

Title: 14/15 PSI - Special Functions 1

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Abstract:

Distributions + Special Functions

- Distributions
- Gamma and Zeta

Distributions + Special Functions

- Distributions
- Gamma and Zeta
- Orthogonal Functions
- Asymptotic Series + Approximating Integrals

Distr

Distributions

Motivation: point charge

$$\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$$

$$\int \rho(\vec{x}) d^3\vec{x} = q$$

not a function no well defined value at $\vec{x} = 0$

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$$\delta(x) \text{ "}" f(x) = \lim_{\sigma \rightarrow 0} f_{\sigma}(x)$$

$$f_{\sigma}(x) = \frac{1}{\sigma} \quad |x| < \sigma$$
$$= 0 \quad |x| \geq \sigma$$

Most important example: Dirac delta "function"

$$\delta(x) \text{ "}" f(x) = \lim_{\sigma \rightarrow 0} f_{\sigma}(x)$$

$$f_{\sigma}(x) = \frac{1}{\sigma} \quad |x| < \frac{\sigma}{2}$$
$$= 0 \quad |x| \geq \frac{\sigma}{2}$$

$$\int_{-\infty}^{\infty} f_{\sigma}(x) dx = 1$$

Definitions: Function space: vector space whose elements

vector space whose elements are
and has a norm which satisfies

positivity $\|f\| \geq 0$ and $\|f\| = 0 \Leftrightarrow f = 0$

function

vector space whose elements are
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positivity $\|f\| \geq 0$ and $\|f\| = 0 \Leftrightarrow f = 0$

triangle inequality $\|f+g\| \leq \|f\| + \|g\|$

linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$

inner product space - norm comes from an inner product

$$\langle f, g \rangle = \int_a^b f^* g \, dx$$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$$L^2[a, b]$$

↑
space of square integrable
functions on (a, b)

duct

In $L^2[-\infty, \infty]$

$$f_1(x) = 0 \quad \|f_1\| = 0$$

$$f_2(x) = \begin{cases} 0 & x \neq 1 \\ \frac{1}{\sqrt{2}} & x = 1 \end{cases} \quad \|f_2\| = 0$$



$$g_\sigma \rightarrow g = \delta$$

for which

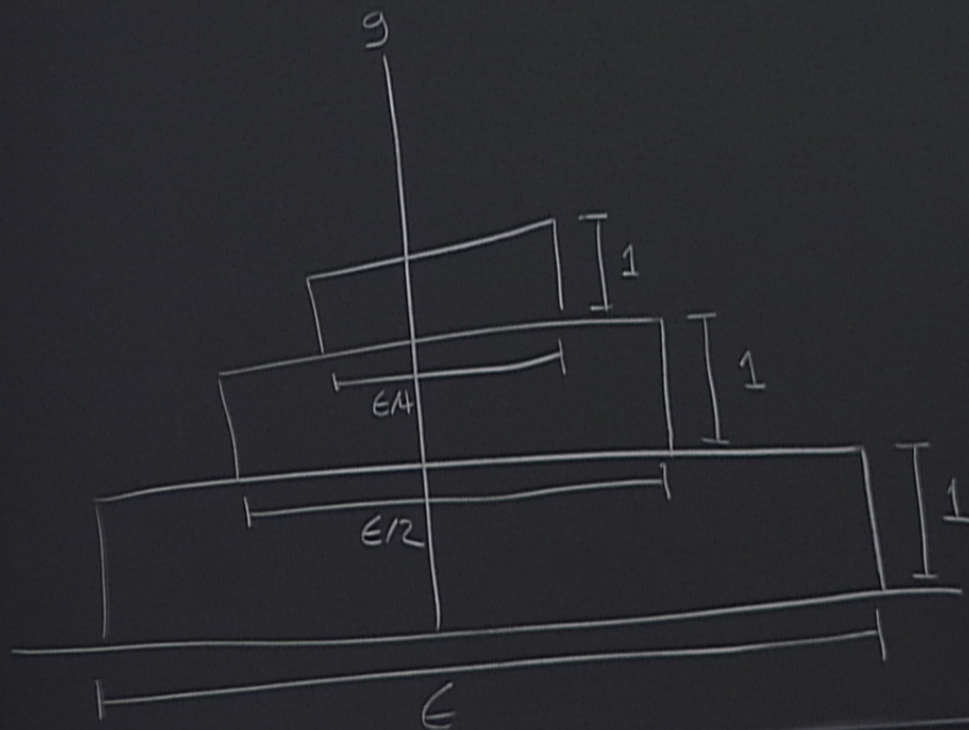
$$\lim_{\sigma \rightarrow 0} g_\sigma(x) = 0 \quad \text{for } x \neq 0$$

