1$
$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$

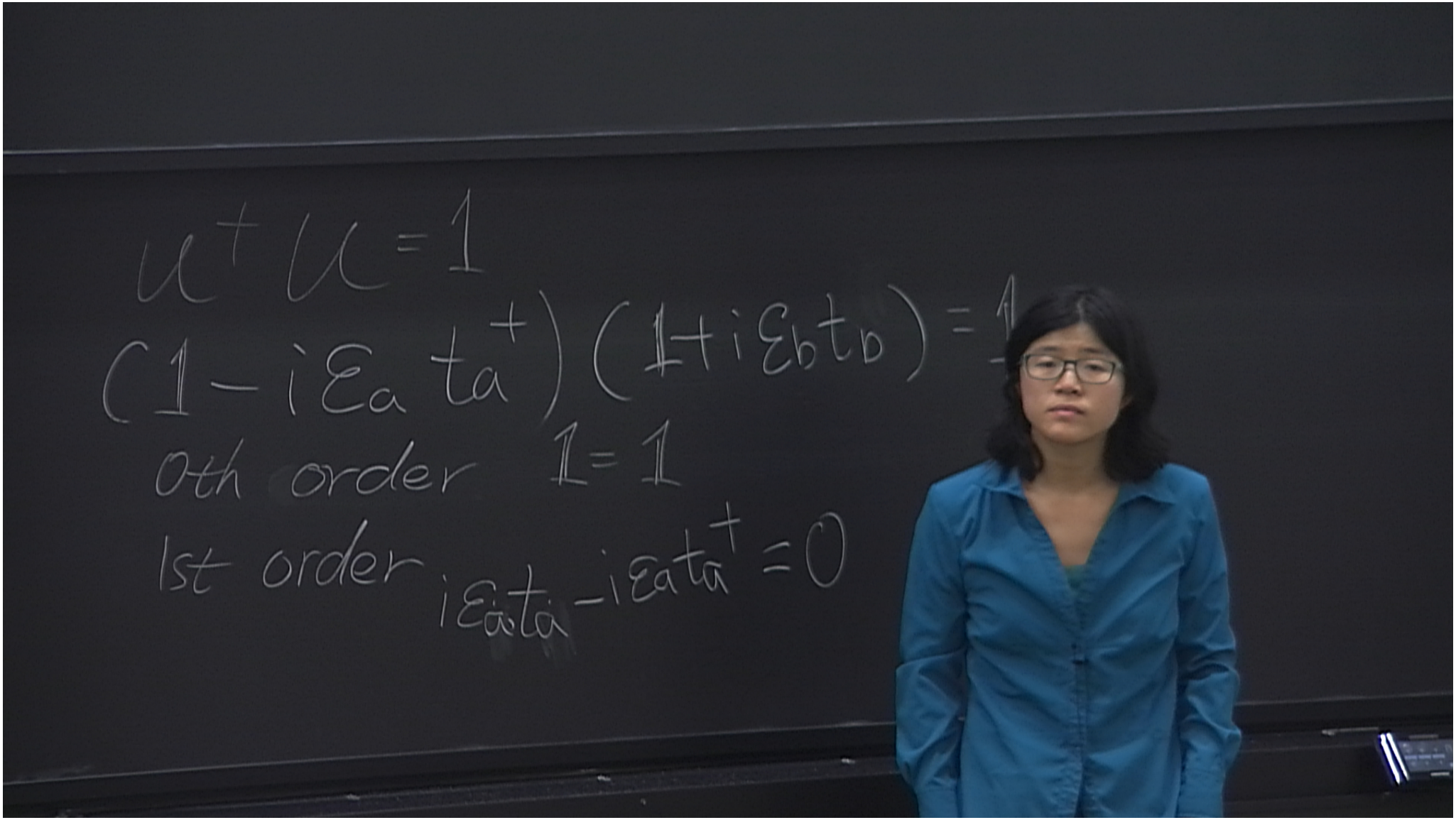

$U^\dagger U = 1$
$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$
0th order $1 = 1$
1st order $i\varepsilon_b t_b - i\varepsilon_a t_a^\dagger$

$$U^\dagger U = 1$$

$$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$$

0th order $\quad 1 = 1$

1st order $\quad i\varepsilon_b t_b - i\varepsilon_a t_a^\dagger = 0$


$U^\dagger U = 1$
$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$
0th order $1 = 1$
1st order $i\varepsilon_a t_a - i\varepsilon_a t_a^\dagger = 0$

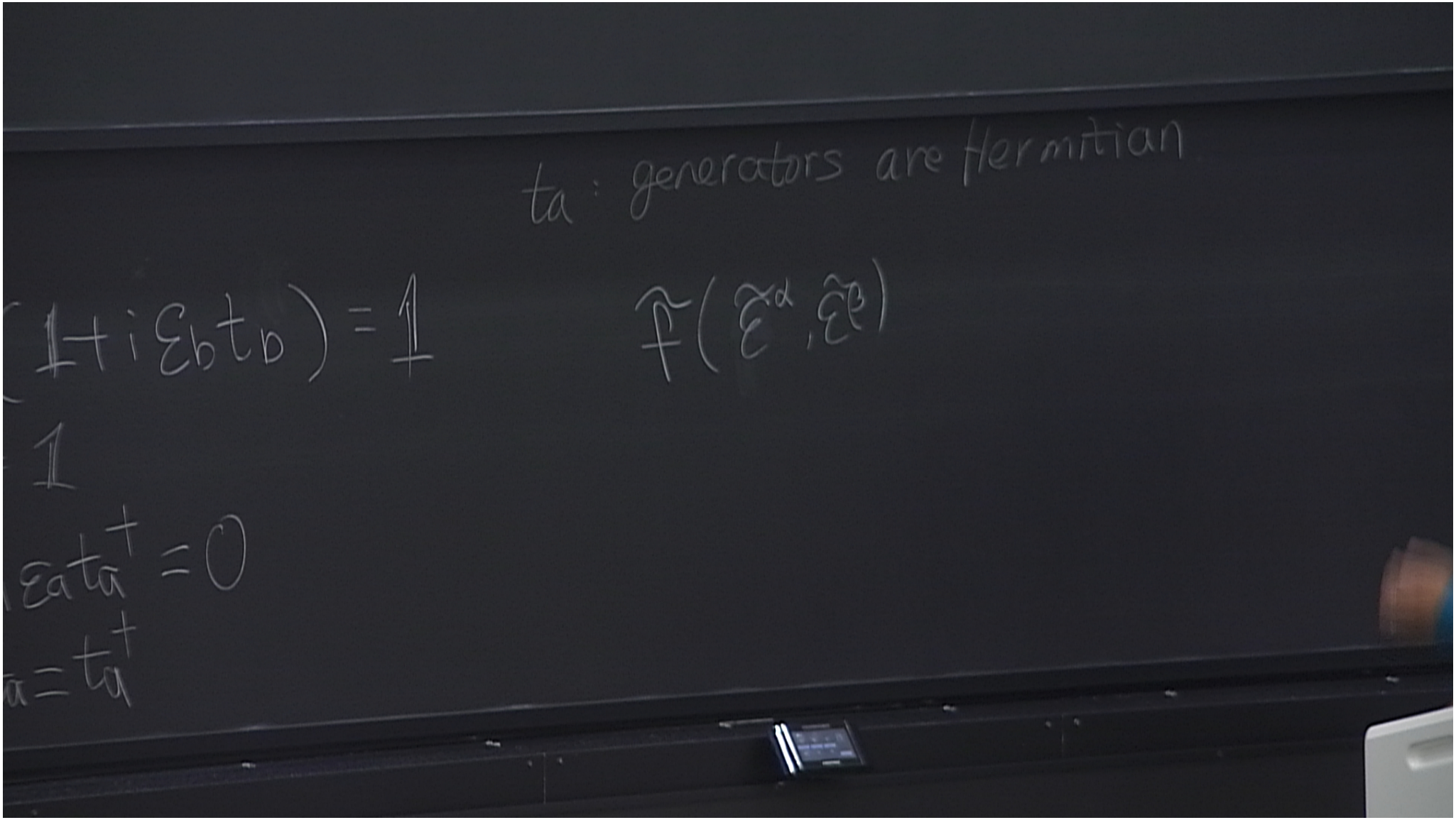
$U^\dagger U = 1$

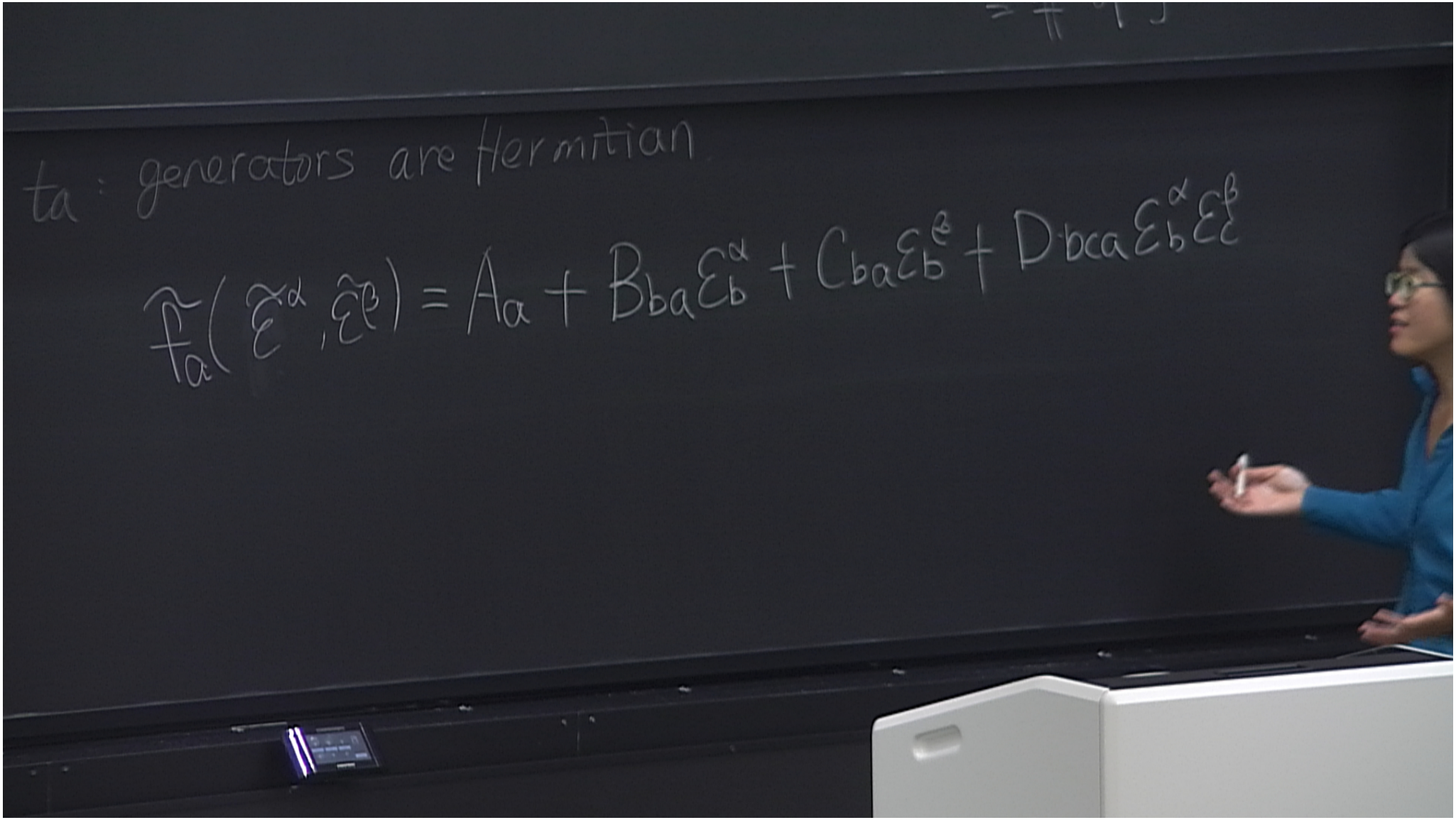
$(1 - i\varepsilon_a t_a^\dagger)(1 + i\varepsilon_b t_b) = 1$

0th order $\mathbb{1} = 1$

1st order $i\varepsilon_a t_a - i\varepsilon_a t_a^\dagger = 0$

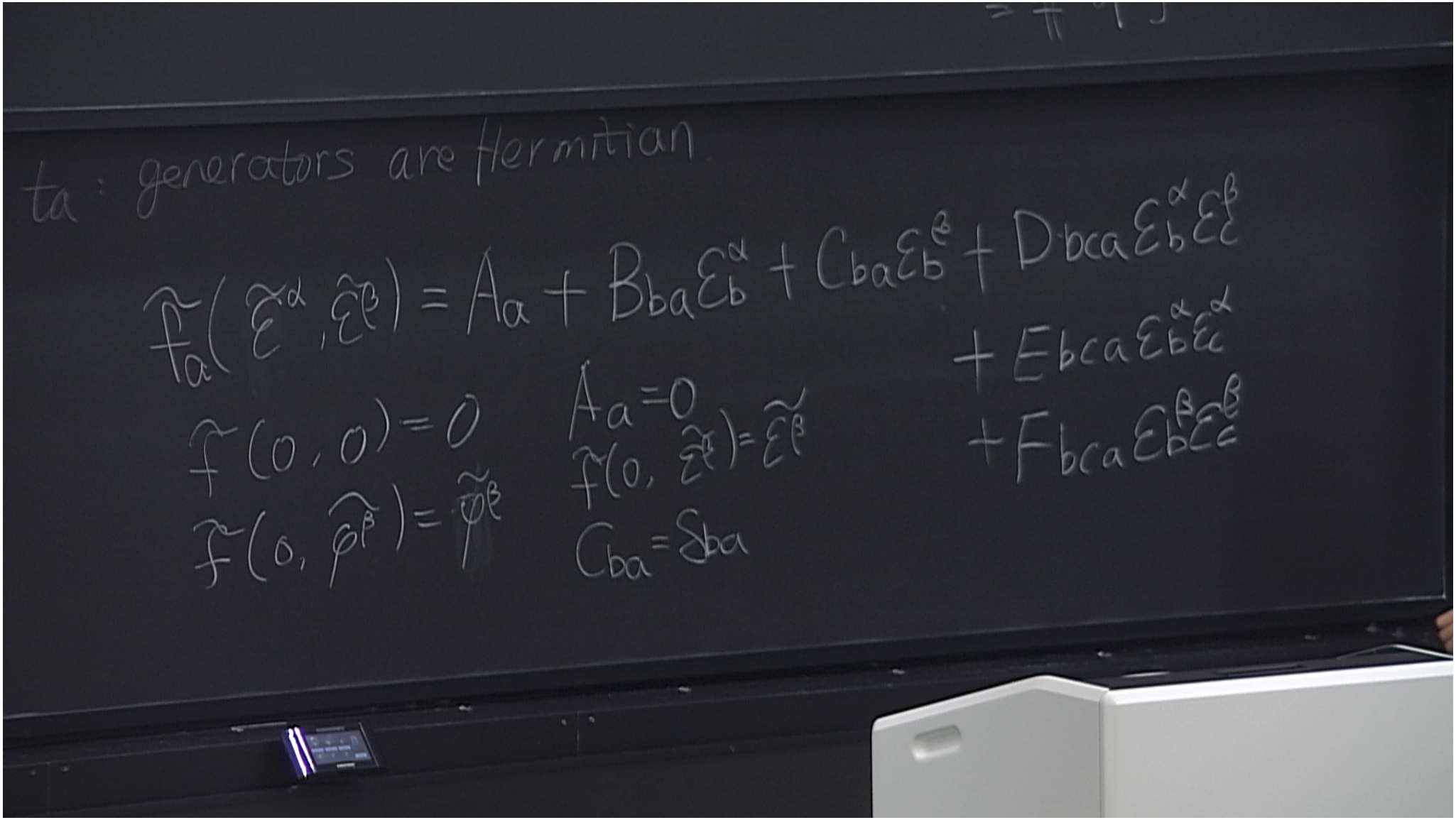
$t_a = t_a^\dagger$

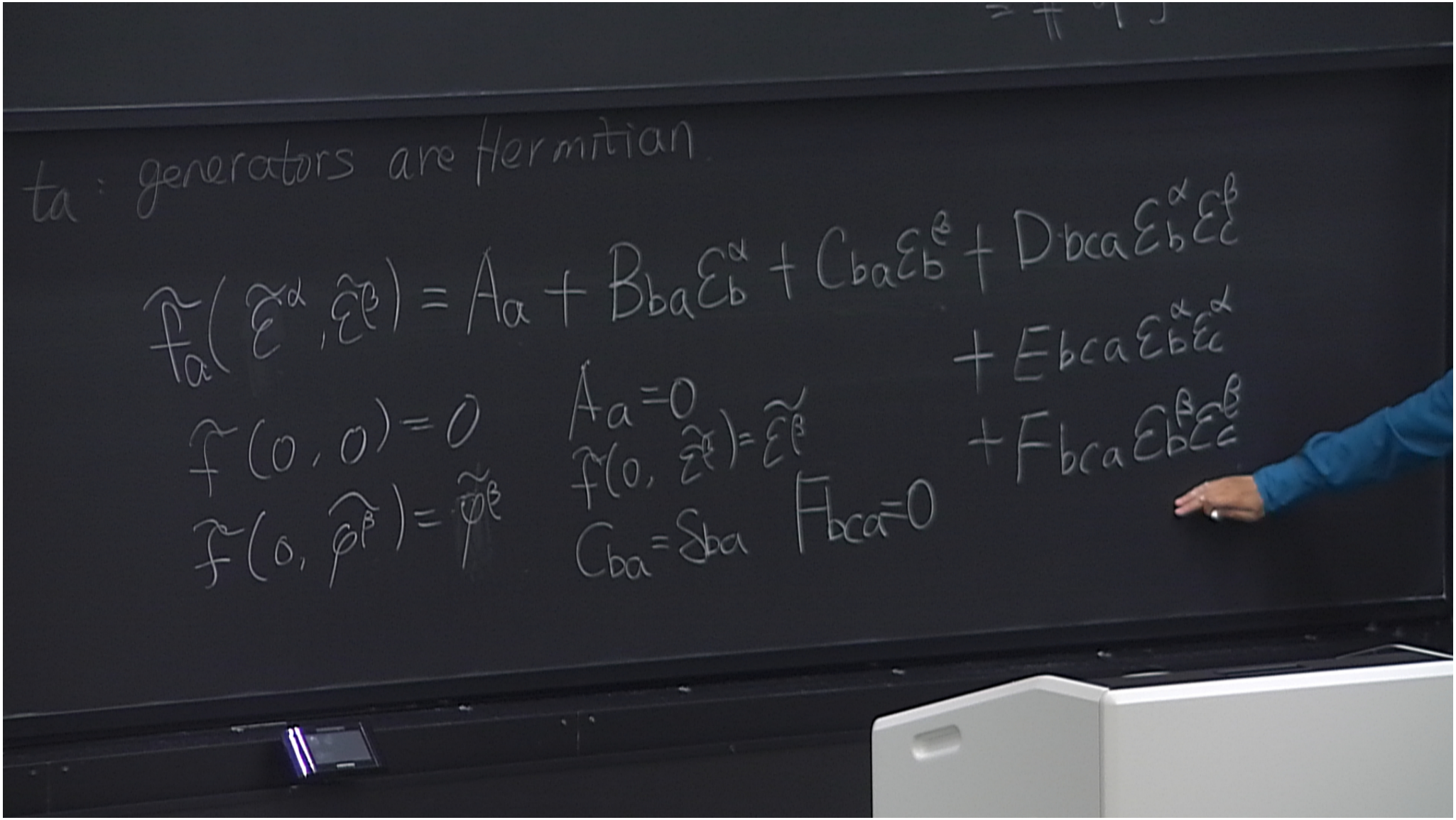

ta: generators are Hermitian
$(1+i\varepsilon_b t_b) = 1$
$\tilde{f}(\tilde{\varepsilon}^a, \tilde{\varepsilon}^b)$
$= 1$
$\varepsilon_a t_a^\dagger = 0$
$= t_a^\dagger$

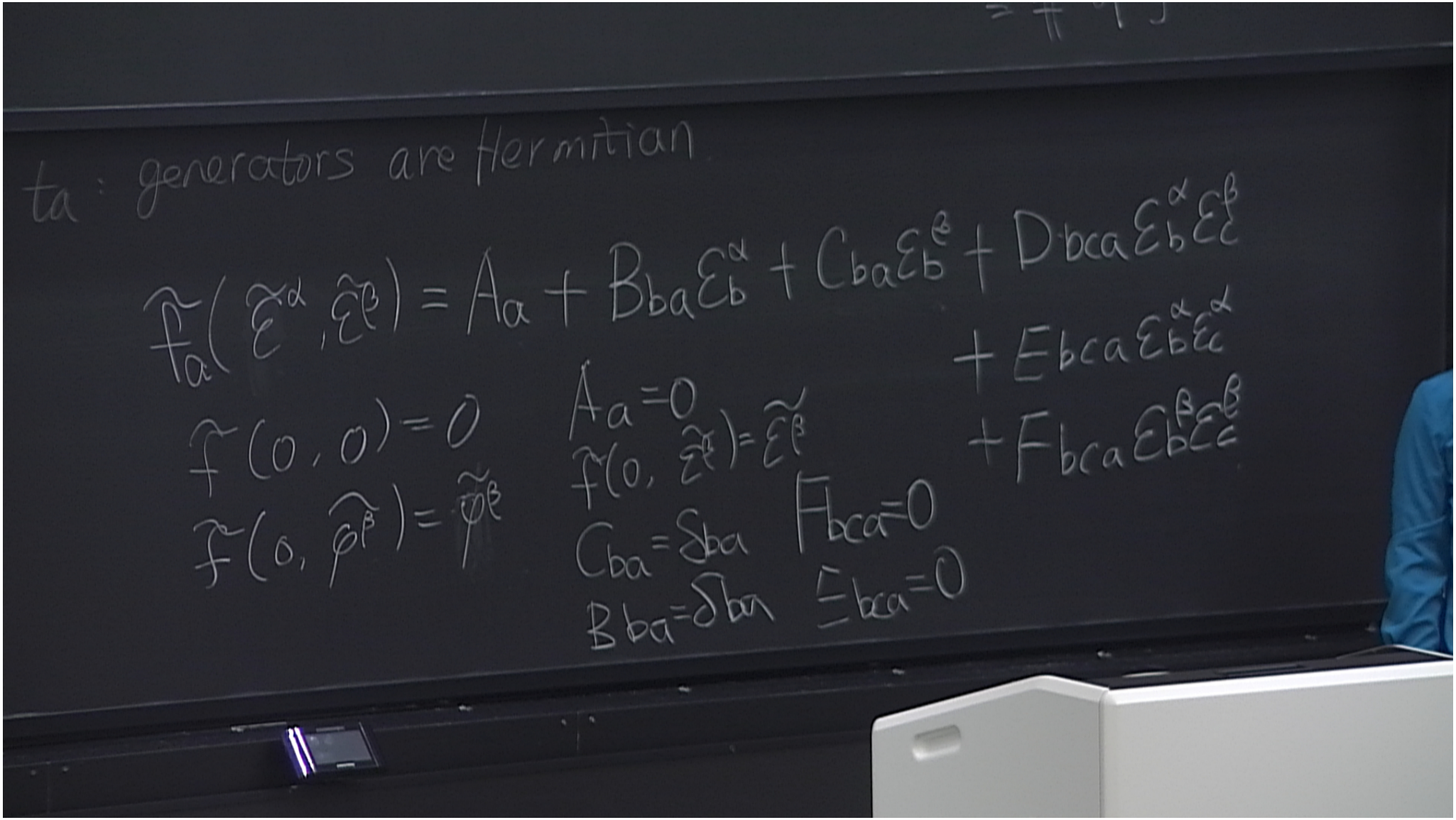

ta: generators are Hermitian
$\tilde{f}_a(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta) = A_a + B_{ba}\varepsilon_b^\alpha + C_{ba}\varepsilon_b^\beta + D_{bca}\varepsilon_b^\alpha\varepsilon_c^\beta$

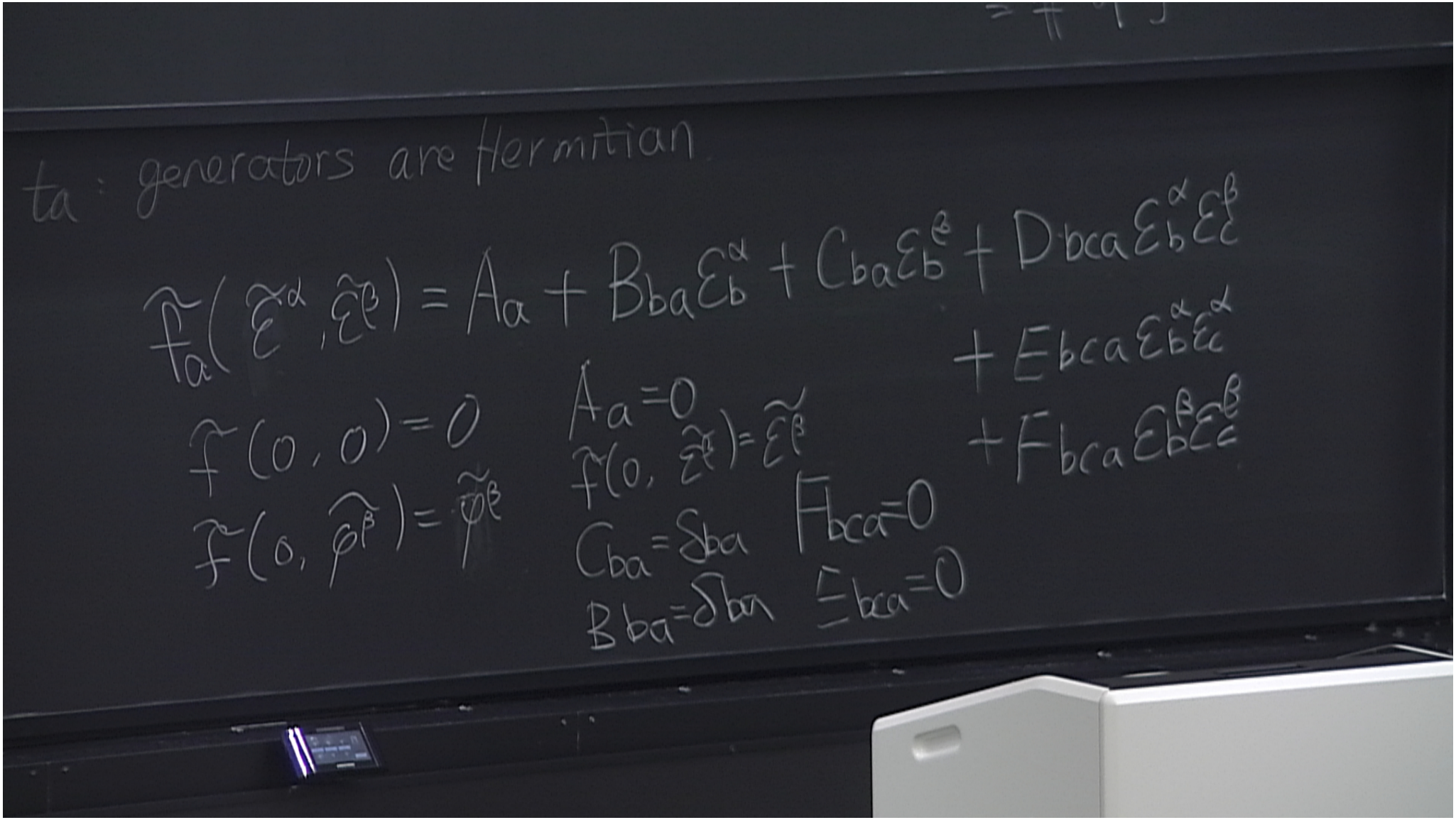
t_a : generators are Hermitian

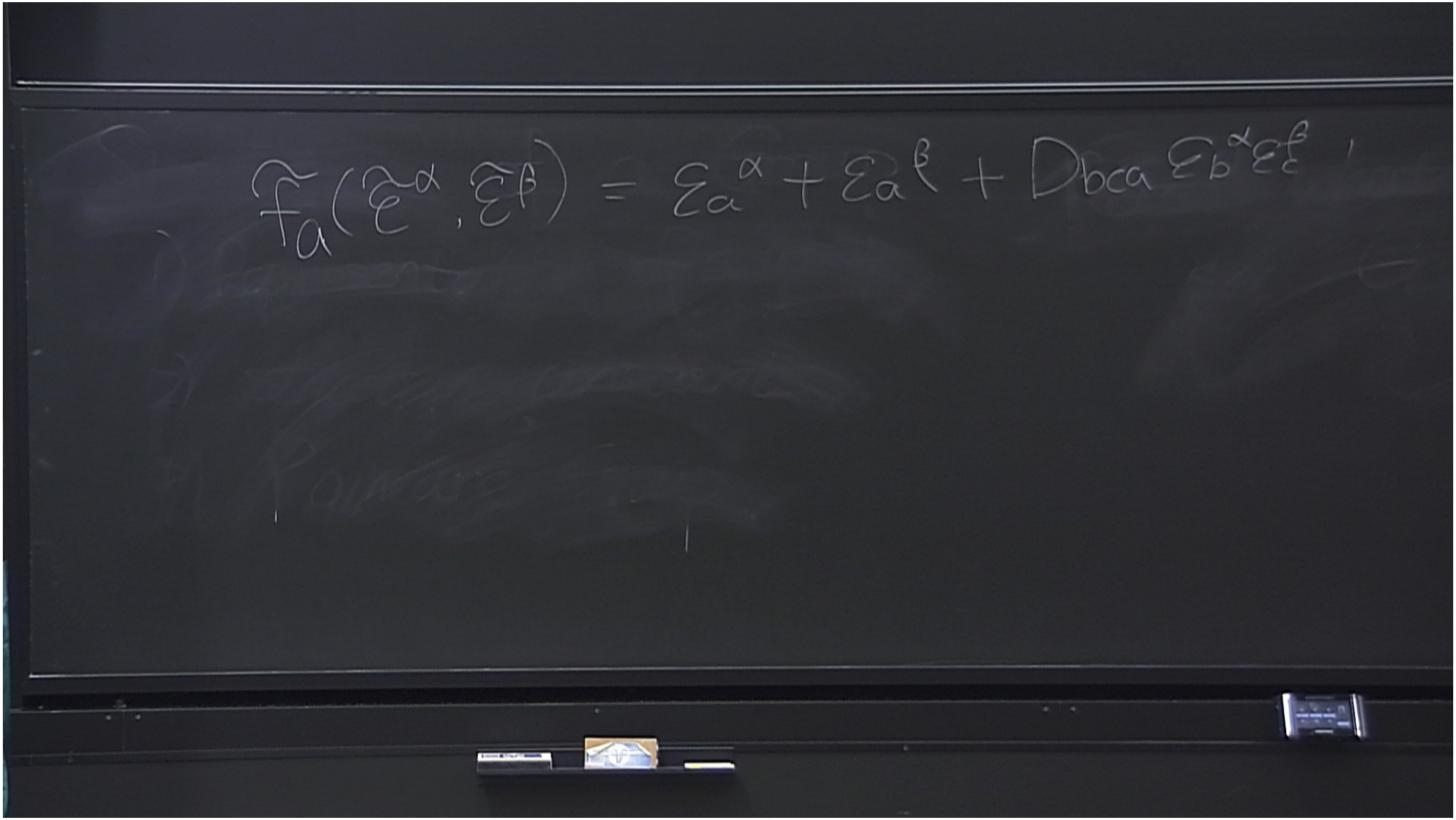
$$\tilde{f}_a(\tilde{\xi}^\alpha, \hat{\xi}^\beta) = A_a + B_{ba}\xi_b^\alpha + C_{ba}\xi_b^\beta + D_{bca}\xi_b^\alpha\xi_c^\beta + E_{bca}\xi_b^\alpha\xi_c^\alpha + F_{bca}\xi_b^\beta\xi_c^\beta$$

