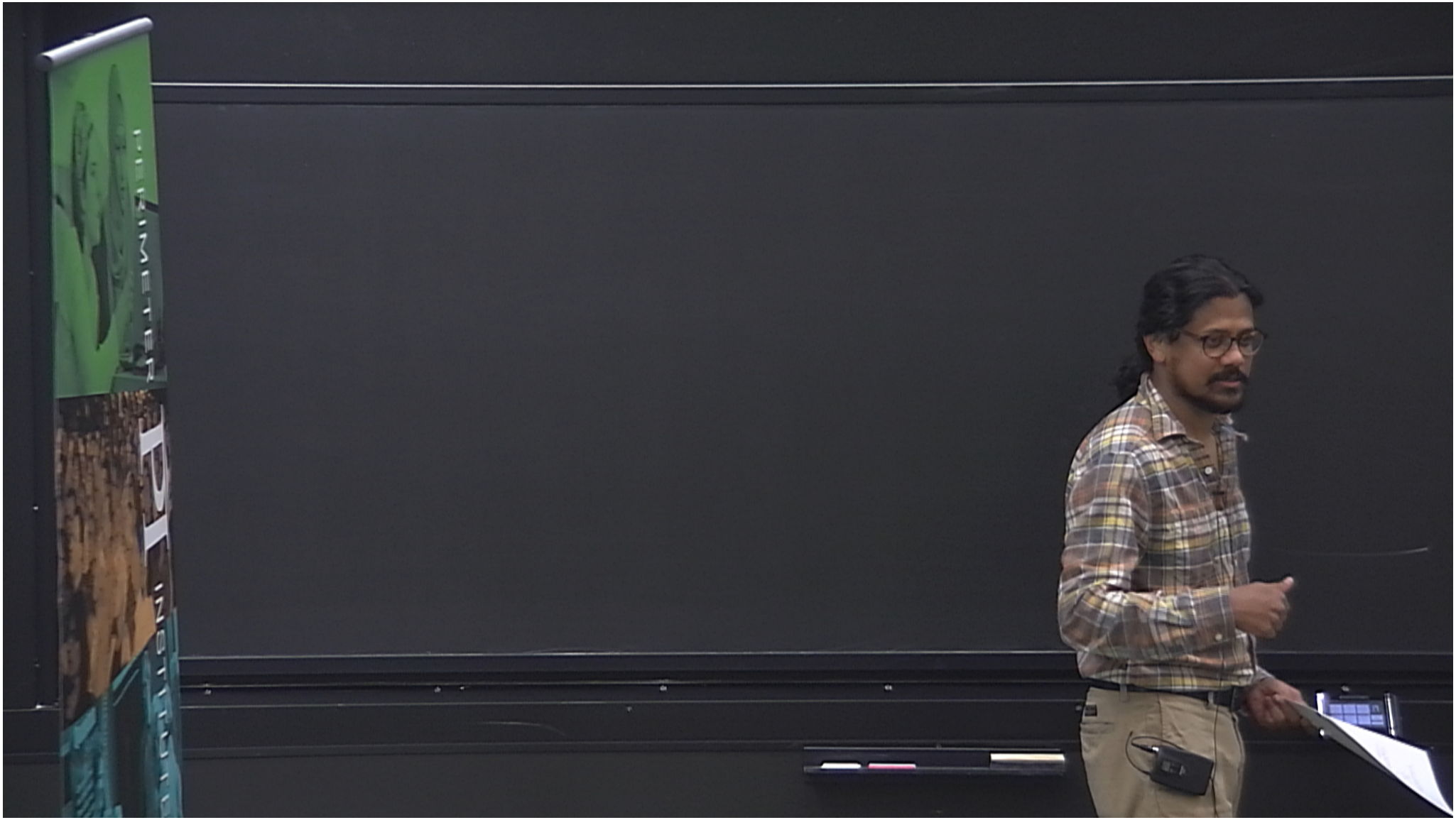
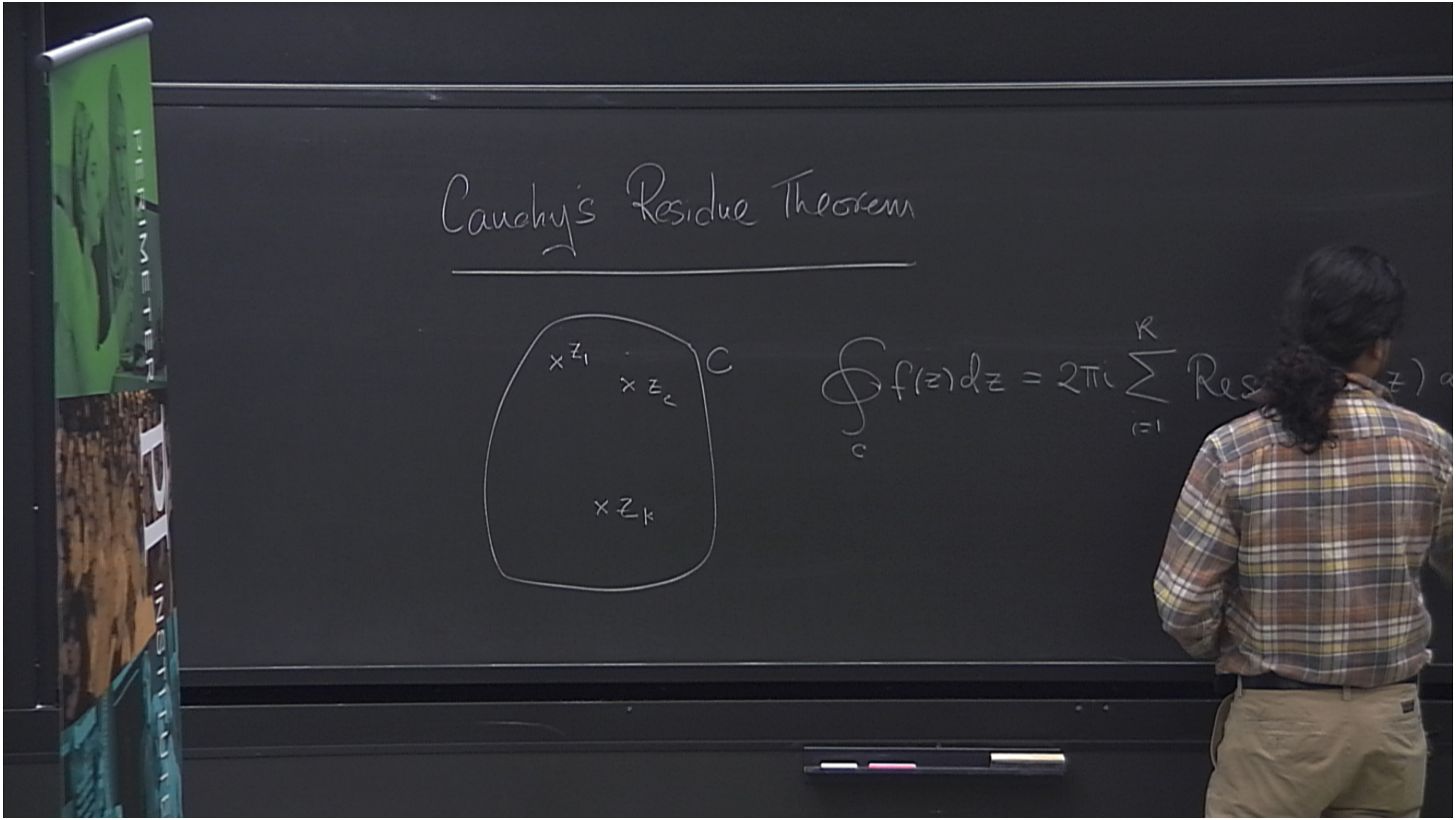
Title: 14/15 PSI - Complex Analysis

Date: Aug 15, 2014 09:00 AM

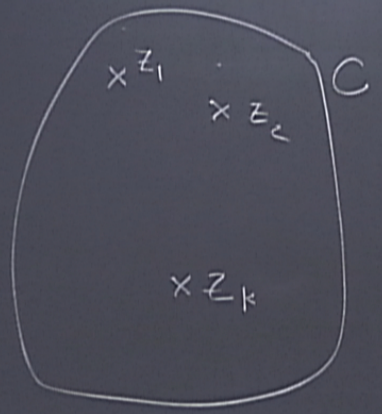
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Abstract:

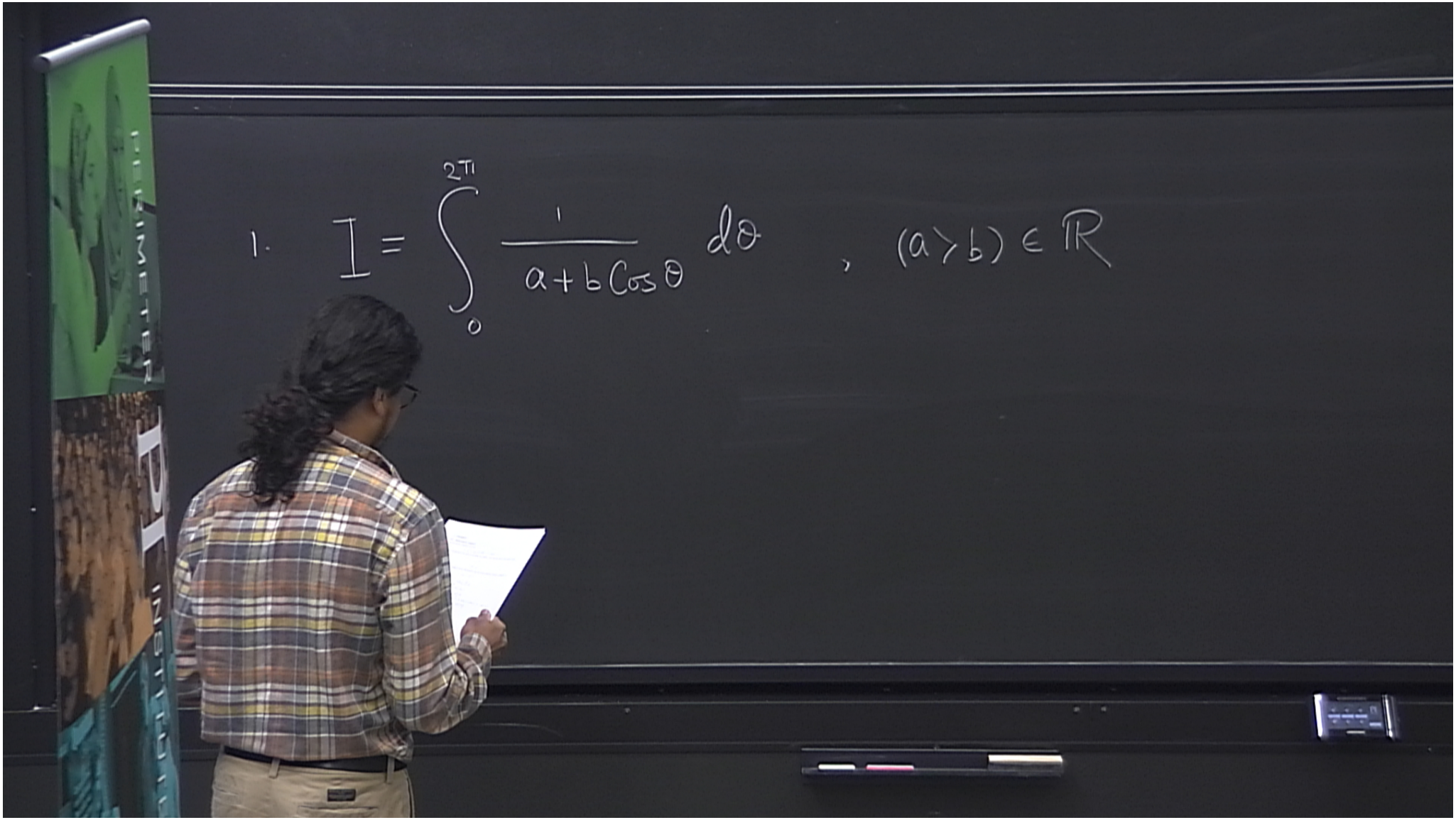

PERIMETER
INSTITUTE

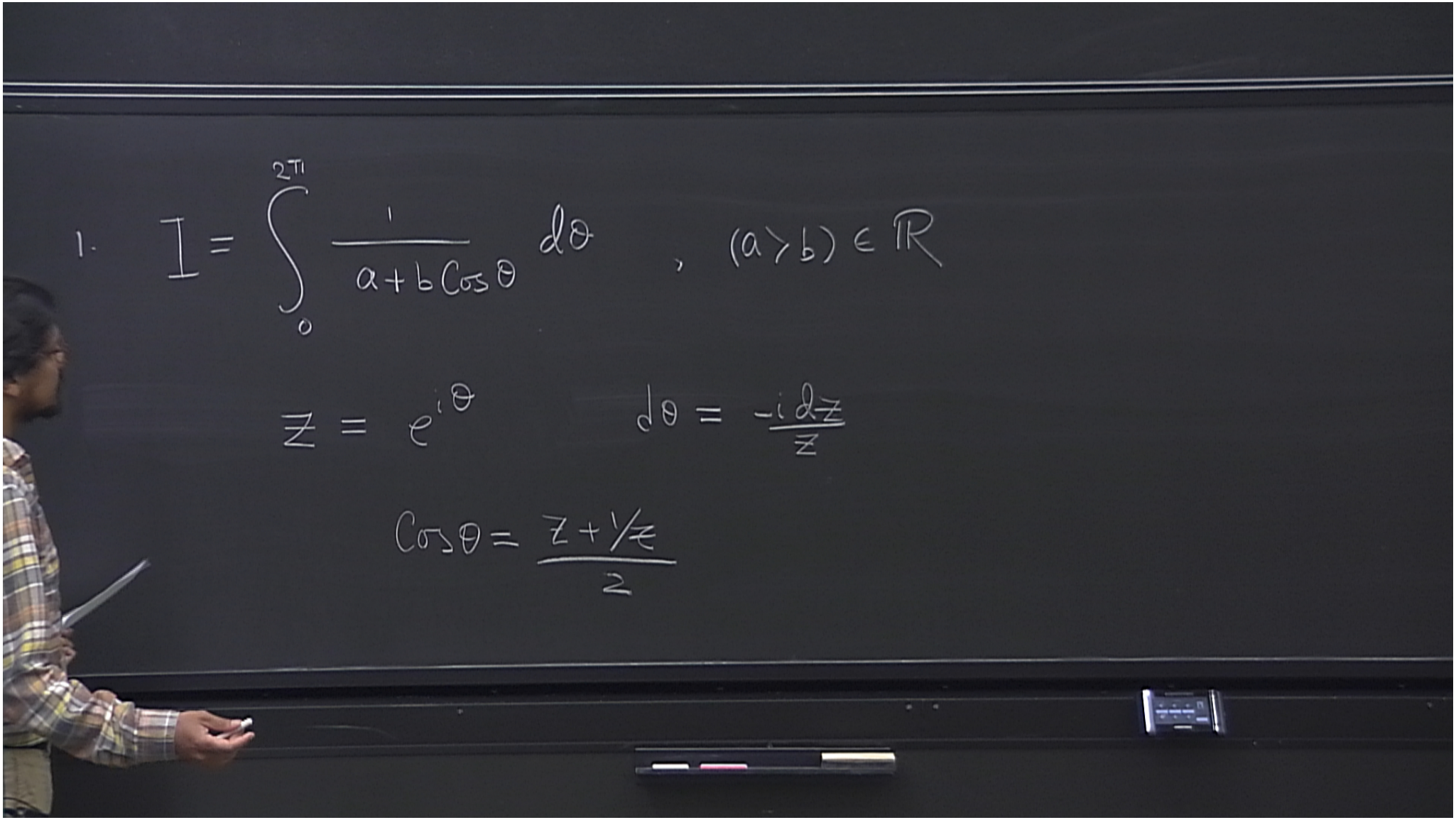

Cauchy's Residue Theorem
z_1
z_2
z_k
C
$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{k} \mathrm{Res}$

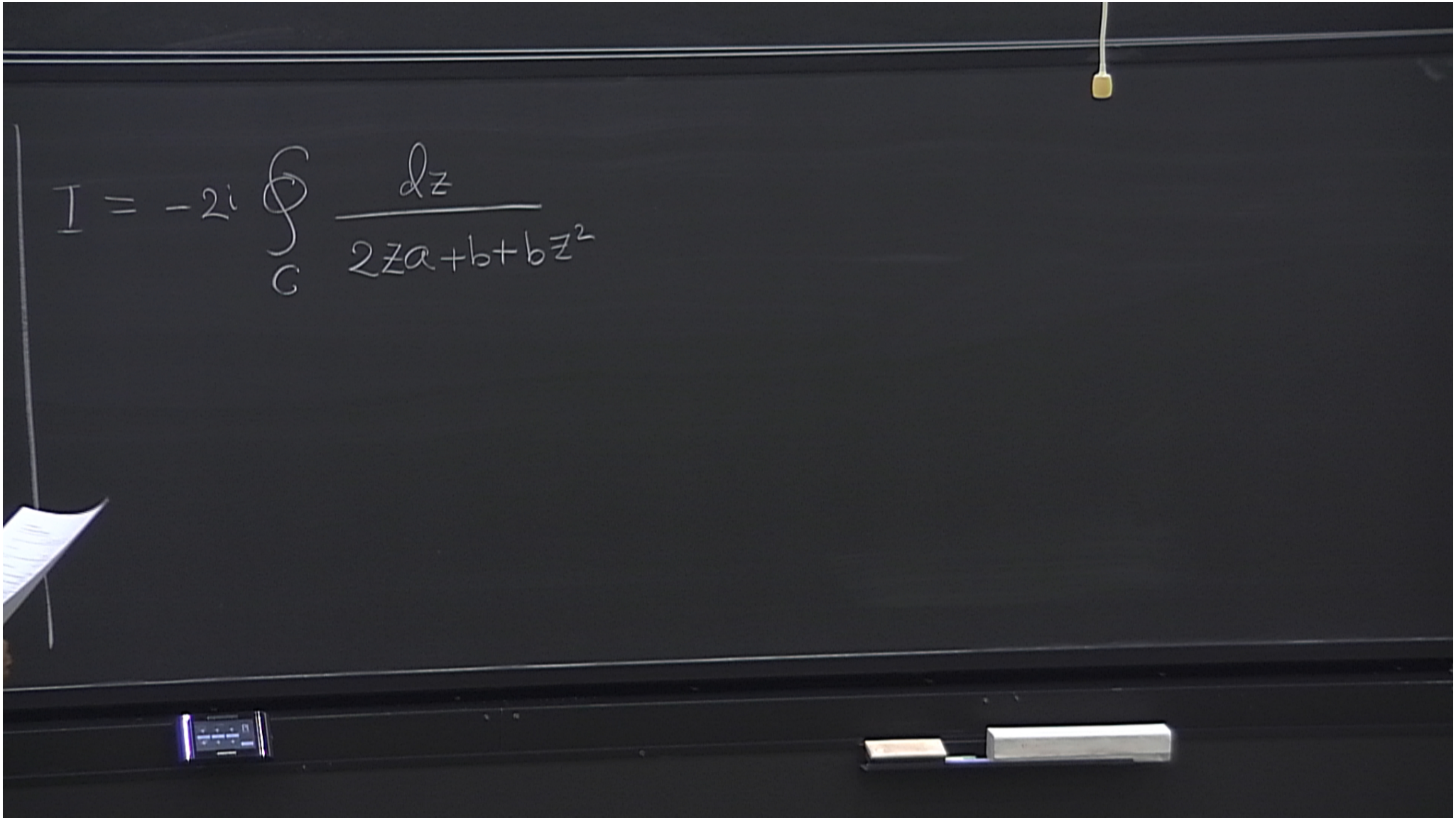
Cauchy's Residue Theorem



$$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{K} \text{Res of } f(z) \text{ at } z = z_i$$

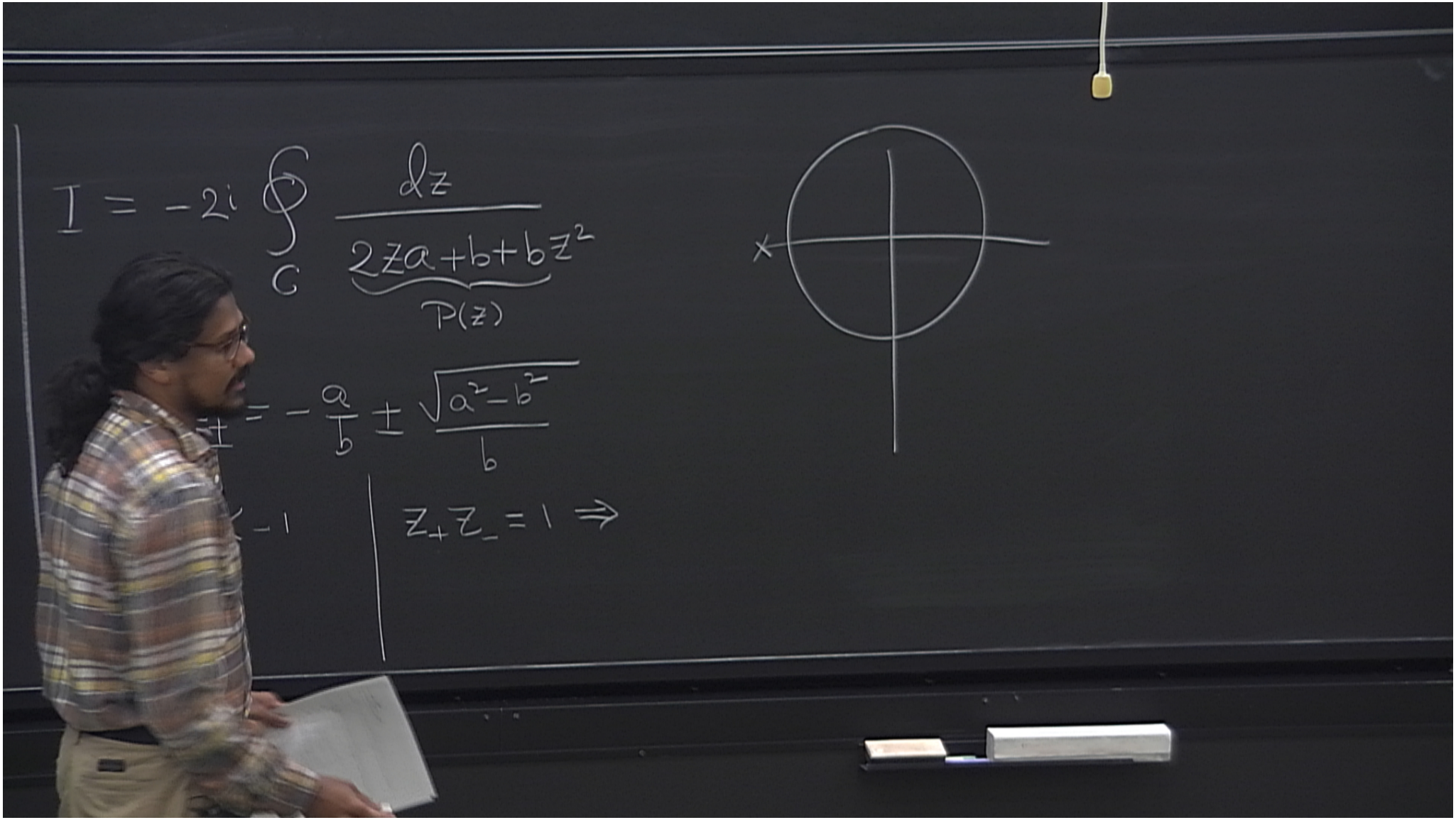

1. $I = \int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$, $(a>b) \in \mathbb{R}$