$$\int_{-\infty}^{\infty} g_\sigma(x) dx = 1 \quad \text{for all } \sigma > 0$$



Choose a sequence $g_\sigma \rightarrow g = \delta$ for which

$$\lim_{\sigma \rightarrow \infty} \int_{-\infty}^{\infty} g_\sigma(x) dx = 0$$



ϵ
g lies inside the boxes so $\int_{-\infty}^{\infty} g(x) dx < A$ area of the boxes

$$A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \dots = 2\epsilon \quad \text{for any } \epsilon$$

$$\int_{-\infty}^{\infty} g(x) dx = 0 \neq \lim_{\sigma \rightarrow 0} \int_{-\infty}^{\infty} g_{\sigma}(x) dx$$

Order matters

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pointwise convergence $\neq$ converge with respect to L^2

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pointwise convergence $\neq$ converge with respect to L^2 norm

Test functions

$\mathcal{D}$

↑
test-function space

set of

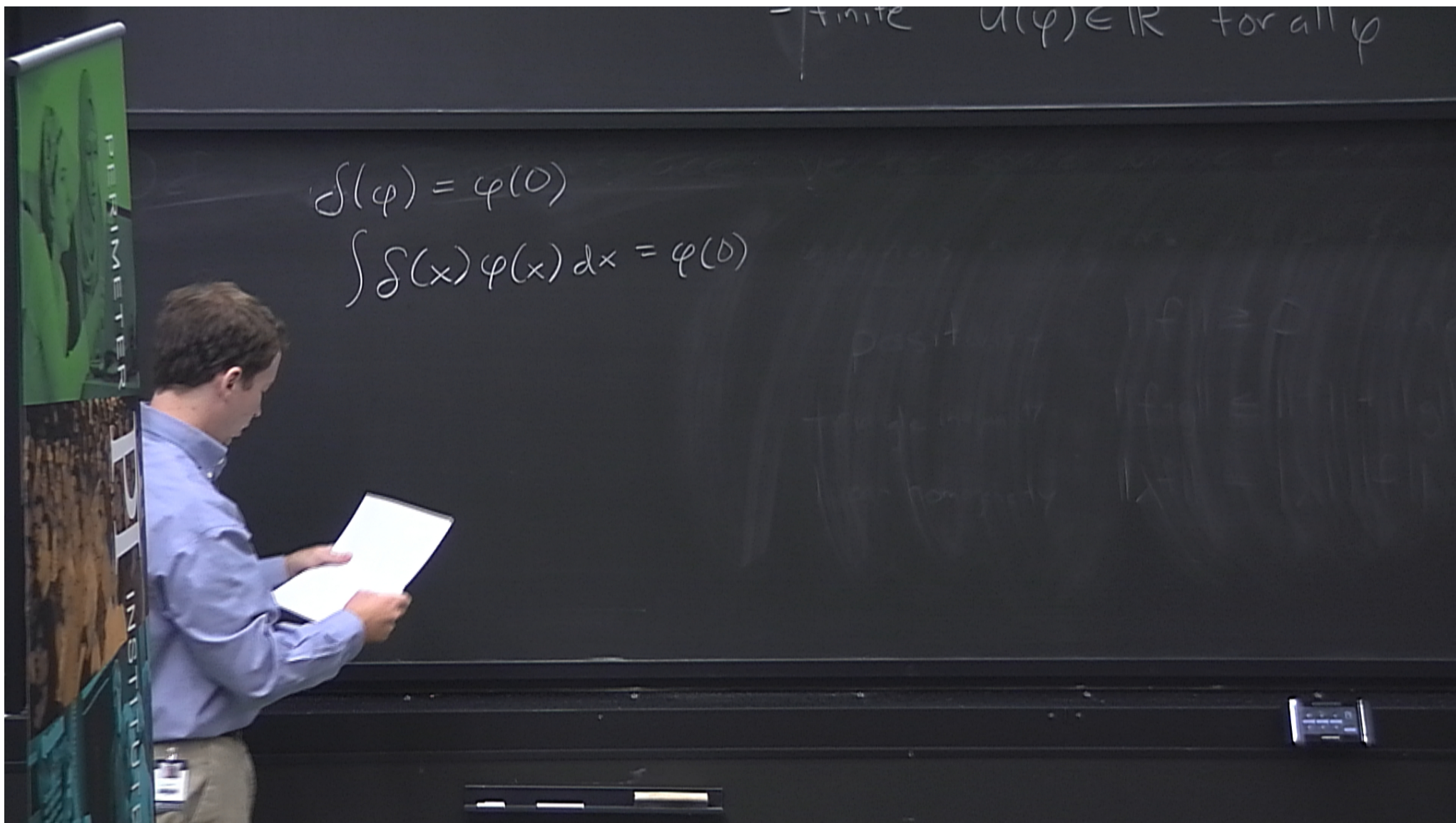
C^∞

↑
continuous
& infinitely
differentiable
functions

with com

Test functions $\mathcal{D}$: set of C^∞ with c
↑
test-function space
↑
continuous & infinitely differentiable functions