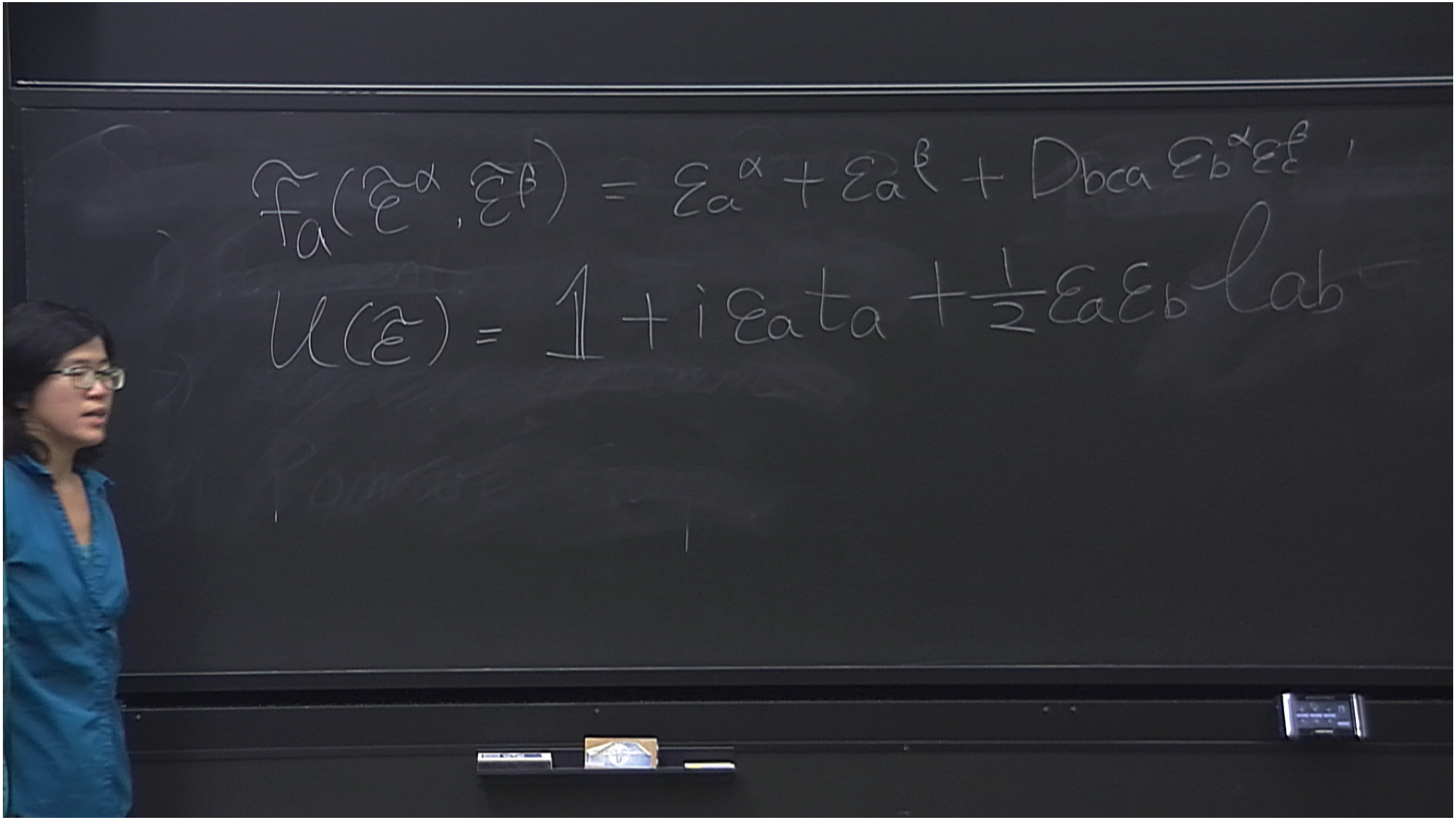

ta: generators are Hermitian
$\tilde{f}_a(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta) = A_a + B_{ba}\varepsilon_b^\alpha + C_{ba}\varepsilon_b^\beta + D_{bca}\varepsilon_b^\alpha\varepsilon_c^\beta$
$+ E_{bca}\varepsilon_b^\alpha\varepsilon_c^\alpha$
$+ F_{bca}\varepsilon_b^\beta\varepsilon_c^\beta$
$\tilde{f}(0,0) = 0$
$\tilde{f}(0, \tilde{\varphi}^\beta) = \tilde{\varphi}^\beta$
$A_a = 0$
$\tilde{f}(0, \tilde{\varepsilon}^\beta) = \tilde{\varepsilon}^\beta$
$C_{ba} = \delta_{ba}$


ta: generators are Hermitian
$\tilde{f}_a(\tilde{\xi}^\alpha, \tilde{\xi}^\beta) = A_a + B_{ba}\xi_b^\alpha + C_{ba}\xi_b^\beta + D_{bca}\xi_b^\alpha\xi_c^\beta$
$+ E_{bca}\xi_b^\alpha\xi_c^\alpha$
$+ F_{bca}\xi_b^\beta\xi_c^\beta$
$\tilde{f}(0,0) = 0$
$\tilde{f}(0, \tilde{\phi}^\beta) = \tilde{\phi}^\beta$
$A_a = 0$
$\tilde{f}(0, \tilde{\xi}^\beta) = \tilde{\xi}^\beta$
$C_{ba} = \delta_{ba}$
$F_{bca} = 0$


ta: generators are Hermitian
$\tilde{f}_a(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta) = A_a + B_{ba}\varepsilon_b^\alpha + C_{ba}\varepsilon_b^\beta + D_{bca}\varepsilon_b^\alpha\varepsilon_c^\beta + E_{bca}\varepsilon_b^\alpha\varepsilon_c^\alpha + F_{bca}\varepsilon_b^\beta\varepsilon_c^\beta$
$\tilde{f}(0,0) = 0$
$\tilde{f}(0, \tilde{\varphi}^\beta) = \tilde{\varphi}^\beta$
$A_a = 0$
$\tilde{f}(0, \tilde{\varepsilon}^\beta) = \tilde{\varepsilon}^\beta$
$C_{ba} = \delta_{ba}$
$B_{ba} = \delta_{ba}$
$F_{bca} = 0$
$E_{bca} = 0$

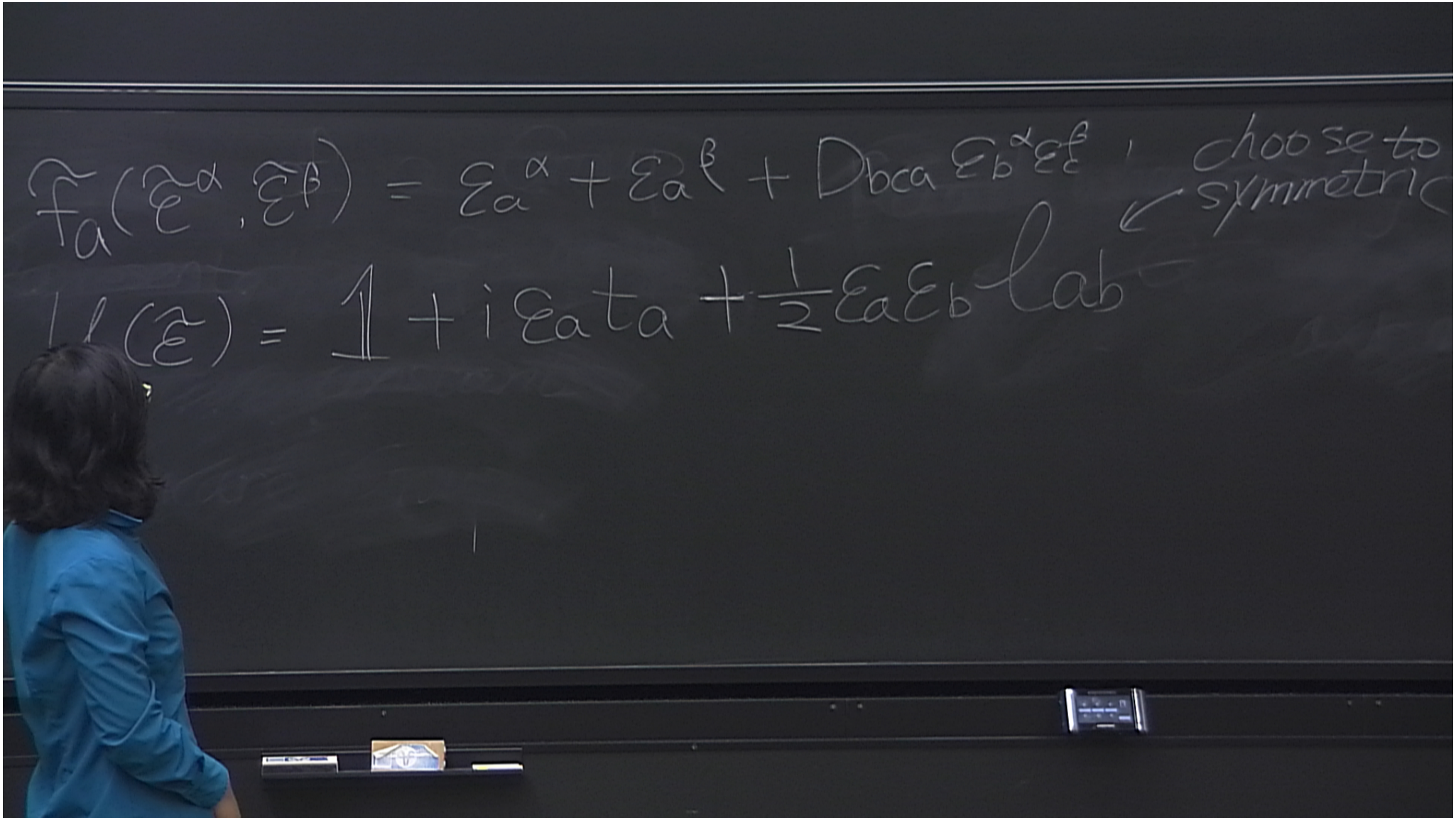

ta: generators are Hermitian
$\tilde{f}_a(\tilde{\xi}^\alpha, \tilde{\xi}^\beta) = A_a + B_{ba}\xi_b^\alpha + C_{ba}\xi_b^\beta + D_{bca}\xi_b^\alpha\xi_c^\beta$
$+ E_{bca}\xi_b^\alpha\xi_c^\alpha$
$+ F_{bca}\xi_b^\beta\xi_c^\beta$
$\tilde{f}(0,0) = 0$
$\tilde{f}(0, \tilde{\phi}^\beta) = \tilde{\phi}^\beta$
$A_a = 0$
$f(0, \tilde{\xi}^\beta) = \tilde{\xi}^\beta$
$C_{ba} = \delta_{ba}$
$B_{ba} = \delta_{ba}$
$F_{bca} = 0$
$E_{bca} = 0$


$\tilde{f}_a(\tilde{\xi}^\alpha, \tilde{\xi}^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta$


$f_a(\varepsilon^\alpha, \varepsilon^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta$,
$U(\varepsilon) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b t_{ab}$

$$f_a(\varepsilon^\alpha, \varepsilon^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta ,$$

$$U(\varepsilon) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b \, l_{ab}$$


$f_a(\varepsilon^\alpha, \varepsilon^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta$; choose to be symmetric
$U(\varepsilon) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b\, l_{ab}$

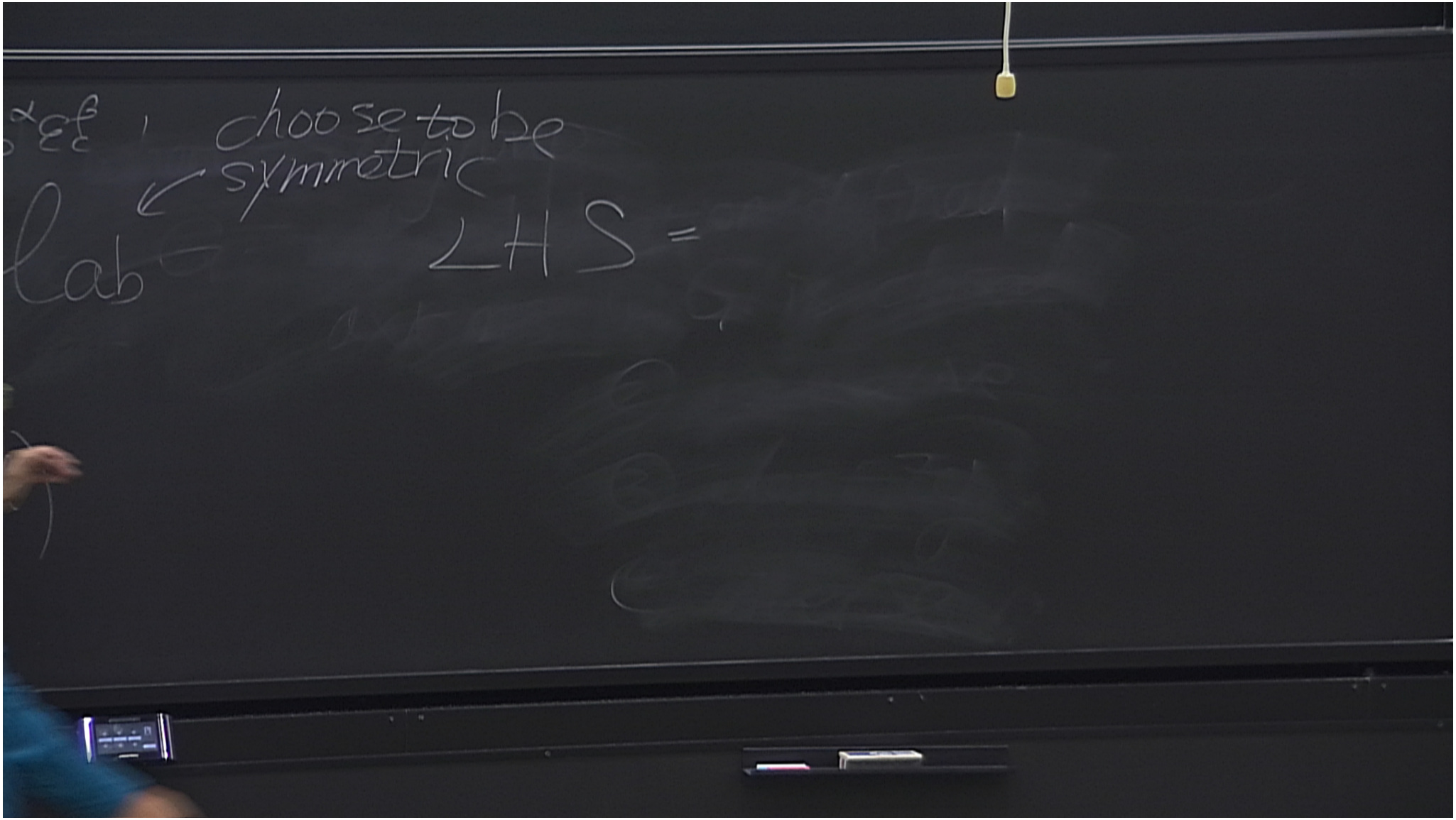
$$f_a(\varepsilon^\alpha, \varepsilon^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta$$; choose to be symmetric

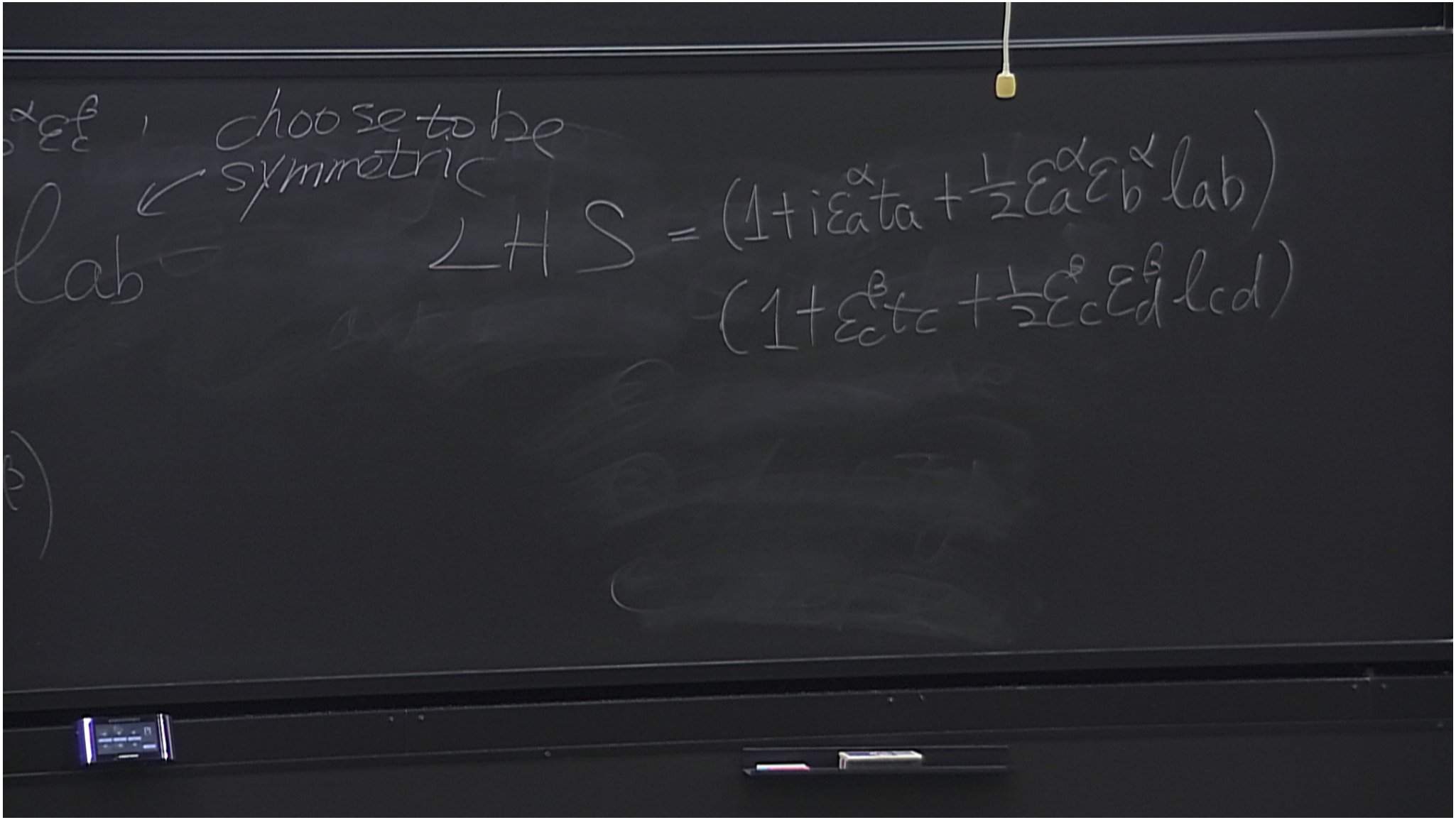
$$U(\varepsilon) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b l_{ab}$$

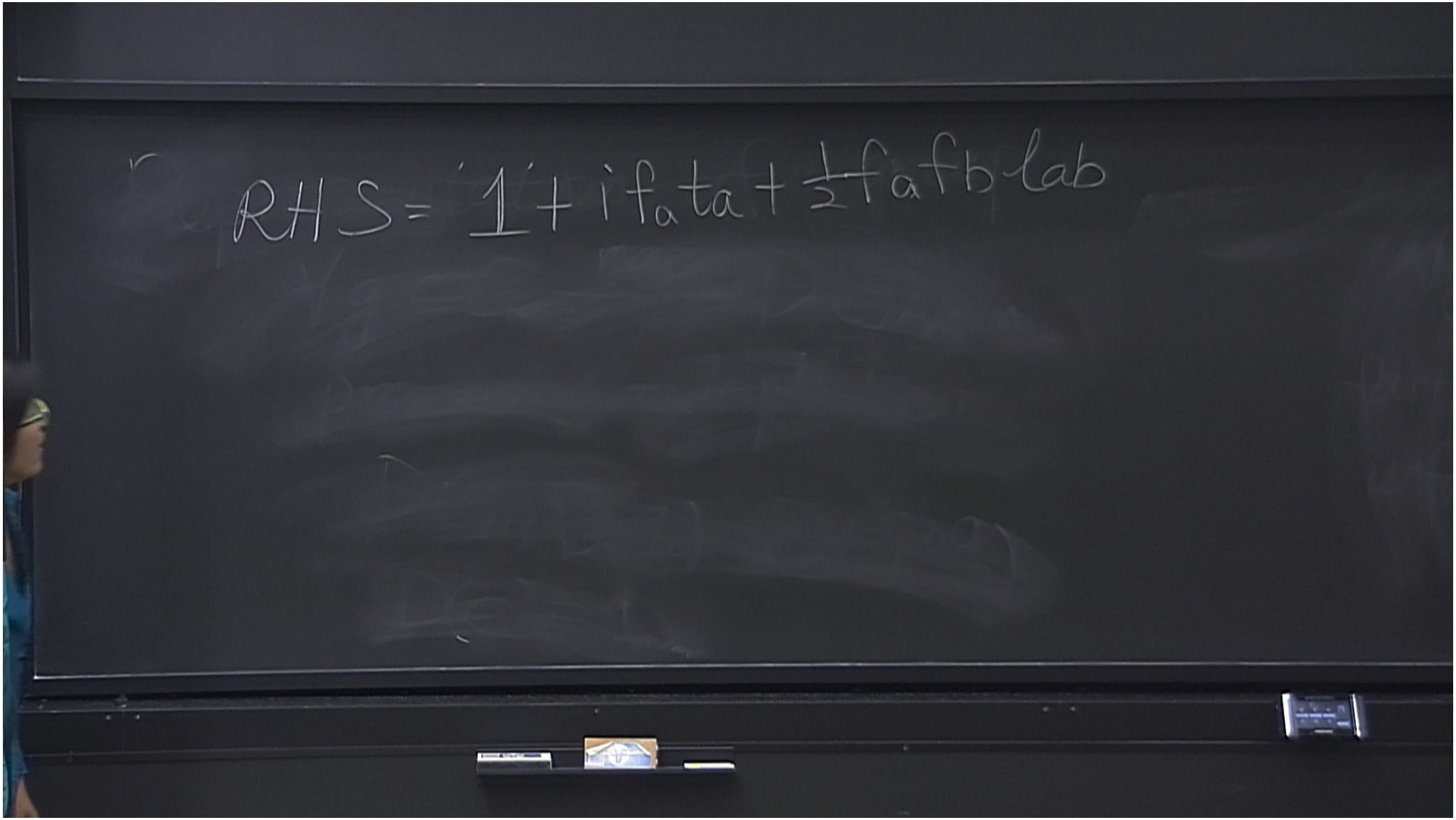
$$f_a(\varepsilon^\alpha, \varepsilon^\beta) = \varepsilon_a{}^\alpha + \varepsilon_a{}^\beta + D_{bca}\,\varepsilon_b{}^\alpha \varepsilon_c{}^\beta$$; choose to be symmetric

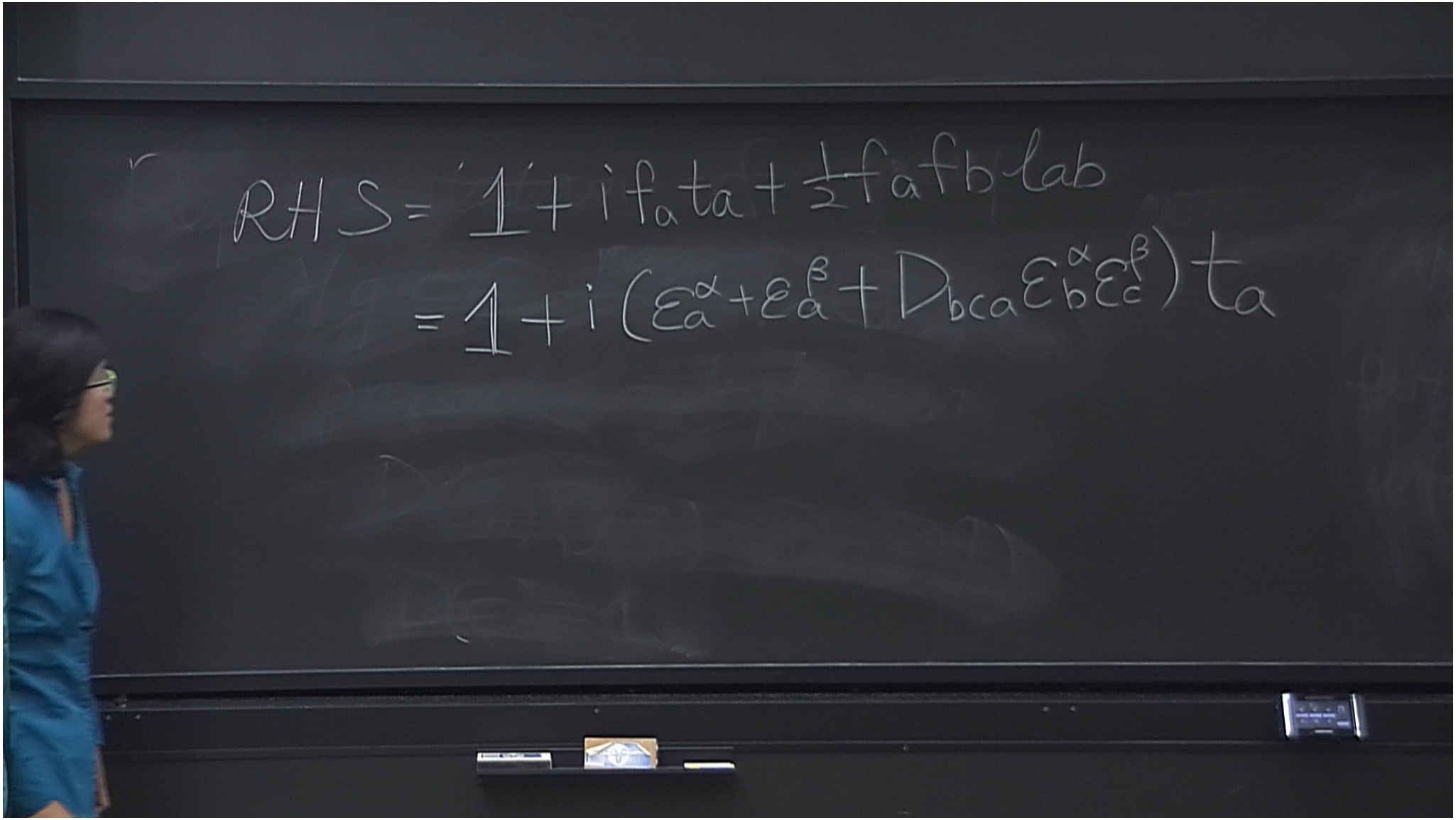
$$U(\varepsilon) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b l_{ab}$$

$$U(\varepsilon^\alpha)\,U(\varepsilon^\beta) = U(f(\varepsilon^\alpha, \varepsilon^\beta))$$


choose to be
symmetric
LHS =


choose to be
symmetric
LHS = (1 + i ε^α_a t_a + ½ ε^α_a ε^α_b l_ab)
(1 + ε^β_c t_c + ½ ε^β_c ε^β_d l_cd)