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$z = e^{i\theta}$
$d\theta = \frac{-i\,dz}{z}$
$\cos\theta = \frac{z+1/z}{2}$


$I = -2i \oint_C \frac{dz}{2za + b + bz^2}$

$$I = -2i \oint_C \frac{dz}{\underbrace{2za + b + bz^2}_{P(z)}}$$

$$z_\pm = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

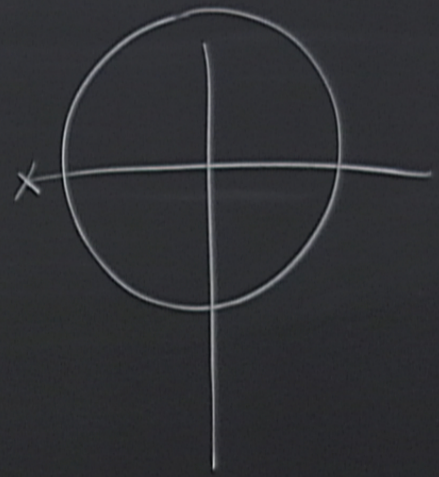

$I = -2i \oint_C \frac{dz}{\underbrace{2za+b+bz^2}_{P(z)}}$
$= -\frac{a}{b} \pm \frac{\sqrt{a^2-b^2}}{b}$
-1
$z_+ z_- = 1 \Rightarrow$

$$I = -2i \oint_C \frac{dz}{\underbrace{2za+b+bz^2}_{P(z)}}$$

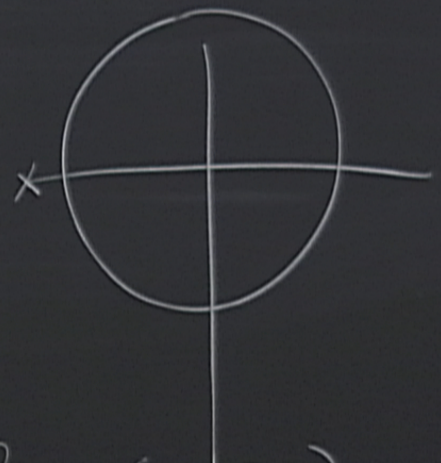
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$$-\frac{a}{b} < -1$$

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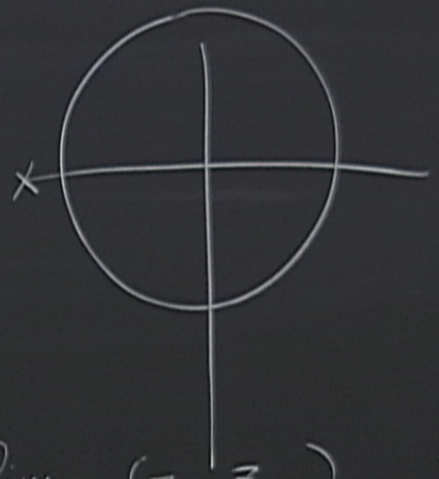
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$$-\frac{a}{b} < -1$$

$$z_+ z_- = 1 \Rightarrow$$

$$\lim_{z \to z_+} (z - z_+) \frac{-2i}{(z - z_+)(z - z_-)} = \frac{-ib}{\sqrt{a^2}}$$

$$= -2i \oint_C \frac{dz}{\underbrace{2za + b + bz^2}_{P(z)}}$$



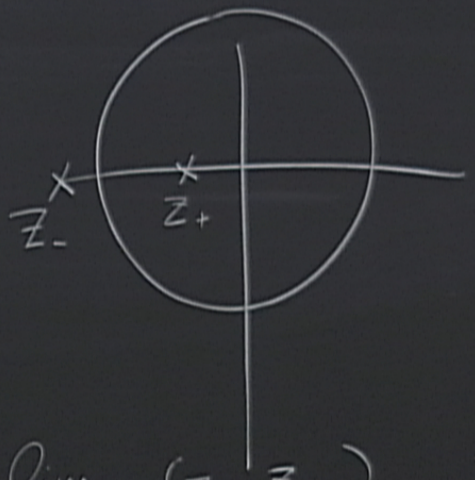
$$I = \frac{2\pi b}{\sqrt{a^2 - b^2}}$$

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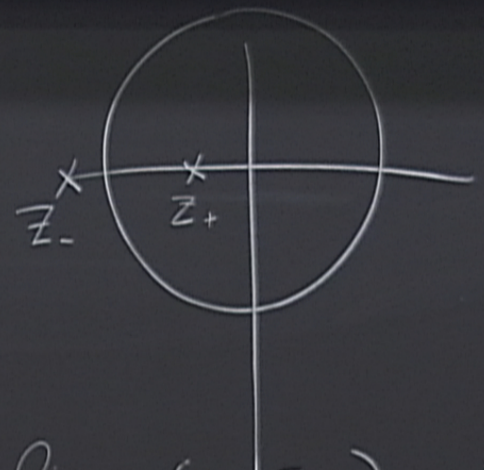
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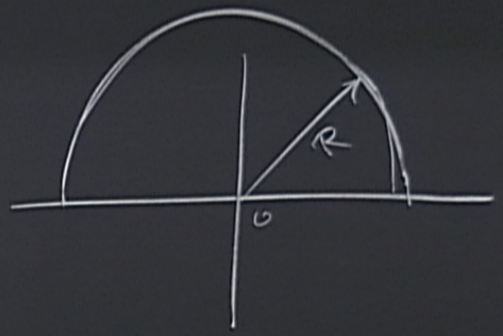
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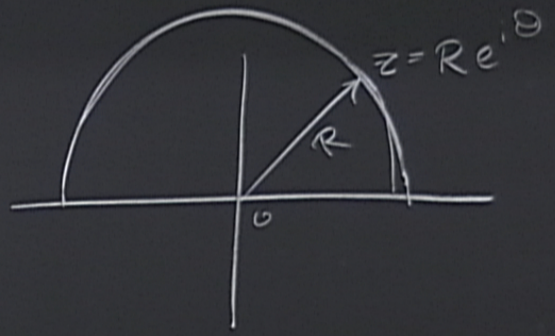
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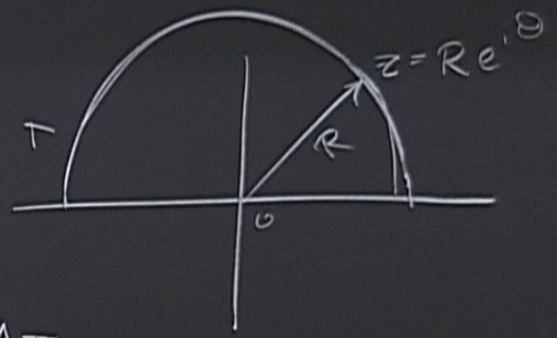
$$|f(z)| \leq \frac{M}{R^k}$$



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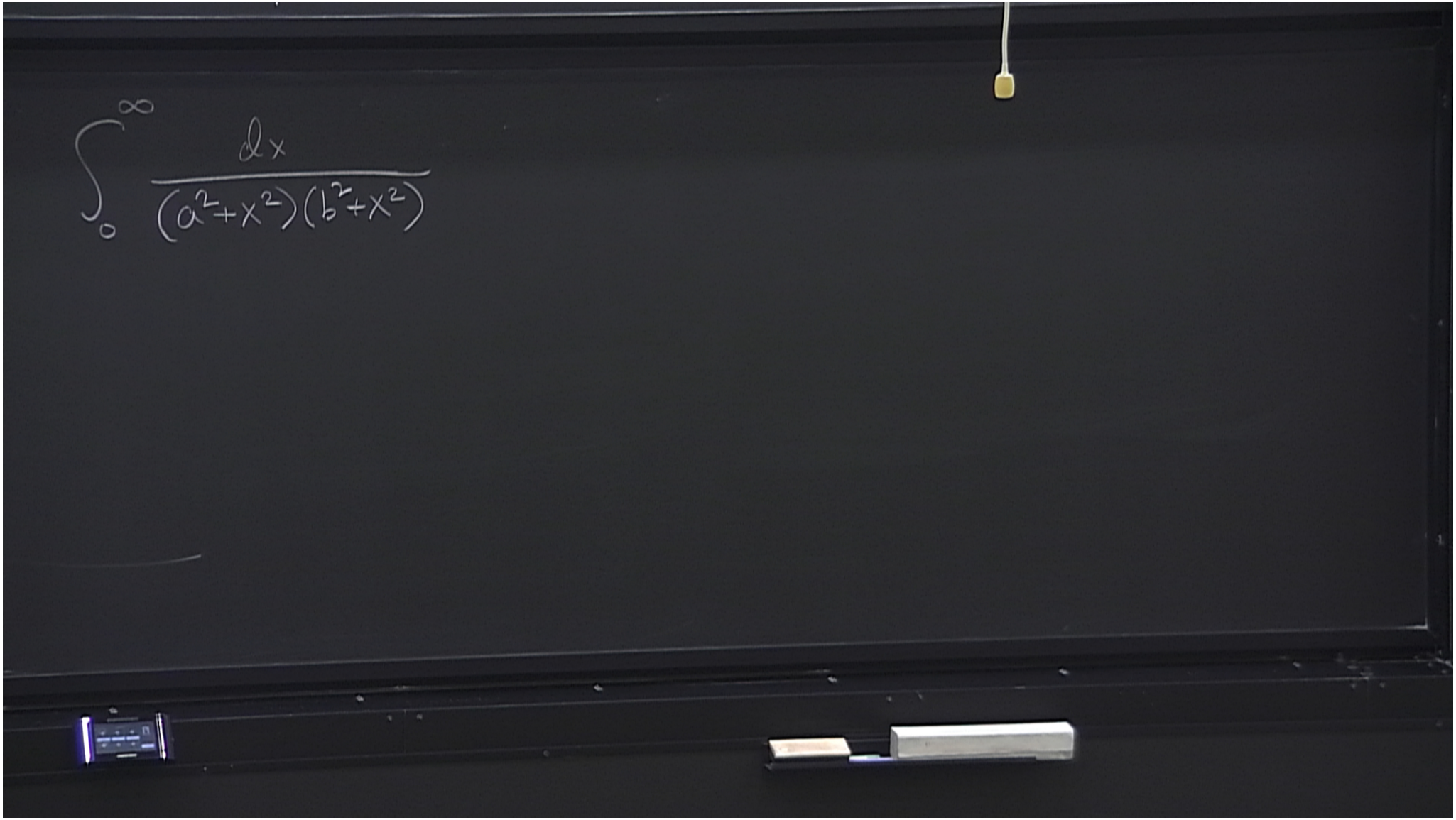
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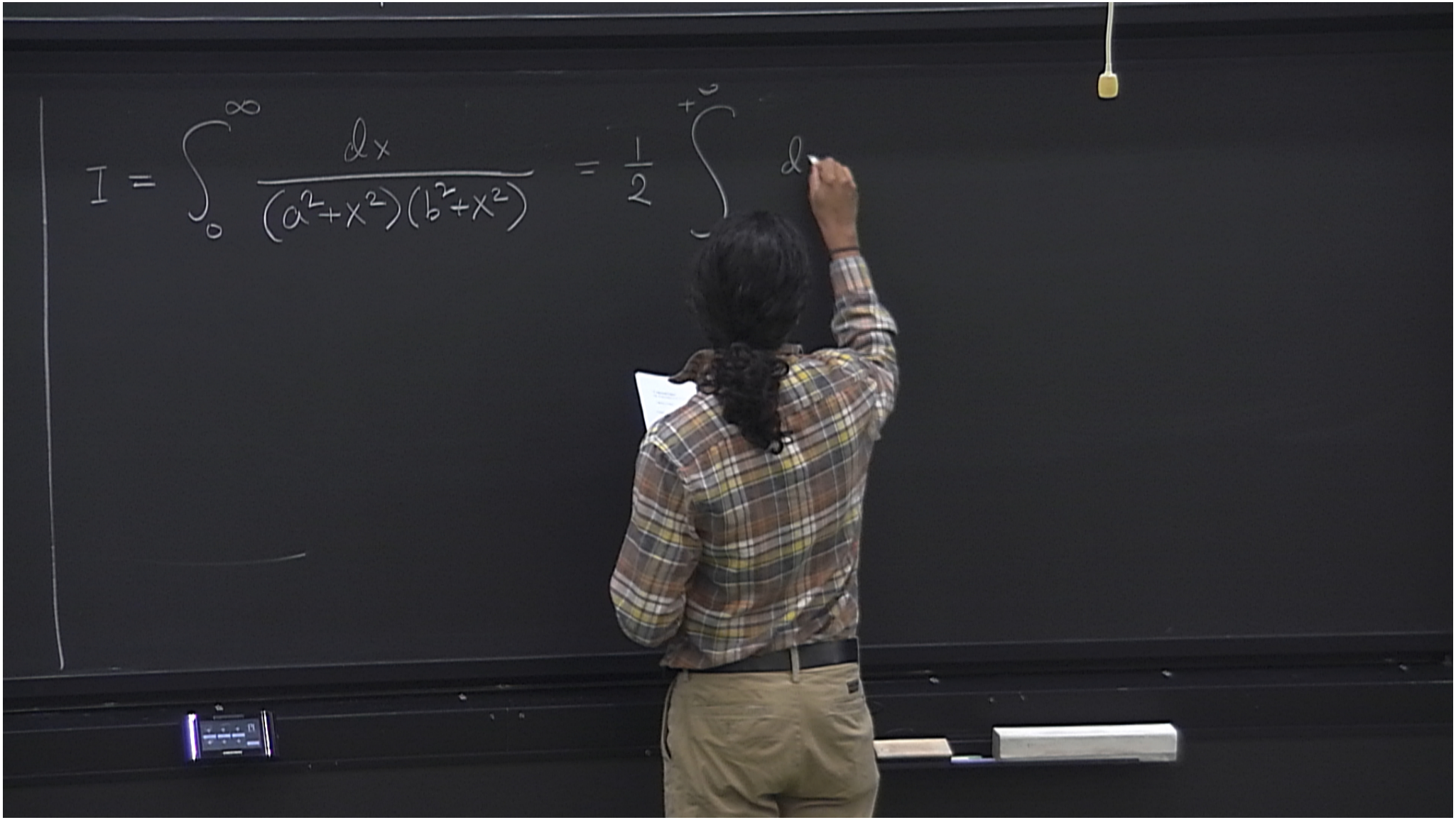


Then $\left| \int_\Gamma f(z)\,dz \right| \leq \frac{M\pi}{R^{k-1}}$

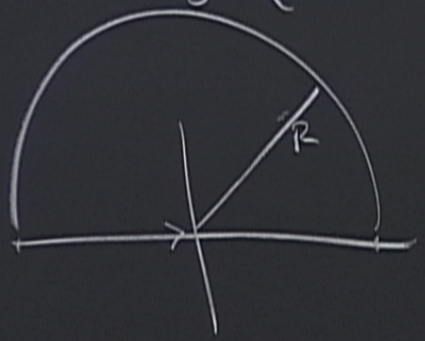
$I = \int_\Gamma$

If $k > 1$ Then $I \to 0$ as $R \to \infty$

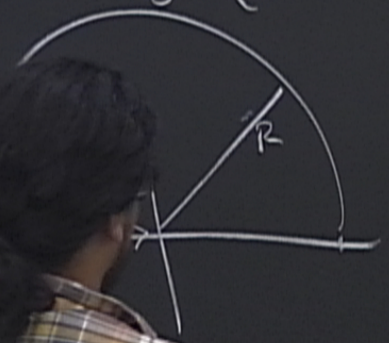

$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$


$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2} \int_{-\infty}^{+\infty} d$

$$I = \int_0^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{\dots}$$



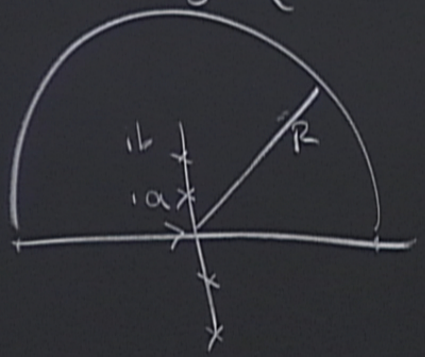
$$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$



$$\frac{1}{2}\,\frac{1}{(a^2+z^2)(b^2+z^2)}$$

$$\pm i a, \pm i b$$

$$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$



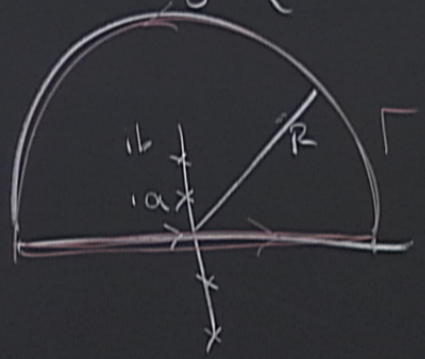
$$\frac{1}{2}\,\frac{1}{(a^2+z^2)(b^2+z^2)}$$

$\pm ia$, $\pm ib$

$R \to \infty$

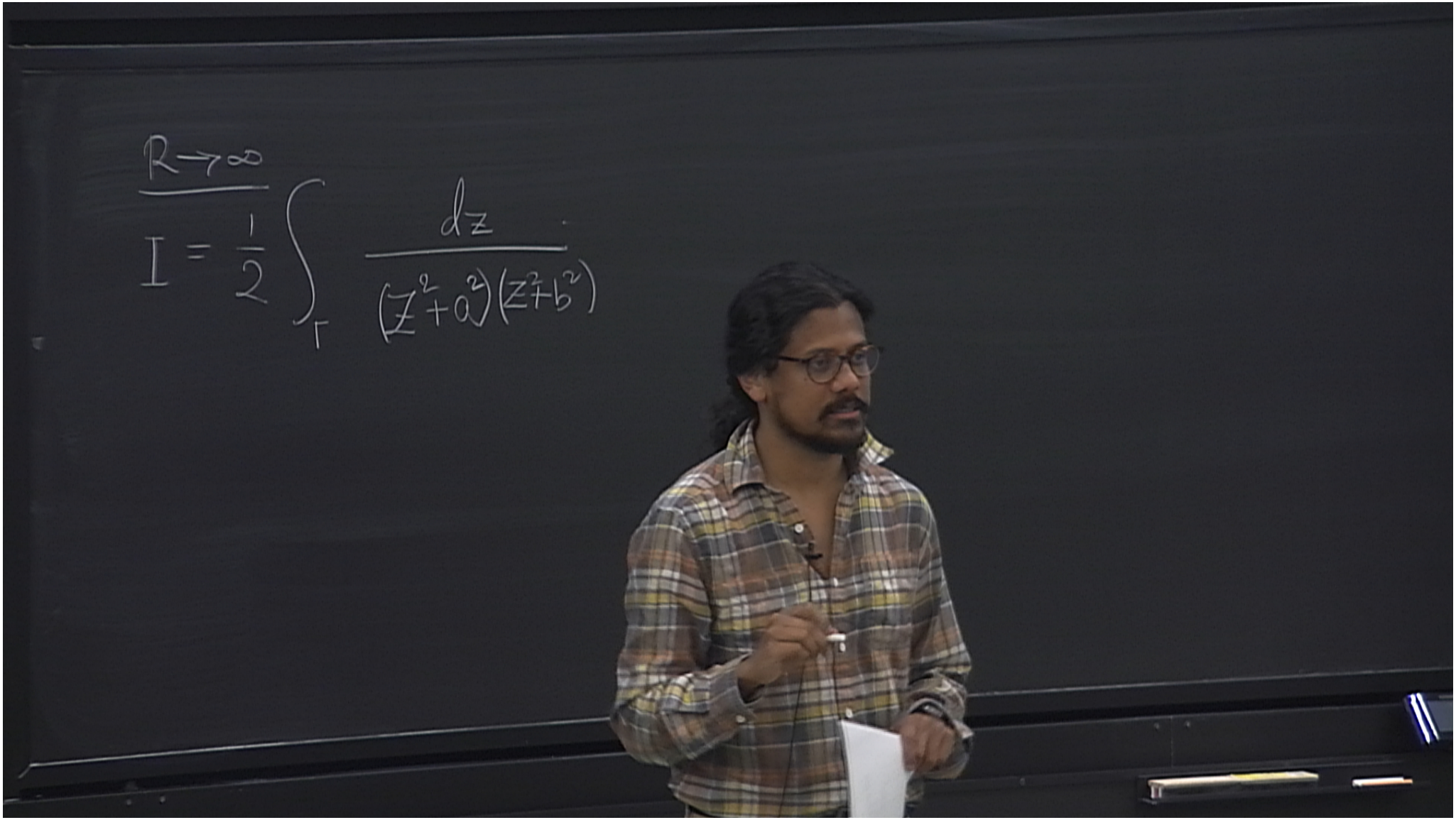
$$I = \frac{1}{2} \int_{\Gamma} \frac{dz}{\quad}$$