$\mathcal{D}(\mathbb{R})$ is a vector space
can define a dual vector space $\mathcal{D}^*$ of linear functions



- finite $u(\varphi) \in \mathbb{R}$ for all φ

$$\delta(\varphi) = \varphi(0)$$

$$\int \delta(x) \varphi(x) dx = \varphi(0)$$

$$(\delta, \varphi) = \varphi(0)$$

pairing notation

finite $u(\varphi) \in \mathbb{R}$ for all φ

$$u(\varphi) = \varphi(0)^2$$

$$v(\varphi) = \sum_{n=1}^m \lambda_n \varphi^{(n)}(0)$$

$$\lambda_n \in \mathbb{R}$$

finite $u(\varphi) \in \mathbb{R}$ for all φ

$$u(\varphi) = \varphi(0)^2 \quad \times \text{ not linear}$$

$$v(\varphi) = \sum_{n=1}^m \lambda_n \varphi^{(n)}(0) \quad \checkmark \text{ linear, continuous, finite}$$

$\lambda_n \in \mathbb{R}$

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$$\lambda_n \in \mathbb{R}$$

Derivatives of Distributions

$$\cancel{u'(x) = \lim_{\epsilon \rightarrow 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}}$$

$u(x)$ may not be defined

$$(\varphi) = \int_{-\infty}^{\infty} u'(x) \varphi(x) dx$$

$$\stackrel{?}{=} \cancel{u(x)\varphi(x)} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x) \varphi'(x) dx$$

ok if u is a function

$$\Rightarrow \int_{-\infty}^{\infty} u(x) \varphi'(x) dx$$

not be defined

$$u'(\varphi) \equiv -u(\varphi')$$

$$u^{(n)}(\varphi) \equiv (-1)^n u(\varphi^{(n)})$$

s a function

X not linear

$n(0)$

✓

linear
continuous
finite

what meant by
"u is a locally integrable function"

$$u(\varphi) = \int f(x) \varphi(x) dx$$

Example: $\textcircled{H}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$\textcircled{H}$ defines a distribution because

$$\textcircled{H}'(\varphi) = \int_{-\infty}^{\infty} \textcircled{H}(x) \varphi(x) dx$$

$$= \int_0^{\infty} \varphi(x) dx$$

finite, continuous, linear

$$\textcircled{H}'(\varphi)$$

Example: $(\ln |x|)'$

$$A = (\ln |x|)'(\varphi) \stackrel{?}{=} - \int_{-\infty}^{\infty} \varphi'(x) \ln |x| dx \quad \text{doesn't converge}$$

$\equiv$

$$\varphi'(\varphi) \stackrel{?}{=} - \int_{-\infty}^{\infty} \varphi'(x) \ln|x| dx \quad \text{doesn't converge near zero}$$

$$\equiv - \lim_{\epsilon \rightarrow 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi'(x) \ln|x| dx$$

$$= \lim_{\epsilon \rightarrow 0} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx + \varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon| \right]$$

$$\varphi'(0) \stackrel{?}{=} - \int_{-\infty}^{\infty} \varphi'(x) \ln|x| dx \quad \text{doesn't converge near zero}$$

$$\equiv - \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi'(x) \ln|x| dx$$

$$\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx + \underbrace{\varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon|}_{\ln|\epsilon| (\varphi(\epsilon) - \varphi(-\epsilon)) \approx \ln|\epsilon| \varphi'(0) 2\epsilon \rightarrow 0}$$

$$+ \left(\int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx \Big]$$

$$)'(\varphi) \stackrel{?}{=} - \int_{-\infty}^{\infty} \varphi'(x) \ln|x| dx \quad \text{doesn't converge near zero}$$

$$\equiv - \lim_{\epsilon \rightarrow 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi'(x) \ln|x| dx$$

$$= \lim_{\epsilon \rightarrow 0} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx + \underbrace{\varphi(\epsilon) \ln|\epsilon| - \varphi(-\epsilon) \ln|\epsilon|}_{\ln|\epsilon|(\varphi(\epsilon) - \varphi(-\epsilon)) \approx \ln|\epsilon| \varphi'(0) 2\epsilon \rightarrow 0} \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x) dx \right] = \left(PV \frac{1}{x} \right) (\varphi)$$