RHS = 1 + i f_a t_a + 1/2 f_a f_b l_ab


RHS = 1 + i f_a t_a + 1/2 f_a f_b l_ab
= 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a

$$S = \underline{1} + i f_a t_a + \frac{1}{2} f_a f_b t_{ab}$$

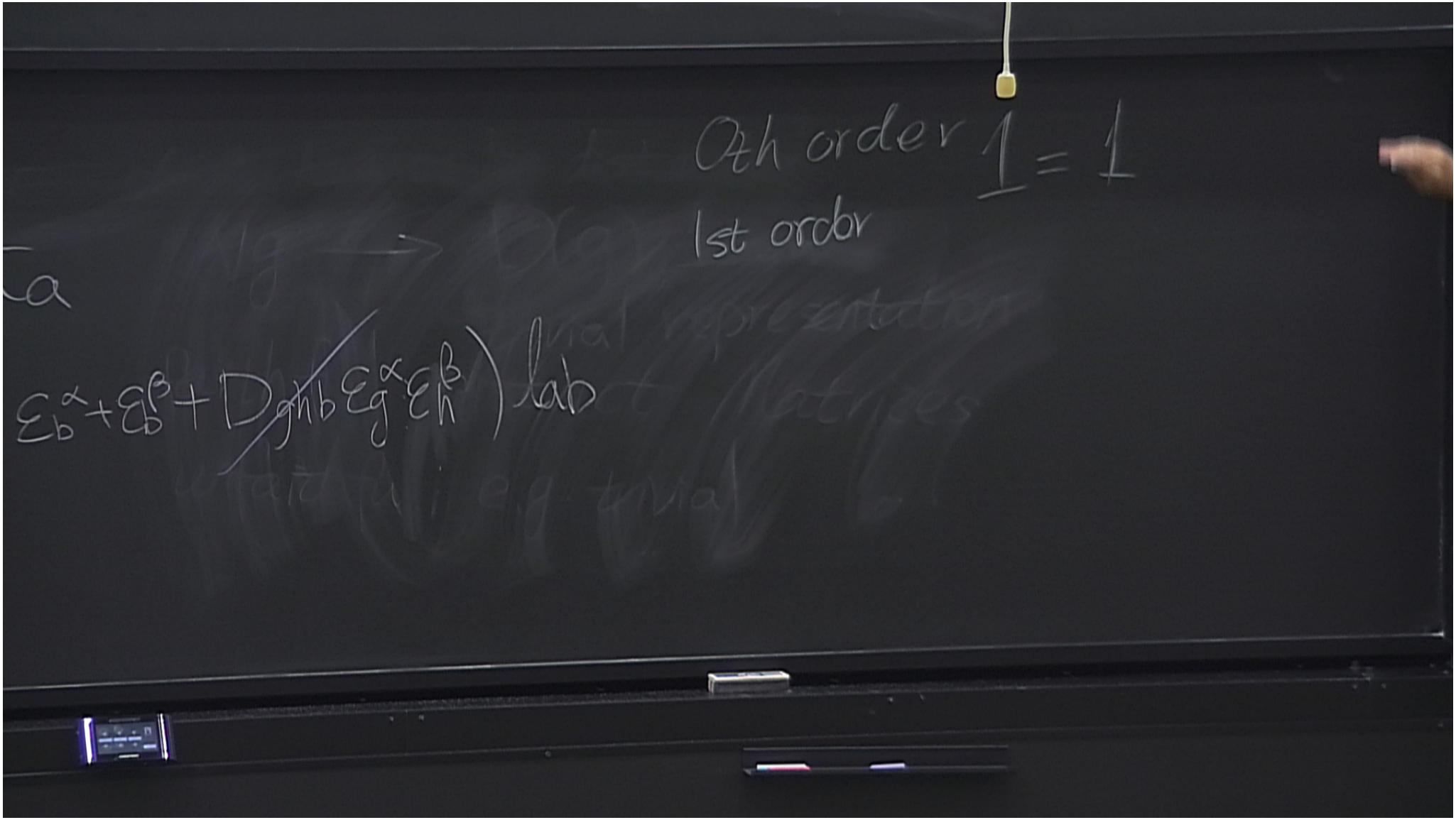
$$= \underline{1} + i\left(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{bca}\varepsilon_b^\alpha \varepsilon_c^\beta\right) t_a$$

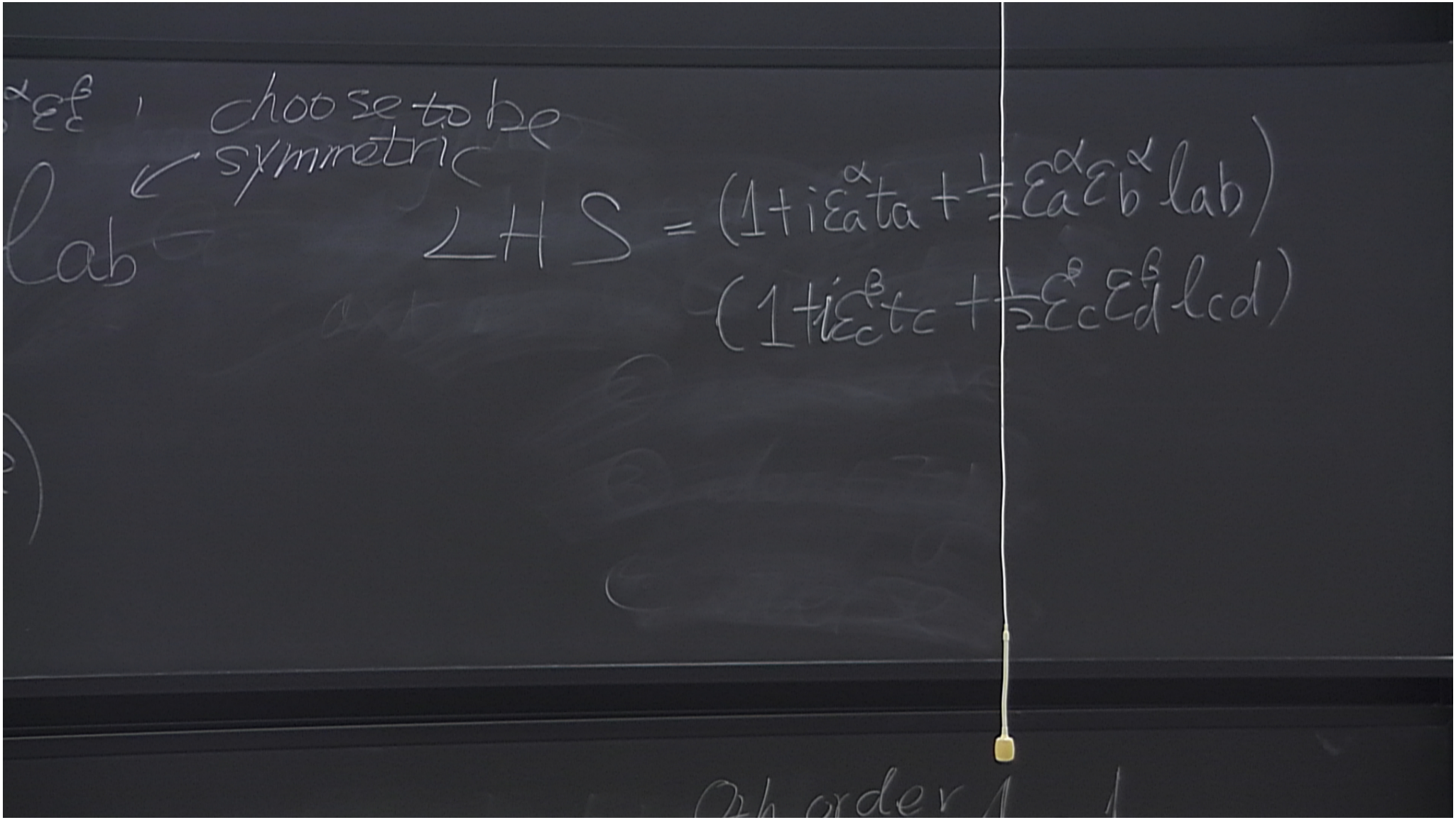
$$+ \frac{1}{2}\left(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{dea}\varepsilon_d^\alpha \varepsilon_e^\beta\right)\left(\varepsilon_b^\alpha + \varepsilon_b^\beta + D_{ghb}\varepsilon_g^\alpha \varepsilon_h^\beta\right) t_{ab}$$

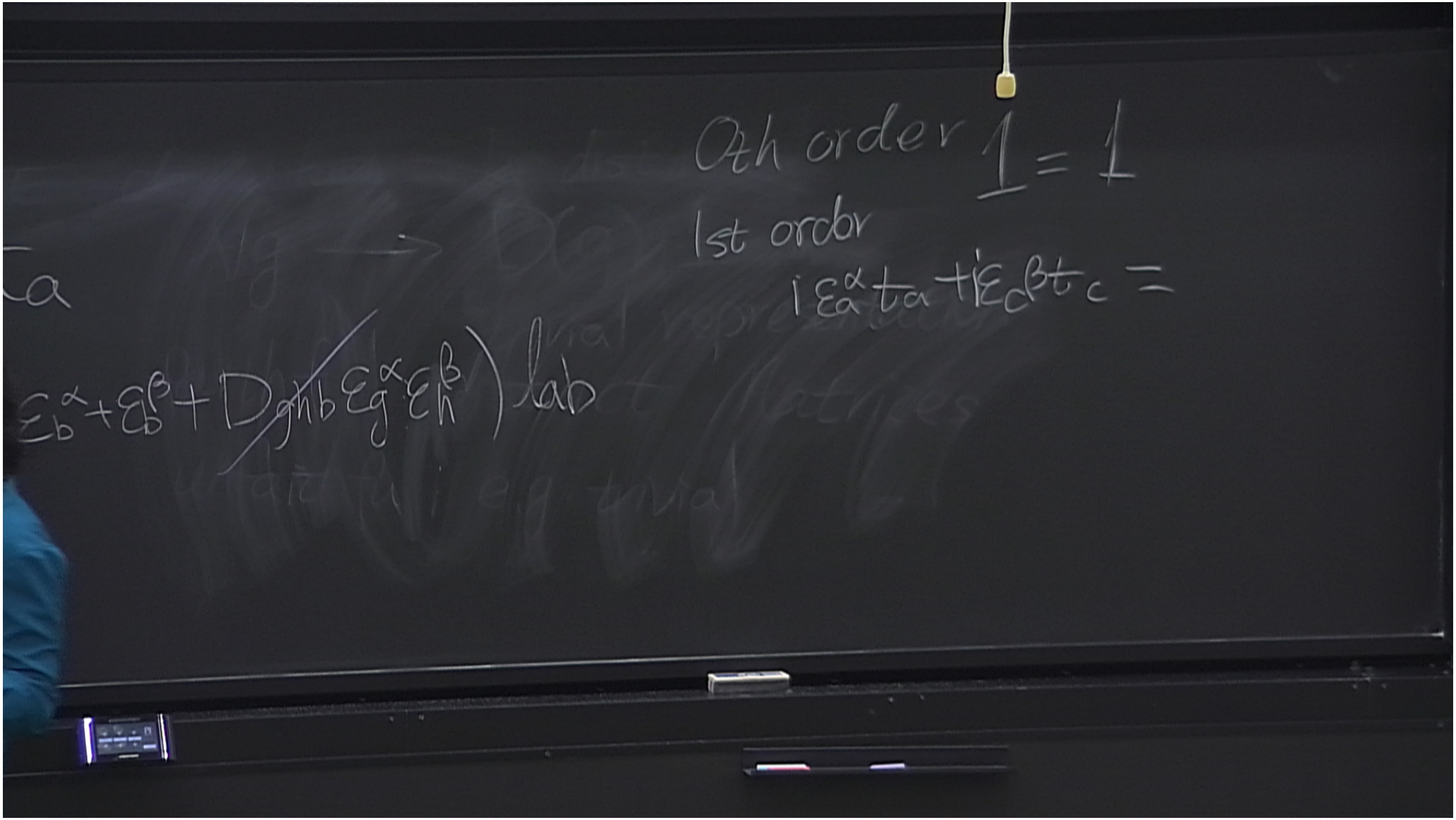
$$S = \underline{1} + i f_a t_a + \frac{1}{2} f_a f_b t_{ab}$$

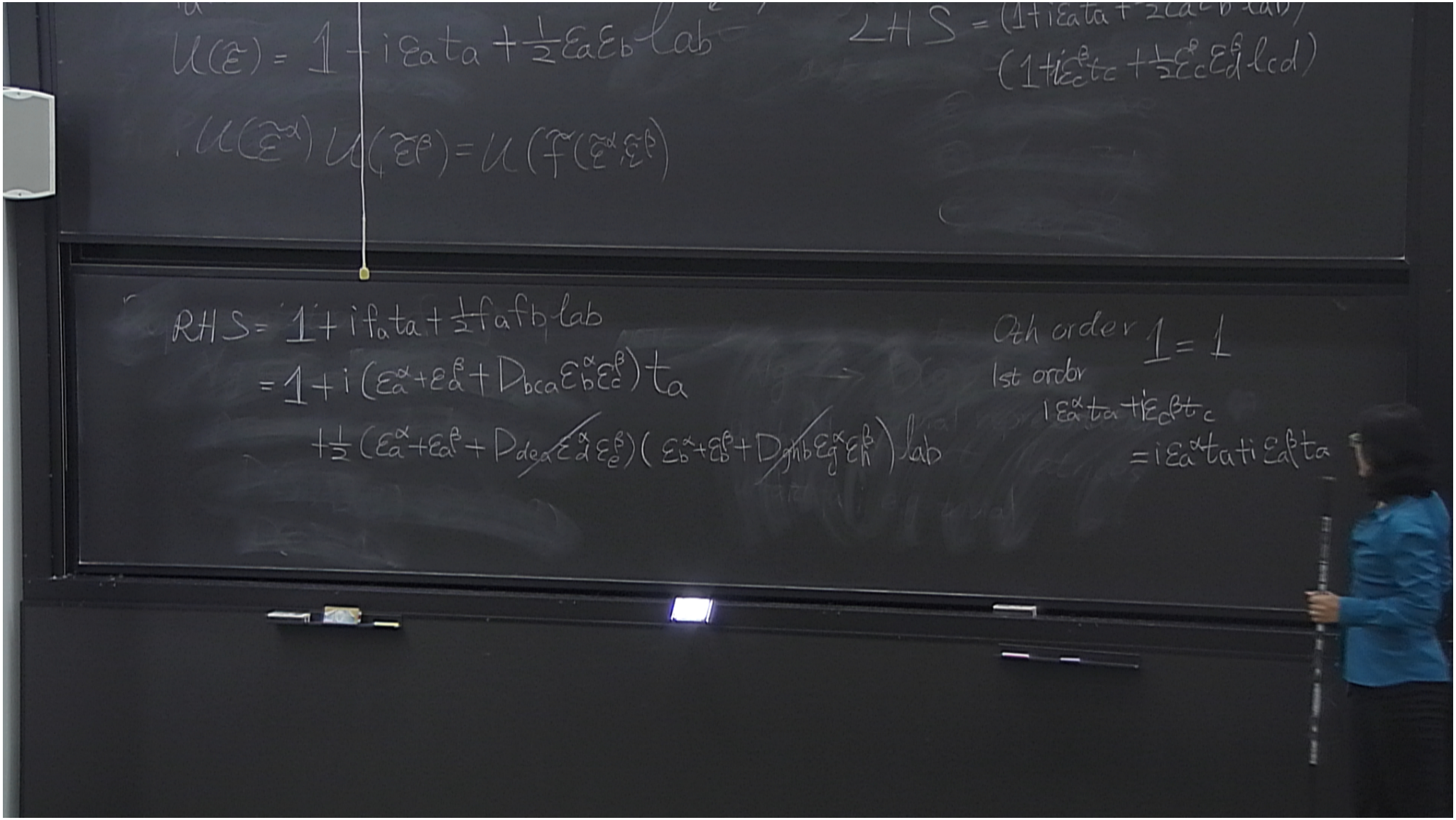
$$= \underline{1} + i\left(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{bca}\varepsilon_b^\alpha \varepsilon_c^\beta\right) t_a$$

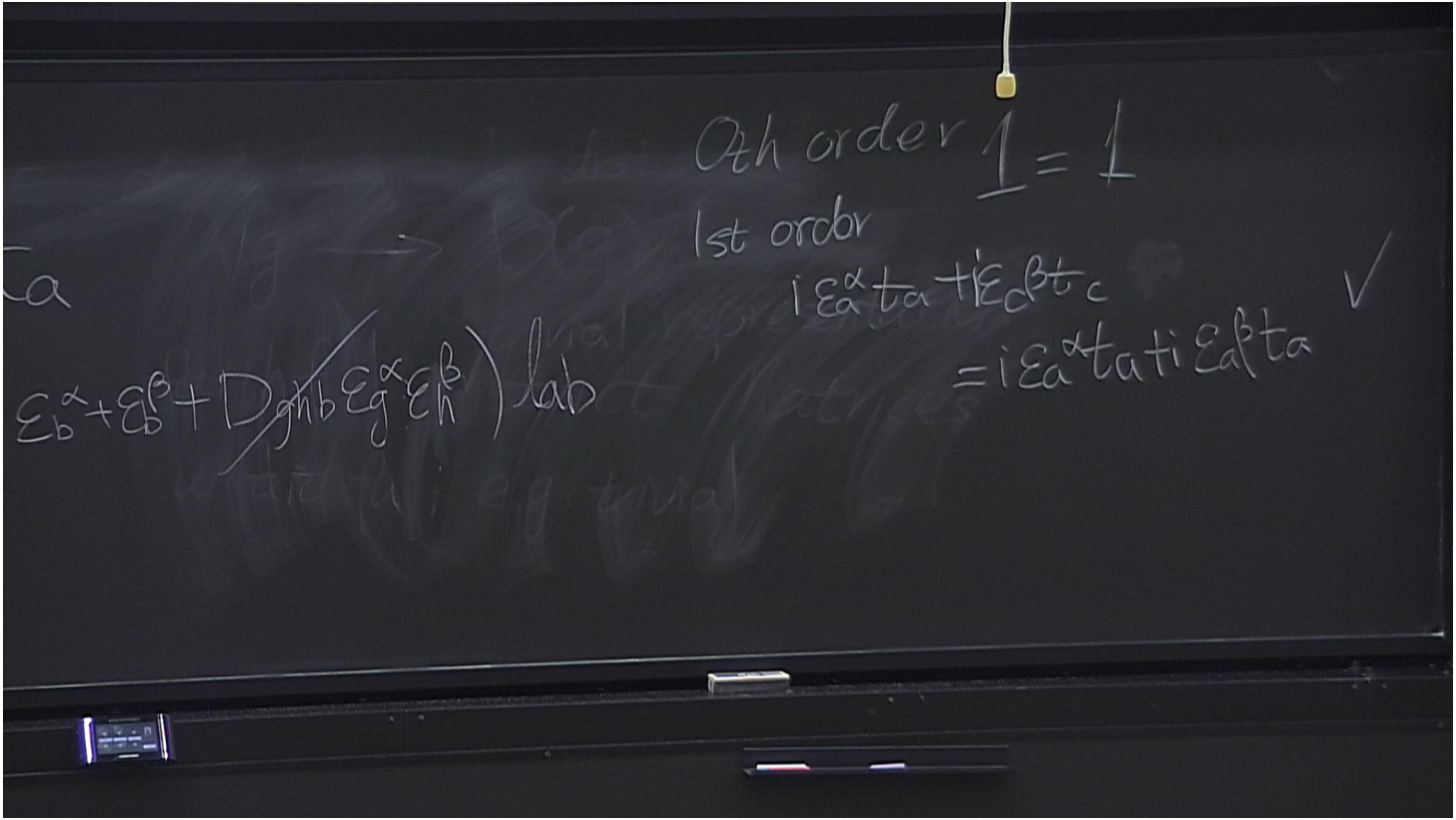
$$+ \frac{1}{2}\left(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{dea}\varepsilon_d^\alpha \varepsilon_e^\beta\right)\left(\varepsilon_b^\alpha + \varepsilon_b^\beta + D_{gnb}\varepsilon_g^\alpha \varepsilon_h^\beta\right) t_{ab}$$

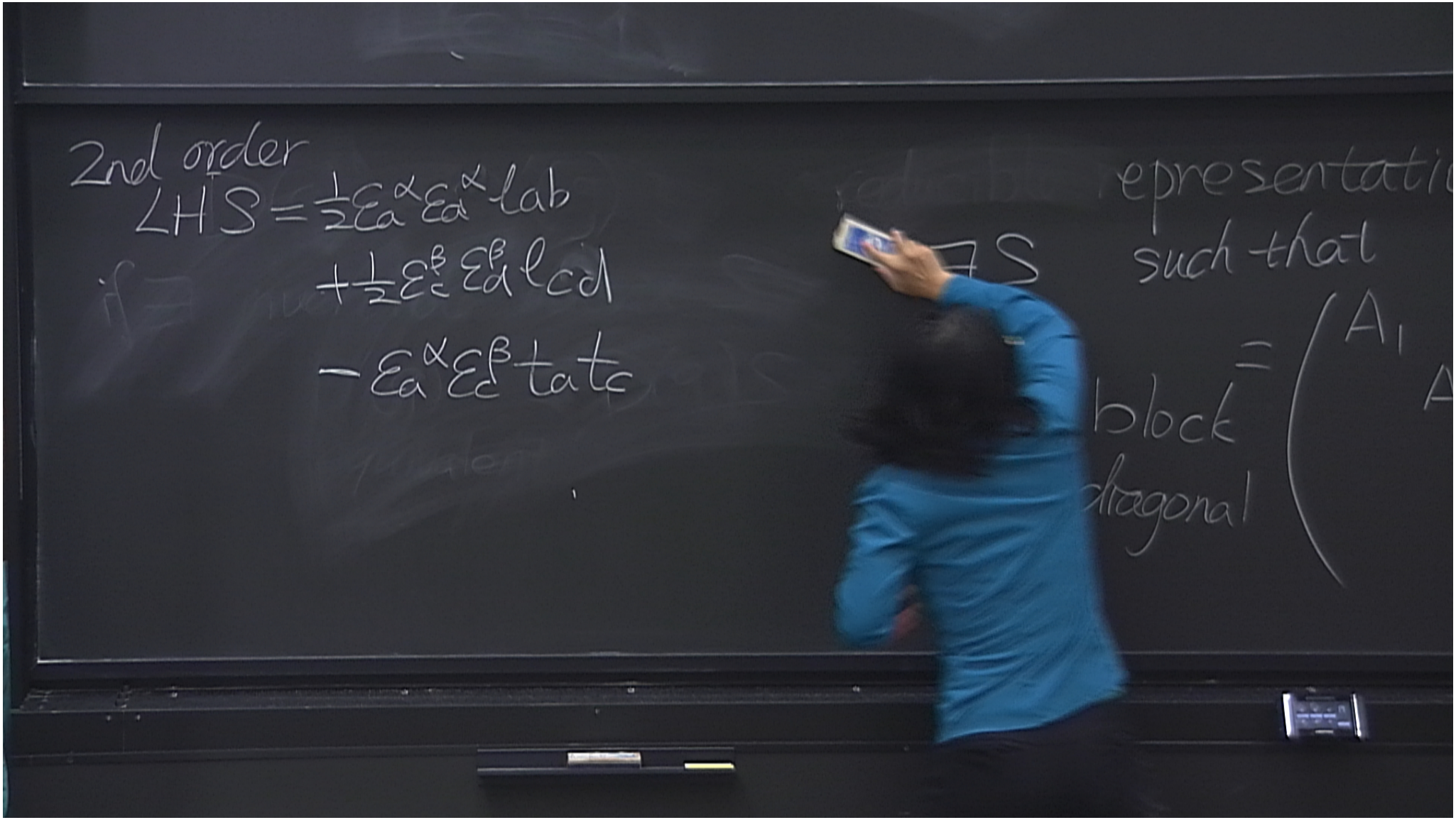

0th order 1 = 1
1st order
$\varepsilon_b^\alpha + \varepsilon_b^\beta + D g^{hb} \varepsilon_g^\alpha \varepsilon_h^\beta) lab$

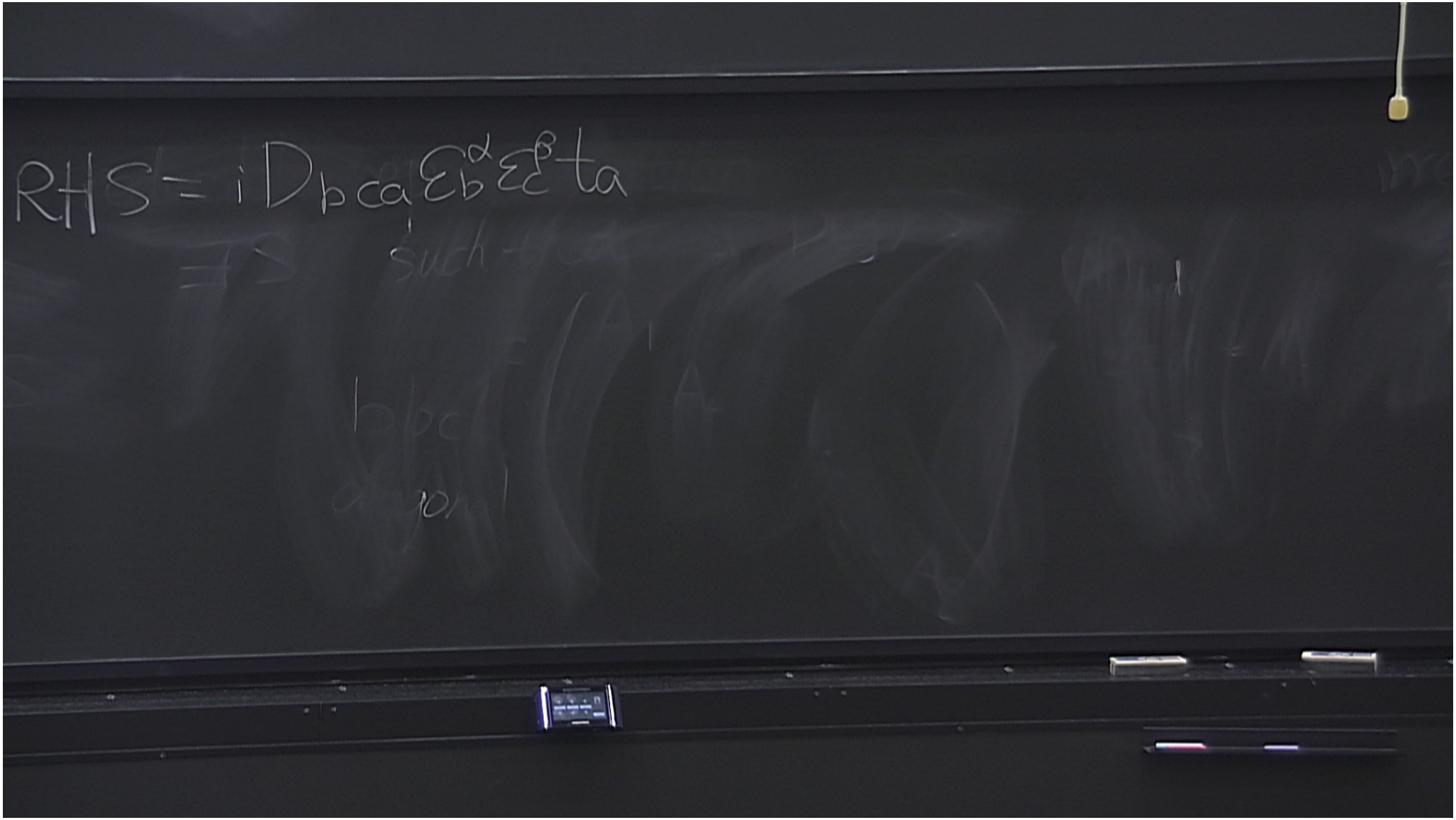

choose to be
symmetric
LHS = (1 + i ε^α_a t_a + ½ ε^α_a ε^α_b l_ab)
(1 + i ε^β_c t_c + ½ ε^β_c ε^β_d l_cd)