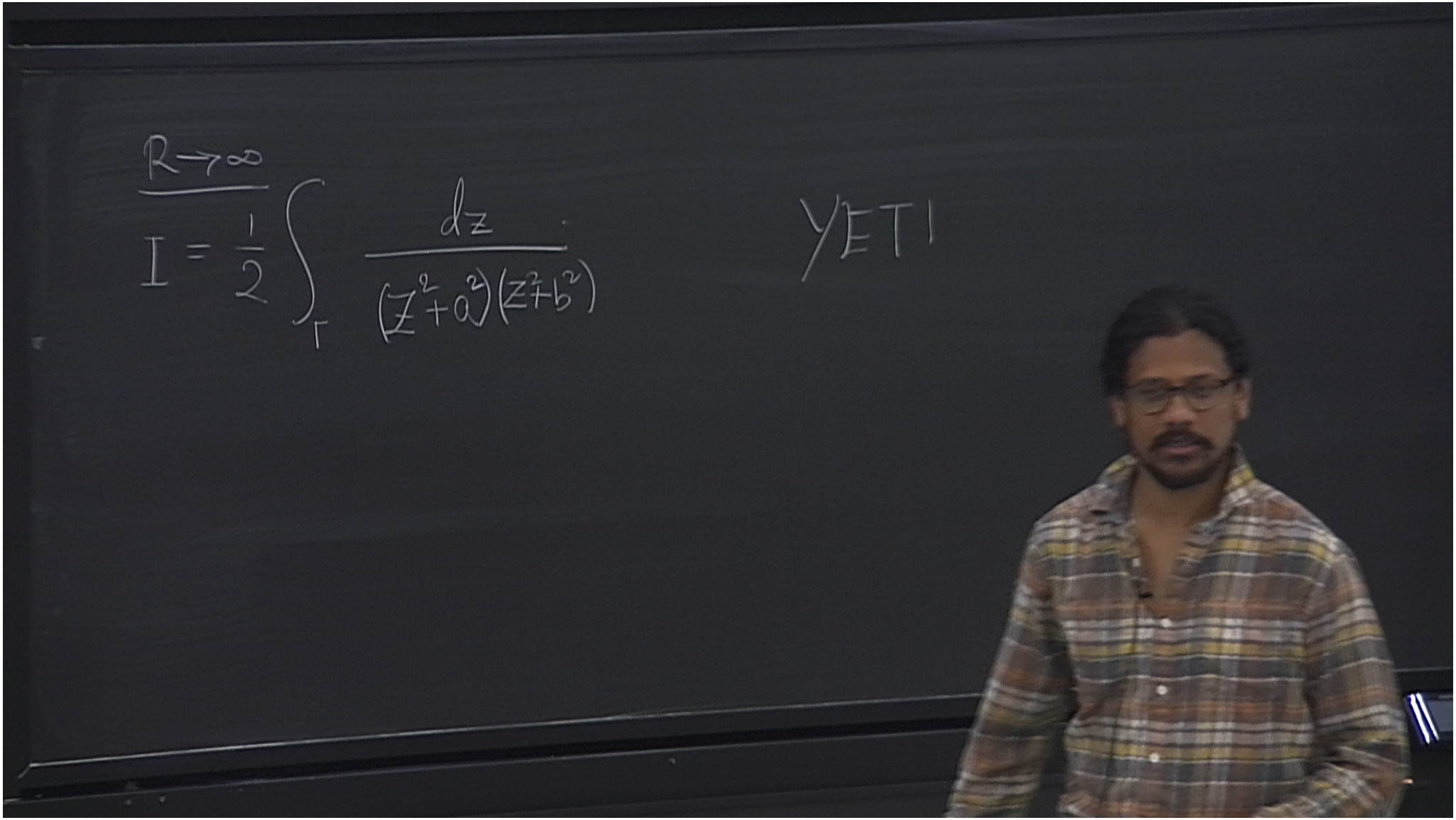
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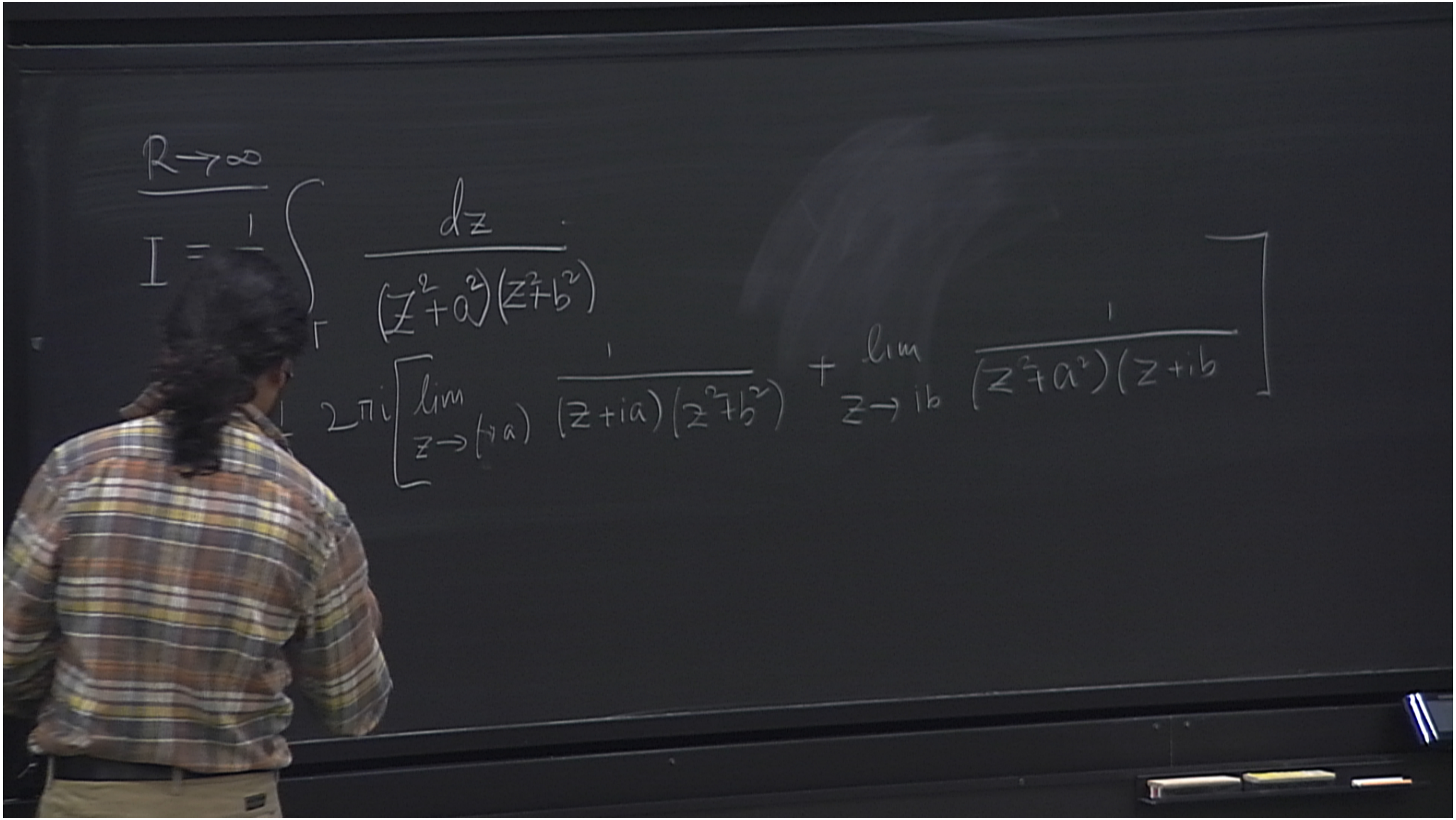

R→∞
I = 1/2 ∫_Γ dz / (z²+a²)(z²+b²)


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YETI

$R \to \infty$

$$I = \frac{1}{2} \int_{\Gamma} \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

$$= \frac{1}{2} 2\pi i \left[\lim_{z \to (+)a} \frac{1}{(z+ia)(z^2+b^2)} + \lim_{z \to} \right.$$


R→∞
I = 1/(2πi) ∫_Γ dz / ((z²+a²)(z²+b²))
2πi [lim_{z→(+)a} 1/((z+ia)(z²+b²)) + lim_{z→ib} 1/((z²+a²)(z+ib

$R \to \infty$

$$I = \frac{1}{2} \int_{\Gamma} \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

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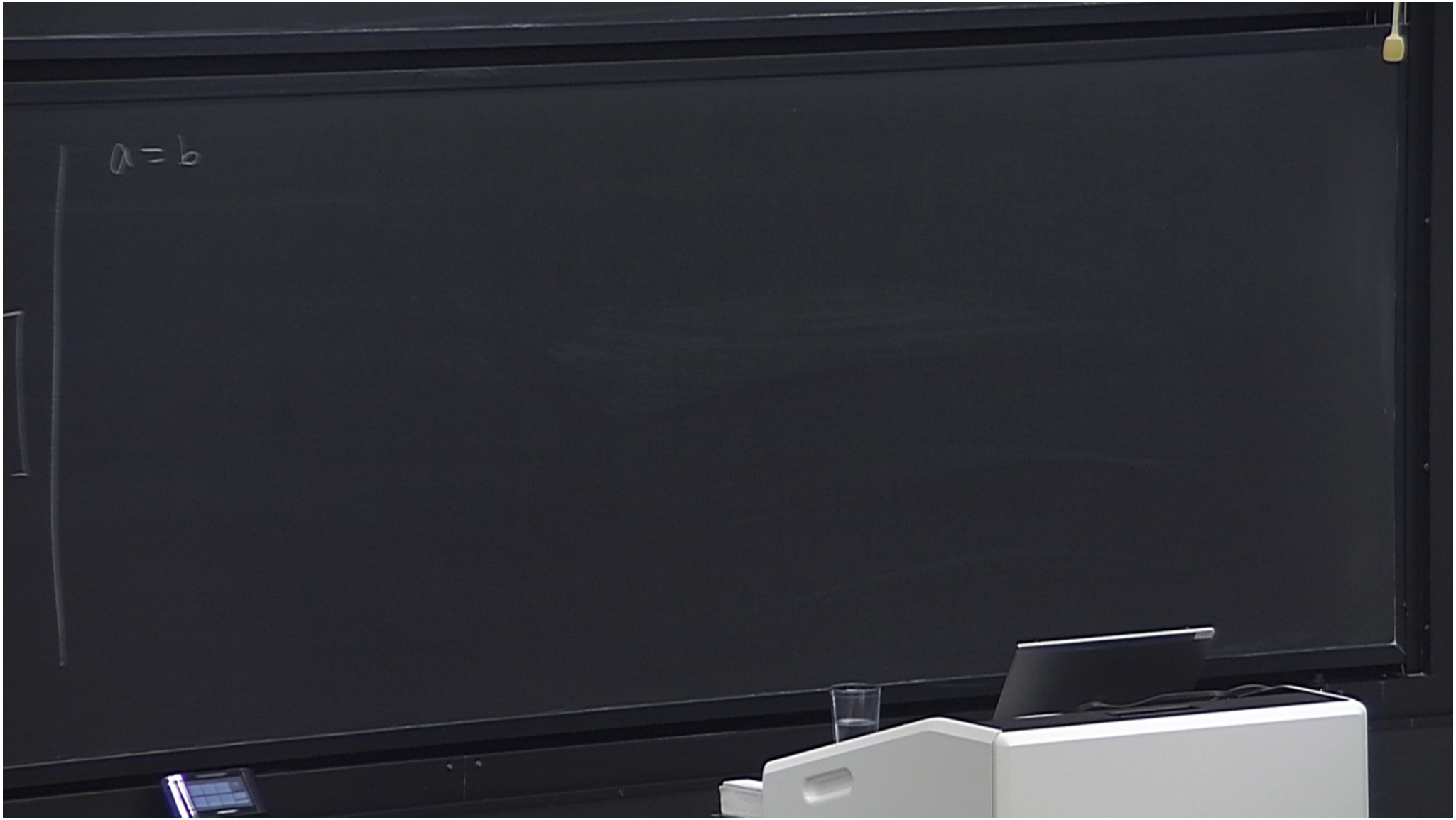
$$= \frac{\pi}{2ab} \frac{1}{(a+b)}$$

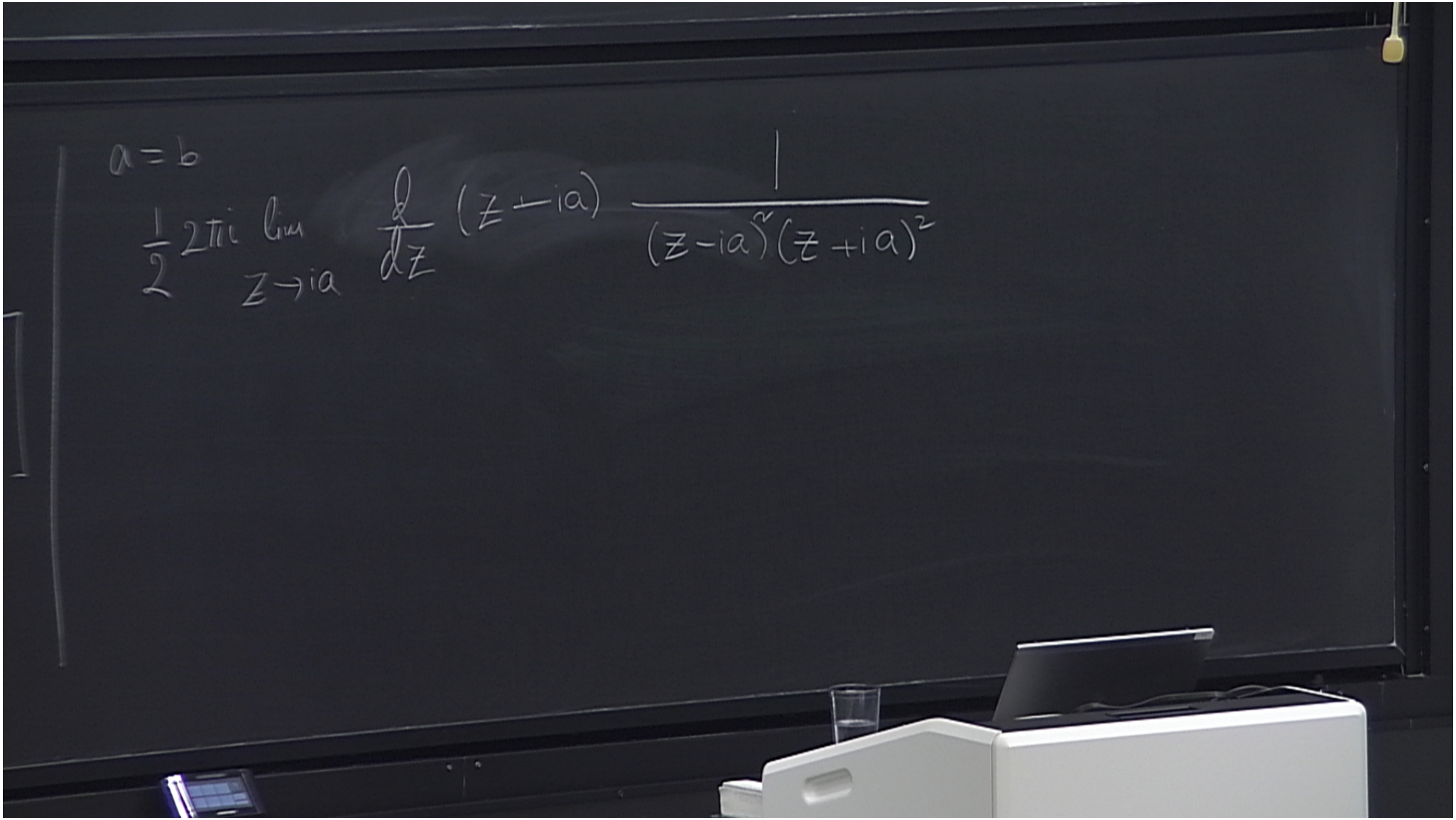
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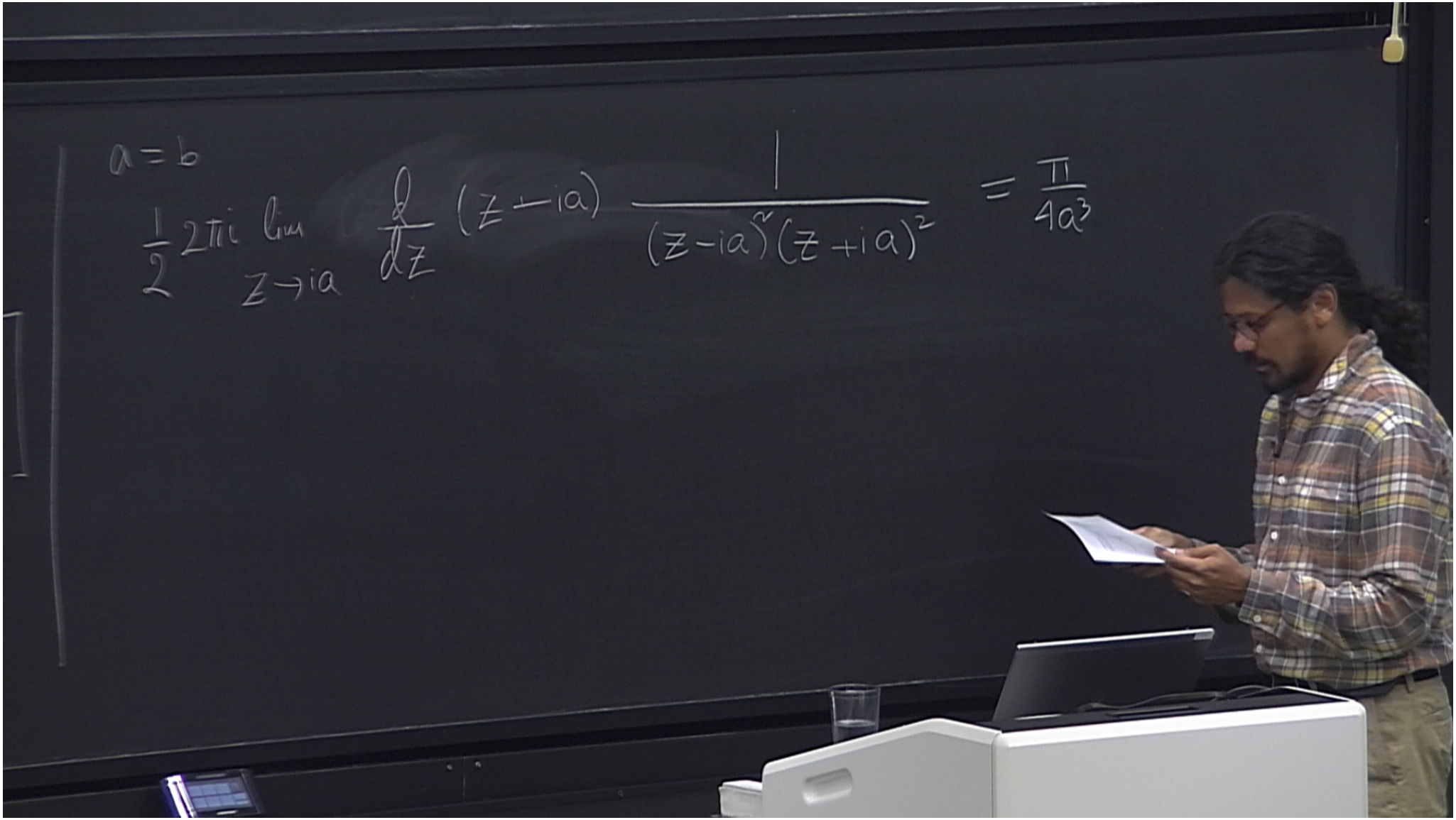
$$I = \frac{1}{2} \int_{\Gamma} \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

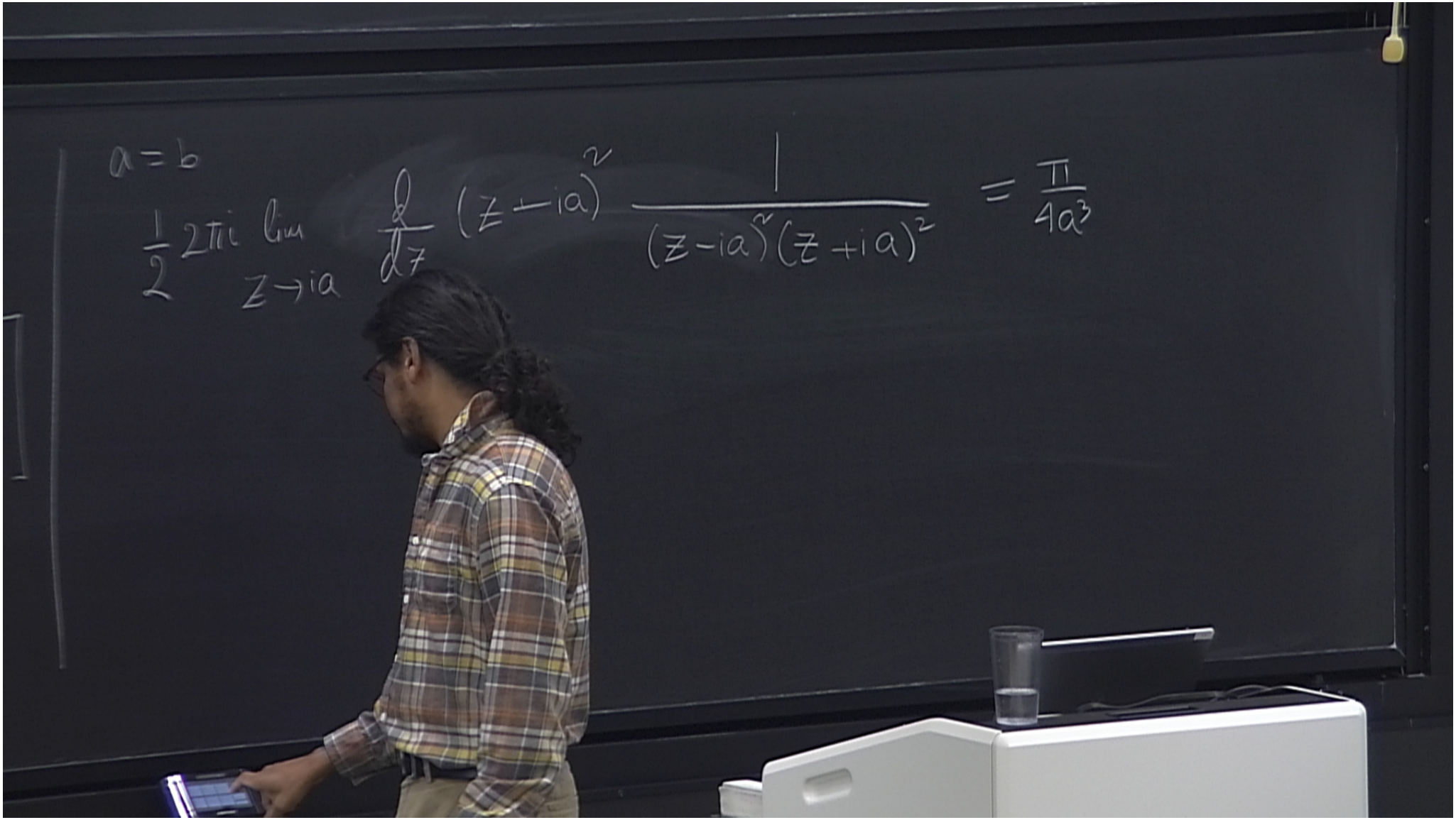
$$= \frac{1}{2} 2\pi i \left[\lim_{z \to (+)a} \frac{1}{(z+ia)(z^2+b^2)} + \lim_{z \to ib} \frac{1}{(z^2+a^2)(z+ib)} \right]$$

