Title: 14/15 PSI - Special Functions 2

Date: Aug 20, 2014 09:00 AM

URL: http://pirsa.org/14080021

Abstract:

Multiplication with a Function

$$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x)\, u(x)\, \varphi(x)\, dx \qquad \text{if } u \text{ is a function}$$

$$= \int_{-\infty}^{\infty} u(x) \big(\psi(x)\, \varphi(x)\big)\, dx$$

$$\boxed{(\psi u)(\varphi) \equiv u(\psi\, \varphi)}$$

sufficient ψ to be a test function

Composition with a function

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\, \varphi(x)\, dx$$

$(g \circ f)(x) = g(f(x))$ $\qquad y = f(x) \qquad x = g(y)$

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(y)\, \underbrace{\varphi(g(y))\, |g'(y)|}_{\text{testfunction}}\, dy$$

$u \circ f$ is guaranteed to exist if

- f is C^∞
- $y = f(x)$ has a unique solution $x = g(y)$
- $g'(y)$ does not change sign

Test function

For specific distributions we can drop some restrictions

$$(\delta \circ f)(\varphi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\varphi)$$

roots of f

works only when all roots of f are simple roots

Application: Functional Derivatives

Example: $$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}]$$

$$= \int dt \left(\frac{1}{2} \dot{q}^2 - V(q) \right)$$

$$\delta S = S[q + \delta q, \dot{q} + \delta \dot{q}] - S[q, \dot{q}]$$

$$= \int dt\, \delta q \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] + \mathcal{O}(\delta q^2)$$

$$= \int dt\, \delta q \, \frac{\delta S}{\delta q(t)}$$

functional derivative

$$\delta S = \sum_i \frac{\partial S}{\partial q_i}\,\delta q_i = \int dt\, \frac{\delta S}{\delta q(t)}\,\delta q(t)$$

$$\delta q = \epsilon\, \phi$$

test function

$$\int \frac{\delta F[f]}{\delta f(x)}\,\phi(x)\,dx = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}$$

Example: $F[f] = f^2(x_0)$

$$\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\epsilon \to 0} \frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\phi(x_0) + \epsilon^2 \phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

$$= 2 f(x_0)\phi(x_0)$$

$$= \int \underbrace{2 f(x_0)\, \delta(x - x_0)} \phi(x)\, dx$$

$$\frac{\delta F}{\delta f(x)} = 2 f(x_0)\, \delta(x - x_0) = 2 f(x)\, \delta(x - x_0)$$

is a function

nction

$$\delta_a(\varphi) \equiv \varphi(a)$$

$$\begin{aligned}\psi\,\delta_a(\varphi) &= \delta_a(\psi\varphi)\\ &= (\psi\varphi)(a)\\ &= \psi(a)\,\varphi(a)\\ &= \psi(a)\,\delta_a(\varphi)\end{aligned}$$

$$x\delta = 0$$

$f(x)=x \quad |x|<c$

$$\begin{aligned}x\,\delta'(\varphi) &= -\delta((x\varphi)')\\ &= -\delta(\varphi) - x\,\delta(\varphi')\\ &= -\delta(\varphi)\end{aligned}$$

$) - f^2(x_0)$

Properties:

Linearity: $$\frac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \frac{\delta F}{\delta f(x)} + \mu \frac{\delta G}{\delta f(x)}$$

Product Rule: $$\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F}{\delta f(x)} G + F \frac{\delta G}{\delta f(x)}$$

Chain Rules: $$\frac{\delta F[g(f)]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta g(f(x))} \frac{dg(f(x))}{df(x)}$$

$$\frac{\delta g[F[f]]}{\delta f(x)} = \frac{dg(F[f])}{dF[f]} \frac{\delta F[f]}{\delta f(x)}$$

Warnings: $u(\varphi)$ and $u \cdot \varphi$ only defined if ϕ is a test function