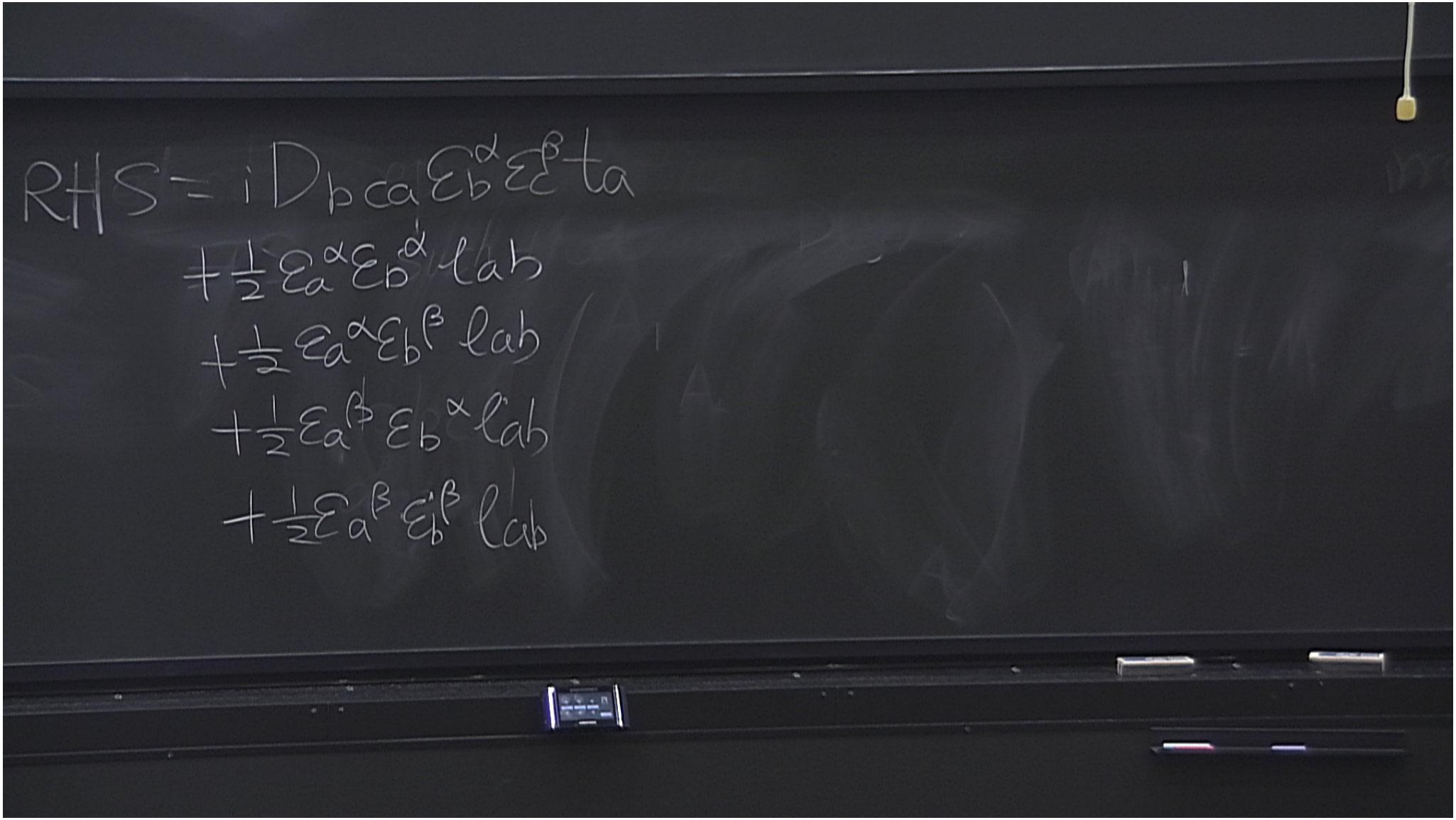

0th order 1 = 1
1st order
$i\varepsilon_a^\alpha t_a + i\varepsilon_c^\beta t_c =$
$\varepsilon_b^\alpha + \varepsilon_b^\beta + D g_{hb} \varepsilon_g^\alpha \varepsilon_h^\beta) lab$

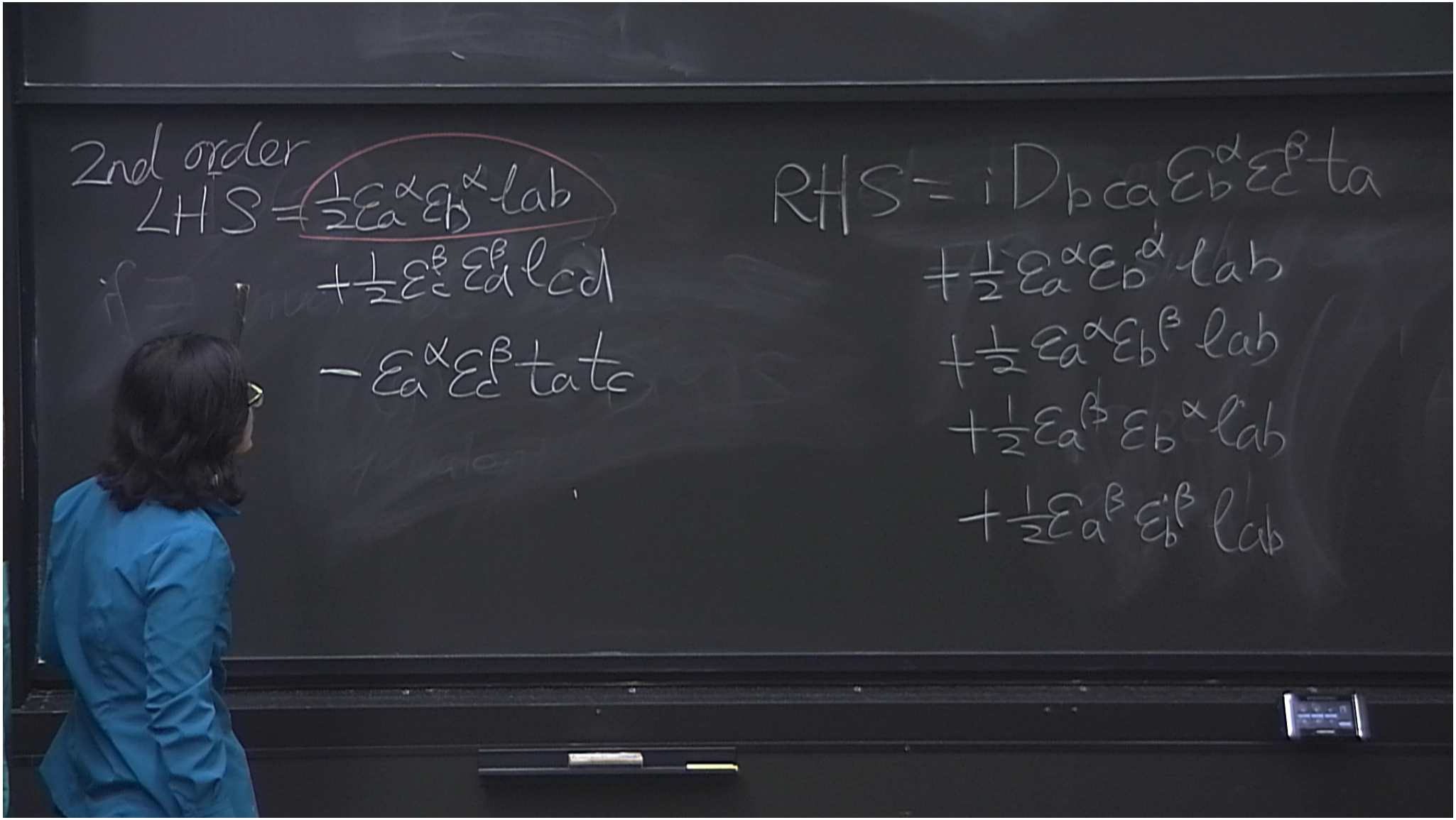

$U(\varepsilon) = 1 + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a\varepsilon_b l_{ab}$
$U(\vec{\varepsilon}^\alpha)\,U(\vec{\varepsilon}^\beta) = U(f^\alpha(\vec{\varepsilon}^\alpha,\vec{\varepsilon}^\beta))$
$LHS = (1 + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a\varepsilon_b l_{ab})$
$(1 + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_c^\beta\varepsilon_d^\beta l_{cd})$
$RHS = 1 + i f_a t_a + \frac{1}{2} f_a f_b l_{ab}$
$= 1 + i(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{bca}\varepsilon_b^\alpha\varepsilon_c^\beta)t_a$
$+ \frac{1}{2}(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{dea}\varepsilon_d^\alpha\varepsilon_e^\beta)(\varepsilon_b^\alpha + \varepsilon_b^\beta + D_{fgb}\varepsilon_f^\alpha\varepsilon_g^\beta) l_{ab}$
0th order $1 = 1$
1st order
$i\varepsilon_a^\alpha t_a + i\varepsilon_c^\beta t_c$
$= i\varepsilon_a^\alpha t_a + i\varepsilon_a^\beta t_a$

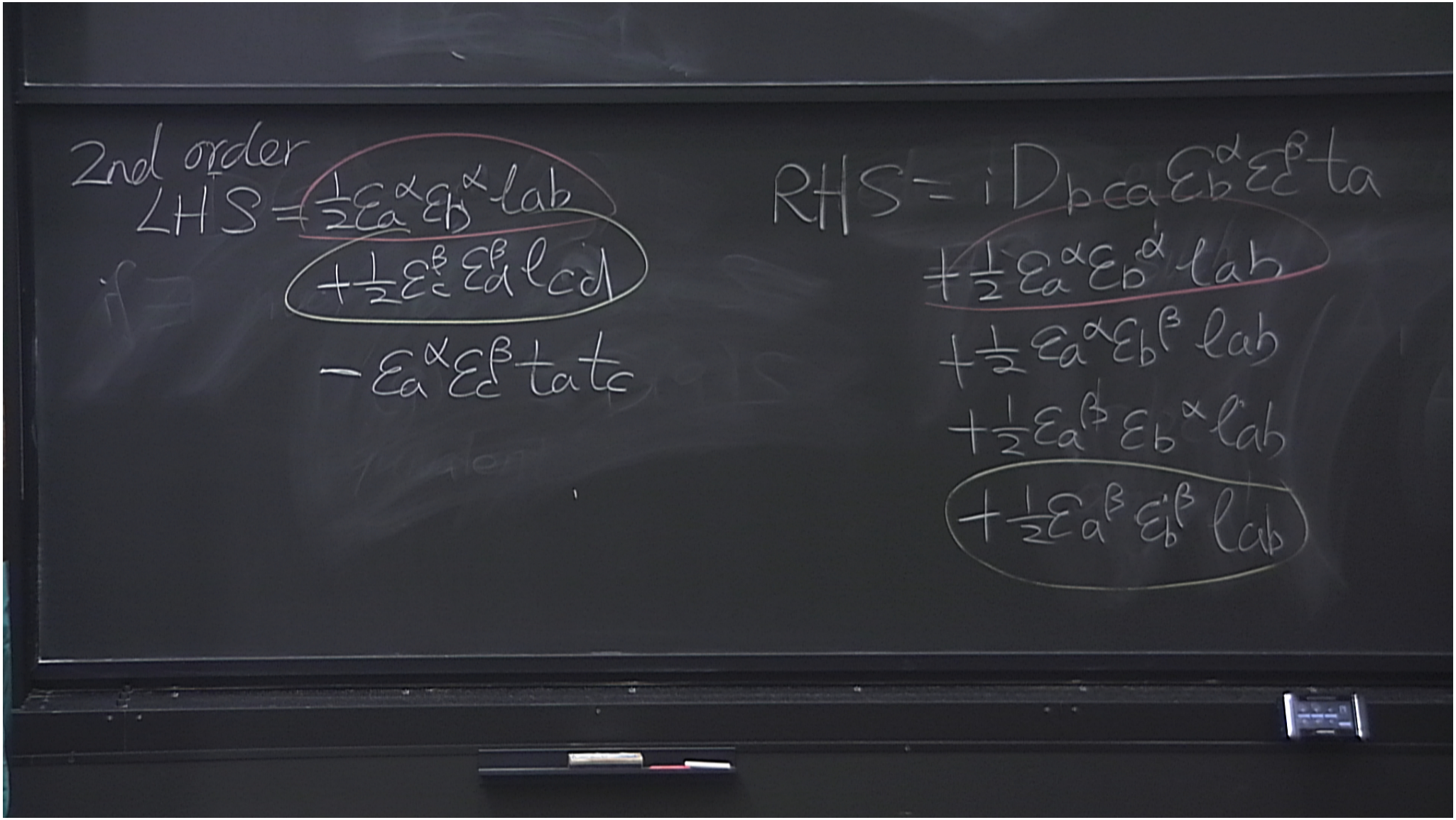

0th order 1 = 1
1st order
$i\varepsilon_a^\alpha t_a + i\varepsilon_c^\beta t_c$
$= i\varepsilon_a^\alpha t_a \, i\varepsilon_\beta t_a$ ✓
$\varepsilon_b^\alpha + \varepsilon_b^\beta + D g^{hb} \varepsilon_g^\alpha \varepsilon_h^\beta) lab$

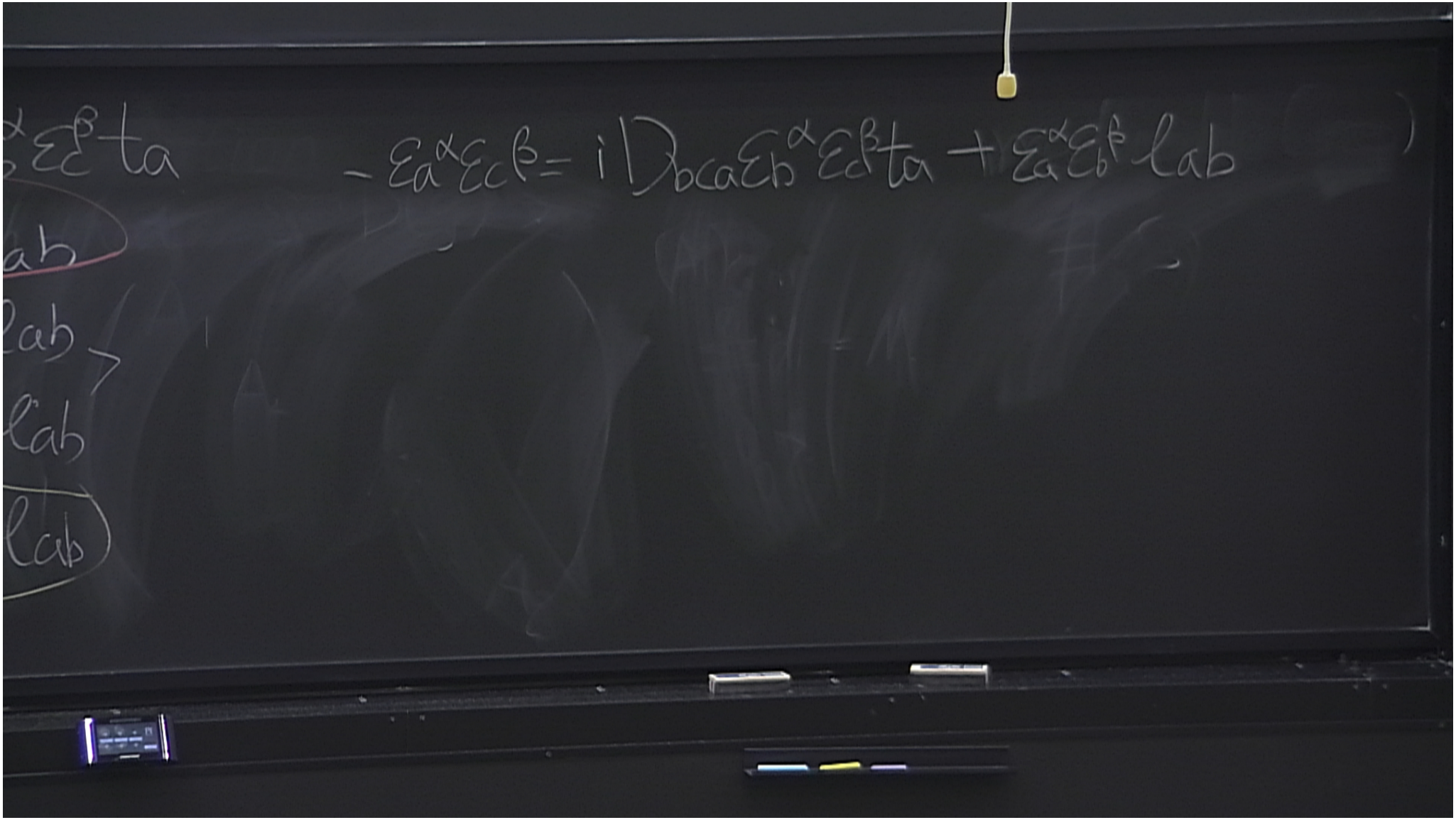

2nd order
LHS = $\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_a{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd}$
$-\varepsilon_a{}^\alpha \varepsilon_c^\beta t_a t_c$
epresentati
such that
S
block
diagonal
A1
A

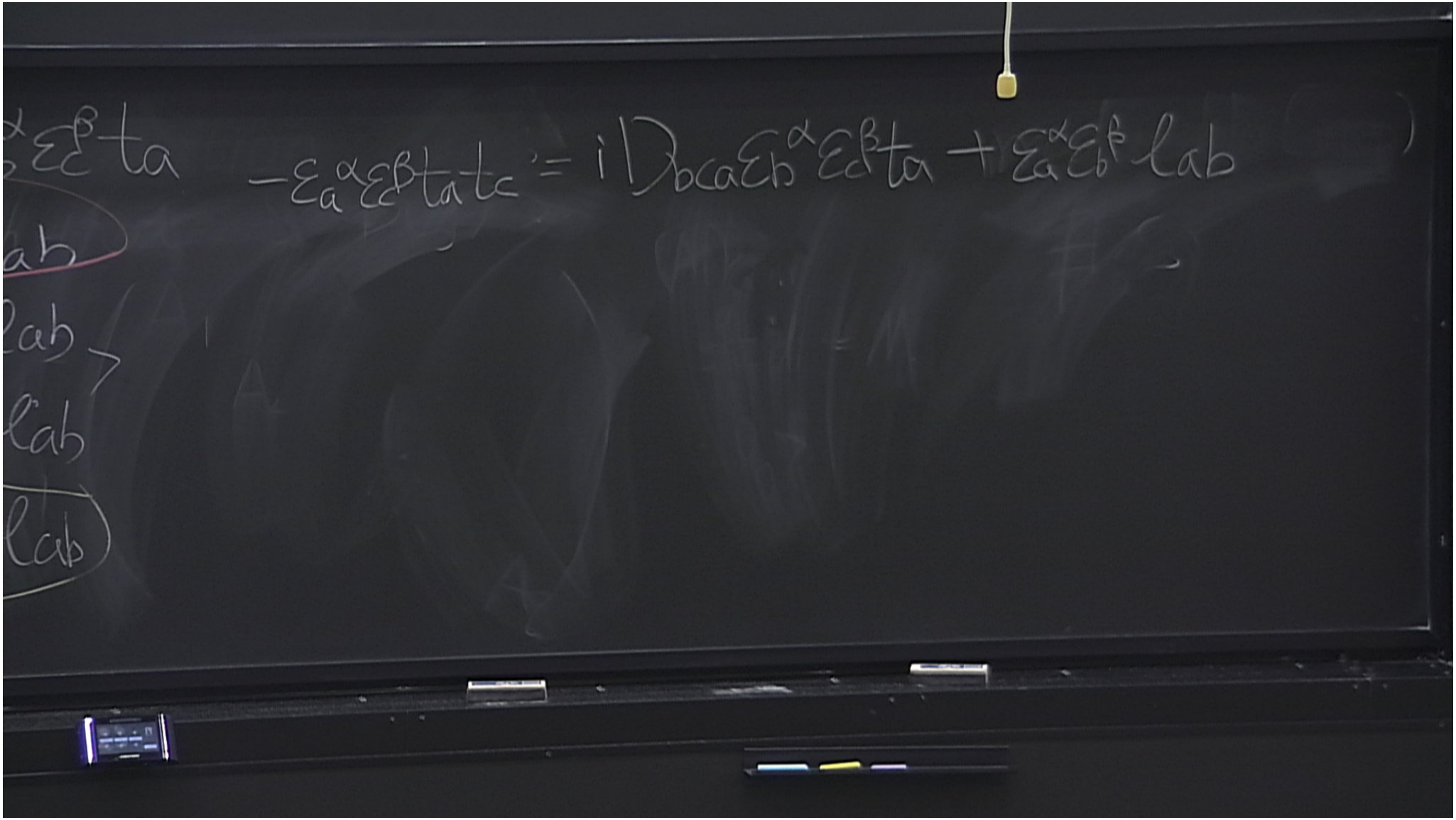

RHS = i D_b c_a ε_b^α ε_c^β t_a

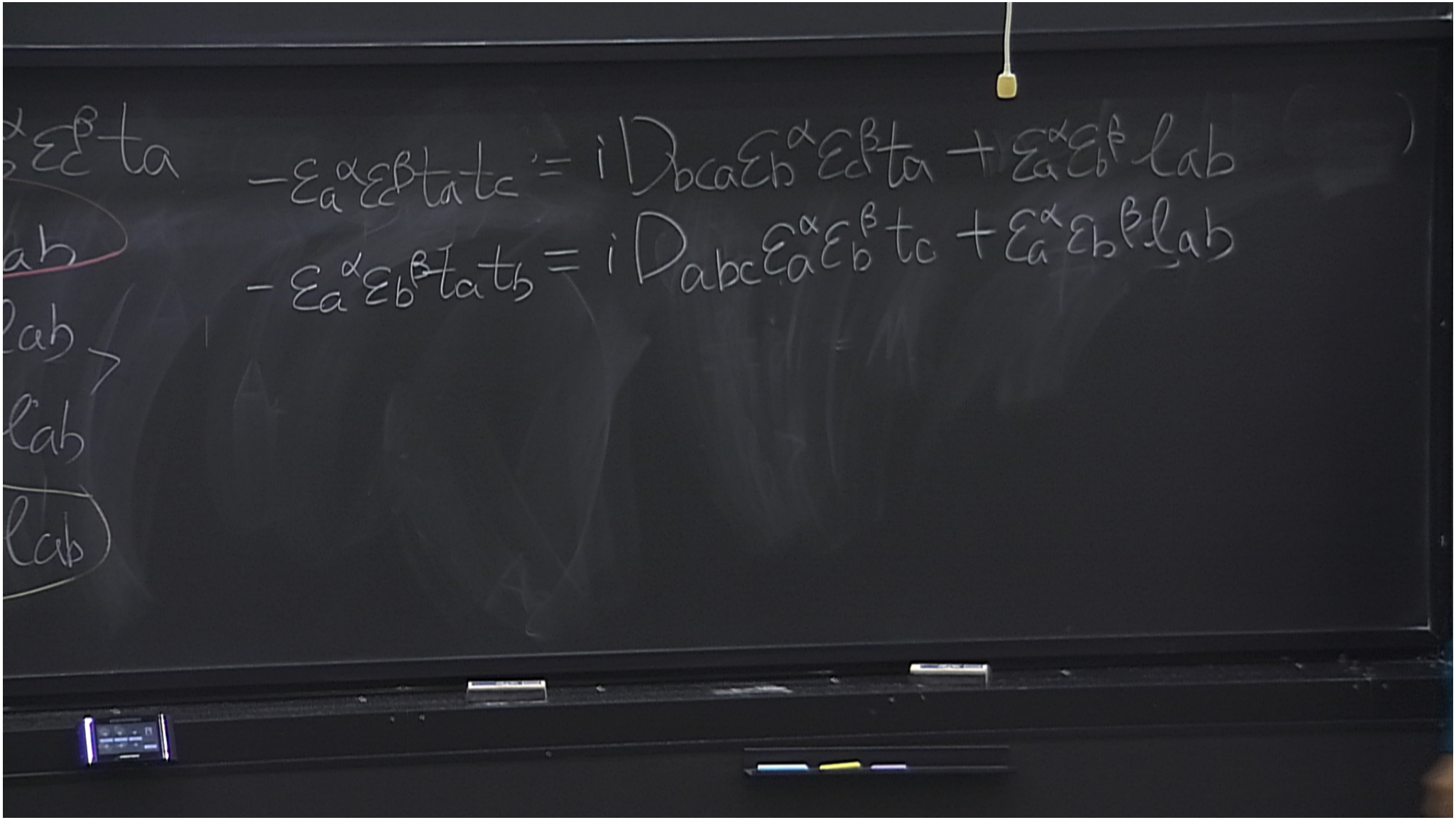

RHS = $iD_b c_a \varepsilon_b^{\alpha} \varepsilon_c^{\beta} t_a$
$+\frac{1}{2} \varepsilon_a^{\alpha} \varepsilon_b^{\dot{\alpha}} l_{ab}$
$+\frac{1}{2} \varepsilon_a^{\alpha} \varepsilon_b^{\beta} l_{ab}$
$+\frac{1}{2} \varepsilon_a^{\dot{\beta}} \varepsilon_b^{\alpha} l_{ab}$
$+\frac{1}{2} \varepsilon_a^{\beta} \varepsilon_b^{\beta} l_{ab}$

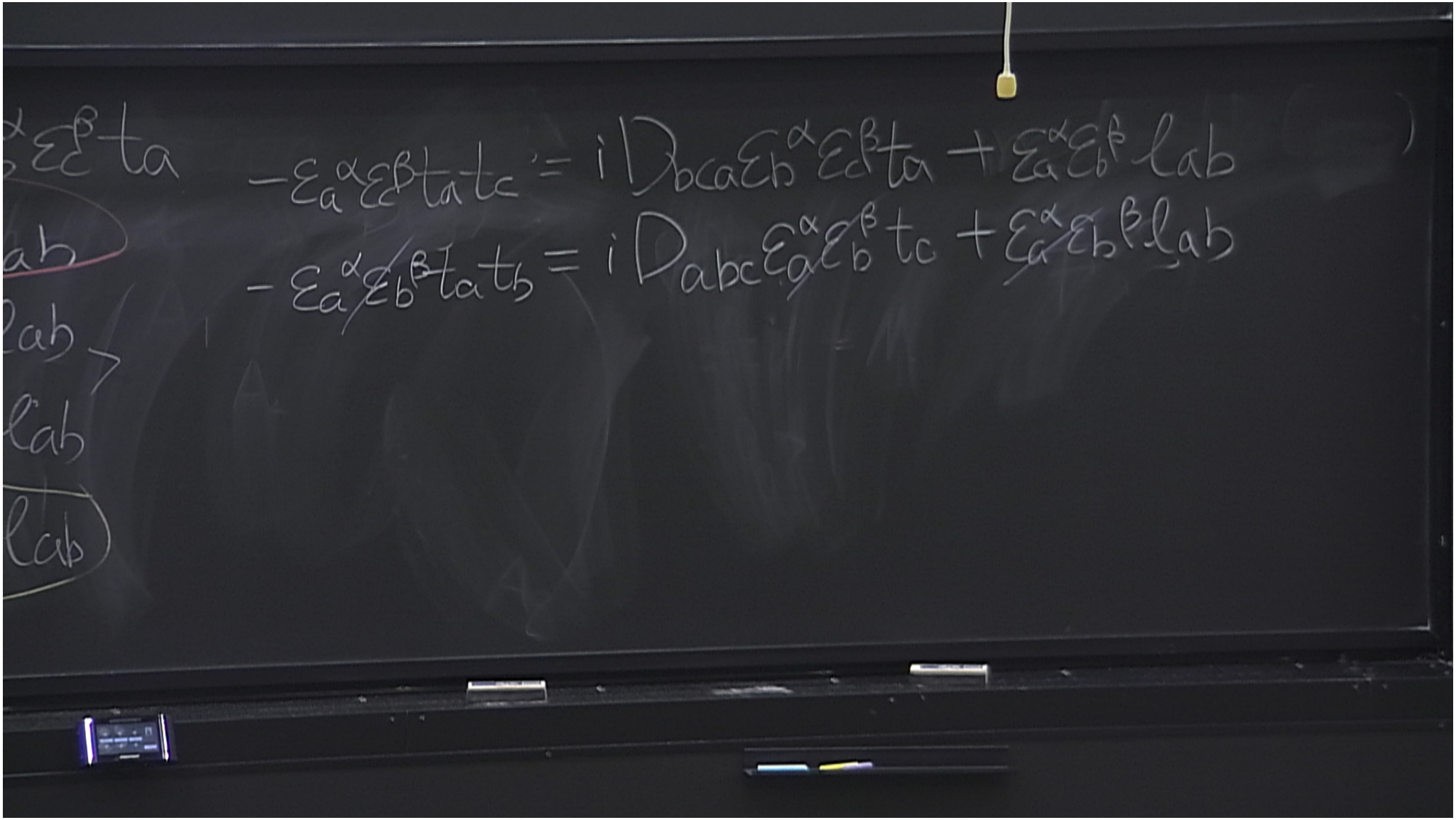

2nd order
LHS = $\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_c{}^\beta \varepsilon_d{}^\beta l_{cd}$
$-\varepsilon_a{}^\alpha \varepsilon_c{}^\beta t_a t_c$
RHS = $iD_b c_a \varepsilon_b{}^\alpha \varepsilon_c{}^\beta t_a$
$+\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\beta \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\beta \varepsilon_b{}^\beta l_{ab}$