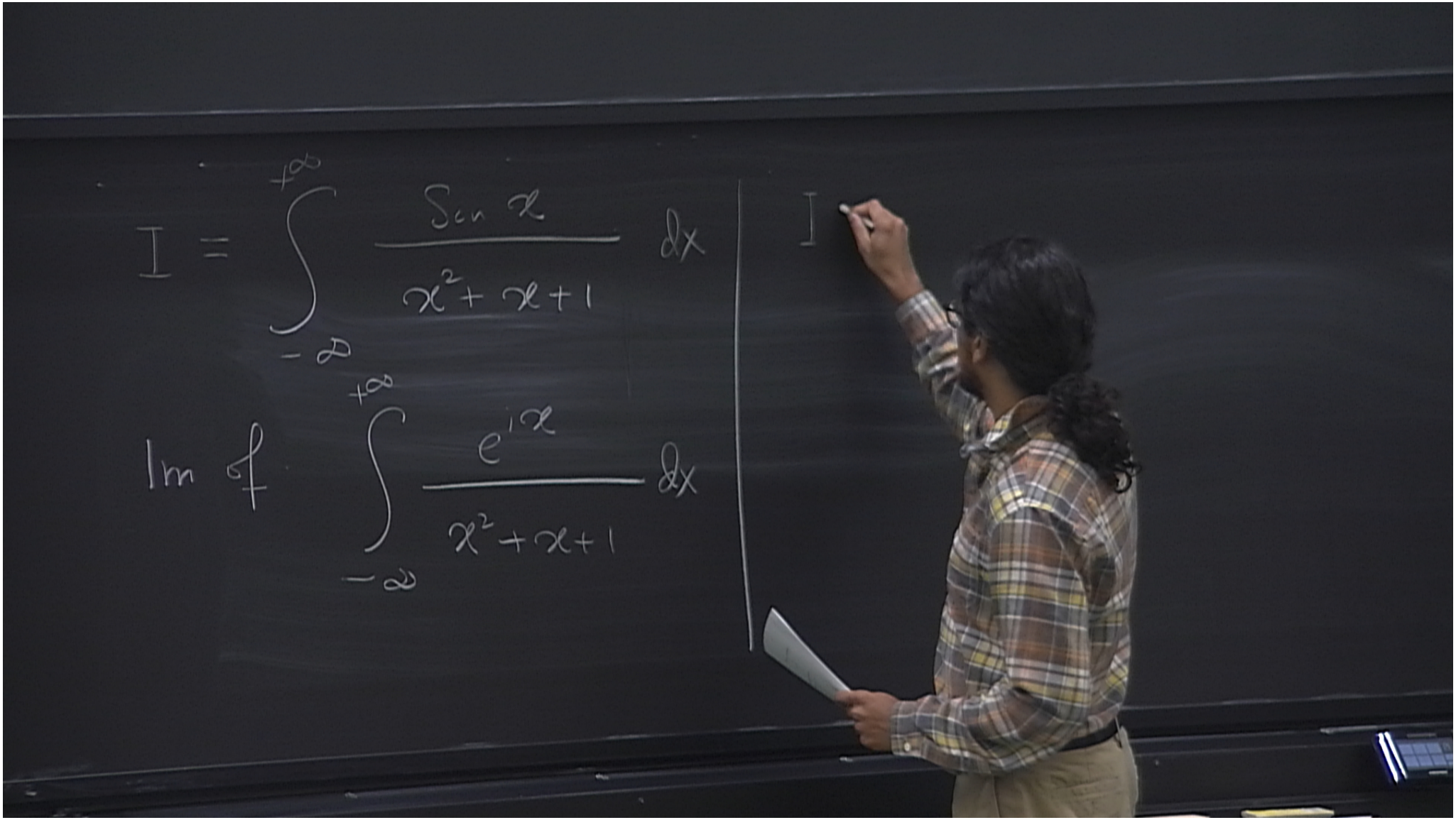
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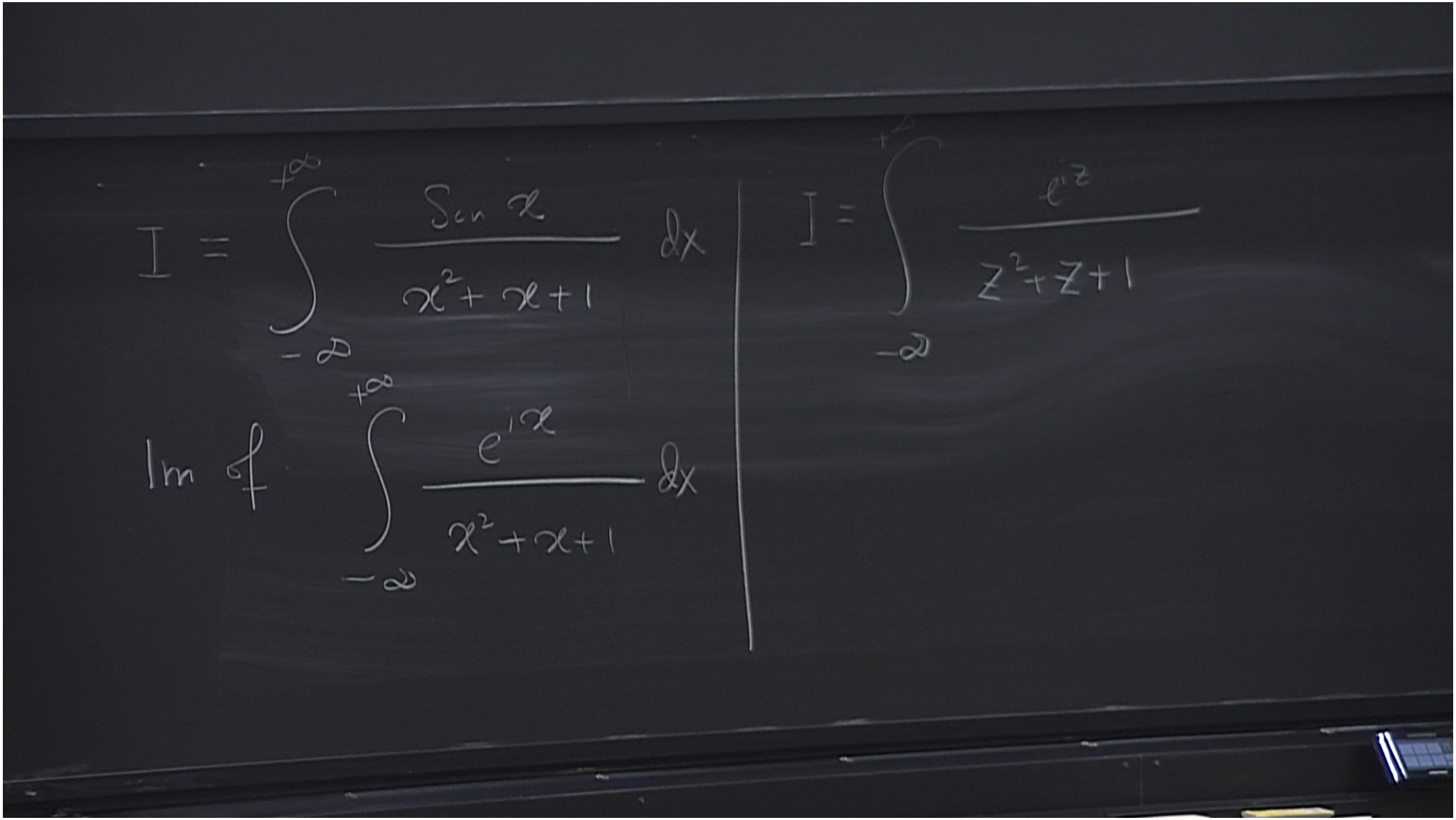

a = b

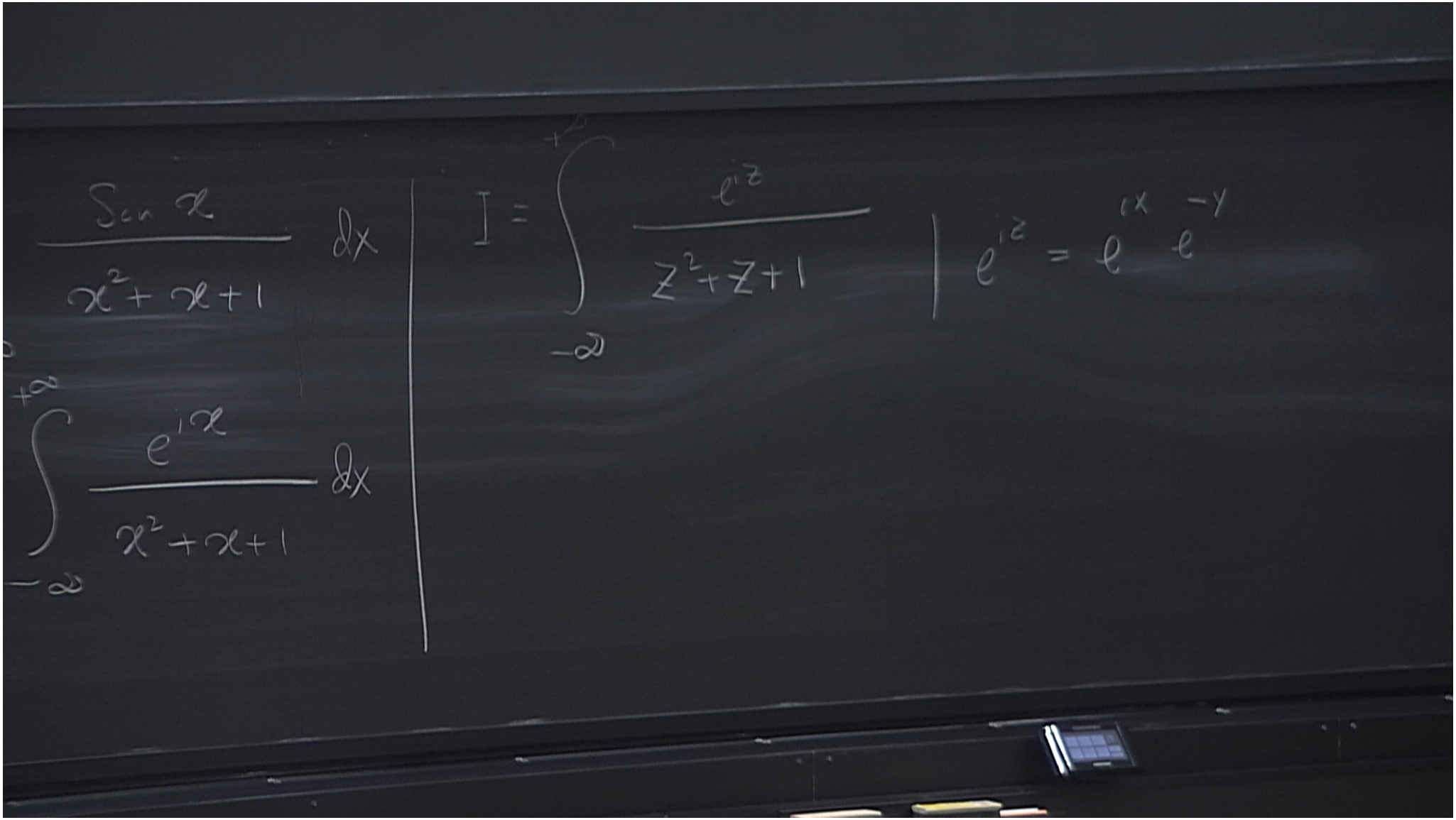

a = b
$\frac{1}{2} 2\pi i \lim_{z \to ia} \frac{d}{dz} (z - ia) \frac{1}{(z-ia)^2 (z+ia)^2}$

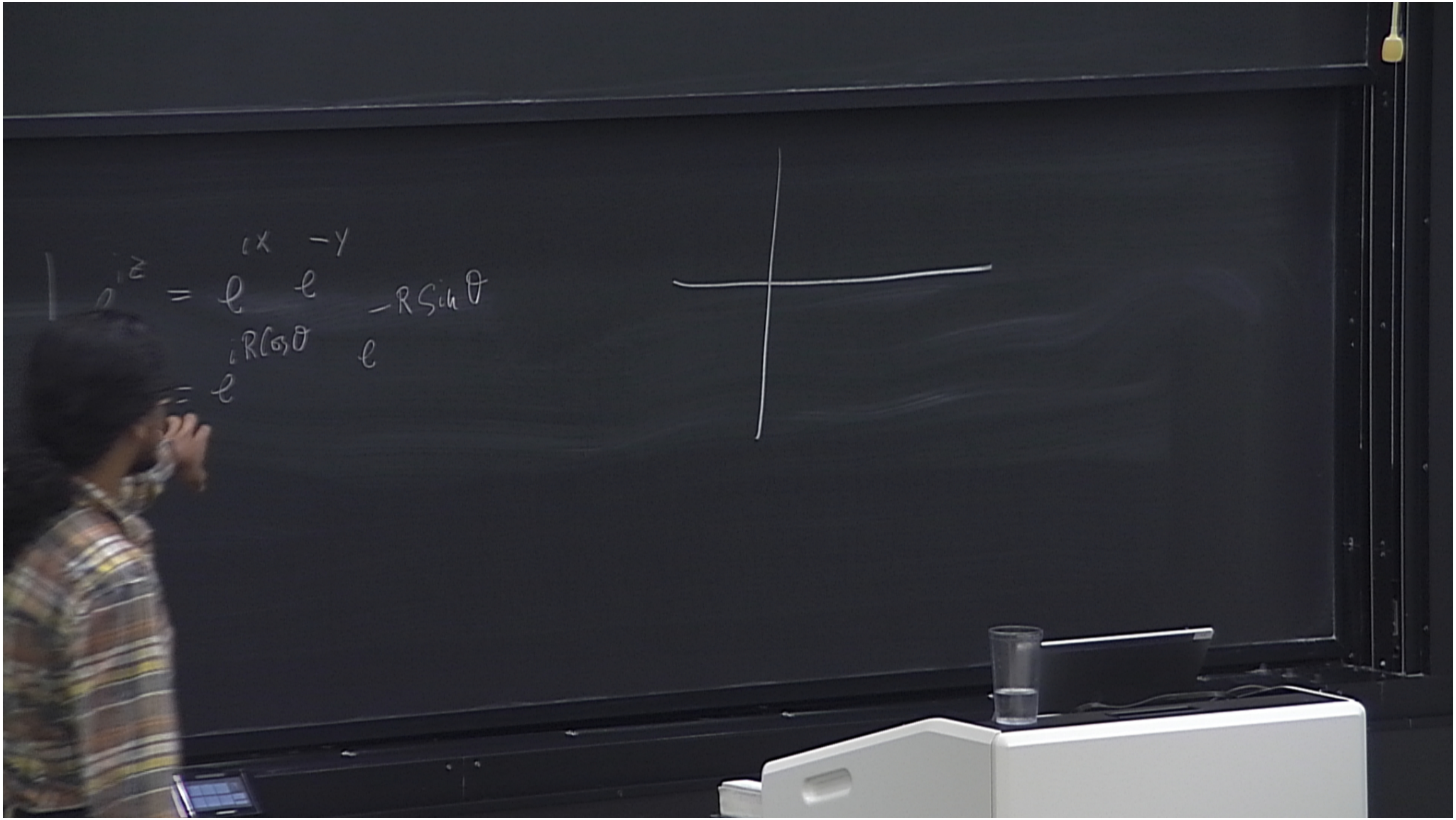

a = b
$\frac{1}{2} 2\pi i \lim_{z \to ia} \frac{d}{dz} (z - ia) \frac{1}{(z-ia)^2 (z+ia)^2} = \frac{\pi}{4a^3}$

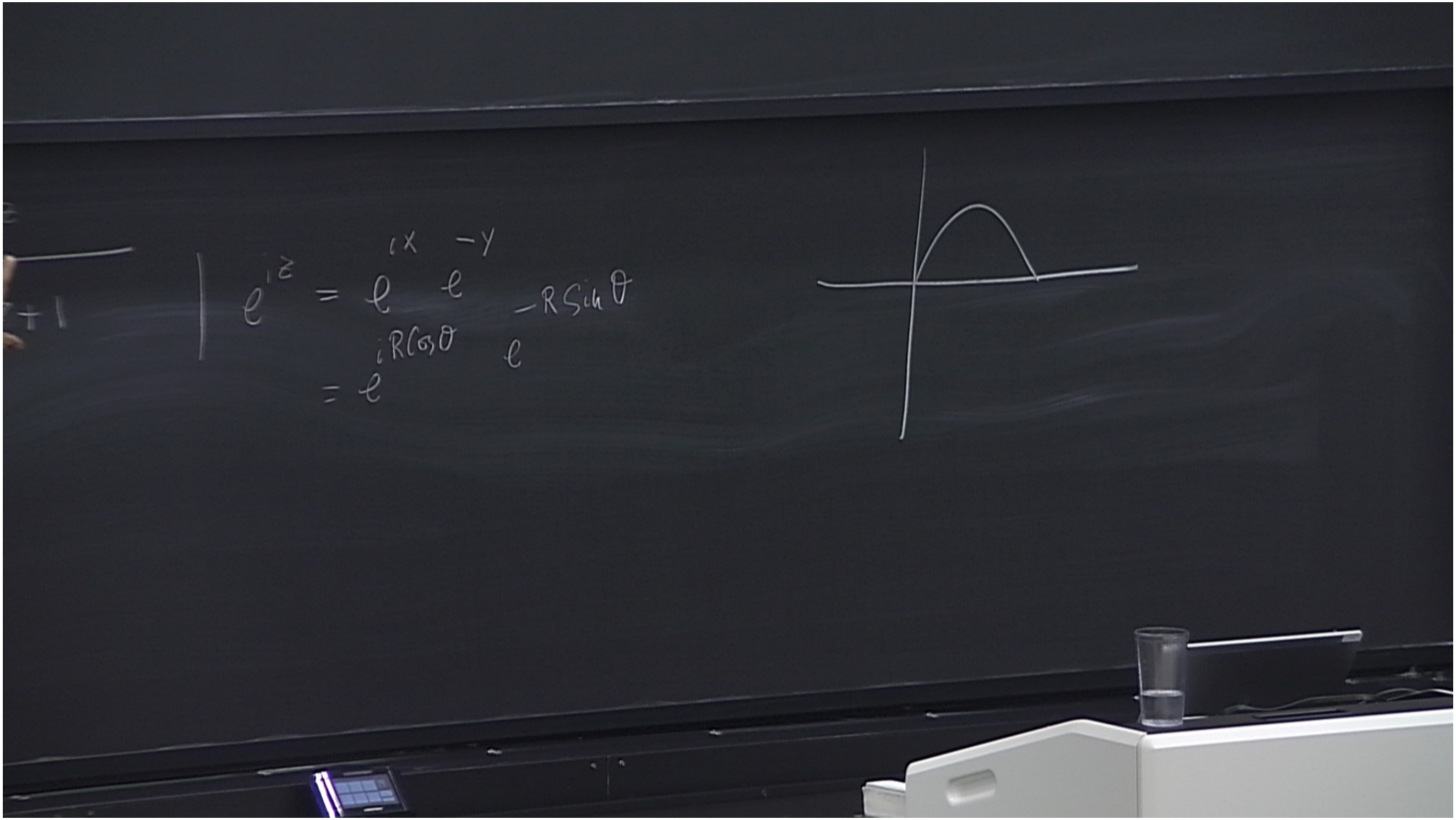

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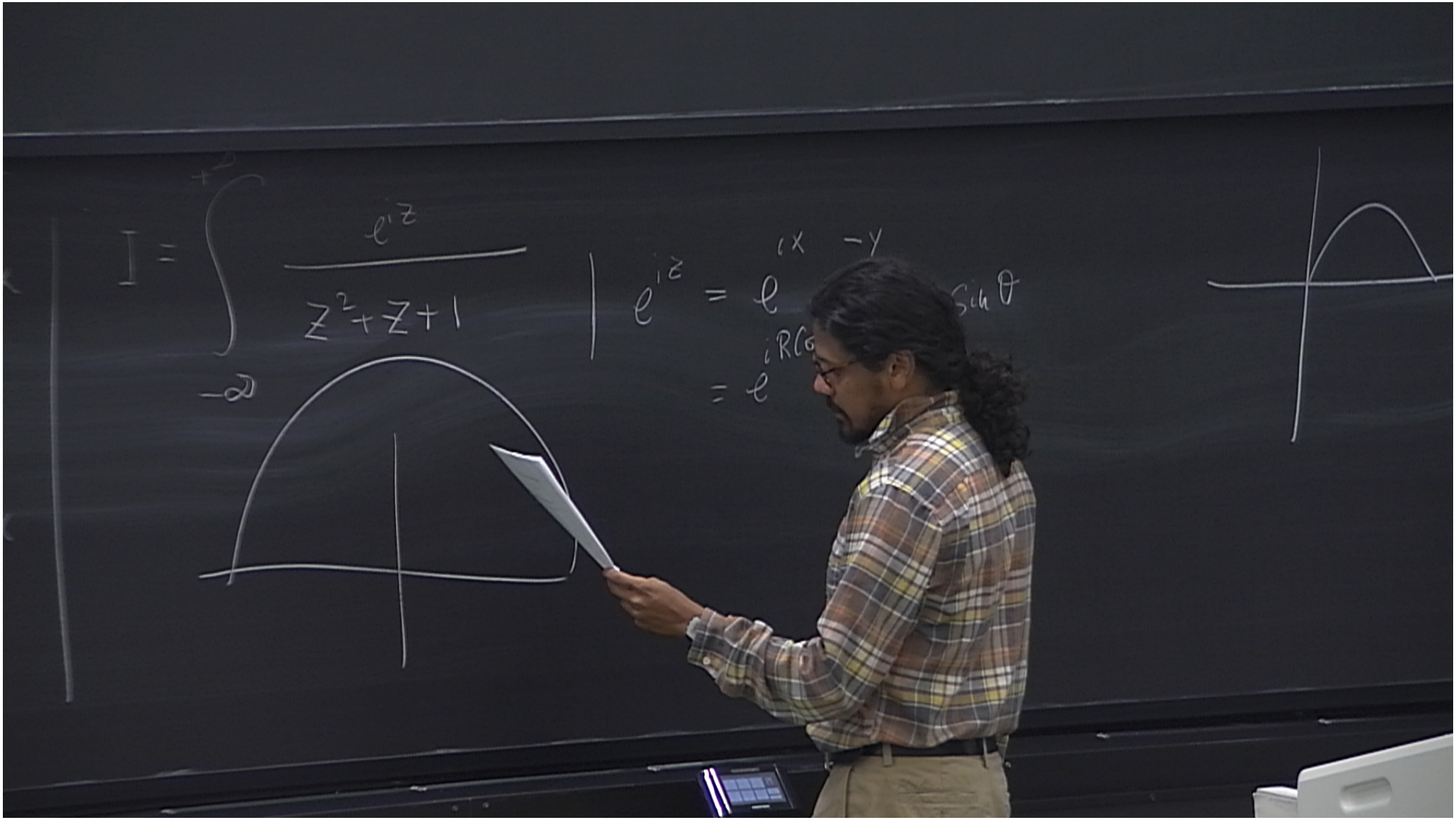