$u \circ \varphi$ difficult to define

linear theory

$f_n(x) \to f(x)$ and $f_n \to f$ not the same

Gamma - generalization of factorial

$$(n+1)! = (n+1)\,n!$$

$$0! = 1$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \mathrm{Re}(z) > 0$$

$$\frac{d}{dt}\left(t^z e^{-t}\right) = z\, t^{z-1} e^{-t} - t^z e^{-t}$$

$$t^z e^{-t}\Big|_{t=0}^{\infty} = z\int_0^\infty t^{z-1} e^{-t}\, dt - \int_0^\infty t^z e^{-t}\, dt$$

Composition with a function

$u \circ f$ is guaranteed t

$$t^{z} e^{-t}\Big|_{t=0}^{\infty} = z\int_0^\infty t^{z-1} e^{-t}\, dt - \int_0^\infty t^{z} e^{-t}\, dt$$

$$0 = z\,\Gamma(z) - \Gamma(z+1)$$

functional equation

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n!$$

Ex

Extend to $\mathrm{Re}(z) < 0$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \qquad \mathrm{Re}\, z+n > 0$$

Let $z = -n + \epsilon + 1$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n)} \underset{\epsilon\to 0}{=} \frac{(-1)^n}{\epsilon\, n!}$$

simple poles at $z=-n$ with residue $\frac{(-1)^n}{n!}$

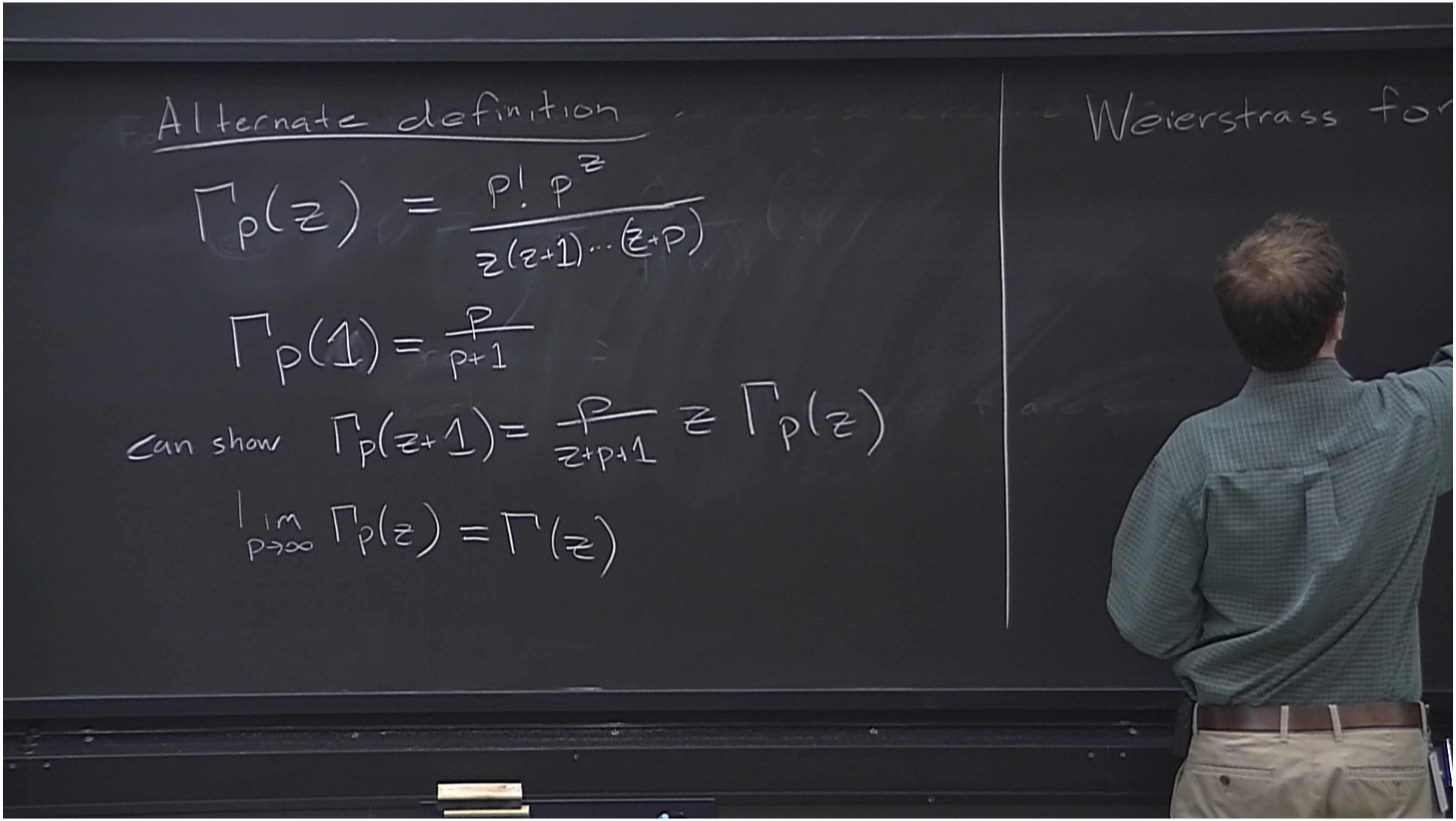
Extend to $\mathrm{Re}(z) < 0$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \qquad \mathrm{Re}\, z + n > 0$$