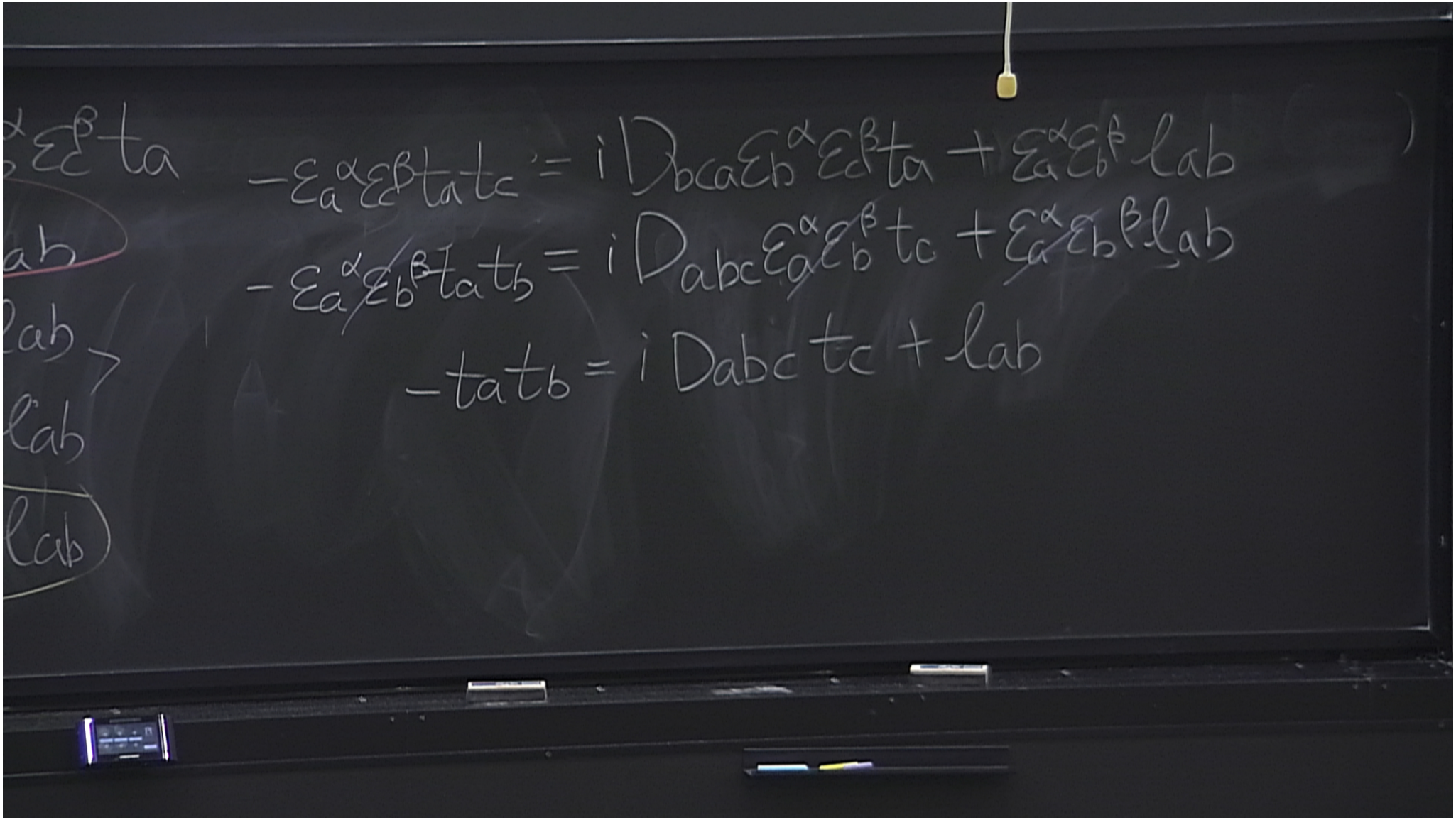

2nd order
LHS = $\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd}$
$-\varepsilon_a{}^\alpha \varepsilon_c^\beta t_a t_c$
RHS = $iD_b e_a \varepsilon_b^\alpha \varepsilon_c^\beta t_a$
$+\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\beta \varepsilon_b{}^\alpha l_{ab}$
$+\frac{1}{2}\varepsilon_a{}^\beta \varepsilon_b^\beta l_{ab}$

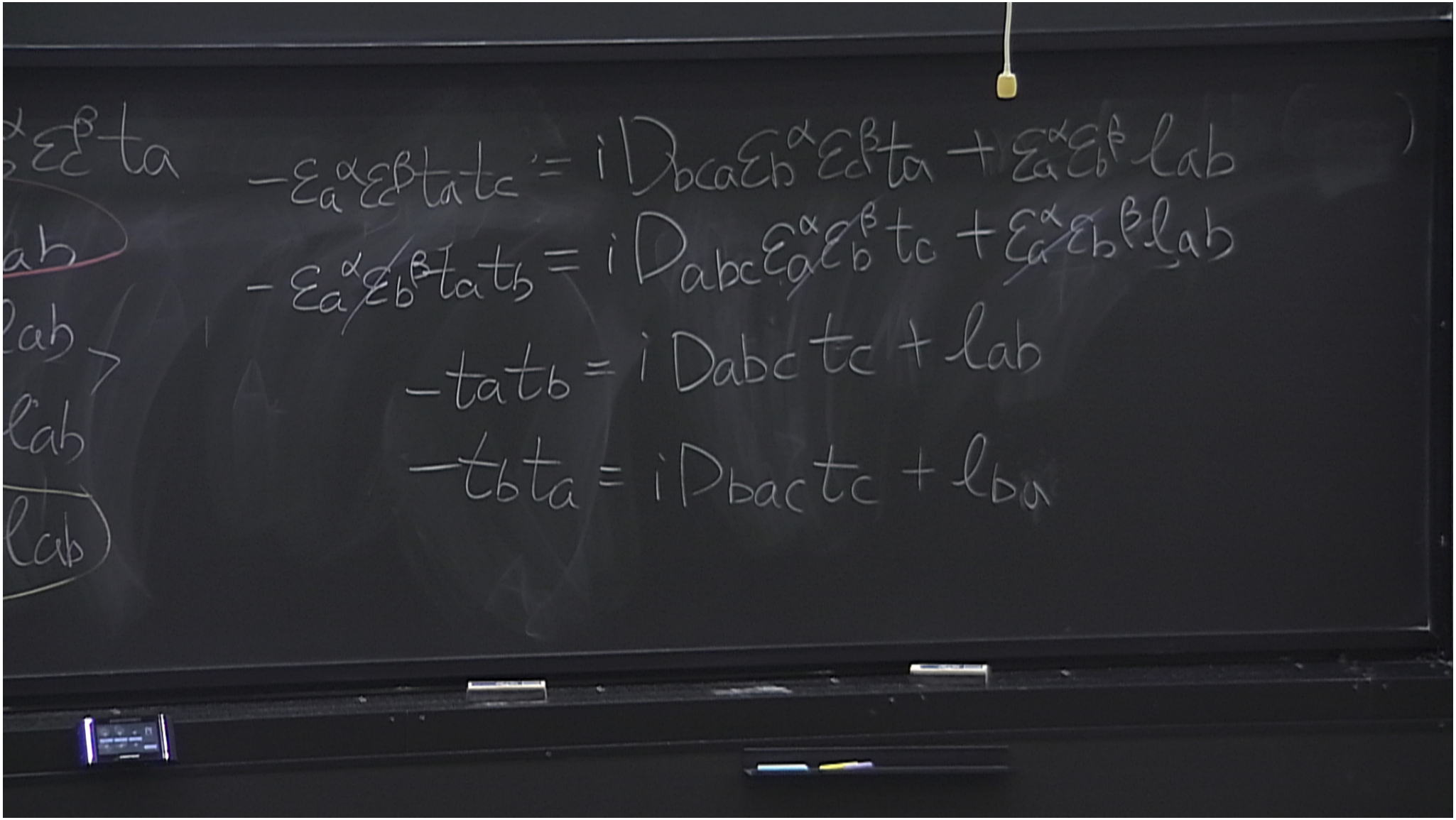

$\varepsilon_c^\beta t_a$
$-\varepsilon_a^\alpha \varepsilon_c^\beta = i f_{bca} \varepsilon_b^\alpha \varepsilon_c^\beta t_a + \varepsilon_a^\alpha \varepsilon_b^\beta l_{ab}$
l_{ab}
l_{ab}
l_{ab}
l_{ab}

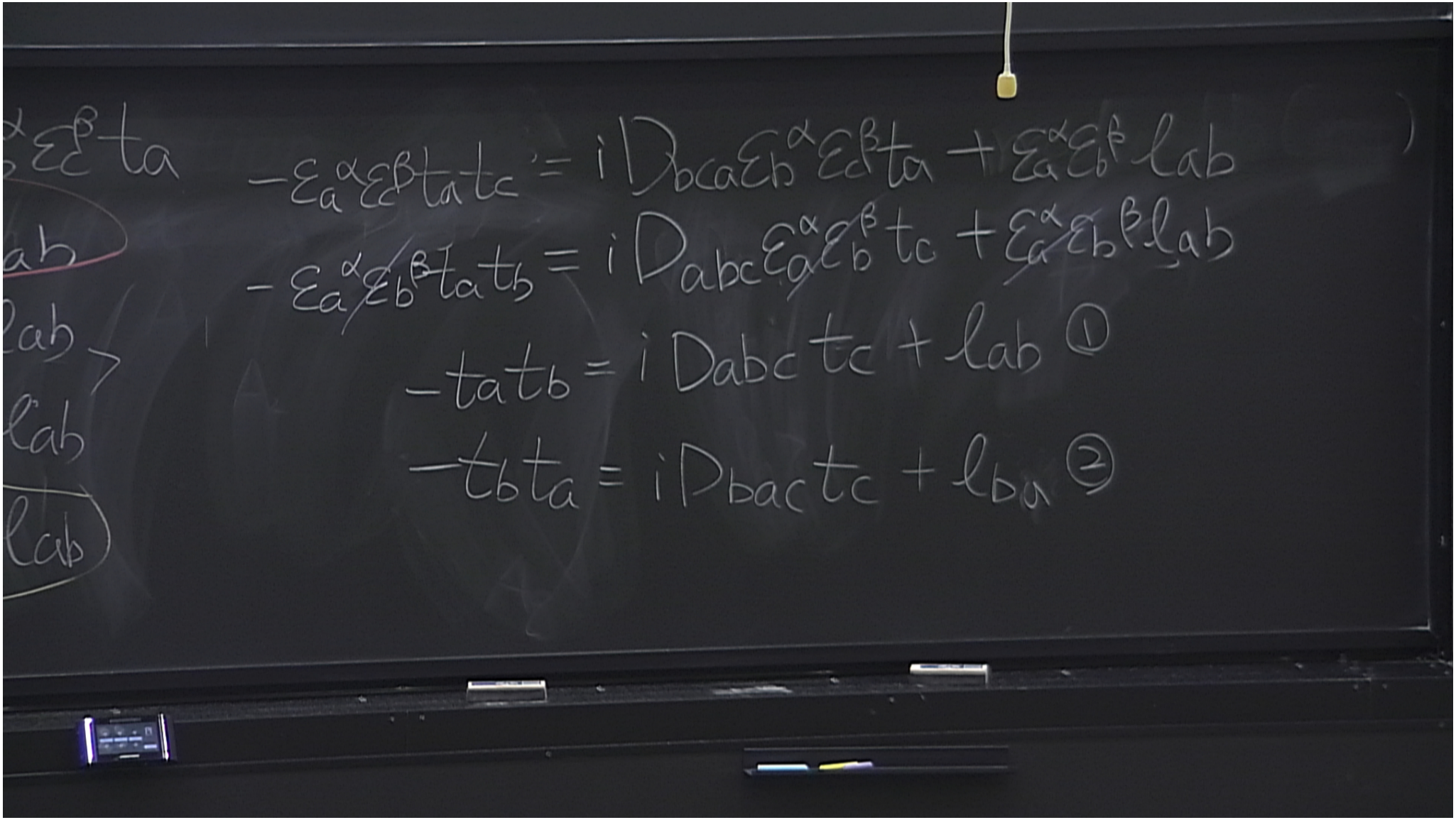

$\varepsilon^{\beta} t_a$
$-\varepsilon_a{}^{\alpha}\varepsilon^{\beta} t_a t_c = i D_{bca}\varepsilon_b{}^{\alpha}\varepsilon_c{}^{\beta} t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
l_{ab}
l_{ab}
l_{ab}
l_{ab}

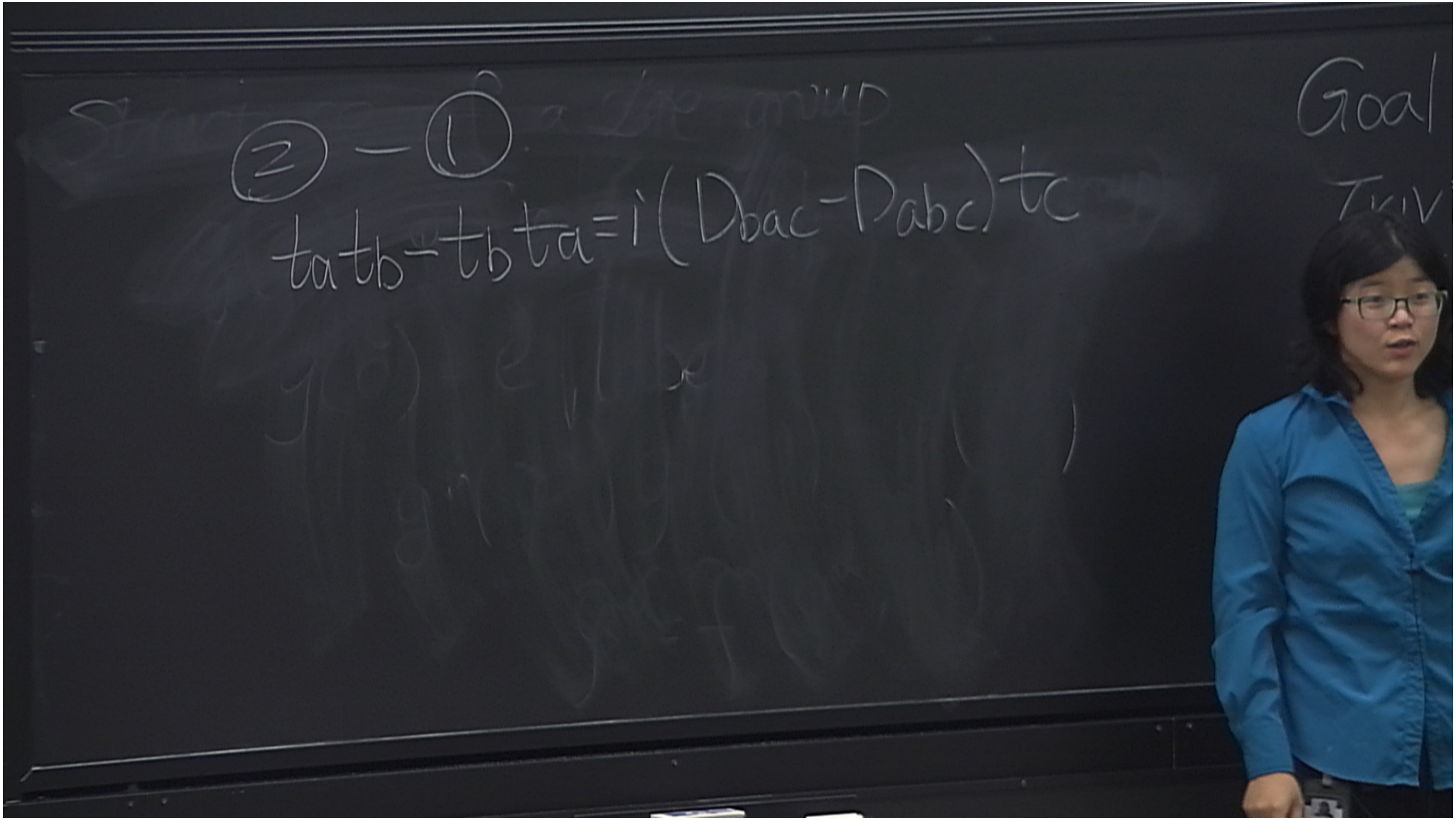

$\varepsilon^{\beta} t_a$
$-\varepsilon_a{}^{\alpha}\varepsilon^{\beta} t_a t_c = i D_{bca}\varepsilon_b{}^{\alpha}\varepsilon^{\beta} t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
$-\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_a t_b = i D_{abc}\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_c + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
ab
lab
lab
lab

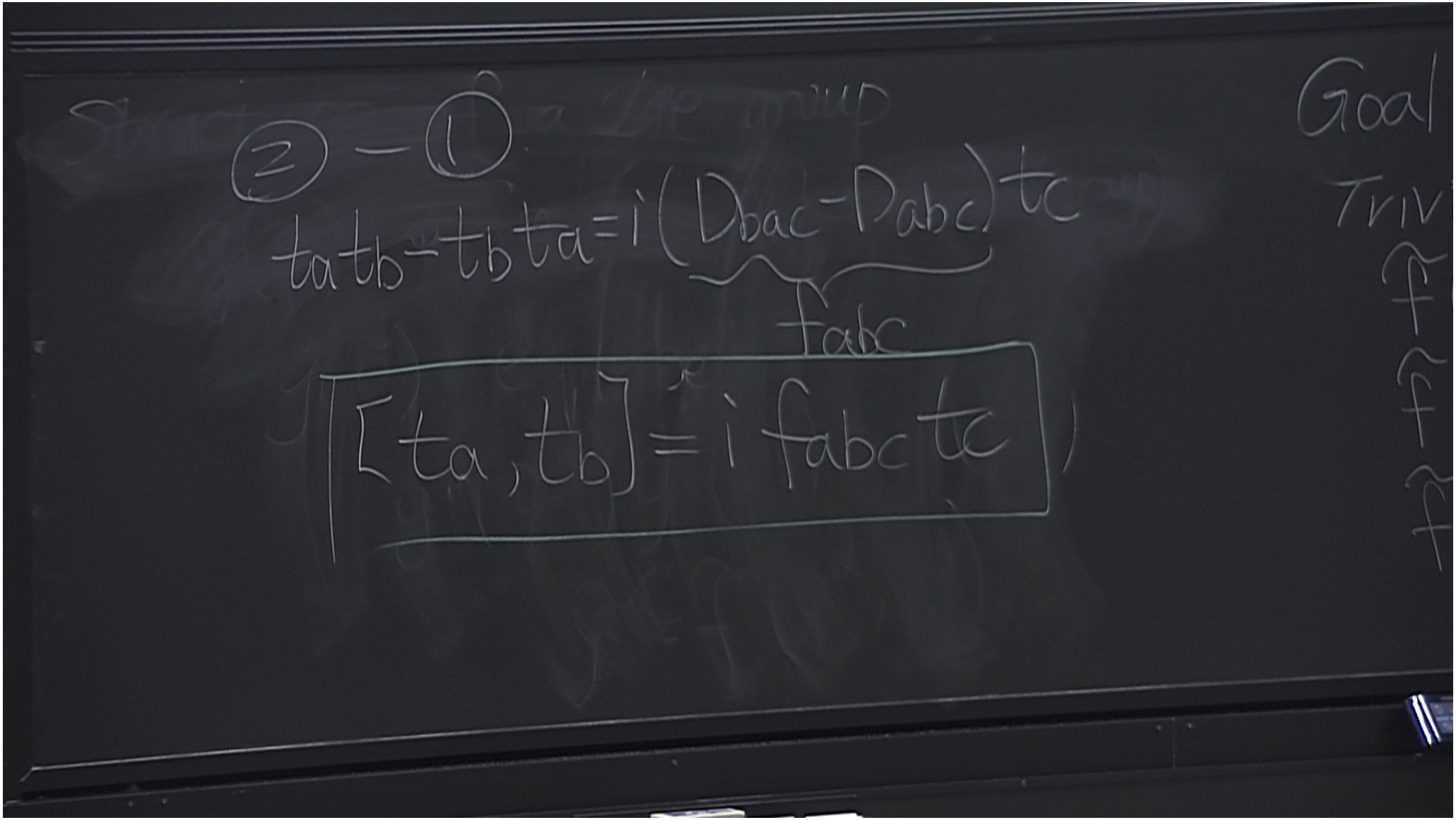

$\varepsilon^{\beta} t_a$
$-\varepsilon_a{}^{\alpha}\varepsilon^{\beta} t_a t_c = i D_{bca}\varepsilon_b{}^{\alpha}\varepsilon^{\beta} t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
$-\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_a t_b = i D_{abc}\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_c + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
ab
lab
lab
lab

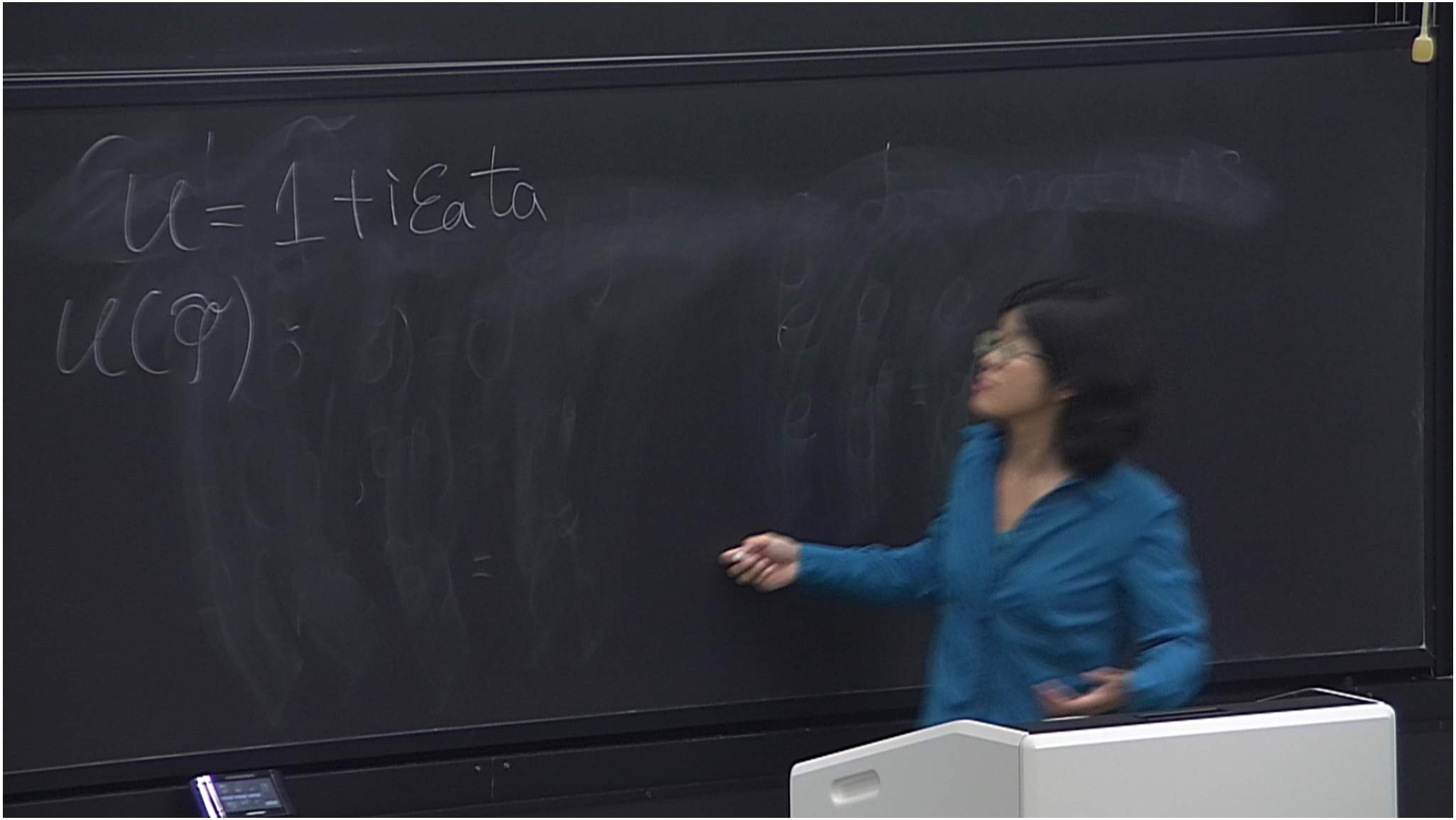

$-\varepsilon_a{}^\alpha \varepsilon_c{}^\beta t_a t_c = i D_{bca} \varepsilon_b{}^\alpha \varepsilon_c{}^\beta t_a + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-\varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_a t_b = i D_{abc} \varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_c + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-t_a t_b = i D_{abc} t_c + l_{ab}$

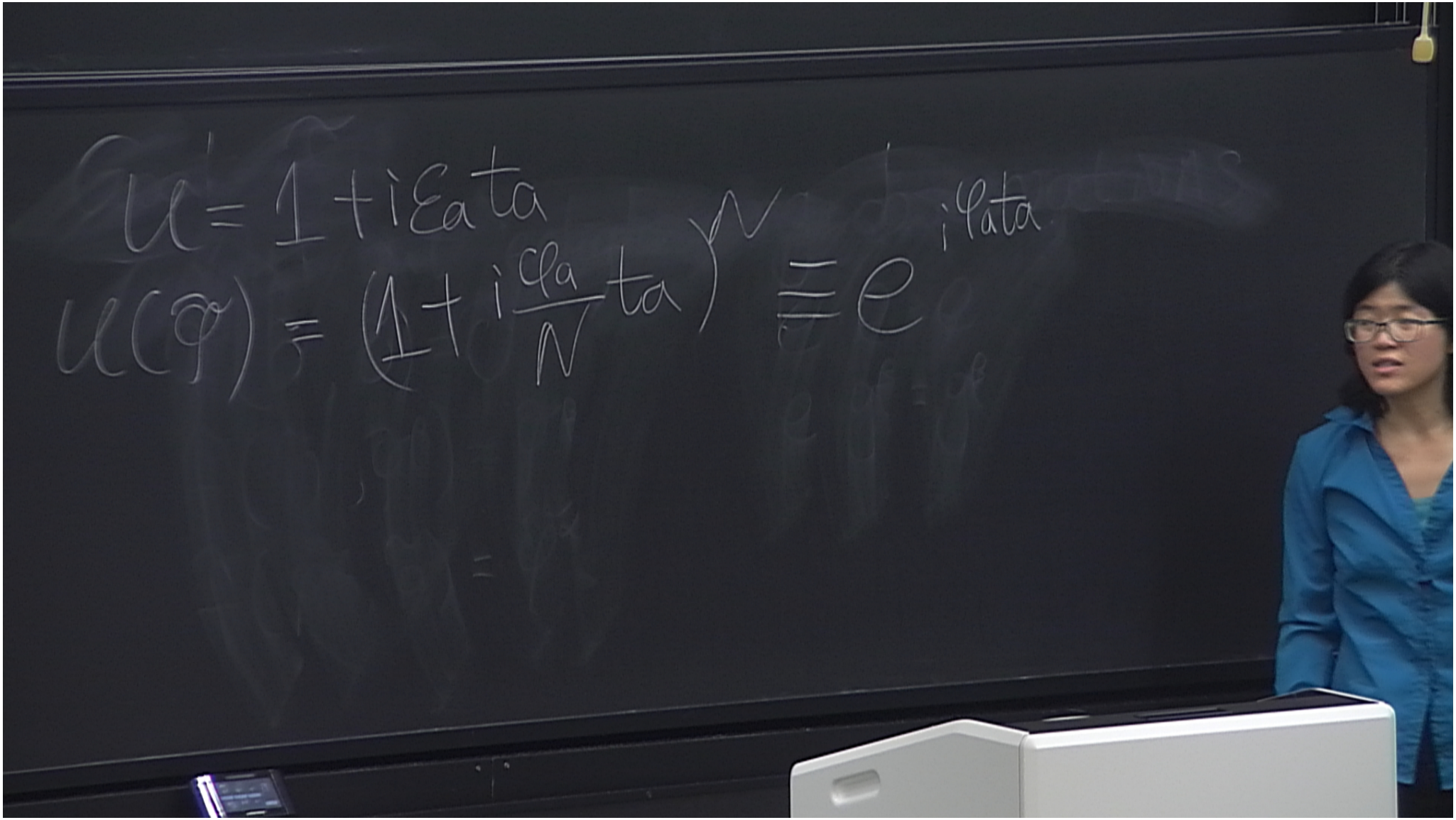

$-\varepsilon_a{}^\alpha \varepsilon_c{}^\beta t_a t_c = i D_{bca} \varepsilon_b{}^\alpha \varepsilon_c{}^\beta t_a + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-\varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_a t_b = i D_{abc} \varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_c + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-t_a t_b = i D_{abc} t_c + l_{ab}$
$-t_b t_a = i D_{bac} t_c + l_{ba}$