$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\,dx$
$\mathrm{Im}\, f \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\,dx$
I


$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\,dx$
$\mathrm{Im}\, f \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\,dx$
$I = \int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+z+1}$


$\frac{\sin x}{x^2+x+1}\,dx$
$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\,dx$
$I = \int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+z+1}$
$e^{iz} = e^{ix}\,e^{-y}$


ix −y
= e e
−R Sin θ
iR Cos θ e
= e


$e^{iz} = e^{ix} e^{-y}$
$= e^{iR\cos\theta} e^{-R\sin\theta}$

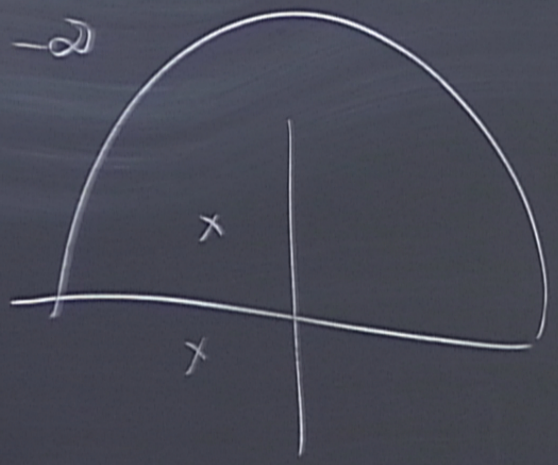

$I = \int \frac{e^{iz}}{z^2+z+1}$
$e^{iz} = e^{ix-y}$
$= e^{iR(\cos\theta}$

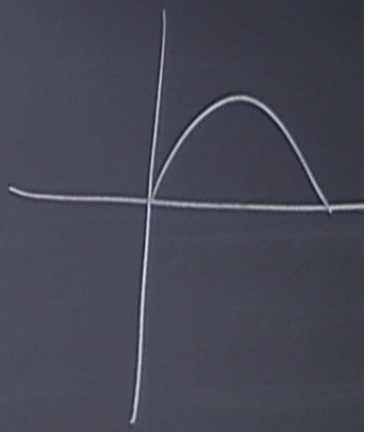
$$I = \int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+z+1}$$

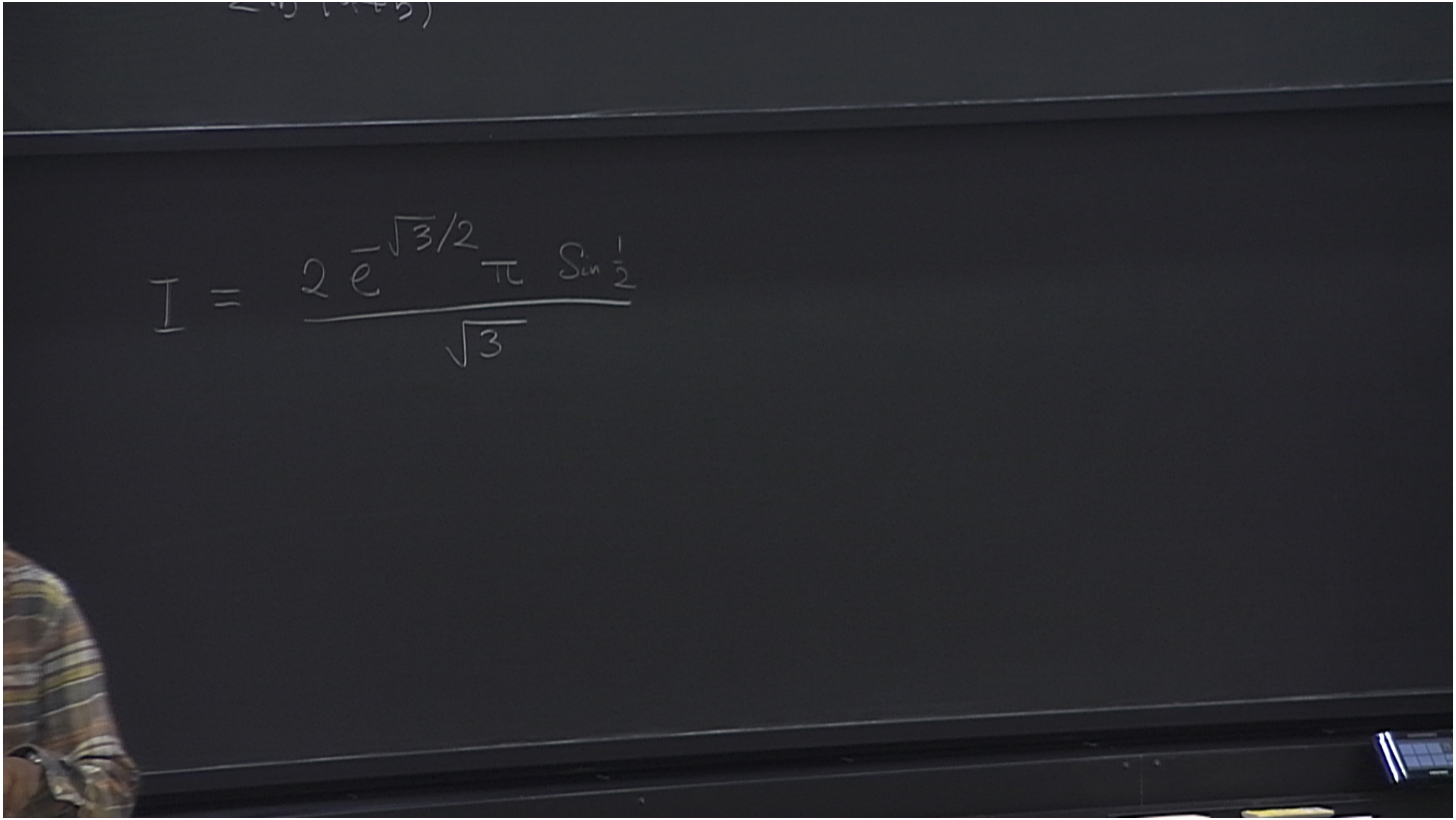
$$e^{iz} = e^{ix}\, e^{-y}$$

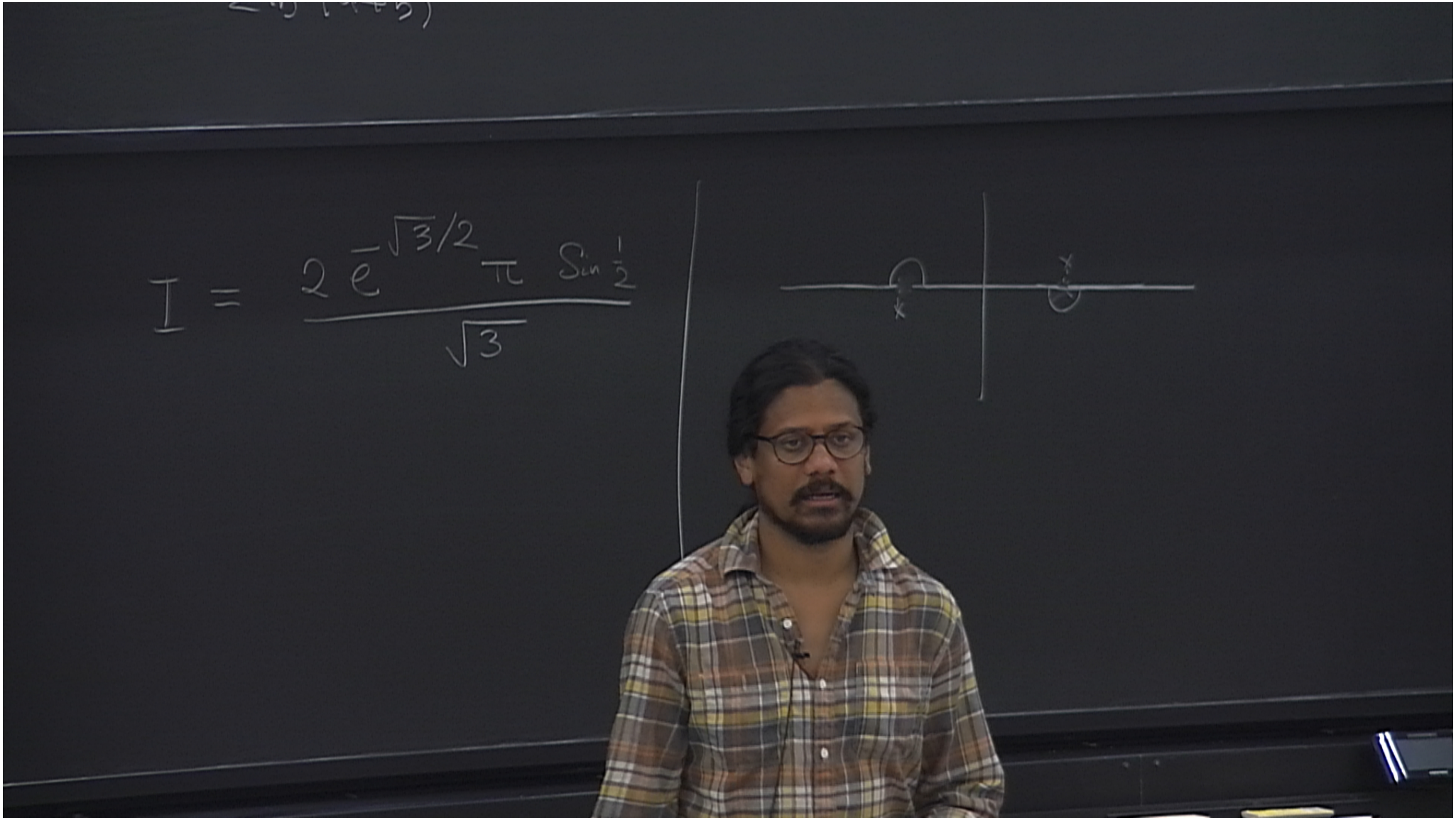
$$= e^{iR\cos\theta}\, e^{-R\sin\theta}$$

$$z_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

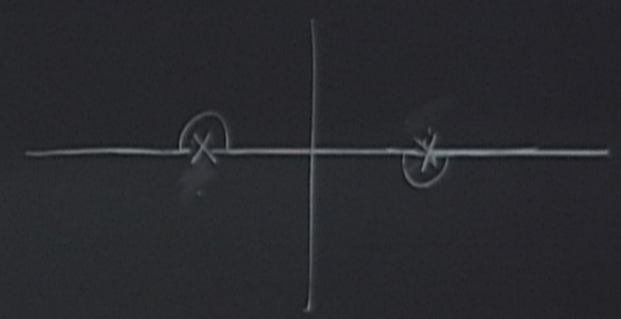




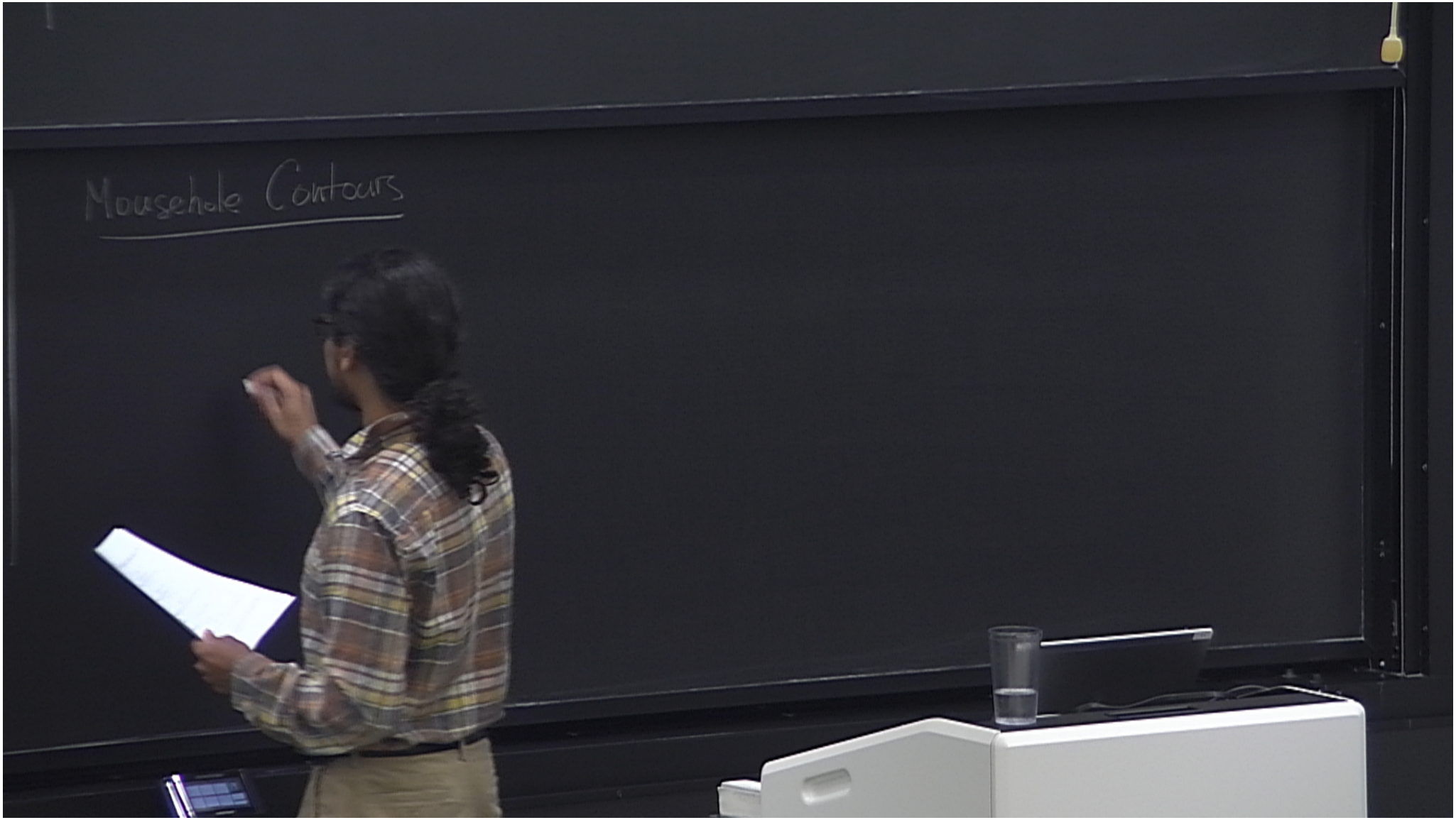

$I = \frac{2 e^{-\sqrt{3}/2} \pi \sin \frac{1}{2}}{\sqrt{3}}$

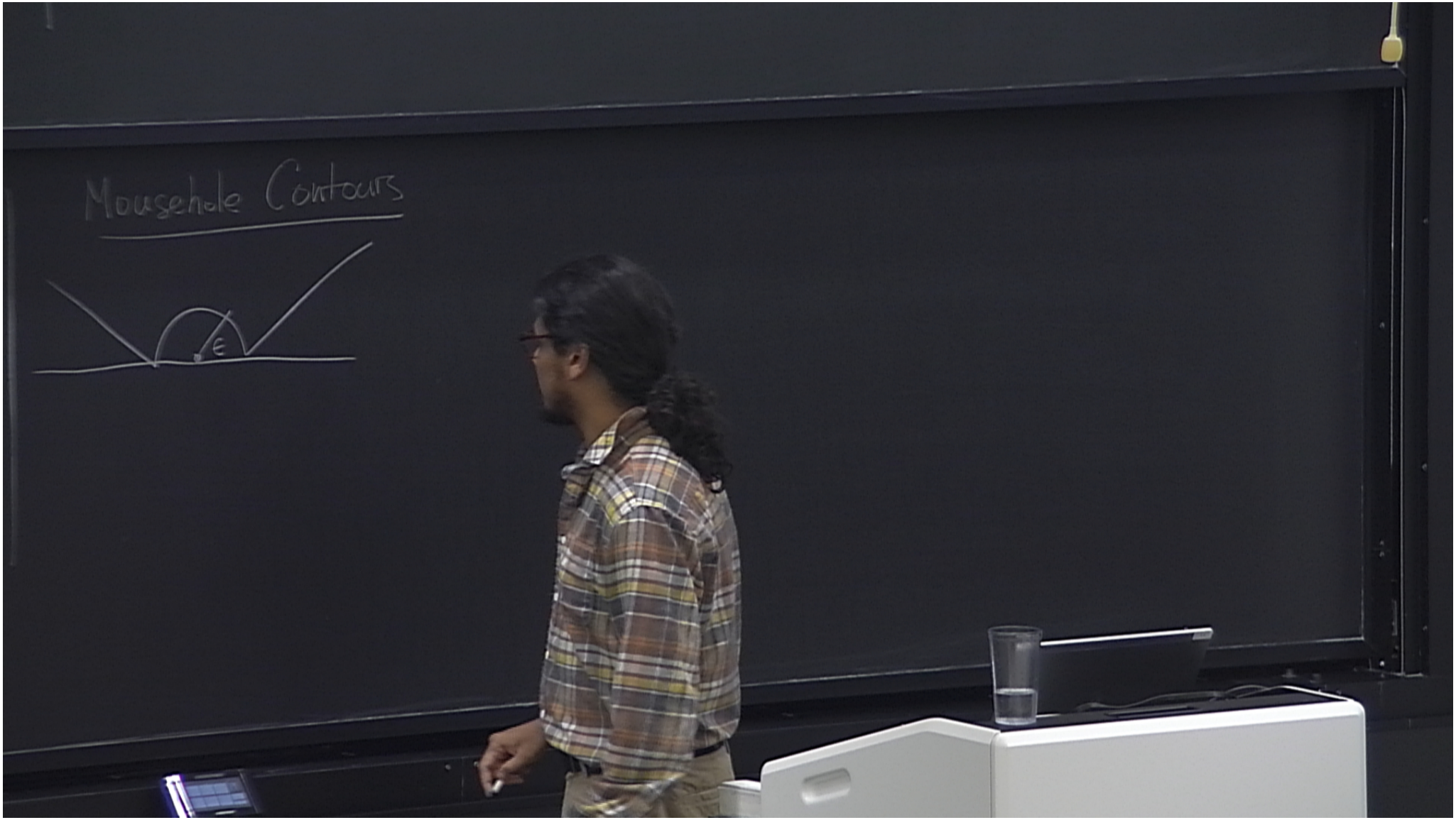

$I = \frac{2 e^{-\sqrt{3}/2} \pi \sin \frac{1}{2}}{\sqrt{3}}$

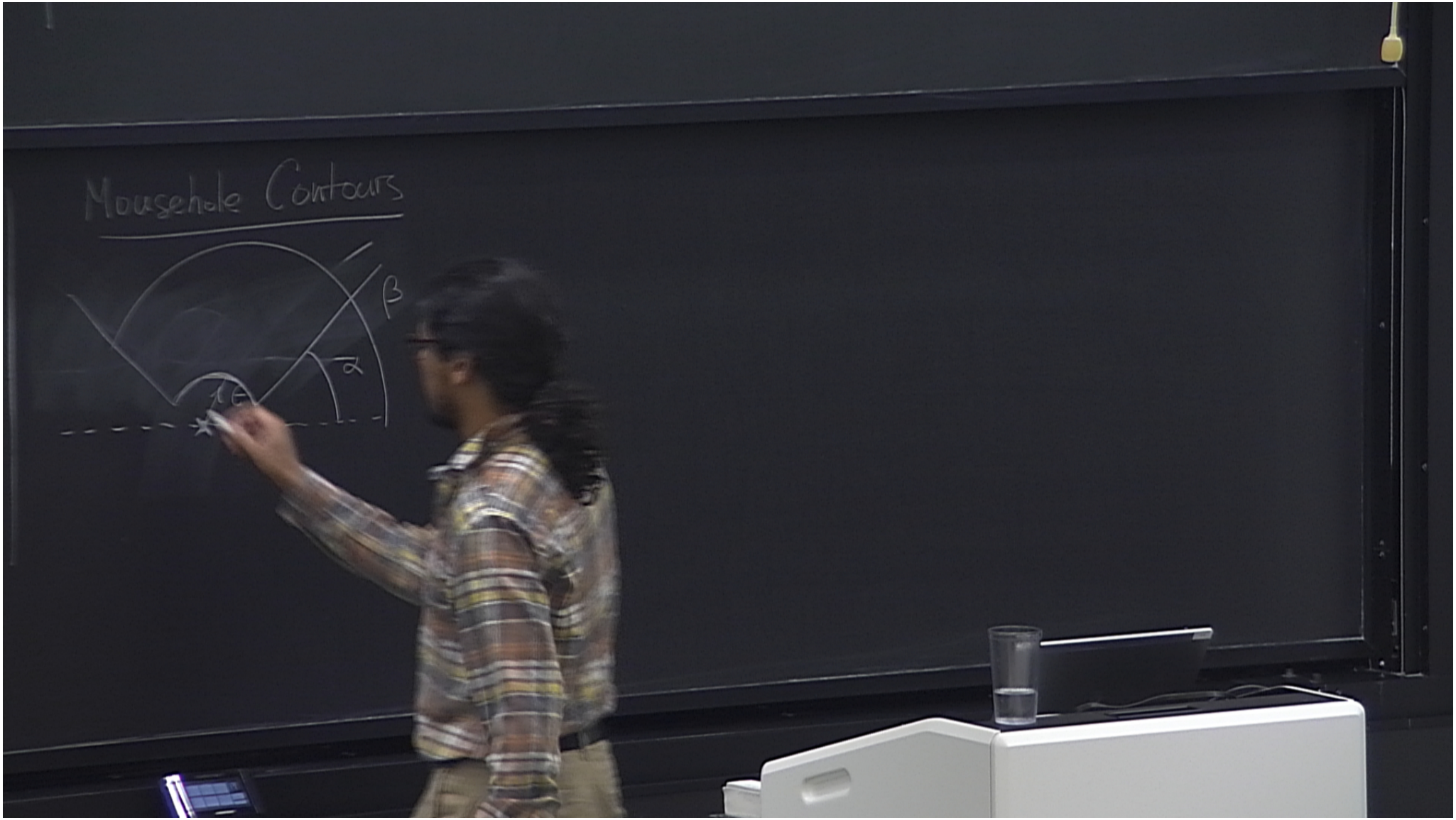
$$I = \frac{2\, e^{-\sqrt{3}/2}\, \pi\, \sin\frac{1}{2}}{\sqrt{3}}$$