Let $z = -n + \epsilon + 1$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n)} \underset{\epsilon \to 0}{=} \frac{(-1)^n}{\epsilon\, n!}$$

simple poles at $z = -n$ with residue $\frac{(-1)^n}{n!}$


Alternate definition
$\Gamma_p(z) = \frac{p!\, p^z}{z(z+1)\cdots(z+p)}$
$\Gamma_p(1) = \frac{p}{p+1}$
can show $\Gamma_p(z+1) = \frac{p}{z+p+1} z \Gamma_p(z)$
$\lim_{p\to\infty} \Gamma_p(z) = \Gamma(z)$
Weierstrass for

Weierstrass formula

$$\Gamma_p(z) = \frac{p^z}{z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{p}\right)}$$

$$= \frac{e^{z(\log p - 1 - \frac{1}{2} - \cdots - 1/p)}\, e^{z + z/2 + \cdots z/p}}{z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{p}\right)}$$

$$\gamma = \lim_{p\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{p} - \log p\right) \approx 0.577$$

$$\boxed{\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}}$$

we found only poles
no zeros

Duplication Formula:

$$\Gamma(z)\,\Gamma\!\left(z+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}}\,\Gamma(2z)$$

Complement Formula:

$$\frac{1}{\Gamma(z)}\,\frac{1}{\Gamma(-z)} = -z^2 \prod_{p=1}^{\infty}\left(1-\frac{z^2}{p^2}\right)$$

$$-z\,\Gamma(-z) = \Gamma(1-z)$$

$$\frac{1}{\Gamma(z)}\,\frac{\cancel{-z}}{\Gamma(1-z)} = +z^2 \prod_{p=1}^{\infty}\left(1-\frac{z^2}{p^2}\right)$$

$$= \frac{\sin(\pi z)}{\pi}$$

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

$$\boxed{\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}}$$

we found only poles
no zeros

Duplication Formula:

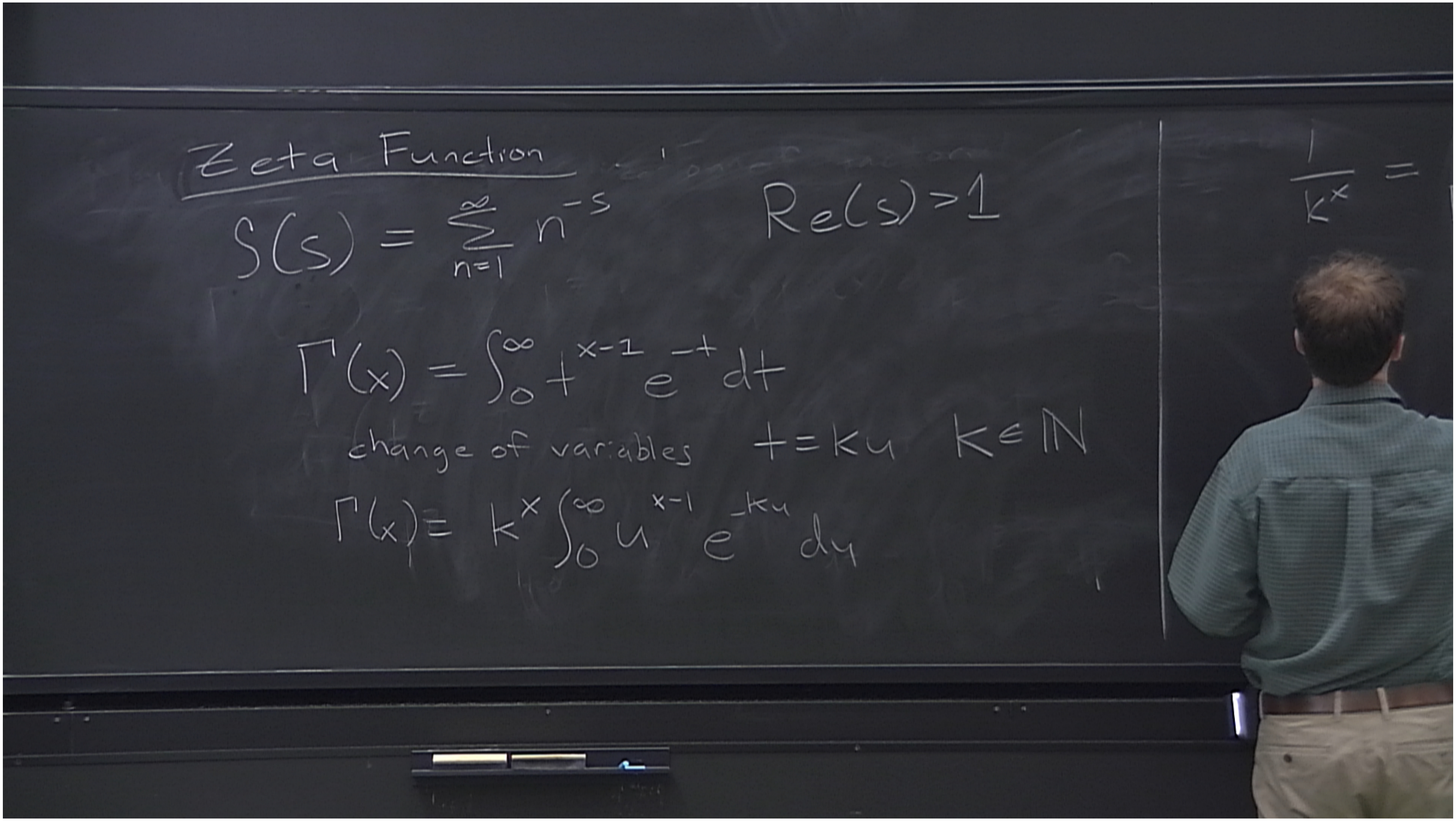
$$\boxed{\Gamma(z)\,\Gamma\!\left(z+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}}\,\Gamma(2z)}$$

Compl

$$\frac{1}{\Gamma(z)}$$

$$\frac{1}{\Gamma(z)}$$

$$\Gamma($$


Zeta Function
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$
$Re(s) > 1$
$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$
change of variables $t = ku$ $k \in \mathbb{N}$
$\Gamma(x) = k^x \int_0^\infty u^{x-1} e^{-ku} du$
$\frac{1}{k^x} =$

$$\frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-ku}\, du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \sum_{k=1}^{\infty} e^{-ku}\, du$$

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \frac{e^{-u}}{1-e^{-u}}\, du$$

$$\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^u - 1}\, du$$

Functional Equation

$$\Lambda(s) = \pi^{-s/2}\,\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

Λ analytic except at 0 and 1

$\zeta(s)$ has a pole at $s=1$ with residue 1

$\zeta(s)$ has

$\zeta(s)$ has zeros at $s = -2, -4, -6$

$\zeta(-1)$ if we know $\zeta(2)$