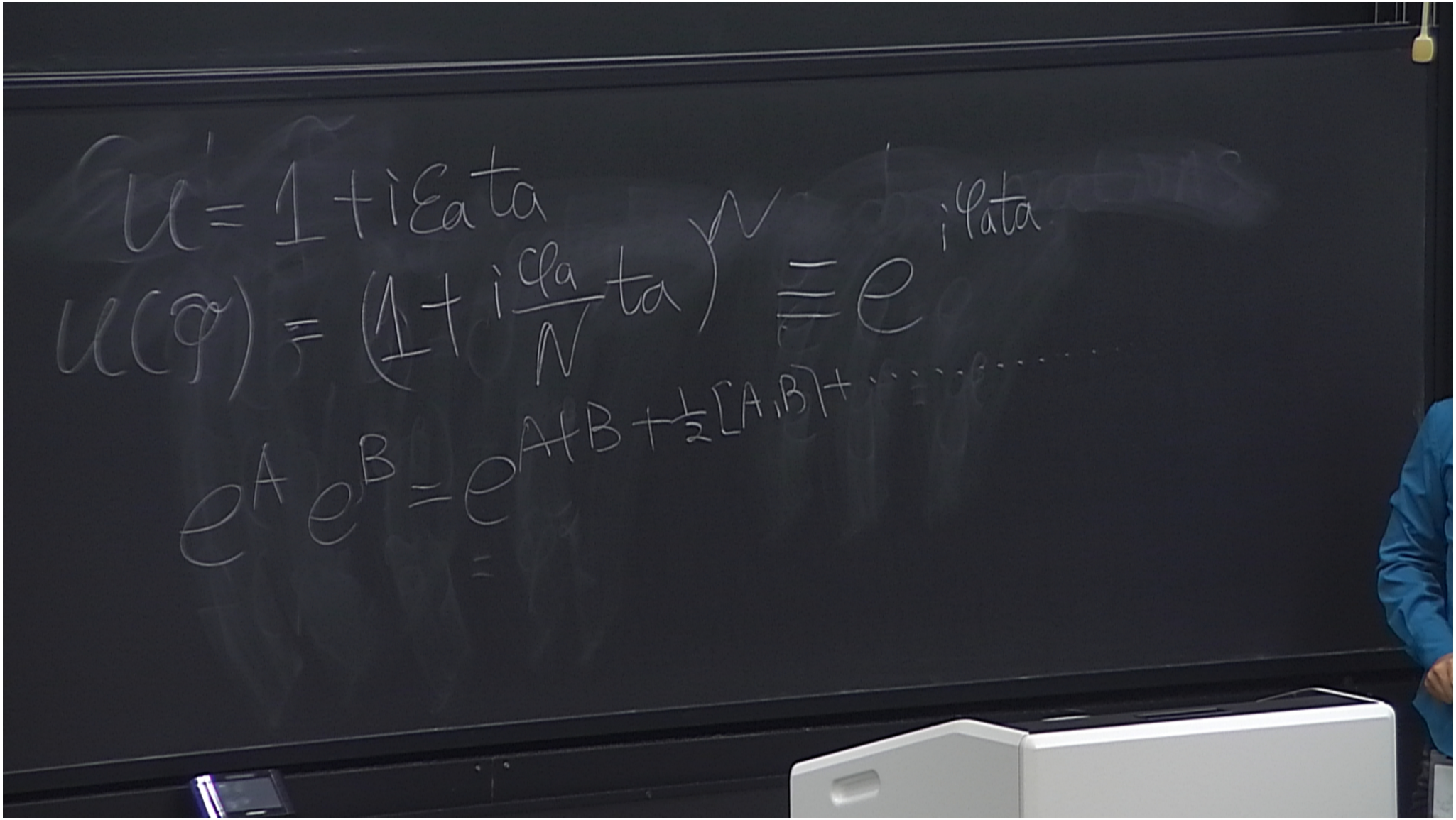

$-\varepsilon_a{}^\alpha \varepsilon_c{}^\beta t_a t_c = i D_{bca} \varepsilon_b{}^\alpha \varepsilon_c{}^\beta t_a + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-\varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_a t_b = i D_{abc} \varepsilon_a{}^\alpha \varepsilon_b{}^\beta t_c + \varepsilon_a{}^\alpha \varepsilon_b{}^\beta l_{ab}$
$-t_a t_b = i D_{abc} t_c + l_{ab}$ ①
$-t_b t_a = i D_{bac} t_c + l_{ba}$ ②

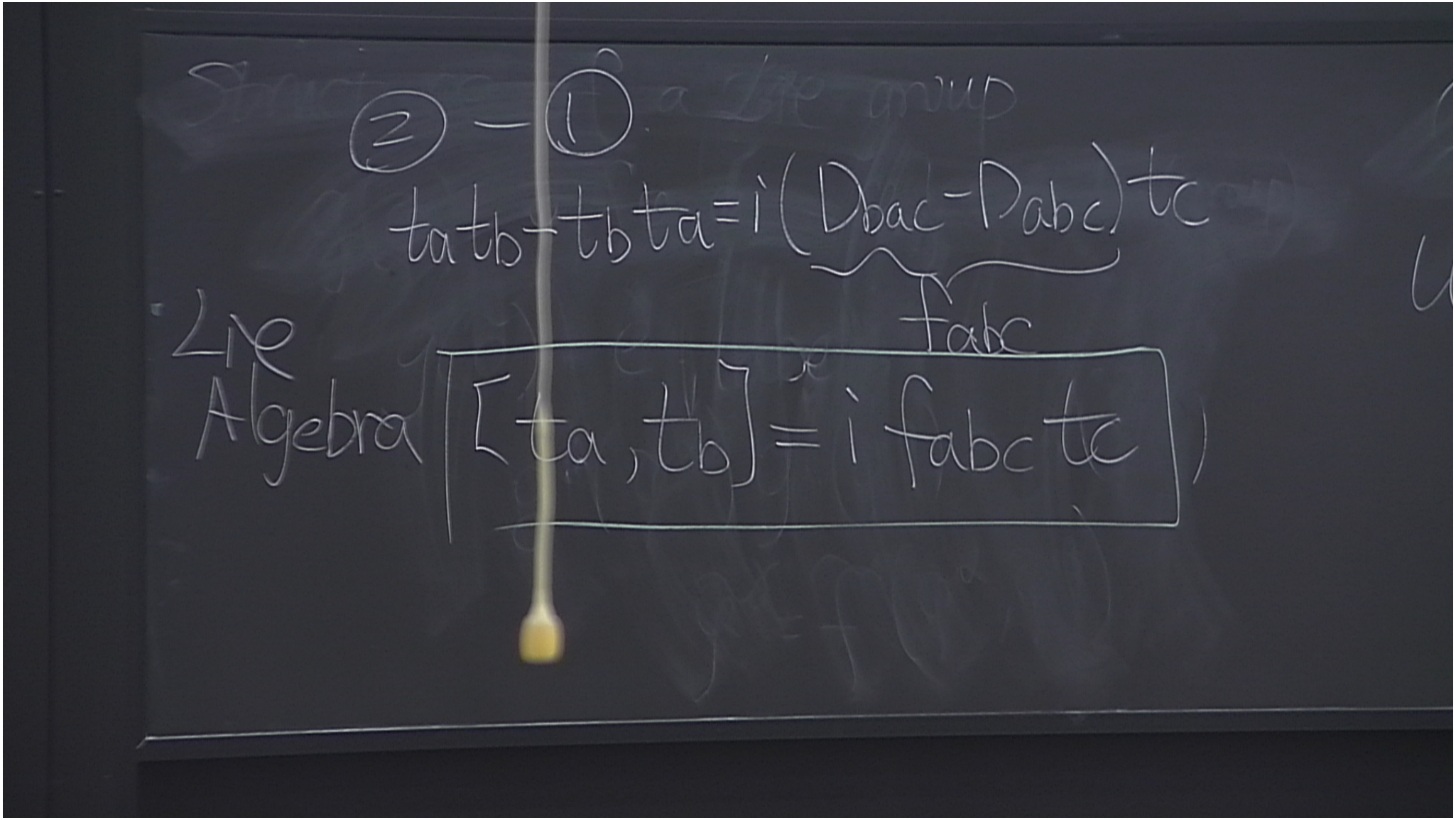

② − ①
$[t_a, t_b] = t_a t_b - t_b t_a = i(D_{bac} - D_{abc}) t_c$
Goal

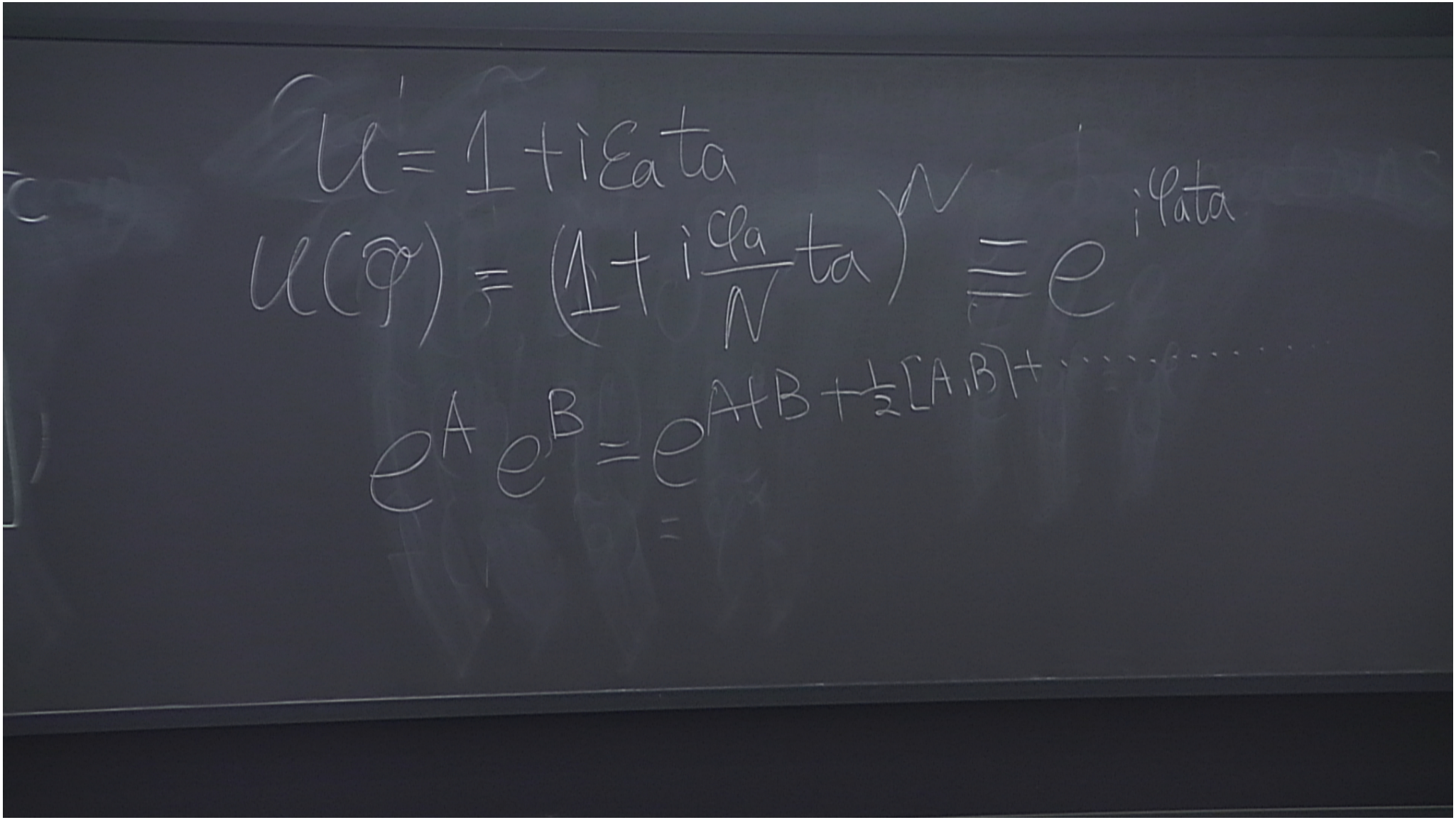

② − ①
$t_a t_b - t_b t_a = i(D_{bac} - D_{abc}) t_c$
f_{abc}
$[t_a, t_b] = i f_{abc} t_c$
Goal

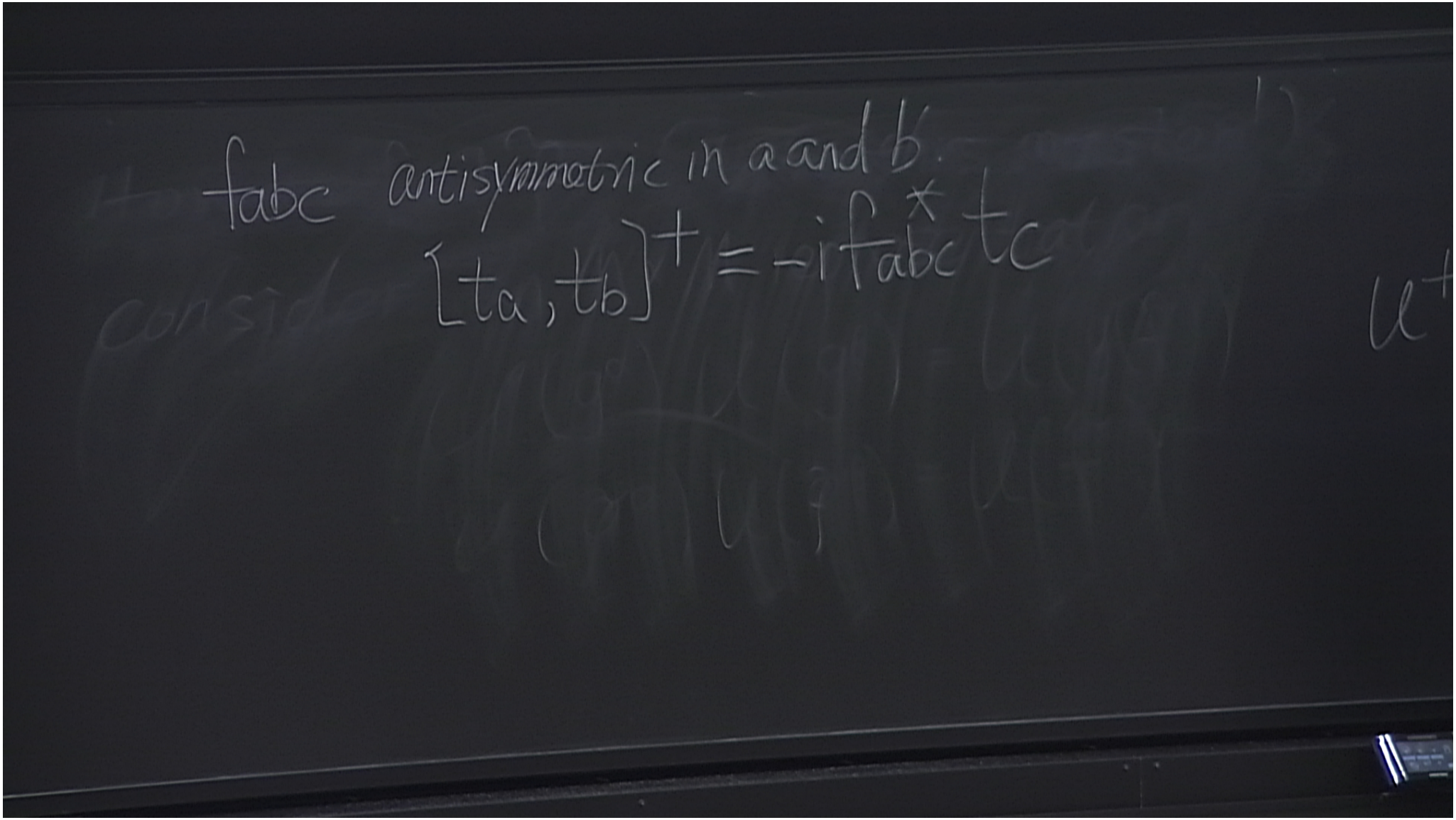

U = 1 + iεa t a
U(g)


$u = 1 + i\varepsilon a t_a$
$U(g) = \left(1 + i\frac{\alpha_a}{N} t_a\right)^N \equiv e^{i\alpha_a t_a}$


$u = 1 + i\varepsilon a t_a$
$u(\varphi) = \left(1 + i\frac{\varphi_a}{N} t_a\right)^N \equiv e^{i\varphi_a t_a}$
$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$


② − ①
$t_a t_b - t_b t_a = i(D_{bac} - D_{abc}) t_c$
f_{abc}
Lie Algebra
$[t_a, t_b] = i f_{abc} t_c$


$U = 1 + i\varepsilon_a t_a$
$U(\varphi) = \left(1 + i\frac{\varphi_a}{N} t_a\right)^N \equiv e^{i\varphi_a t_a}$
$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$


f^{abc} antisymmetric in a and b.
$[t^a, t^b]^\dagger = -i f^{abc} t^c$

f_{abc} antisymmetric in a and b.

$$[t_a, t_b]^\dagger = -i f_{abc}^* t_c$$

$$= \{t_a t_b - t_b t_a\}^\dagger$$

$$= t_b^\dagger t_a^\dagger - t_a^\dagger t_b^\dagger$$

$$=$$

f^{abc} antisymmetric in a and b.

$$[t^a, t^b]^\dagger = -i f^{abc} t^c$$

$$= \{t^a t^b - t^b t^a\}^\dagger$$

$$= t^{b\dagger} t^{a\dagger} - t^{a\dagger} t^{b\dagger}$$

$$= [t^b, t^a]$$

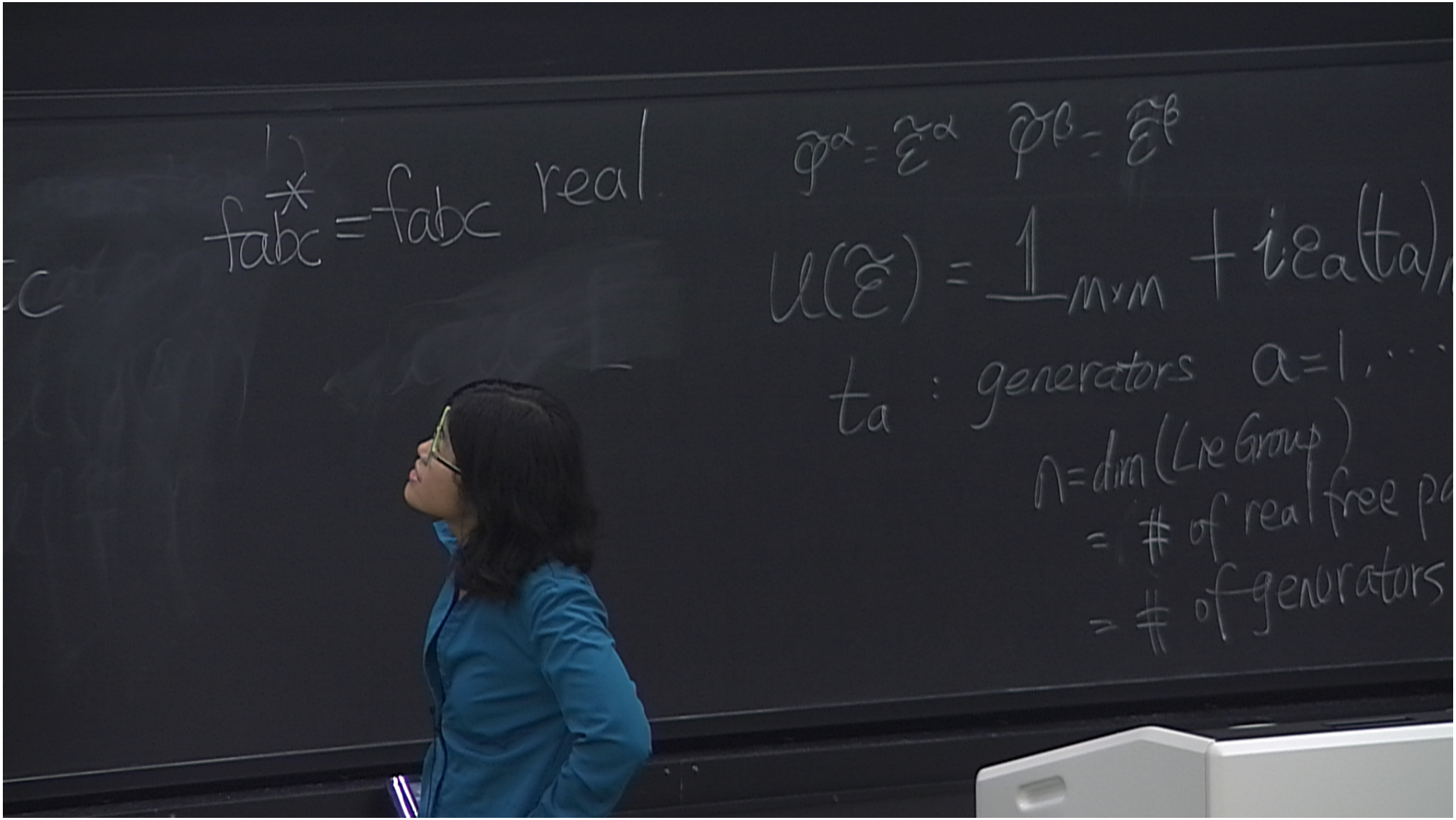
f^{abc} antisymmetric in a and b.

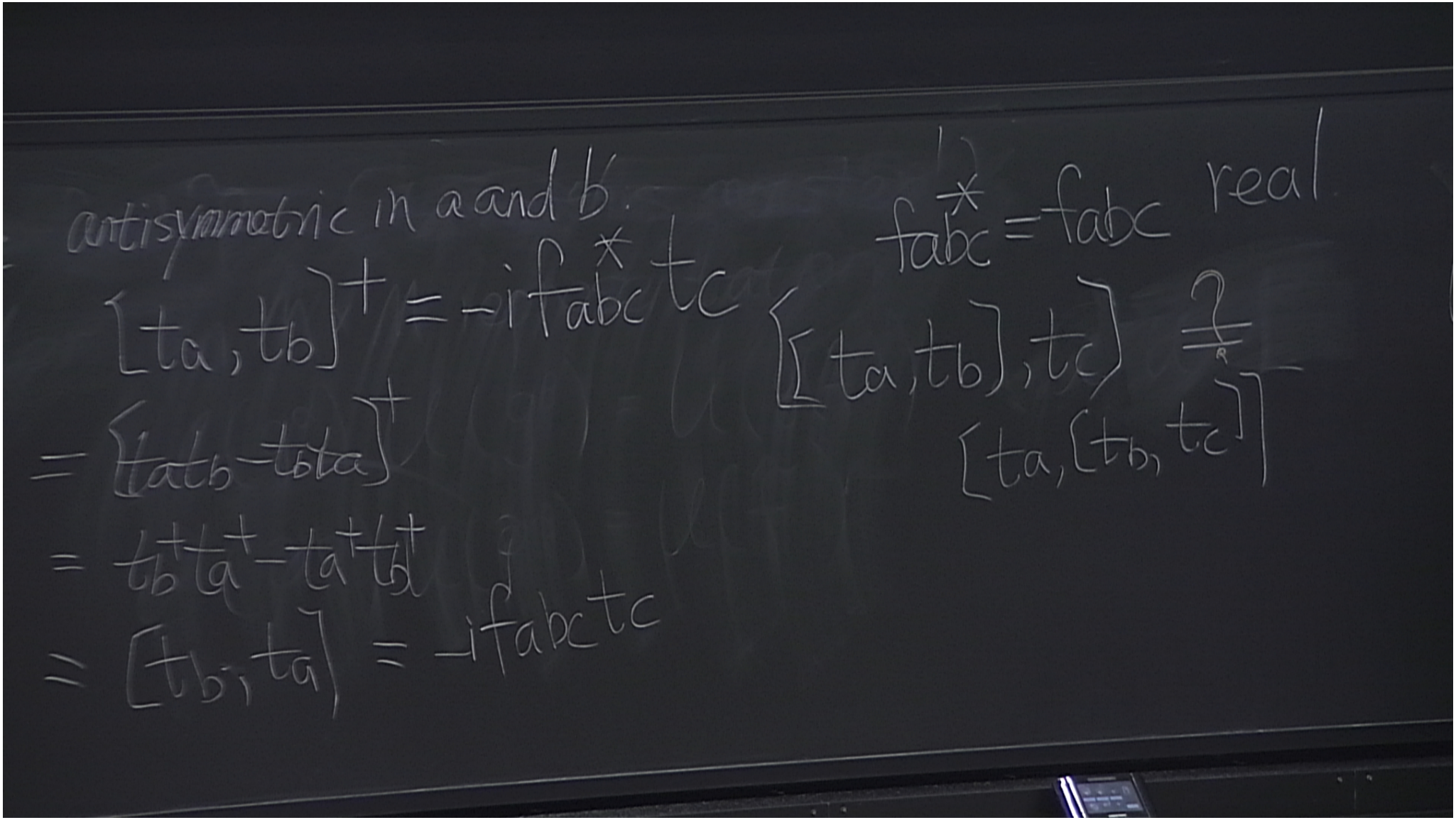
$$[t^a, t^b]^\dagger = -i f^{abc} t^c$$

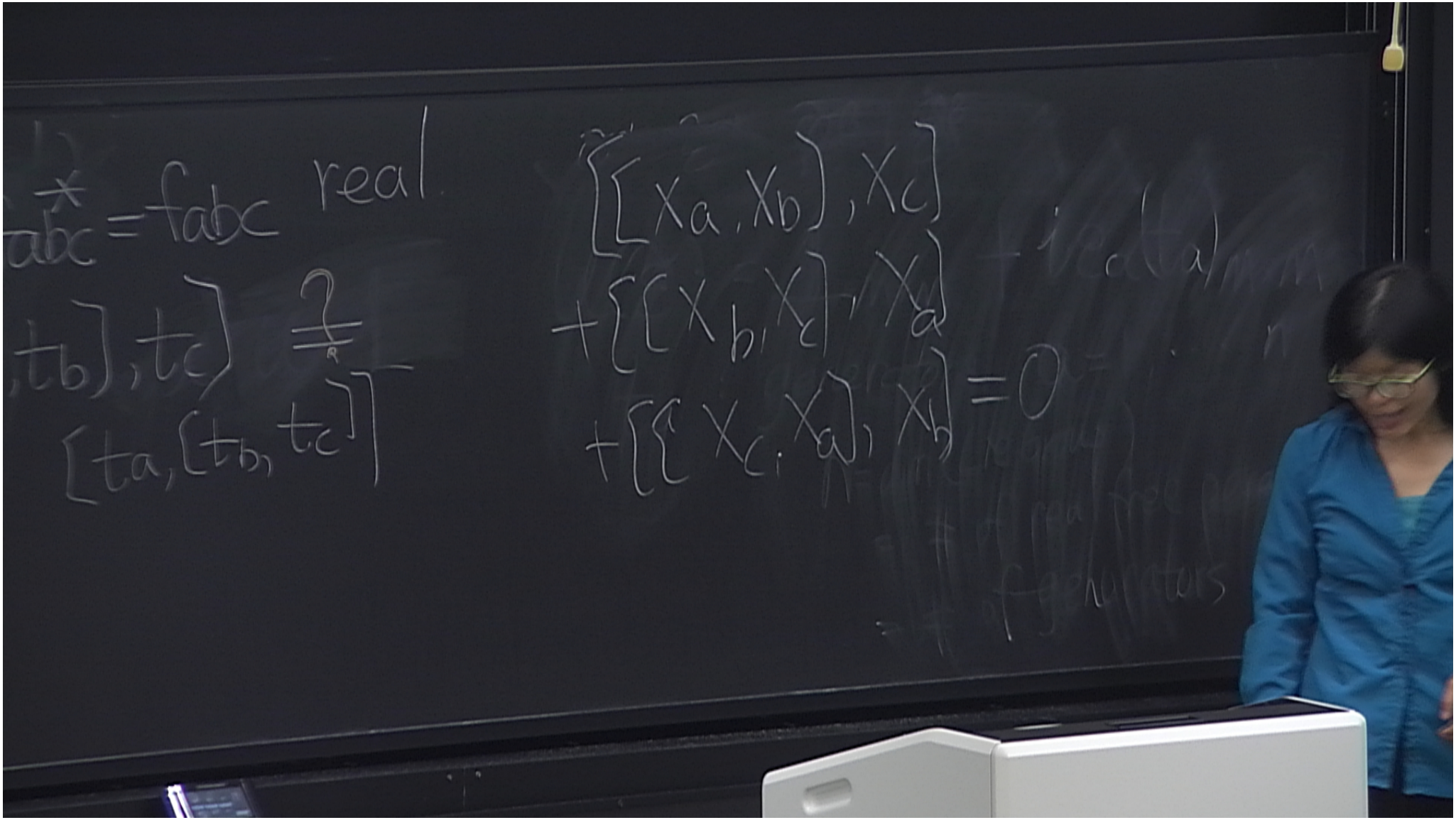
$$= \{t^a t^b - t^b t^a\}^\dagger$$

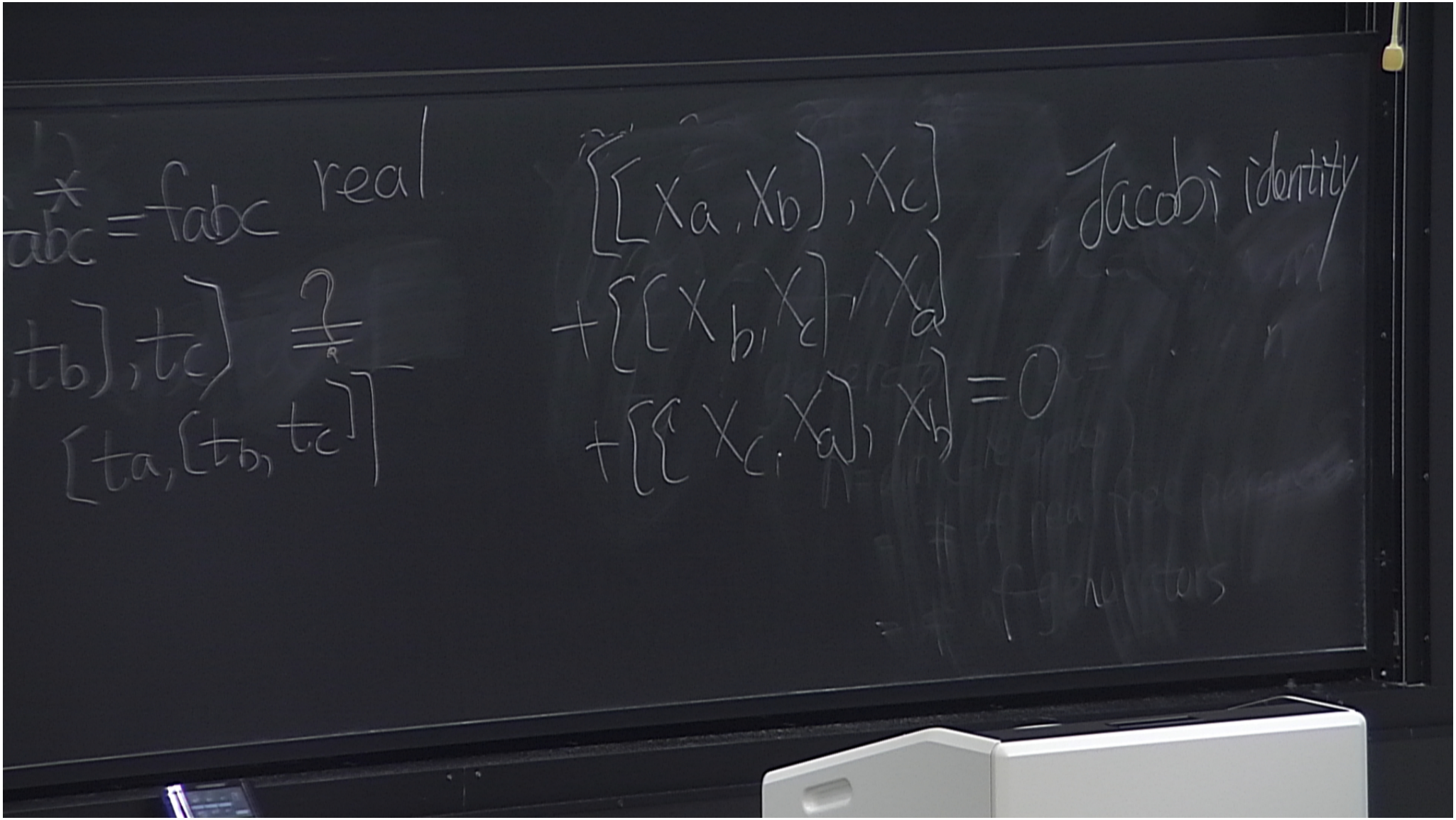
$$= t^{b\dagger} t^{a\dagger} - t^{a\dagger} t^{b\dagger}$$