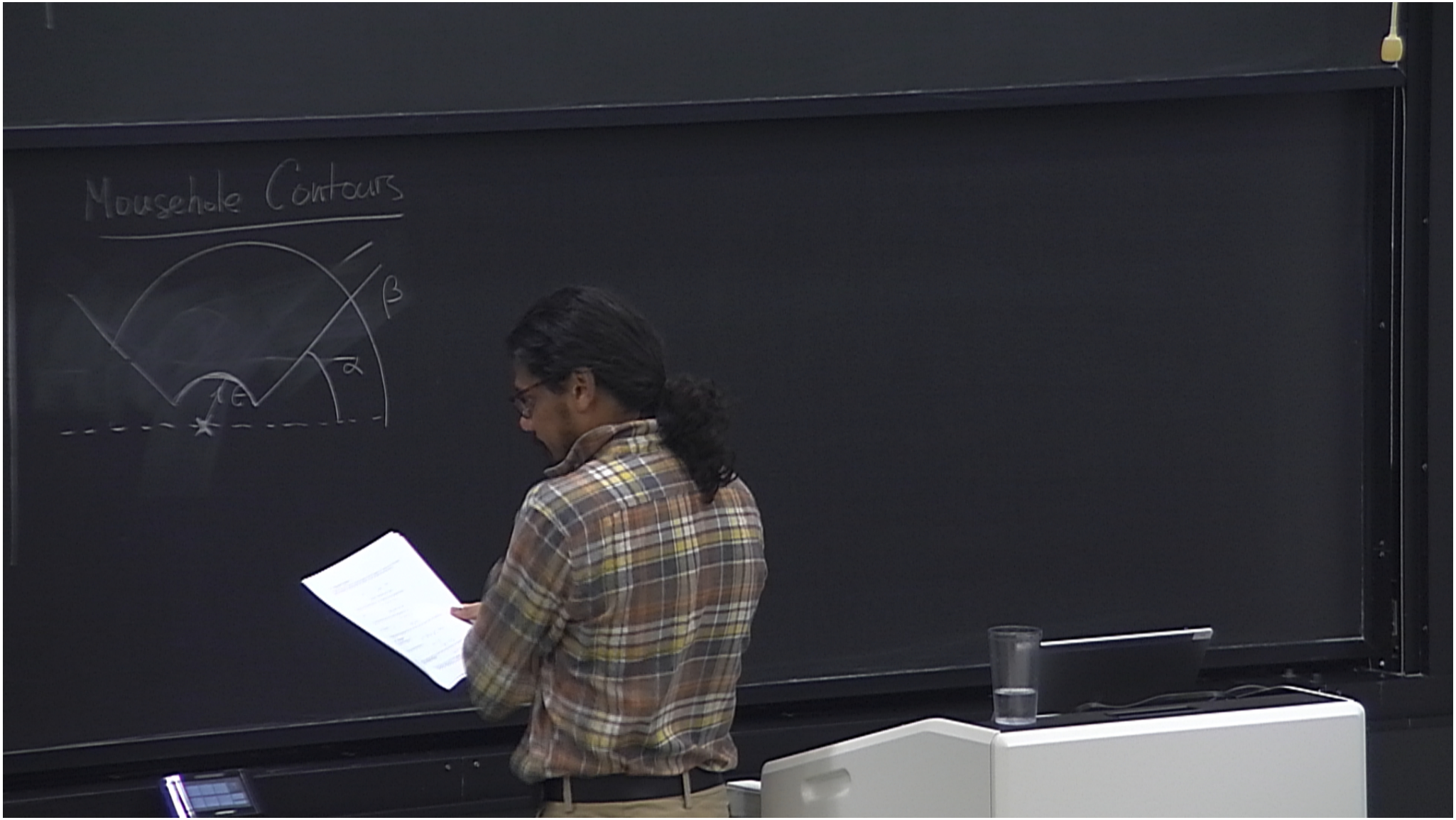


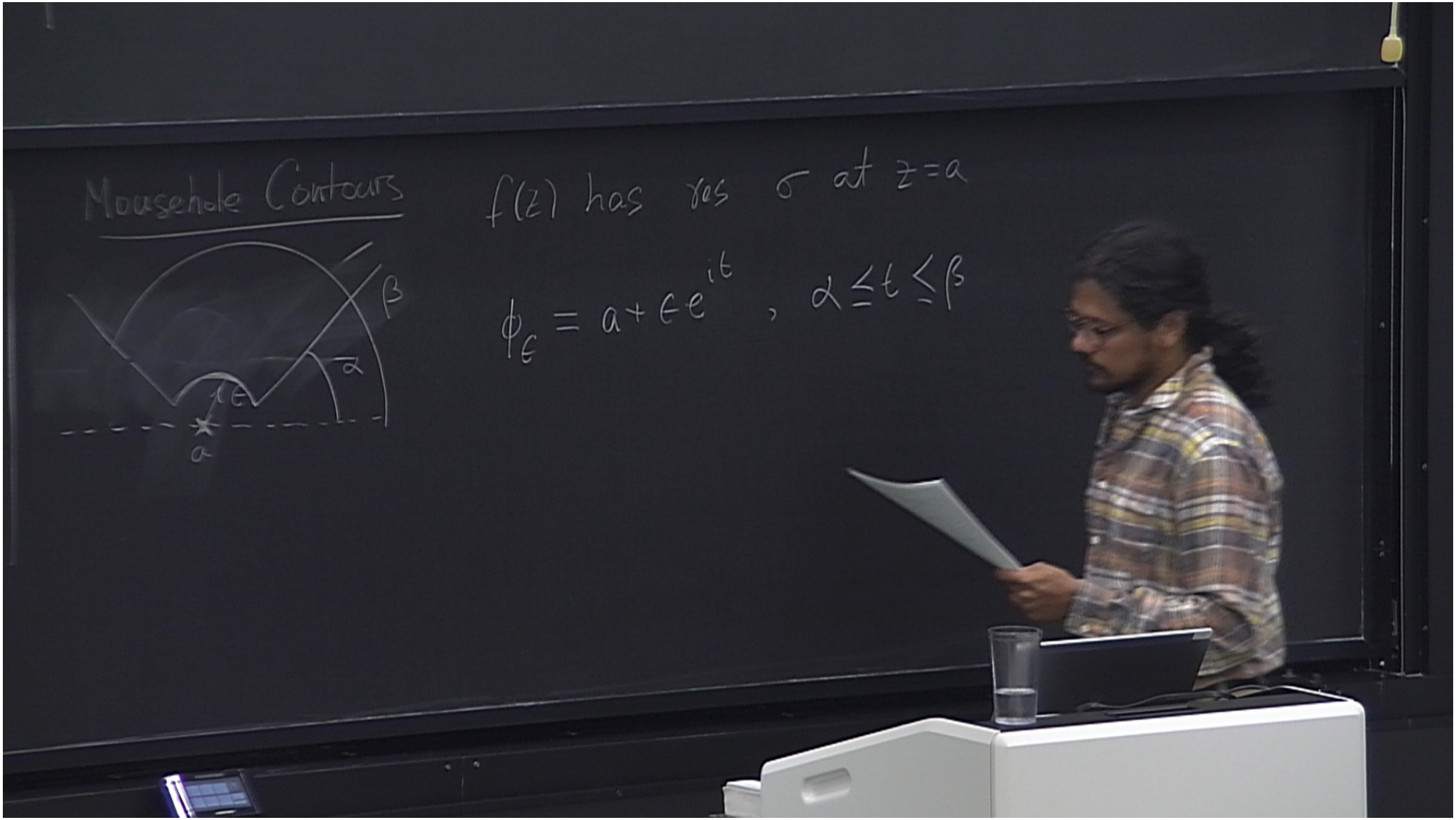
M


Mousehole Contours


Mousehole Contours
ϵ

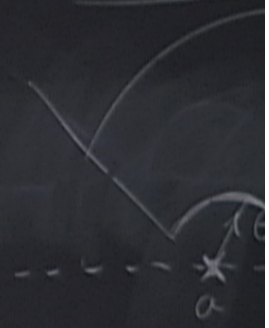

Mousehole Contours
β
α


Mousehole Contours
β
α


Mousehole Contours
f(z) has res σ at z=a
$\phi_\epsilon = a + \epsilon e^{it}$, $\alpha \le t \le \beta$
β
α
ε
a

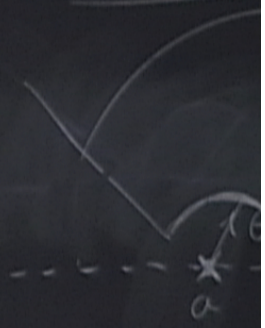
$$\lim_{\epsilon \to 0} \int_{\phi_c} f(z)\,dz = (\beta - \alpha)\,i\sigma$$

Mousehole



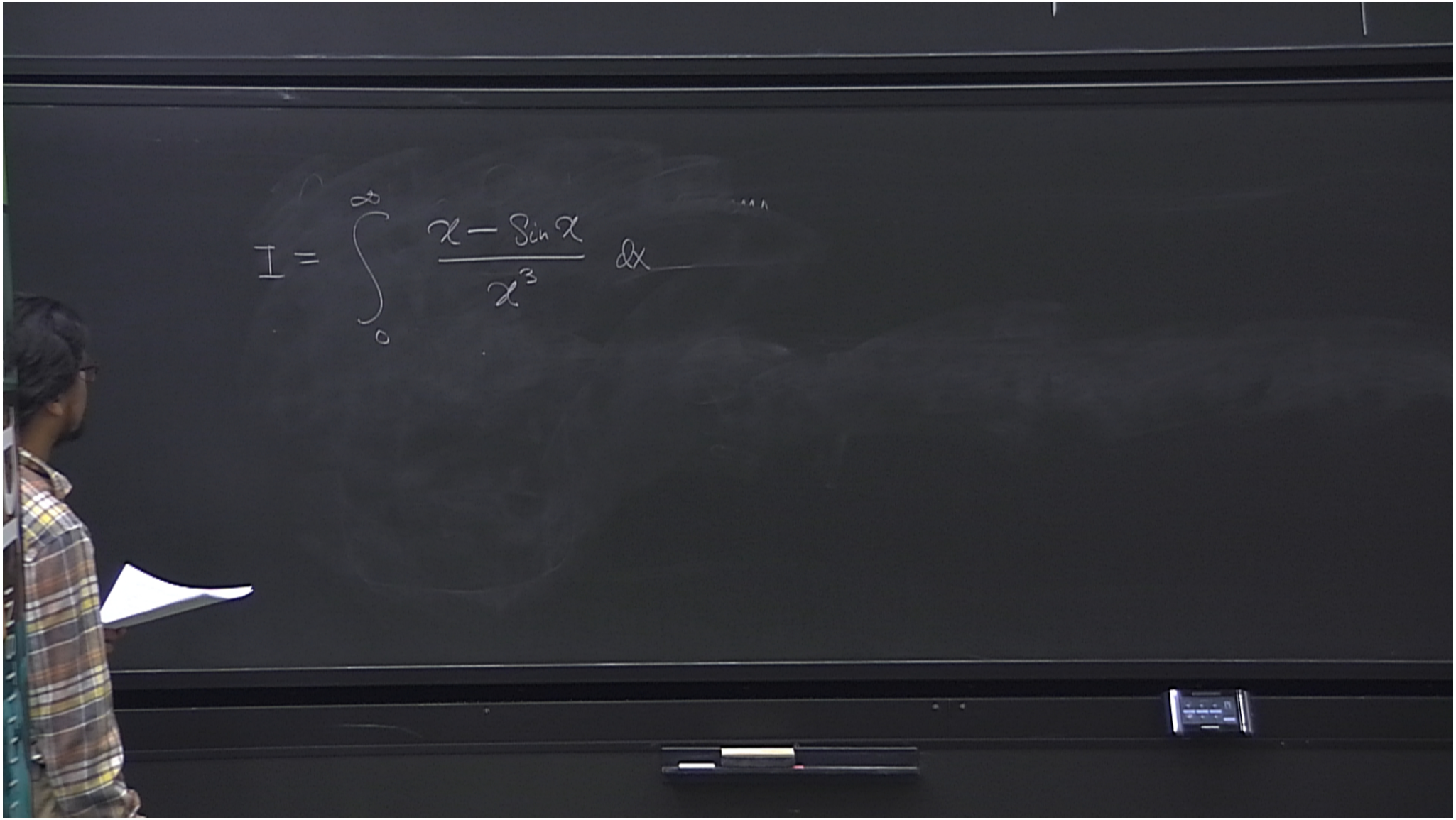
$$\lim_{\epsilon \to 0} \int_{\phi_\epsilon} f(z)\,dz = (\beta - \alpha)\,i\sigma$$

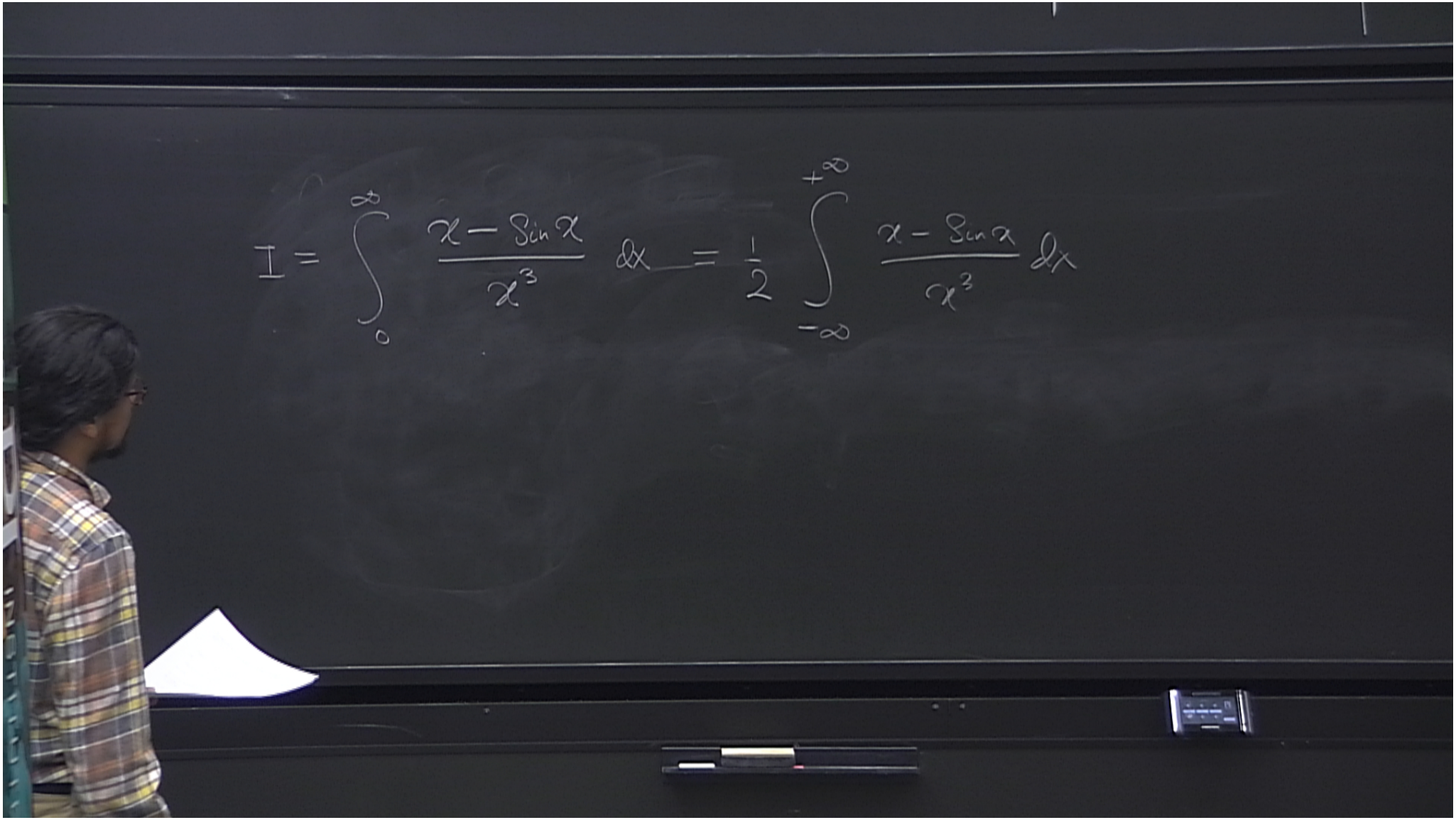
Mousehole



Proof $f(z)$ has a simple pole at $z = a$

$$g(z) = f(z) - \frac{\sigma}{(z-a)}$$

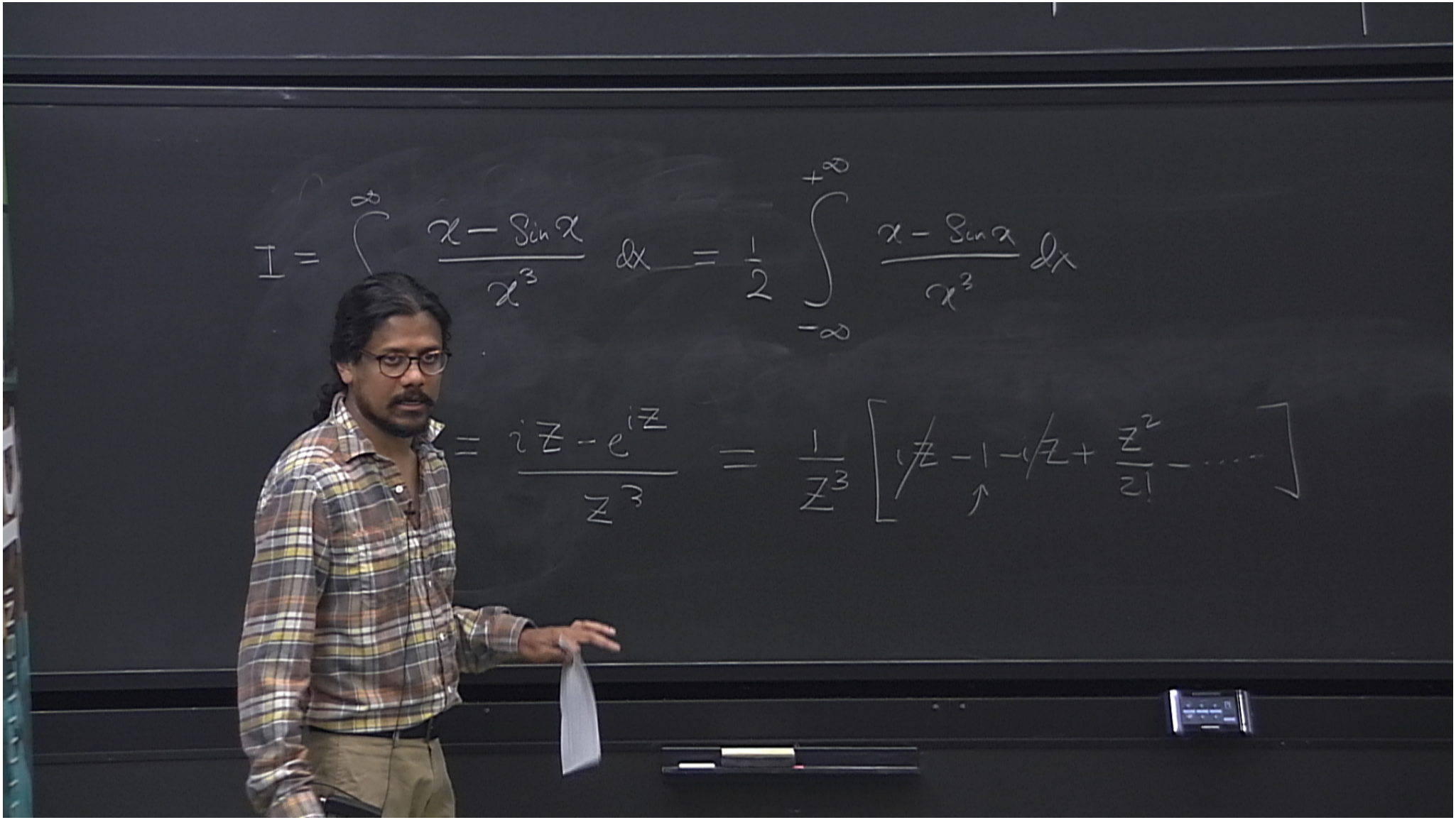

$I = \int_0^{\infty} \frac{x - \sin x}{x^3} \, dx$

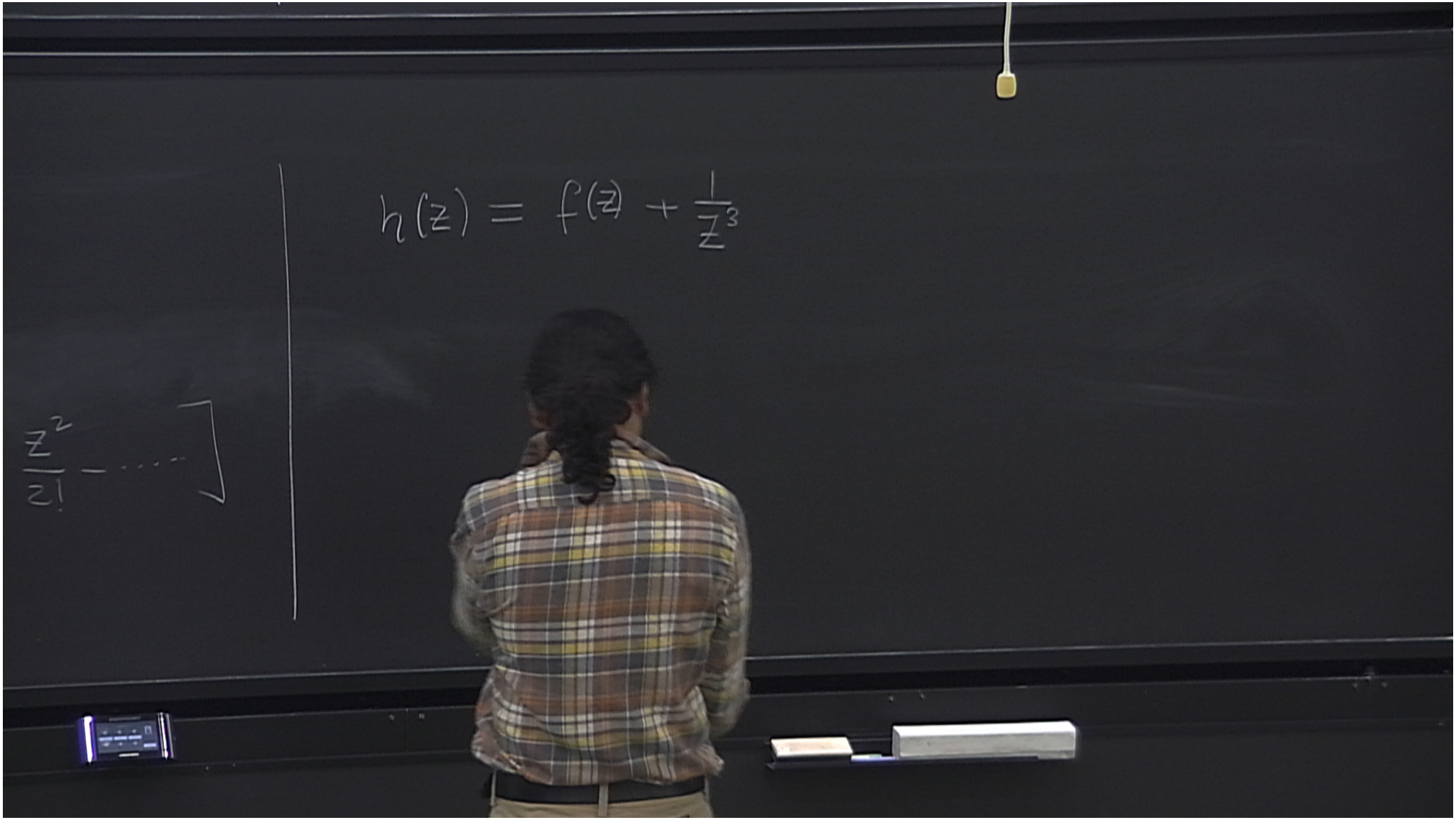

$I = \int_0^{\infty} \frac{x - \sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3}\,dx$