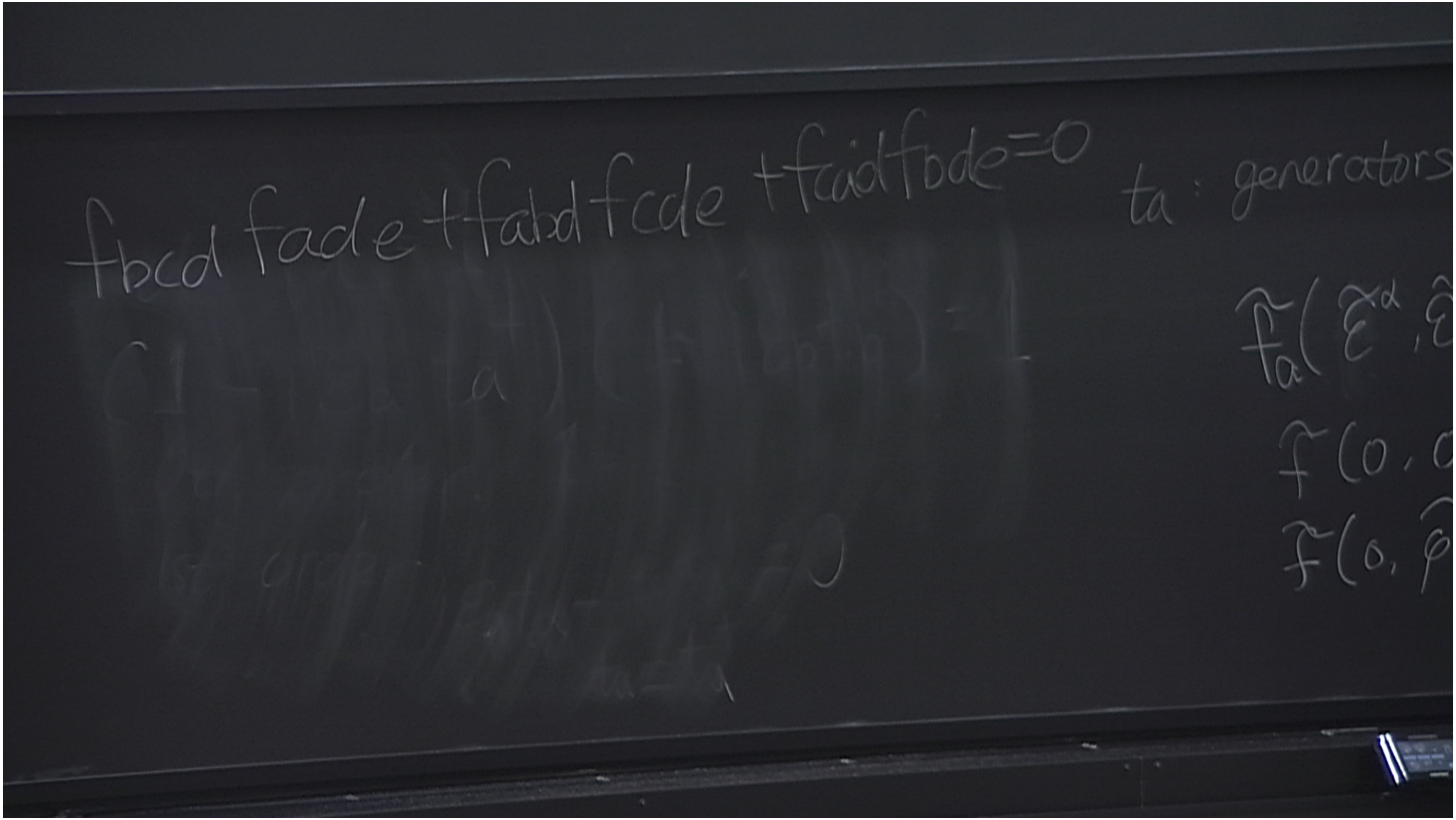
$$= [t^b, t^a] = -i f^{abc} t^c$$

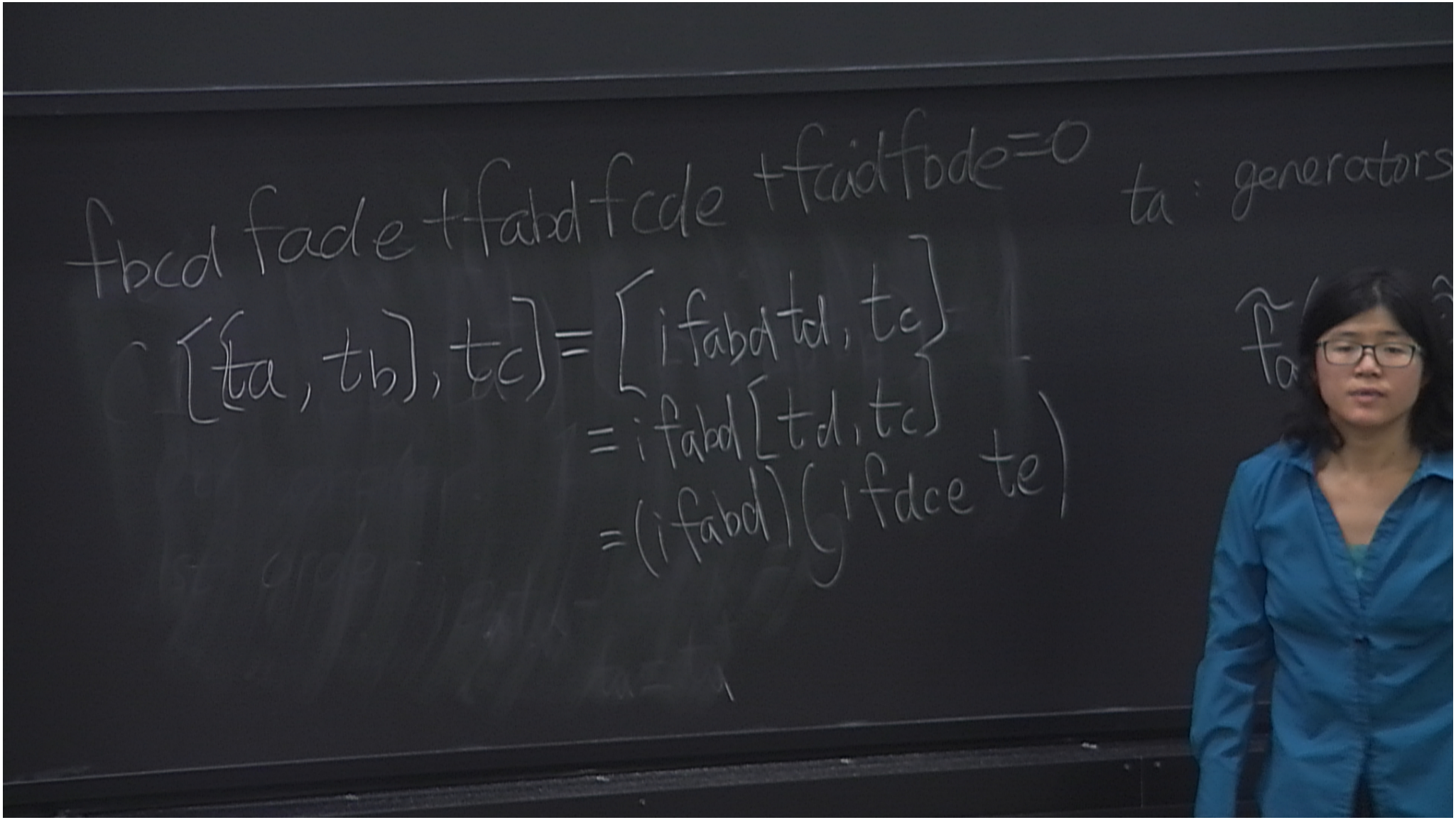

$f^{abc} = -f^{abc}$ real
$\varphi^\alpha = \varepsilon^\alpha$ $\varphi^\beta = \varepsilon^\beta$
$U(\varepsilon) = \mathbb{1}_{n\times n} + i\varepsilon_a (t_a)$
t_a : generators $a = 1, \cdots$
n = dim (Lie Group)
= # of real free pa
= # of generators

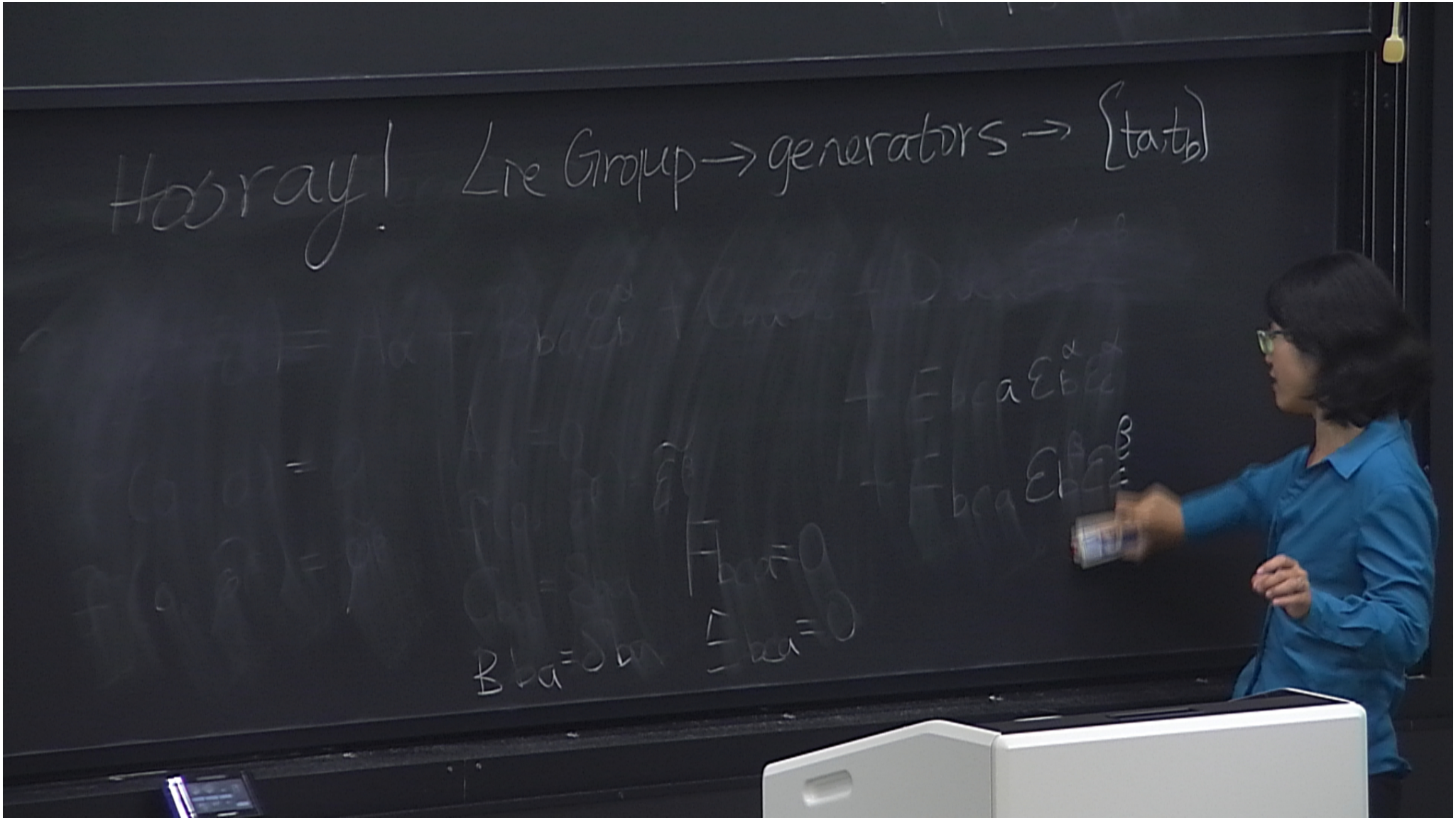

antisymmetric in a and b.
$[t_a, t_b]^\dagger = -i f^*_{abc} t_c$
$= [t_a t_b - t_b t_a]^\dagger$
$= t_b^\dagger t_a^\dagger - t_a^\dagger t_b^\dagger$
$= [t_b, t_a] = -i f_{abc} t_c$
$f^*_{abc} = f_{abc}$ real
$[[t_a, t_b], t_c] \overset{?}{=} [t_a, [t_b, t_c]]$

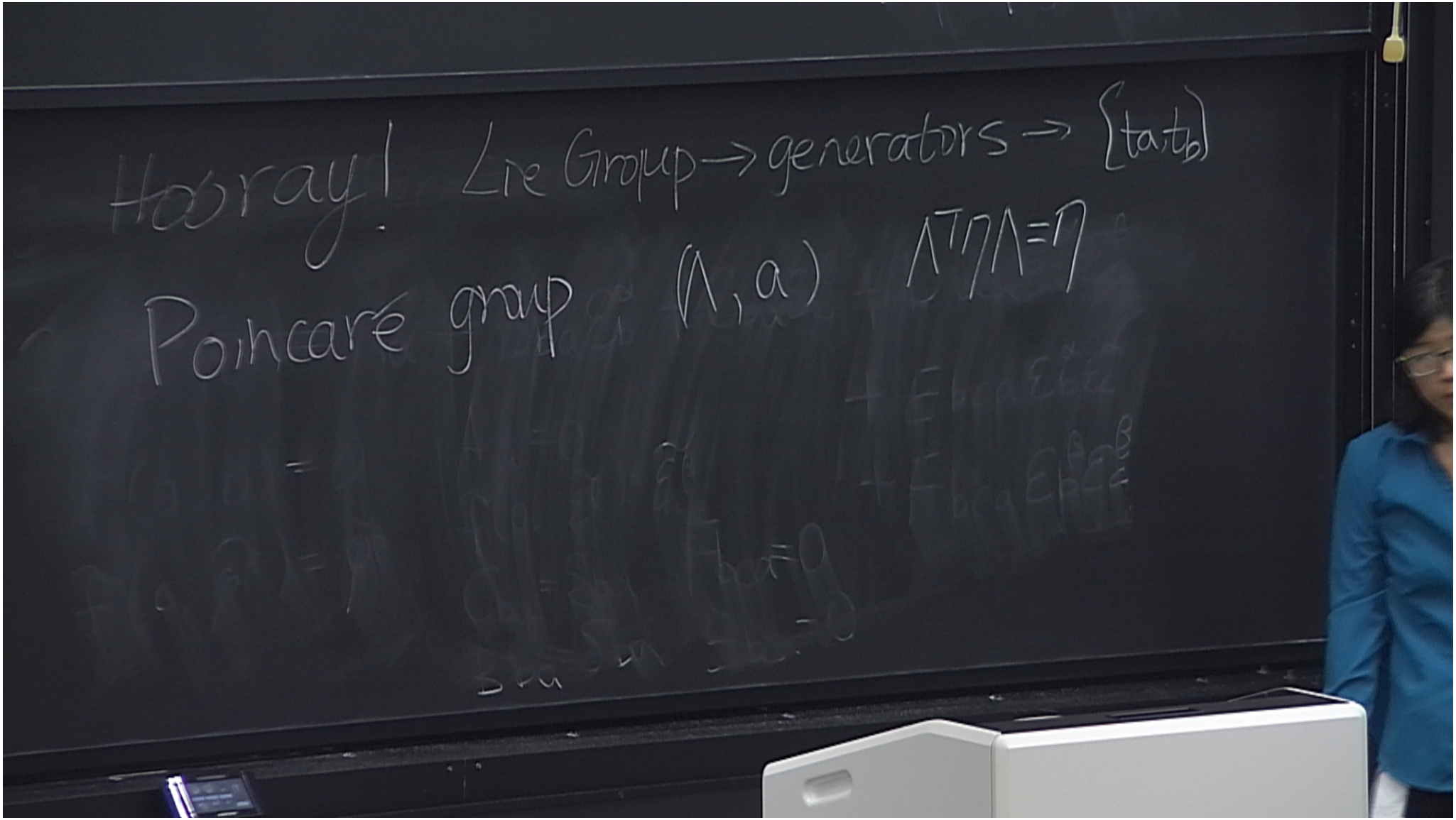

real
[[Xa, Xb], Xc]
+[[Xb, Xc], Xa]
+[[Xc, Xa], Xb] = 0

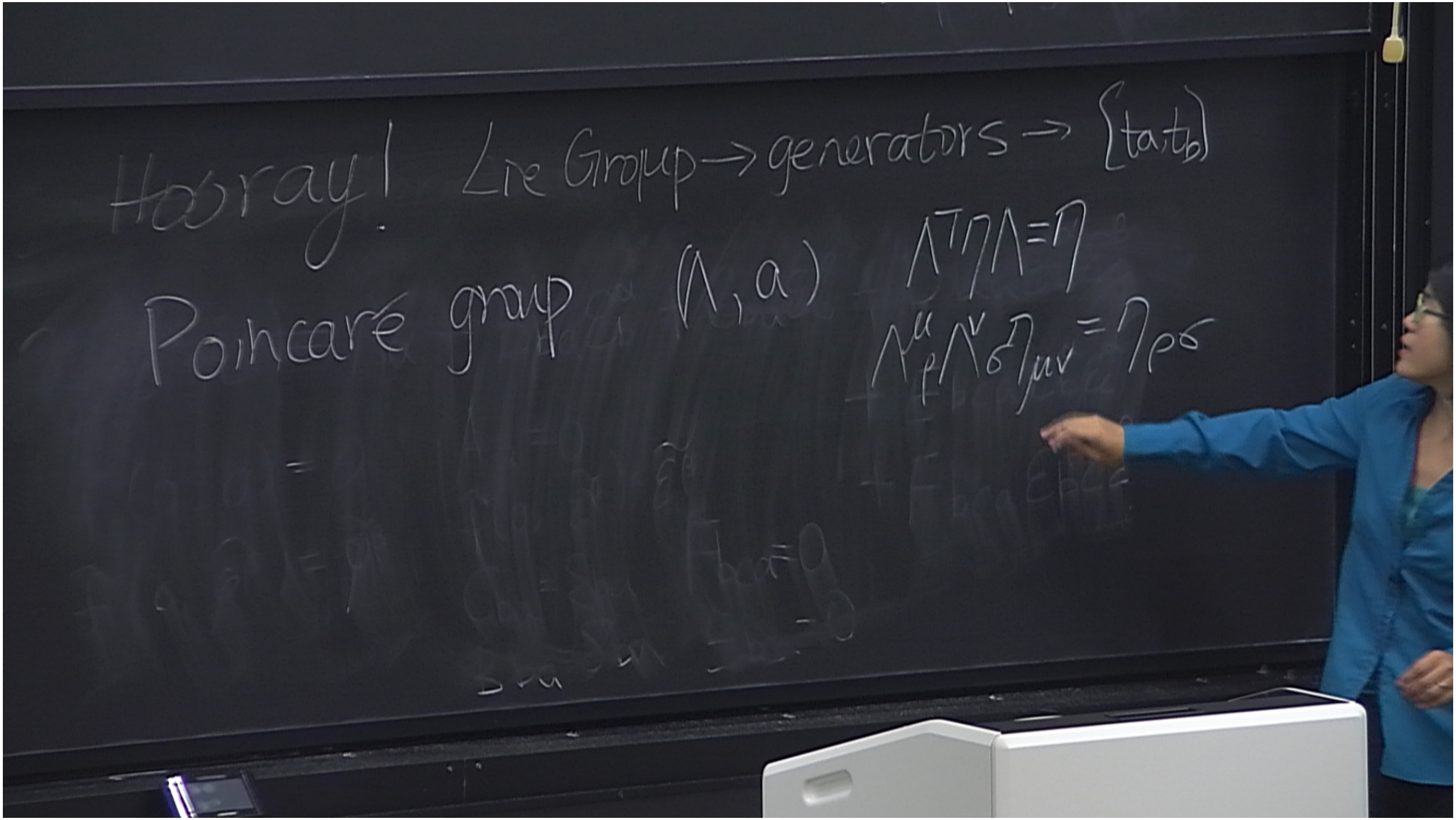

real
Jacobi identity

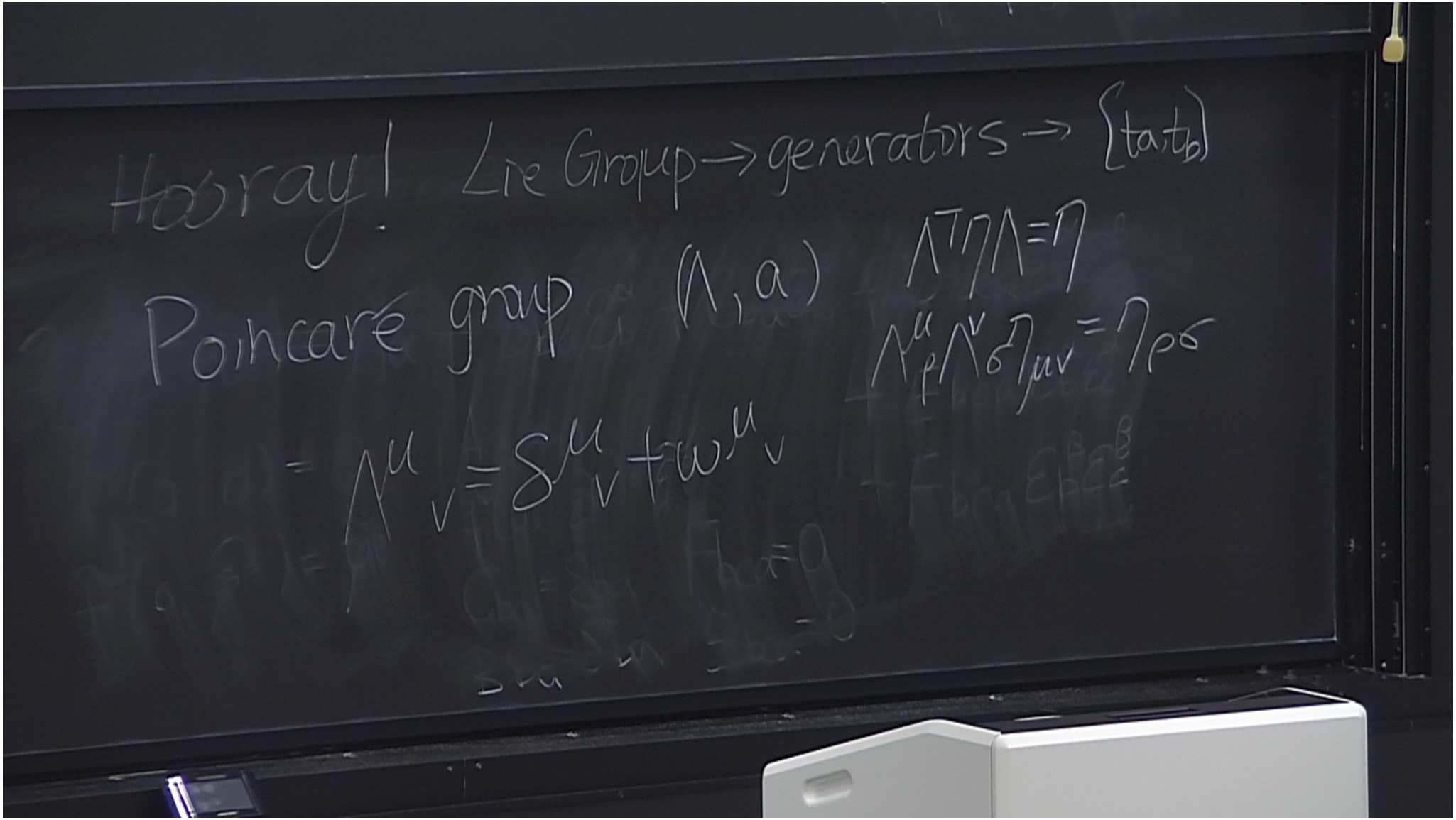

$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$
t_a : generators

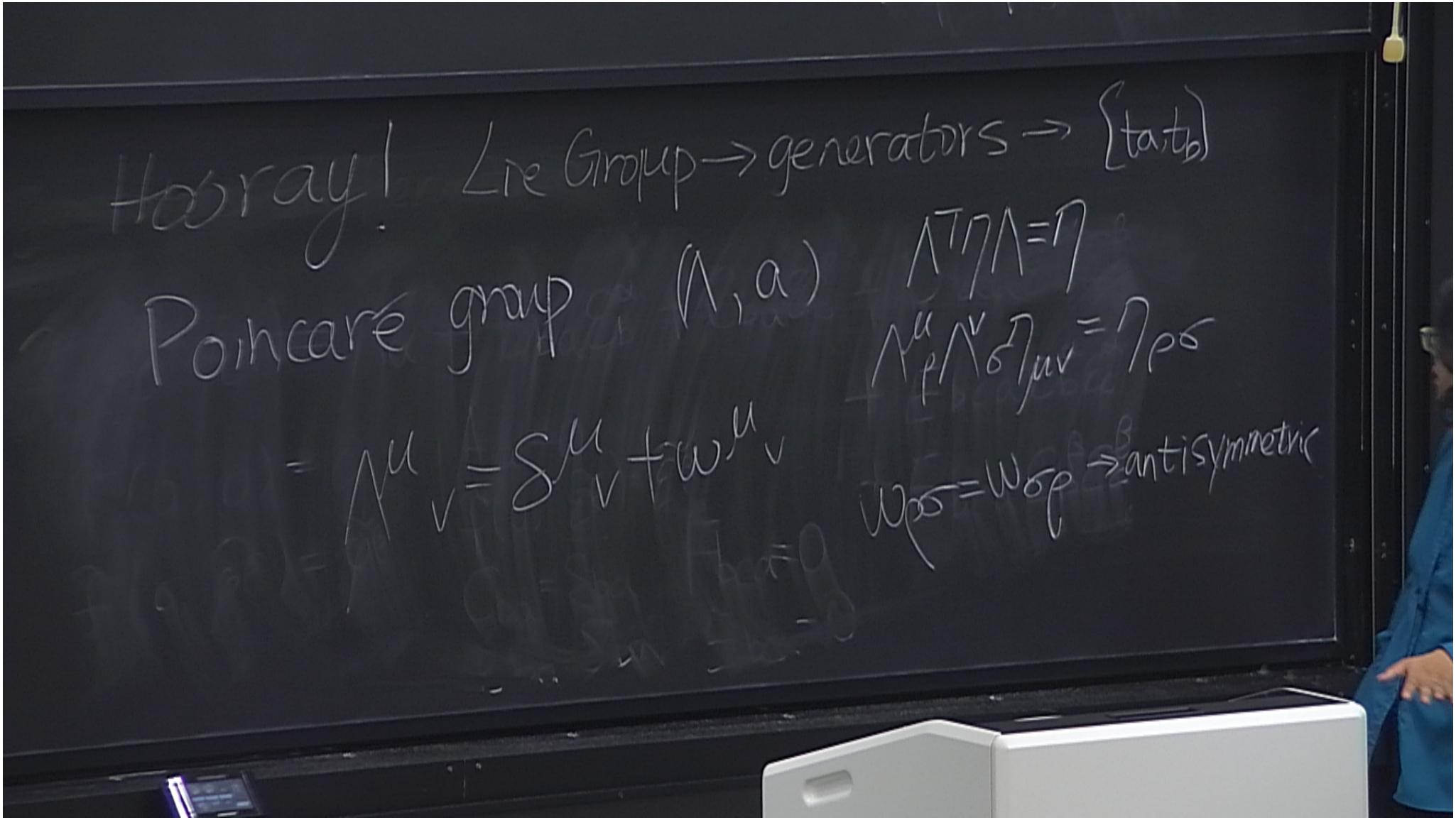

fbcd fade + fabd fce + fcid fode = 0
[[ta, tb], tc] = [i fabd td, tc]
= i fabd [td, tc]
= (i fabd)(i fdce te)
ta : generators