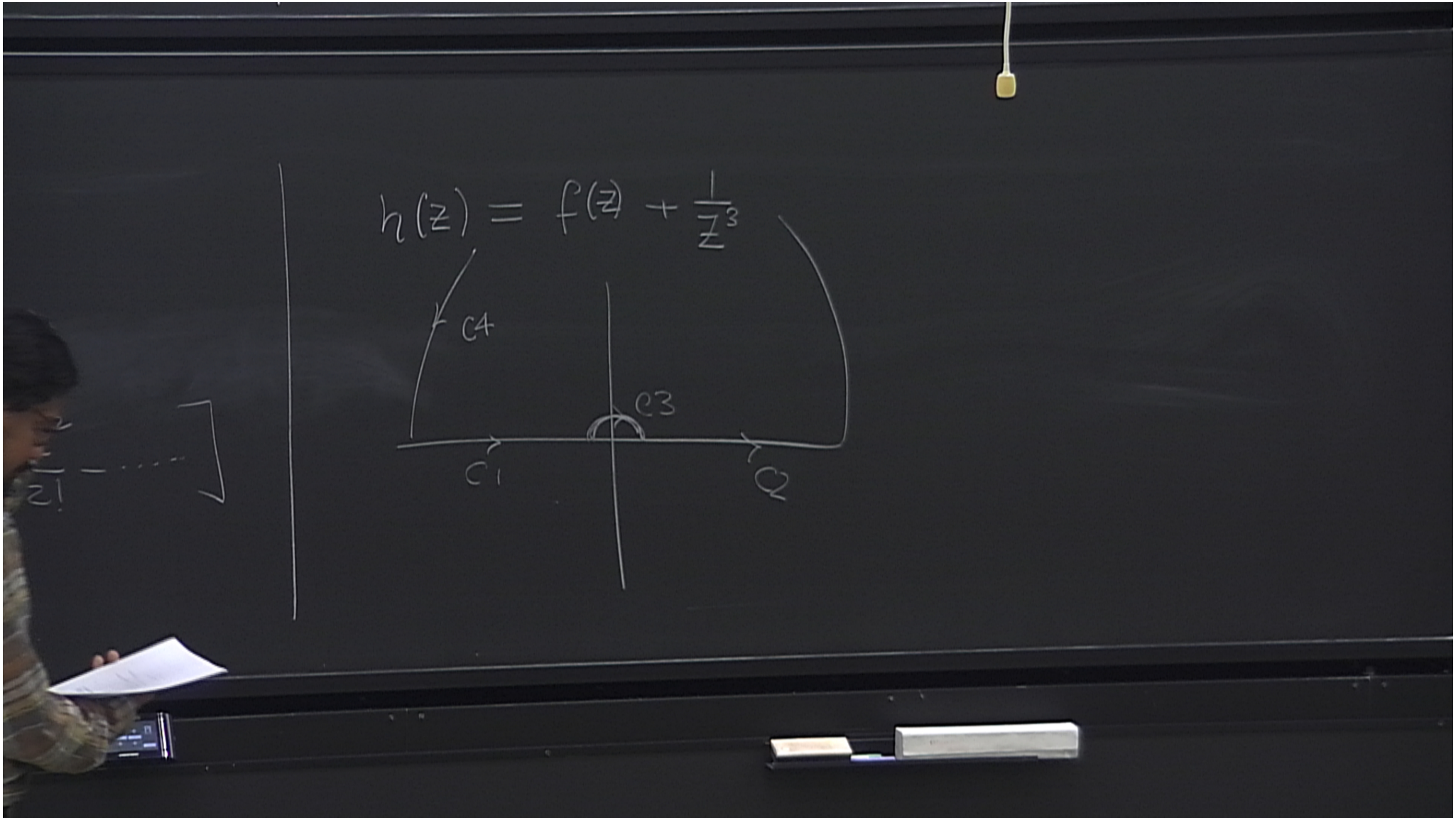
$$I = \int_0^{\infty} \frac{x - \sin x}{x^3}\, dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3}\, dx$$

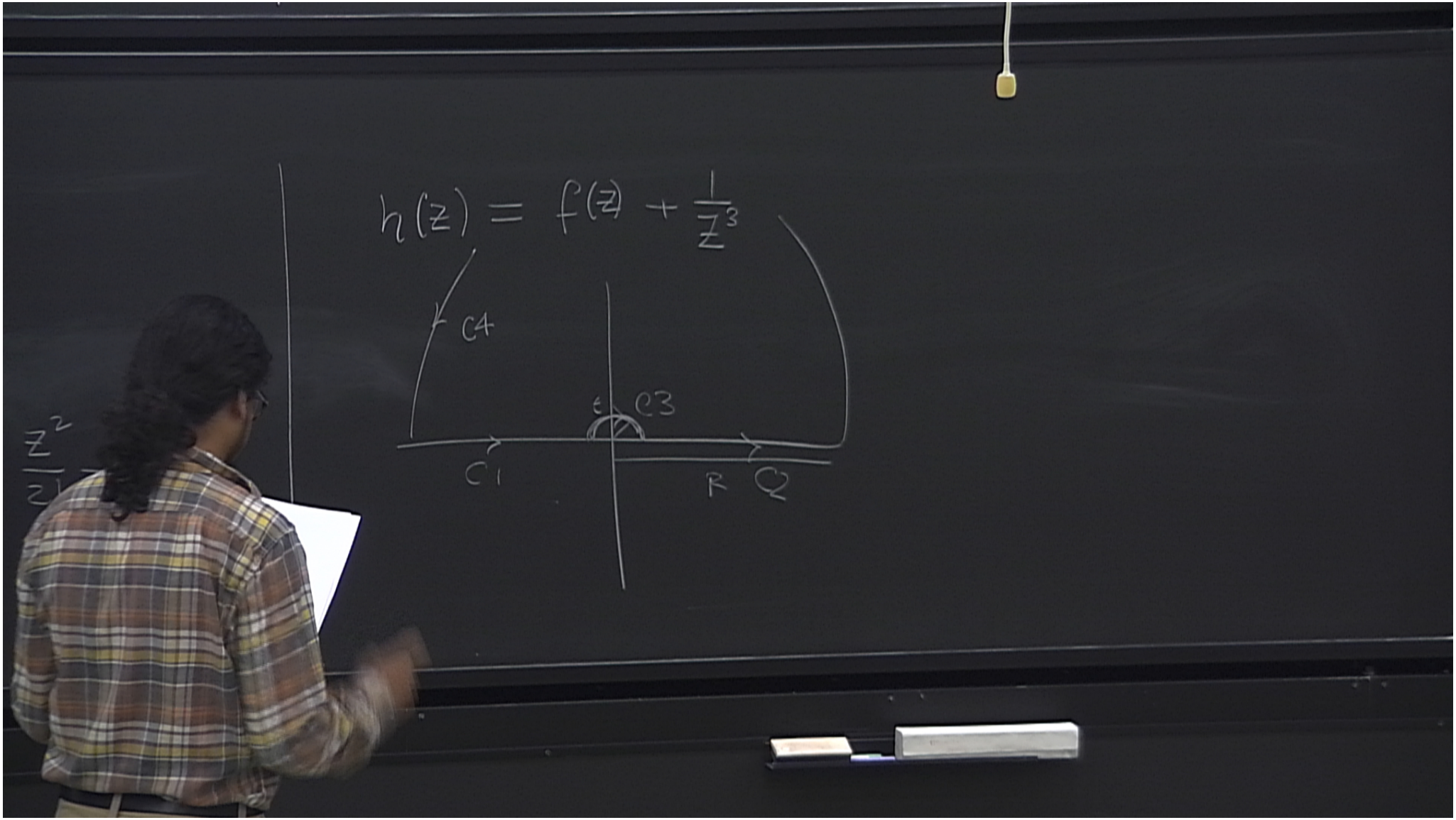
$$f(z) = \frac{iz - e^{iz}}{z^3}$$

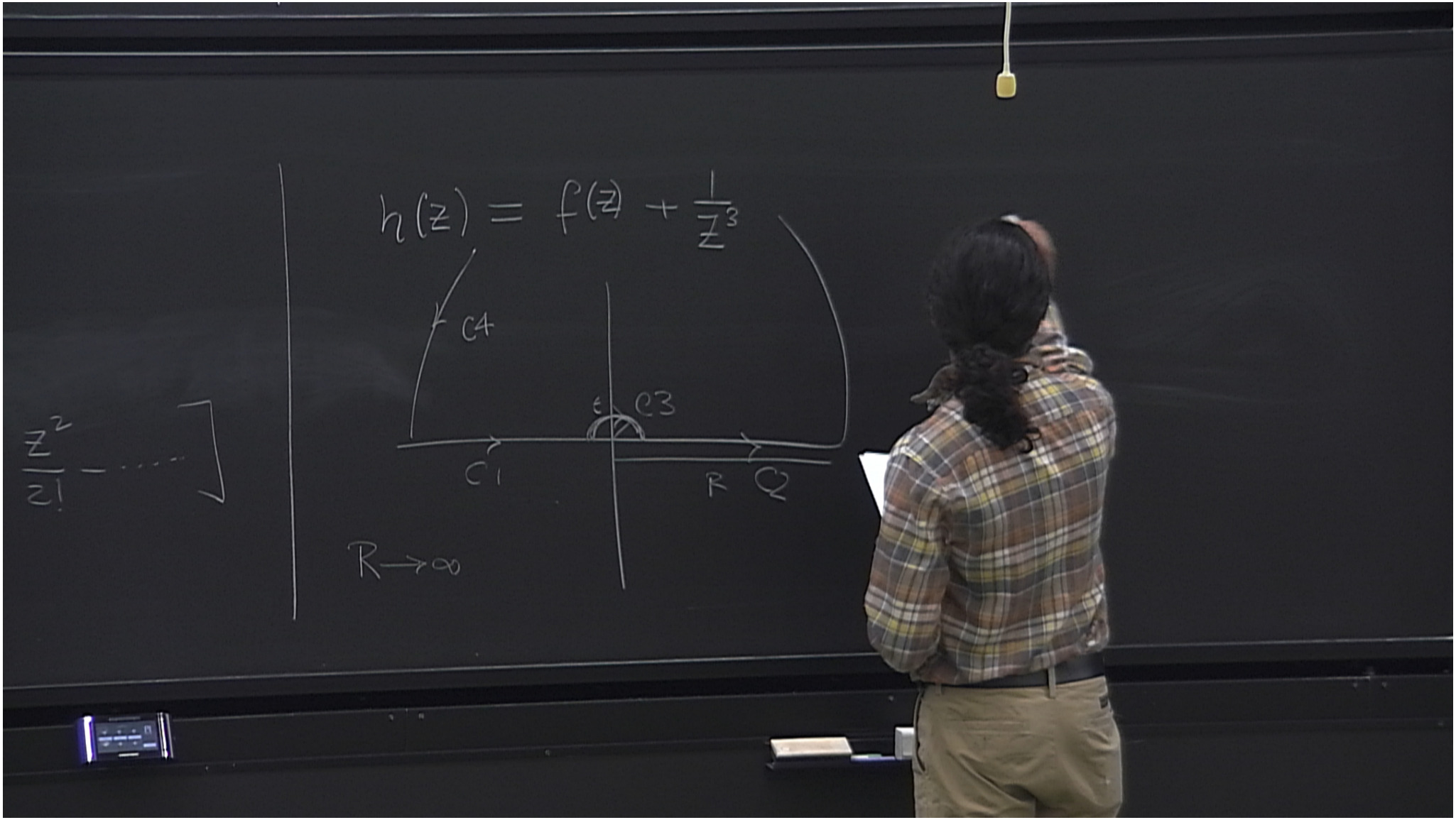


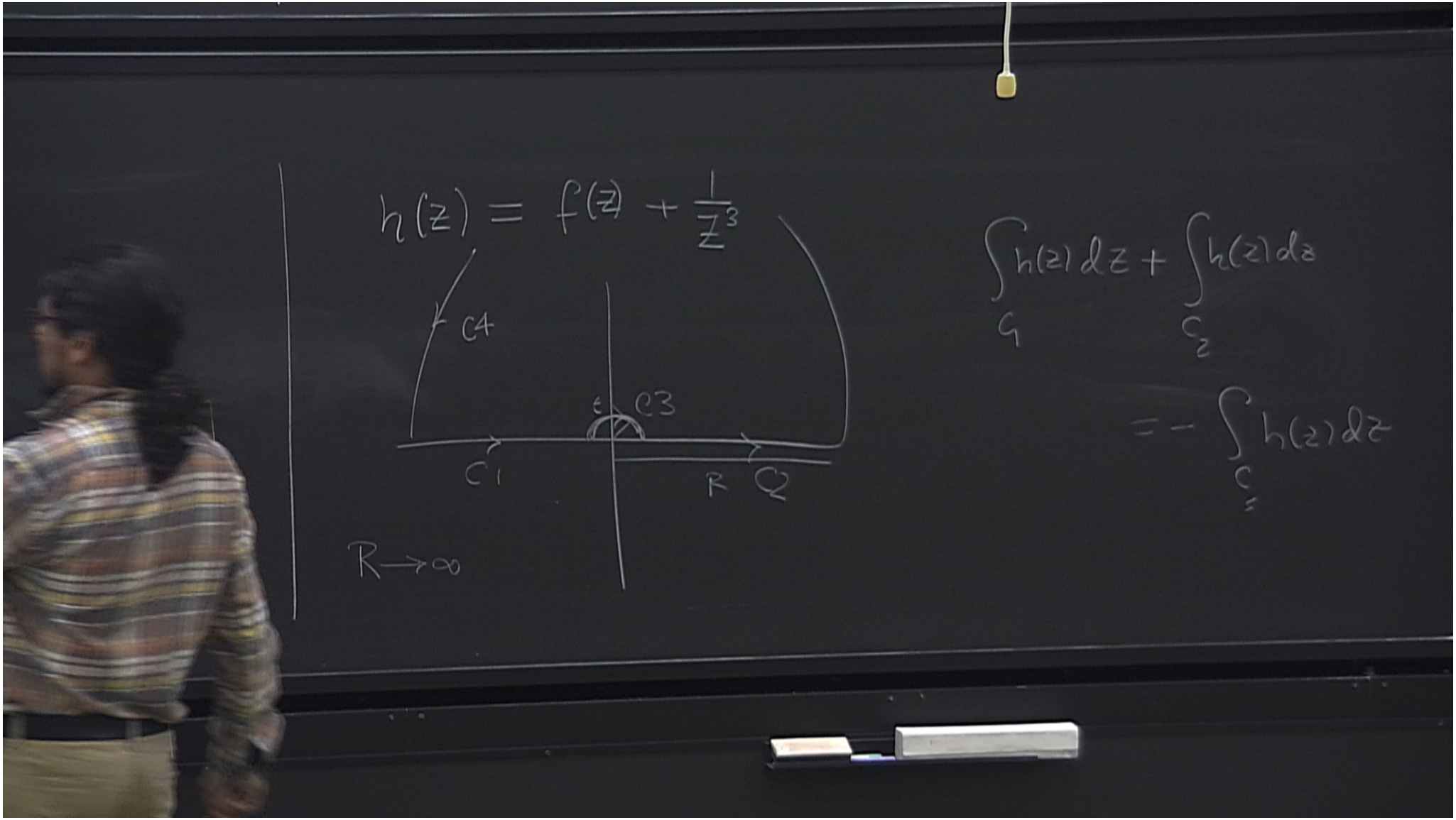

$I = \int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3} dx$
$= \frac{iz - e^{iz}}{z^3} = \frac{1}{z^3} \left[\cancel{iz} - 1 - \cancel{\frac{iz}{2}} + \frac{z^2}{2!} - \cdots \right]$


$h(z) = f(z) + \frac{1}{z^3}$
$\frac{z^2}{2!}$

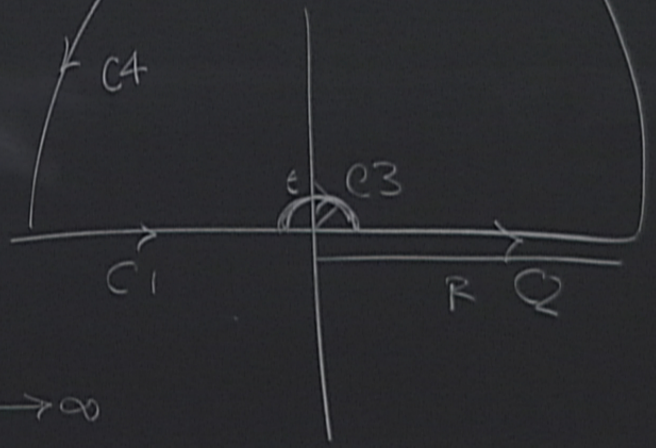

$h(z) = f(z) + \frac{1}{z^3}$
C4
C3
C1
C2


$h(z) = f(z) + \frac{1}{z^3}$
C4
C3
C1
R
C2


$h(z) = f(z) + \frac{1}{z^3}$
C4
C3
C1
C2
R
$R \to \infty$
$\frac{z^2}{2!}$


$h(z) = f(z) + \frac{1}{z^3}$
C4
C3
C1
R
C2
$R \to \infty$
$\int_{C_1} h(z)\,dz + \int_{C_2} h(z)\,dz$
$= -\int_{C_3} h(z)\,dz$