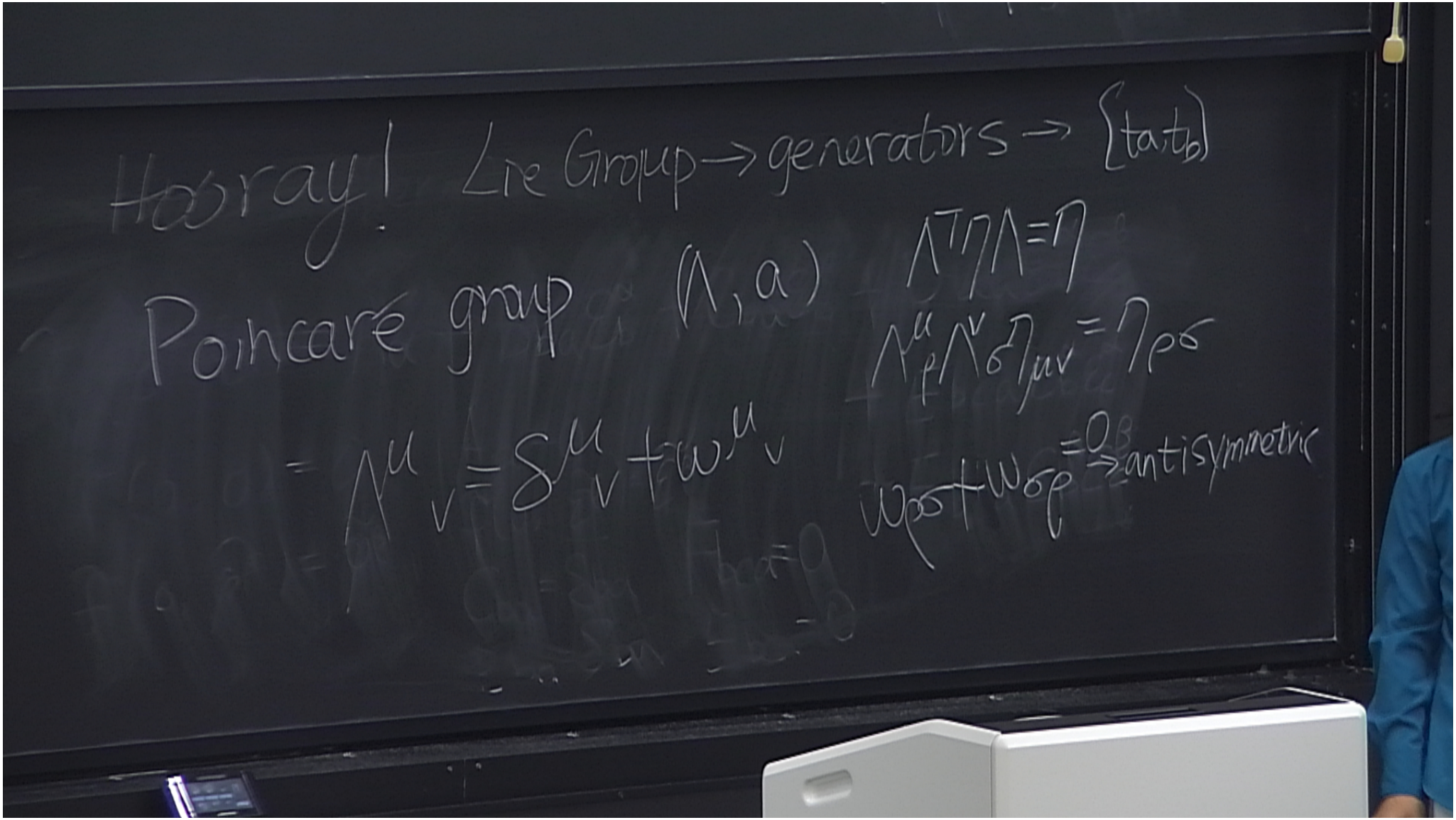

Hooray! Lie Group → generators → {ta, tb}

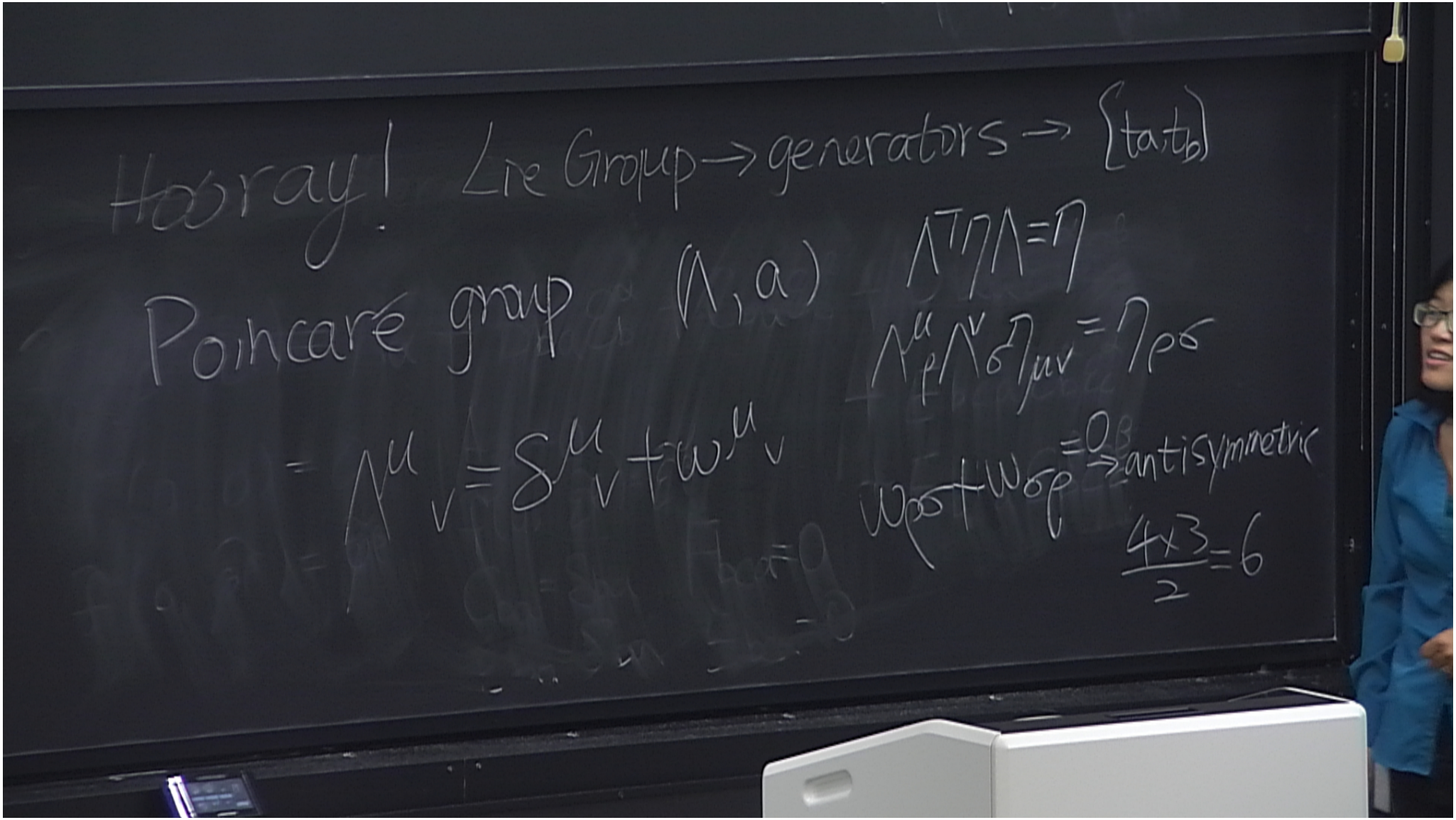

Hooray! Lie Group → generators → {ta, tb}
Poincare group (Λ, a) ΛᵀηΛ = η

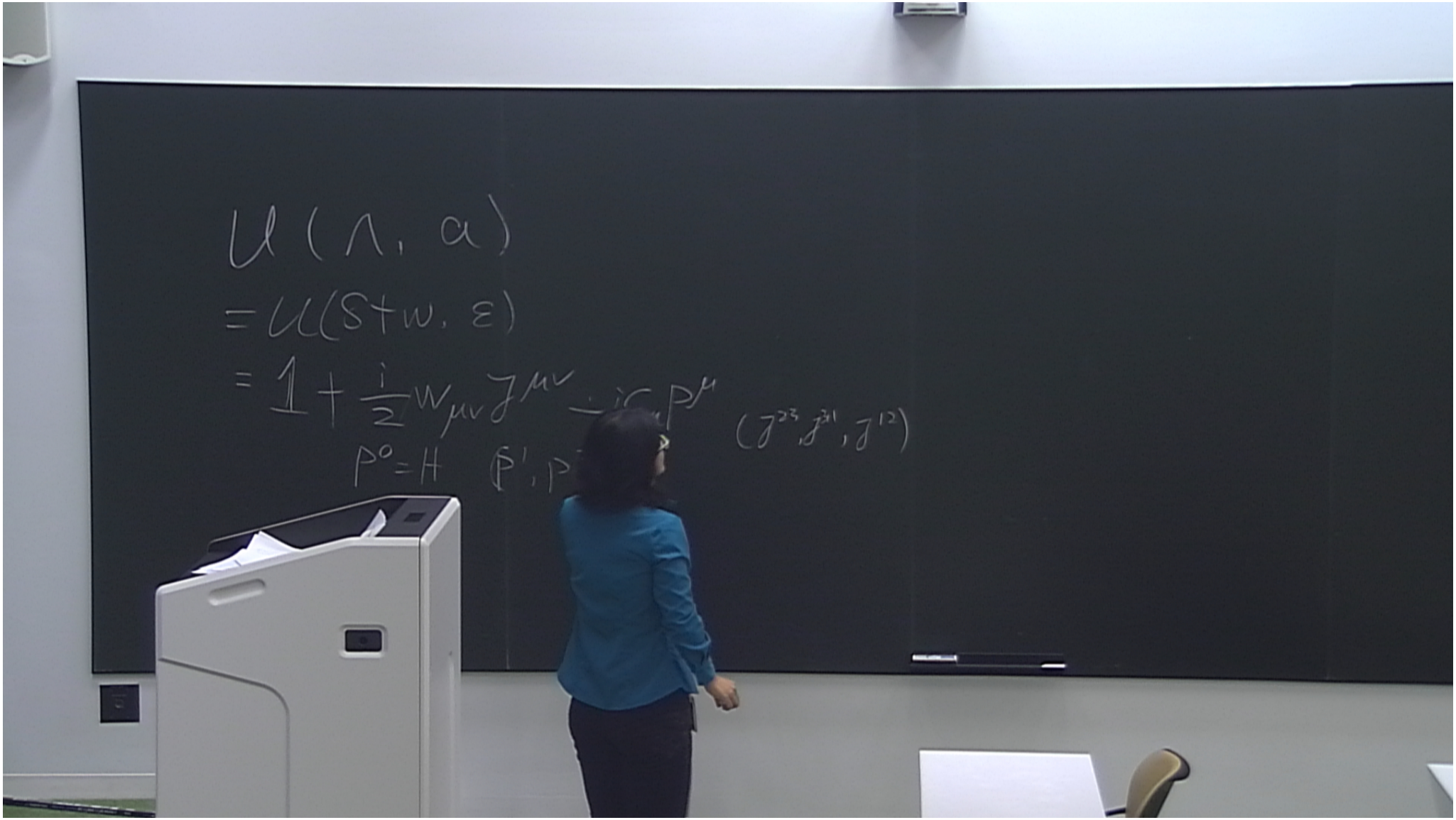

Hooray! Lie Group → generators → {ta, tb}
Poincare group (Λ, a)
$\Lambda^T \eta \Lambda = \eta$
$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$

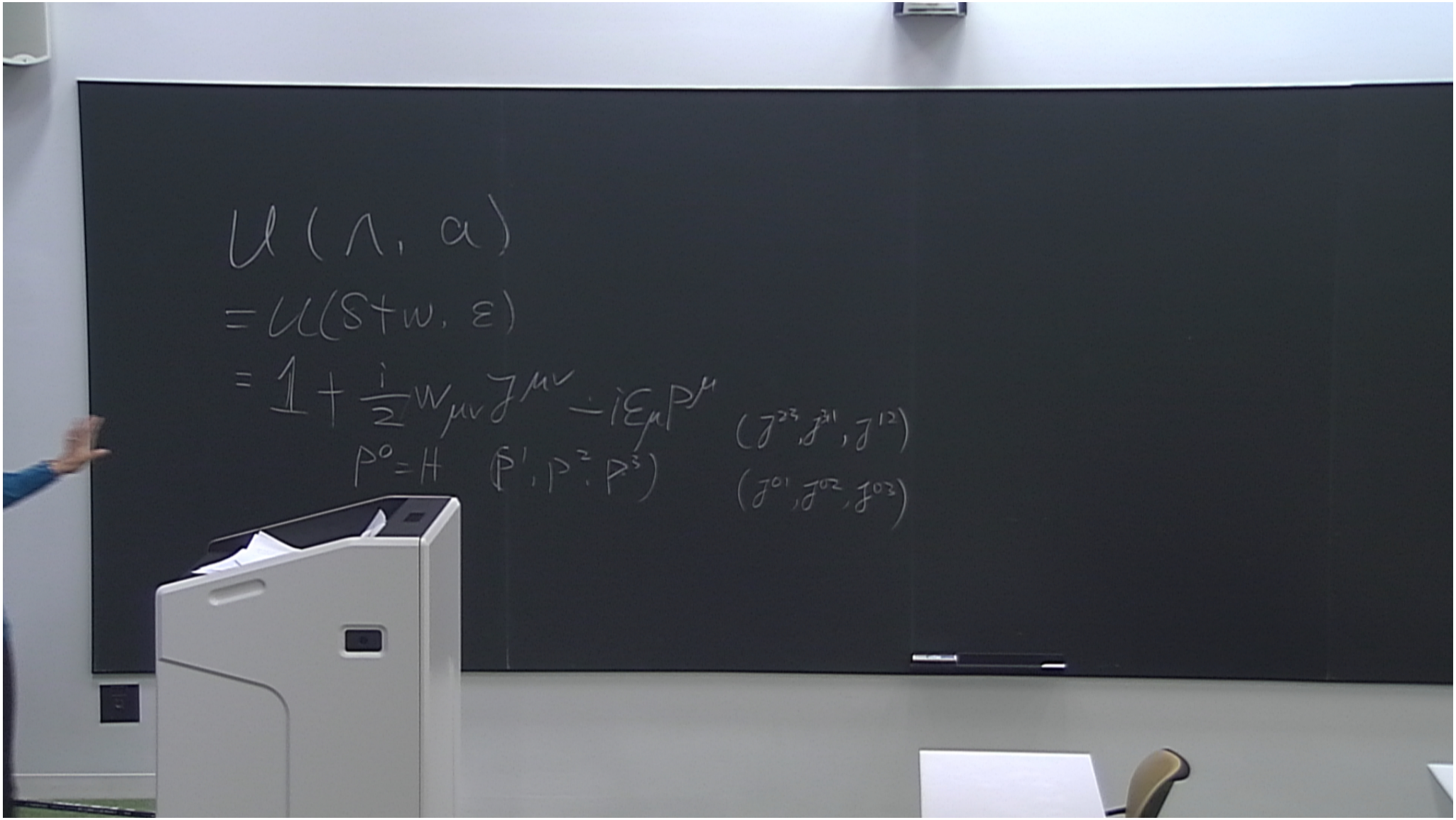

Hooray! Lie Group → generators → {ta, tb}
Poincare group (Λ, a)
Λ^T η Λ = η
Λ^μ_ρ Λ^ν_σ η_μν = η_ρσ
Λ^μ_ν = δ^μ_ν + ω^μ_ν

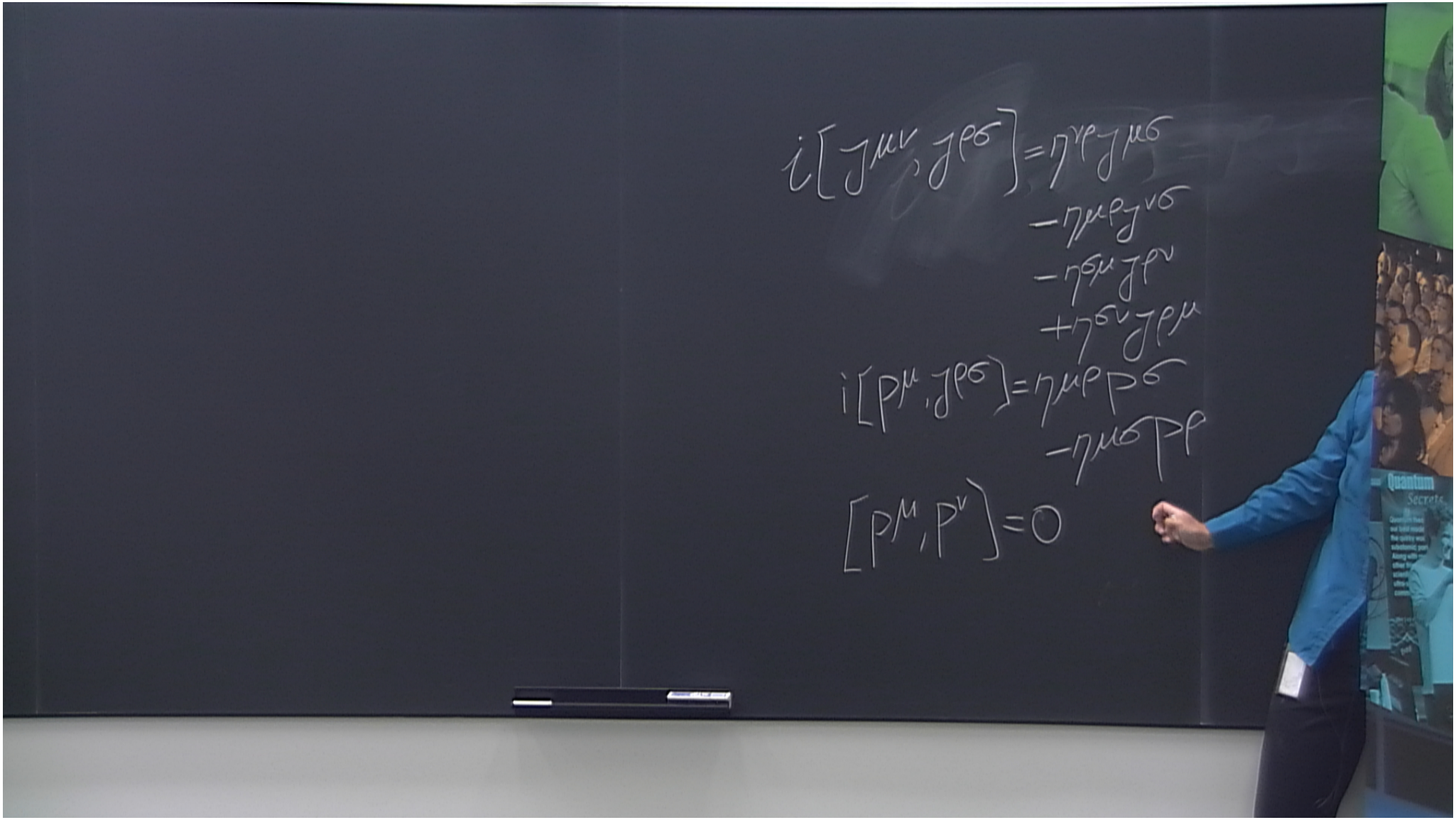

Hooray! Lie Group → generators → {ta, tb}
Poincare group (Λ, a)
Λ^μ_ν = δ^μ_ν + ω^μ_ν
ω_ρσ = ω_σρ ⇒ antisymmetric

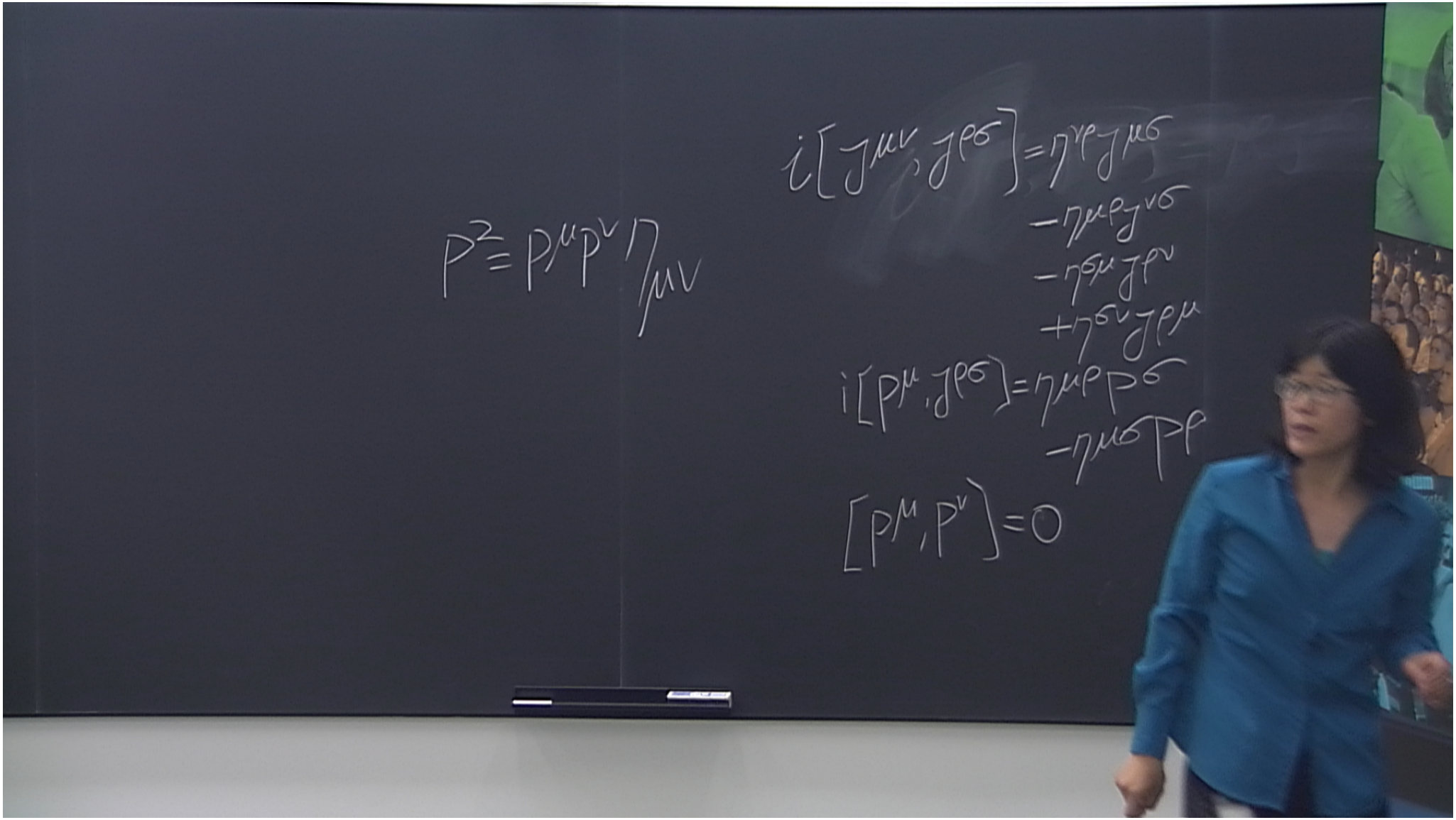

Hooray! Lie Group → generators → [ta, tb]
Poincare group (Λ, a)
Λ^μ_ν = δ^μ_ν + ω^μ_ν
Λ^T η Λ = η
Λ^μ_ρ Λ^ν_σ η_μν = η_ρσ
ω_ρσ + ω_σρ = 0
⇒ antisymmetric

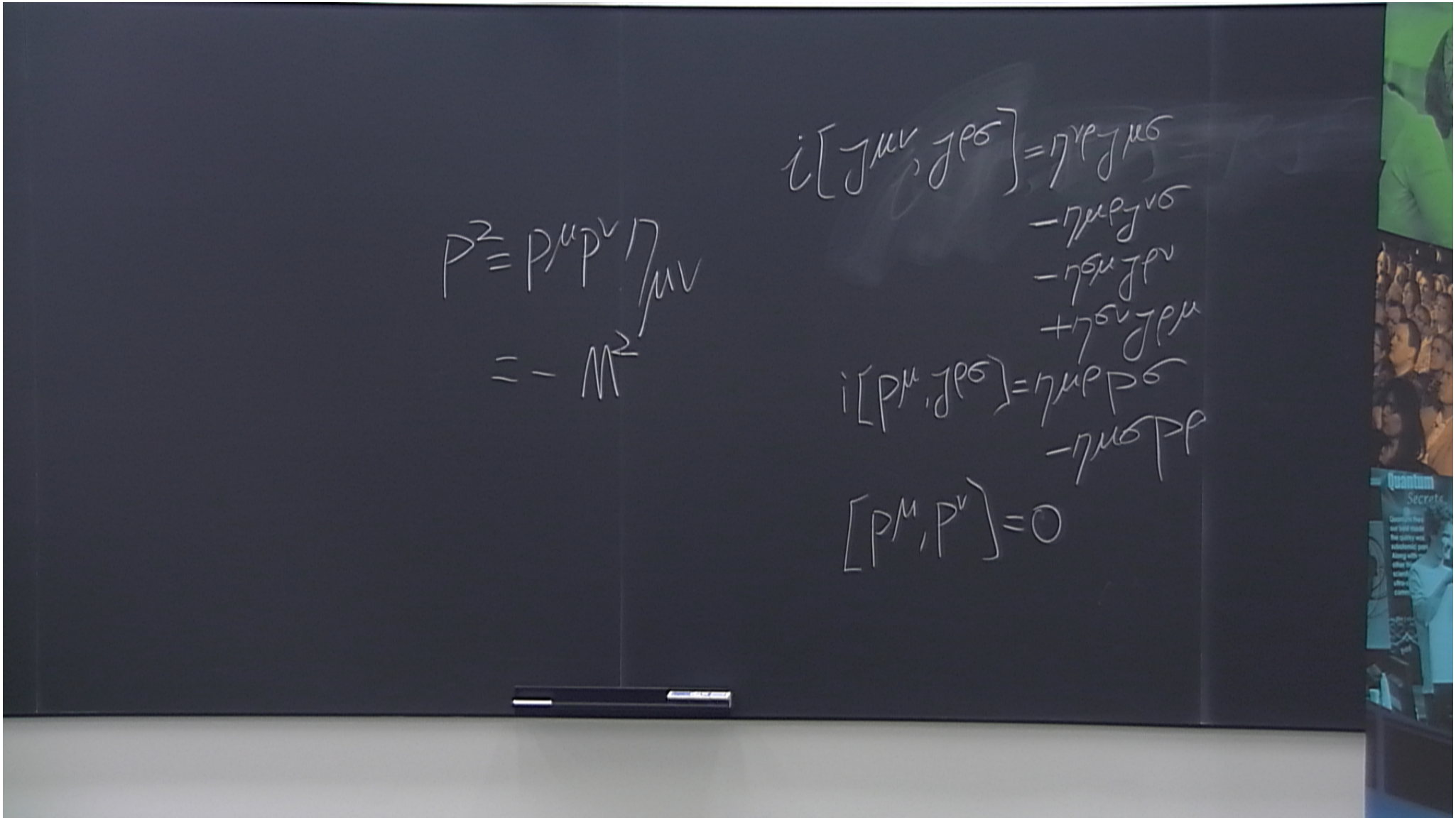

Hooray! Lie Group → generators → {ta, tb}
Poincare group (Λ, a)
$\Lambda^T \eta \Lambda = \eta$
$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$
$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$
$\omega_{\rho\sigma} + \omega_{\sigma\rho} = 0 \Rightarrow$ antisymmetric
$\frac{4 \times 3}{2} = 6$


U(Λ, a)
= U(Stw, ε)
= 1 + (i/2) w_μν J^μν − i ε_μ P^μ
P^0 = H
(J^23, J^31, J^12)


$U(\Lambda, a)$
$= U(Stw, \varepsilon)$
$= 1 + \frac{i}{2} w_{\mu\nu} J^{\mu\nu} - i\varepsilon_\mu P^\mu$
$P^0 = H \quad (P^1, P^2, P^3)$
(J^{23}, J^{31}, J^{12})
(J^{01}, J^{02}, J^{03})


$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma}$
$- \eta^{\mu\rho} J^{\nu\sigma}$
$- \eta^{\sigma\mu} J^{\rho\nu}$
$+ \eta^{\sigma\nu} J^{\rho\mu}$
$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma$
$- \eta^{\mu\sigma} P^\rho$
$[P^\mu, P^\nu] = 0$
Quantum
Secrets


$P^2 = P^\mu P^\nu \eta_{\mu\nu}$
$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma}$
$- \eta^{\mu\rho} J^{\nu\sigma}$
$- \eta^{\sigma\mu} J^{\rho\nu}$
$+ \eta^{\sigma\nu} J^{\rho\mu}$
$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma$
$- \eta^{\mu\sigma} P^\rho$
$[P^\mu, P^\nu] = 0$


$P^2 = P^\mu P^\nu \eta_{\mu\nu}$
$= -M^2$
$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma}$
$- \eta^{\mu\rho} J^{\nu\sigma}$
$- \eta^{\sigma\mu} J^{\rho\nu}$
$+ \eta^{\sigma\nu} J^{\rho\mu}$
$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma$
$- \eta^{\mu\sigma} P^\rho$
$[P^\mu, P^\nu] = 0$
Quantum
Secrets