$h(z) = f(z) + \frac{1}{z^3}$



$R \to \infty$

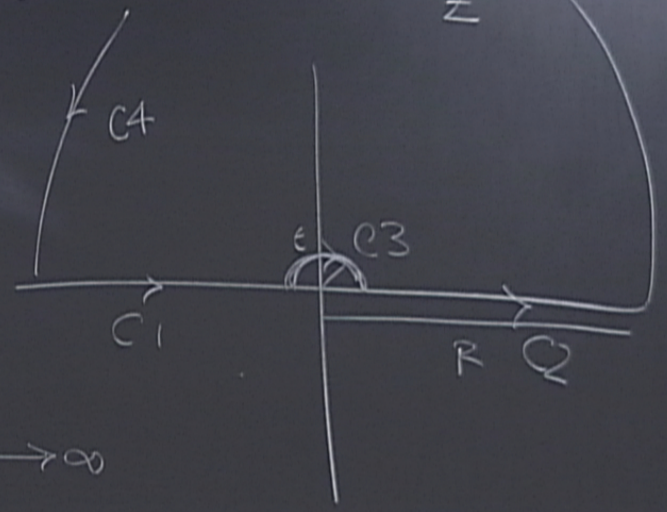
z^2 ……

$$\int_{C_1} h(z)\,dz + \int_{C_2} h(z)\,dz$$

$$= -\int_{C_3} h(z)\,dz$$

$$= \frac{(\pi - 0)i}{2}$$

$$h(z) = f(z) + \frac{1}{z^3}$$



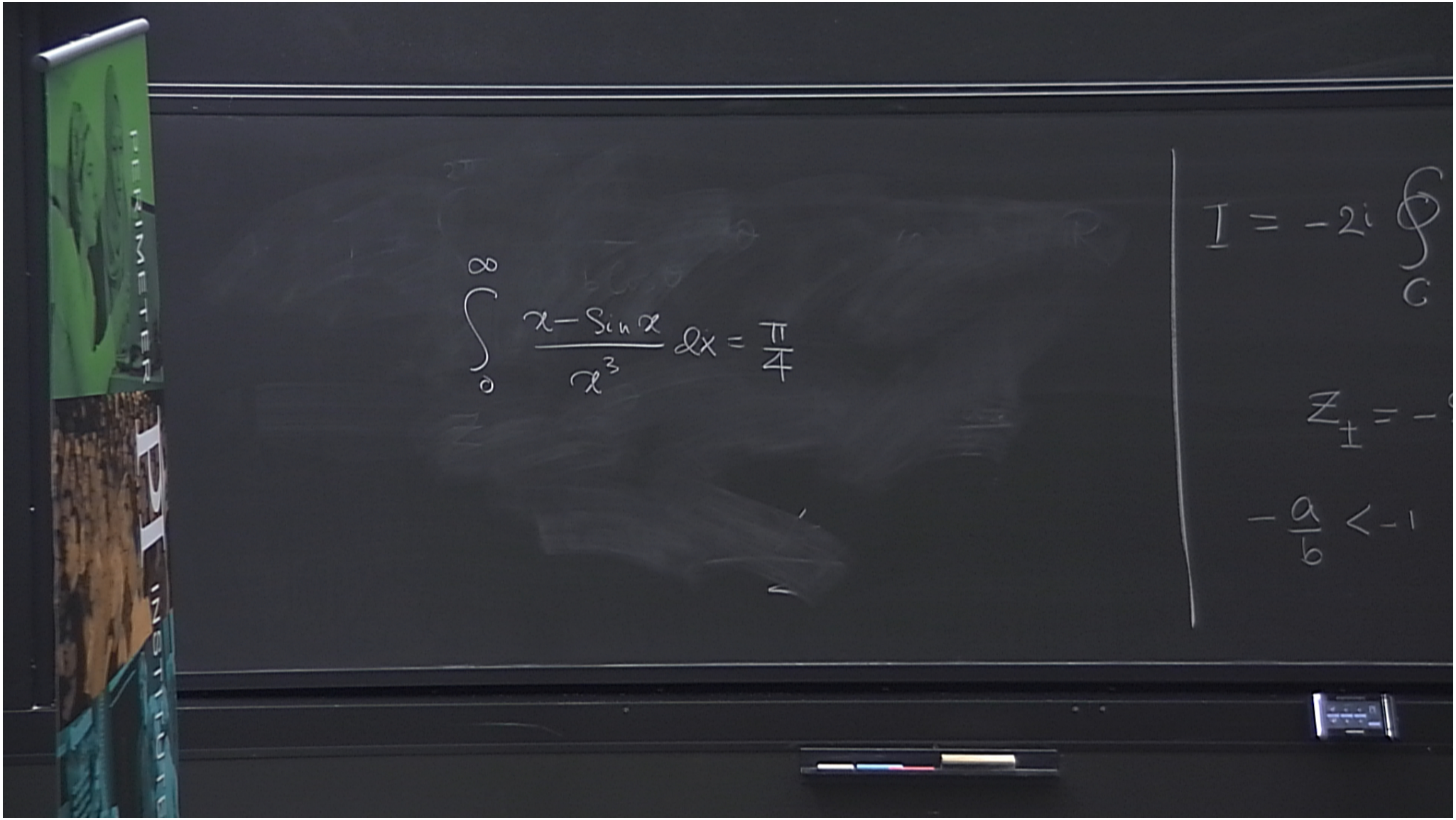
$R \to \infty$

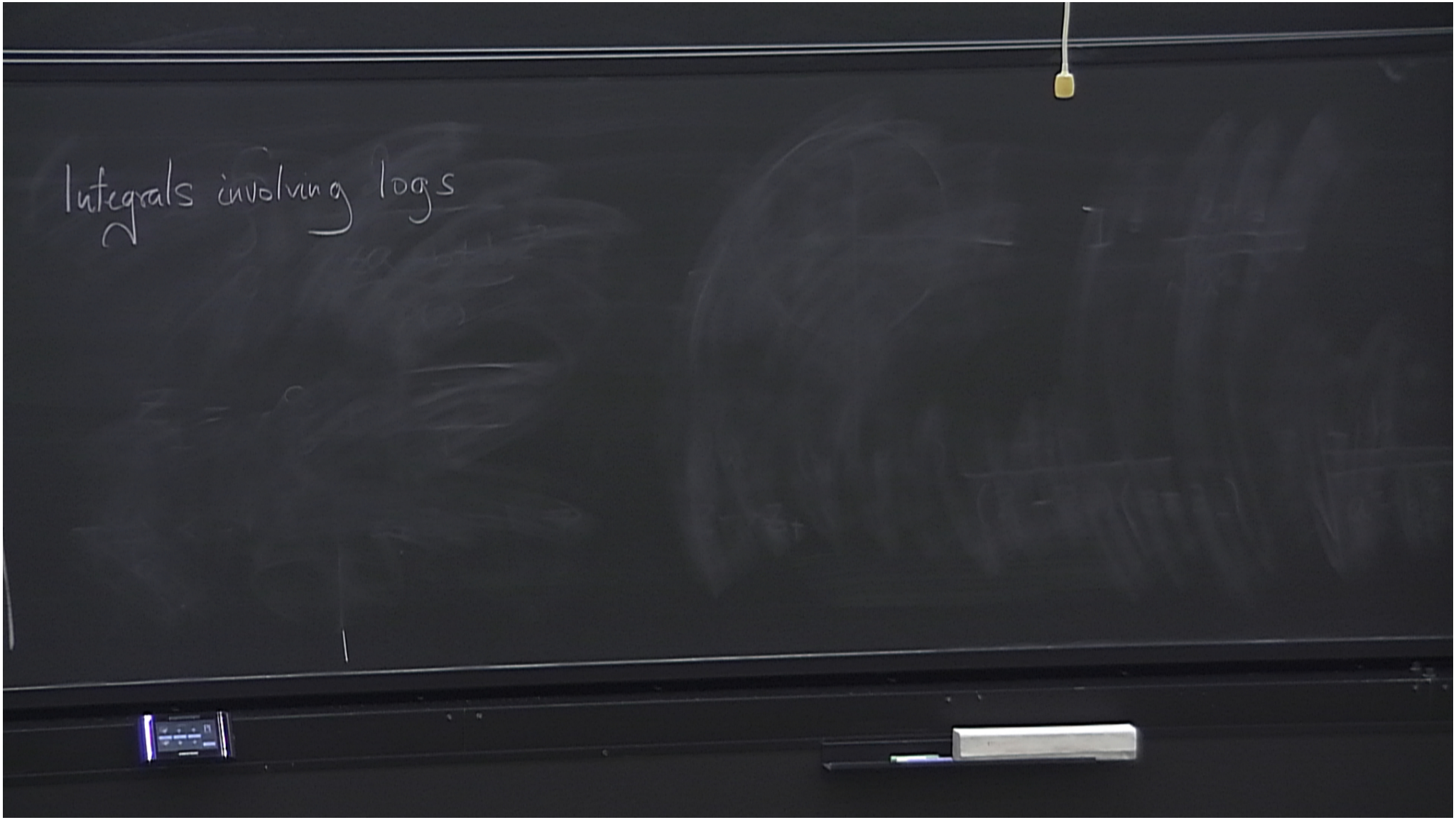
$$\frac{z^2}{2!} - \cdots$$

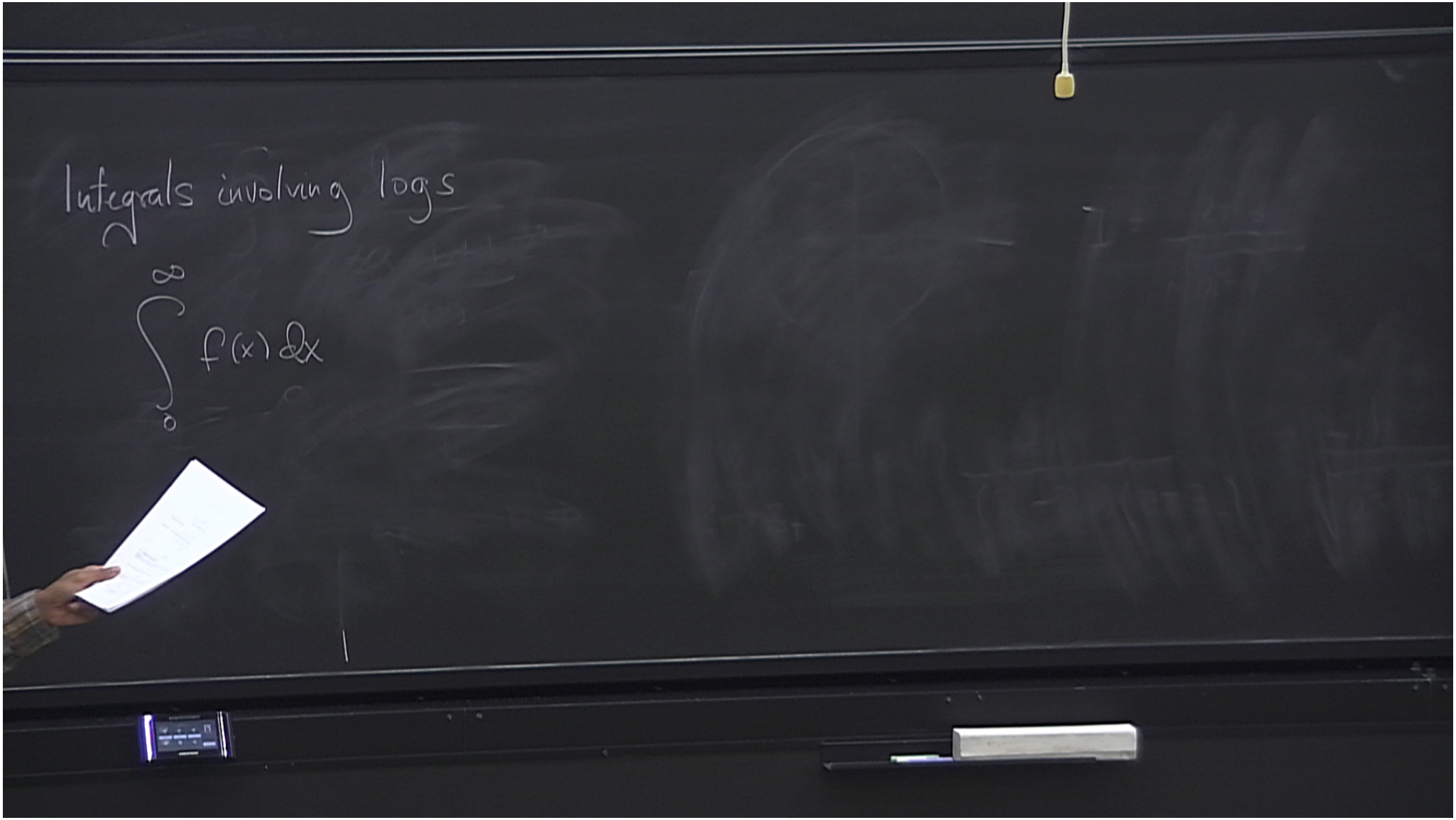
$$\int_{C_1} h(z)\,dz + \int_{C_2} h(z)\,dz$$

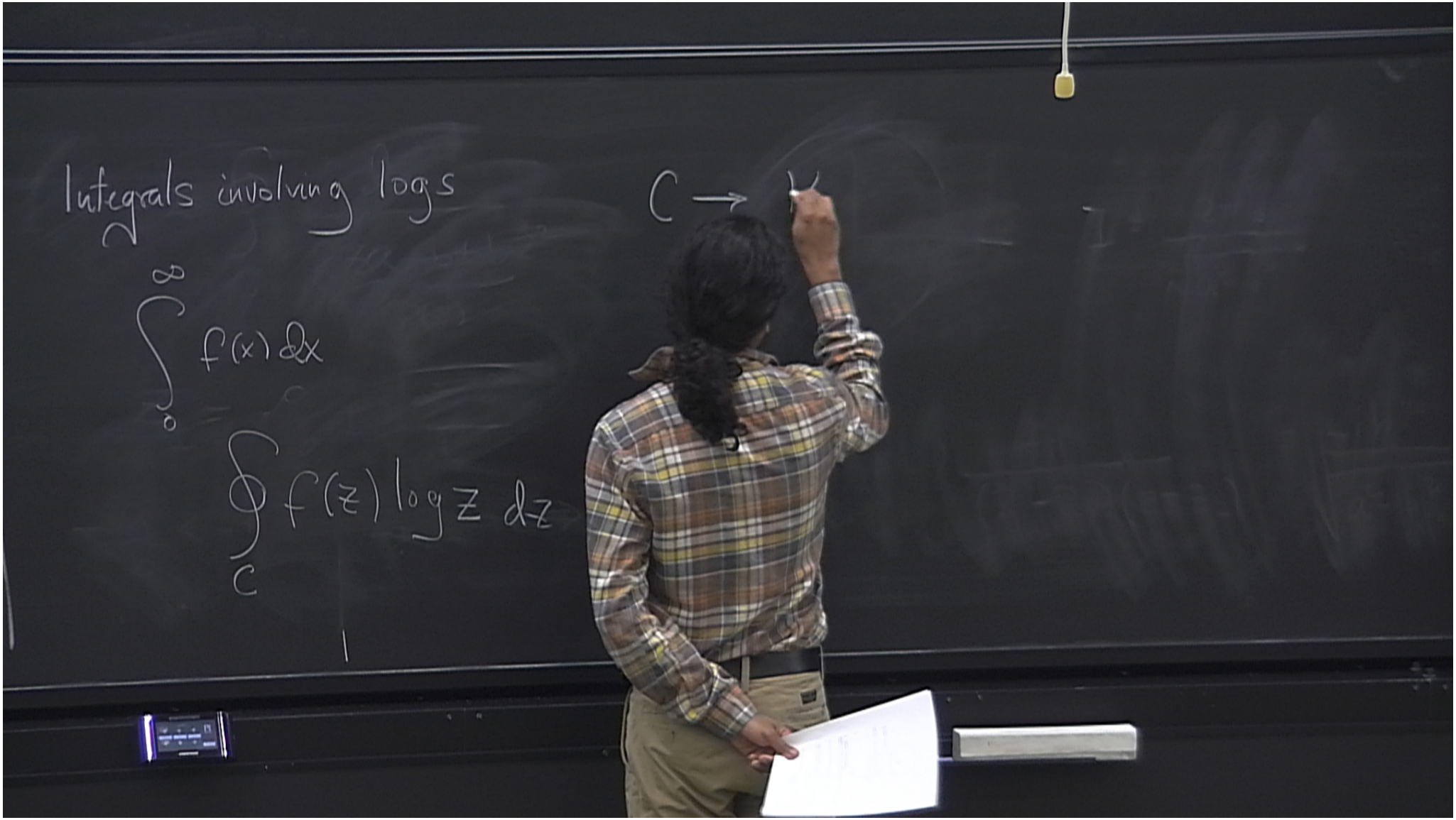
$$= -\int_{C_3} h(z)\,dz$$

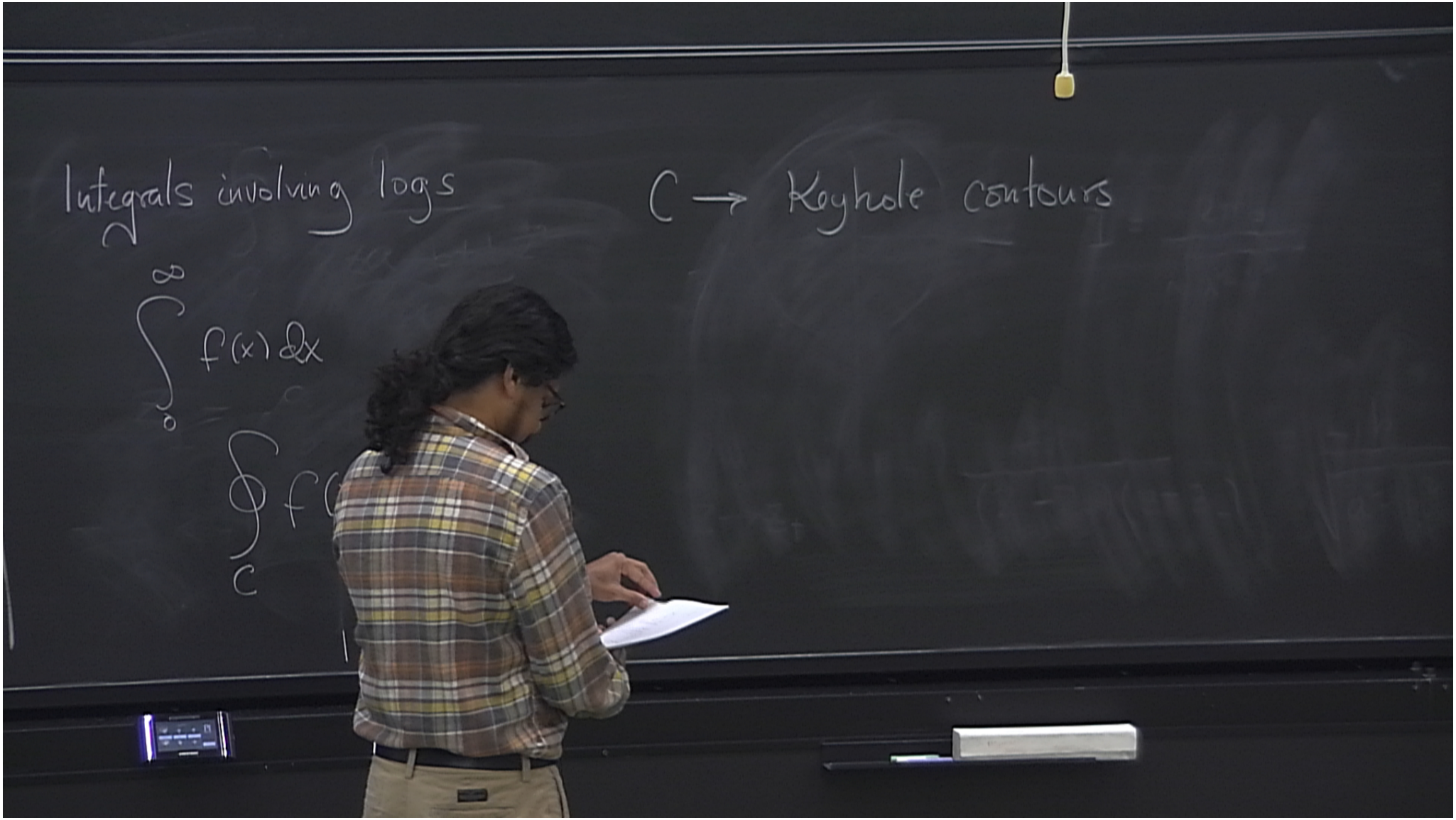
$$= \frac{(\pi - 0)i}{2}$$

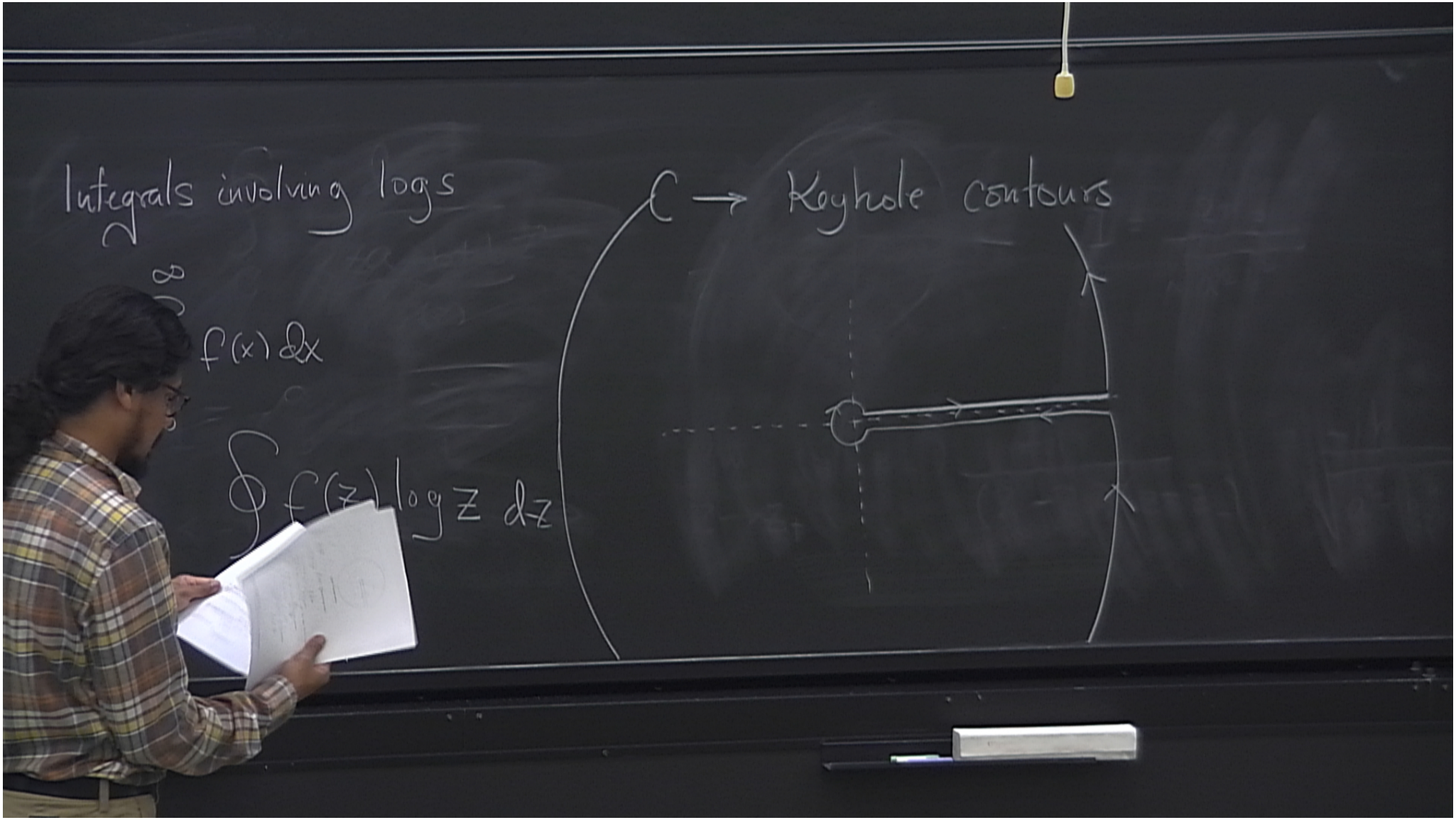

$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \frac{\pi}{4}$
$I = -2i \oint_C$
$z_1 = -$
$-\frac{a}{b} < -1$


Integrals involving logs


Integrals involving logs
$\int_0^\infty f(x)\,dx$


Integrals involving logs
C →
$\int_0^\infty f(x)\,dx$
$\oint_C f(z)\log z\,dz$


Integrals involving logs
C → Keyhole contours
$\int_0^\infty f(x)\,dx$
$\oint_C f($

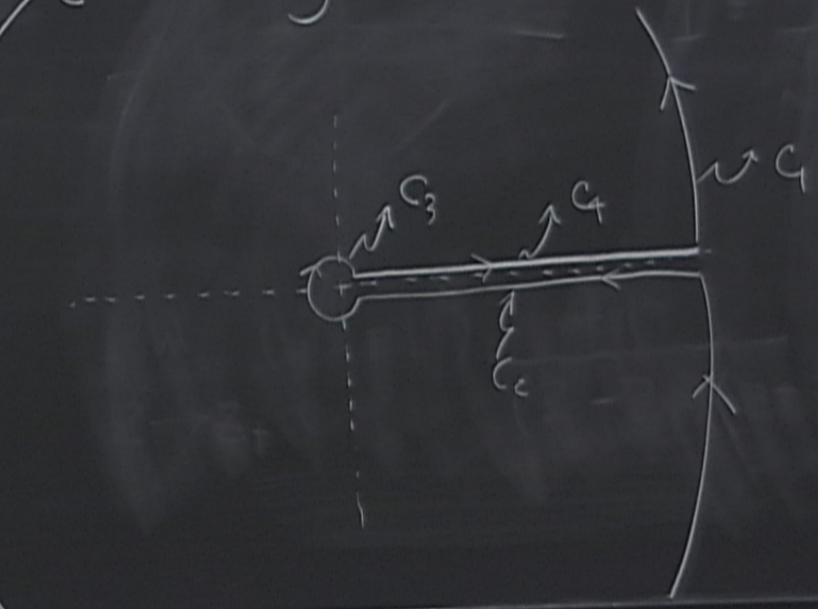

Integrals involving logs
$\int_{-\infty}^{\infty} f(x)\,dx$
$\oint f(z) \log z \, dz$
C → Keyhole contours

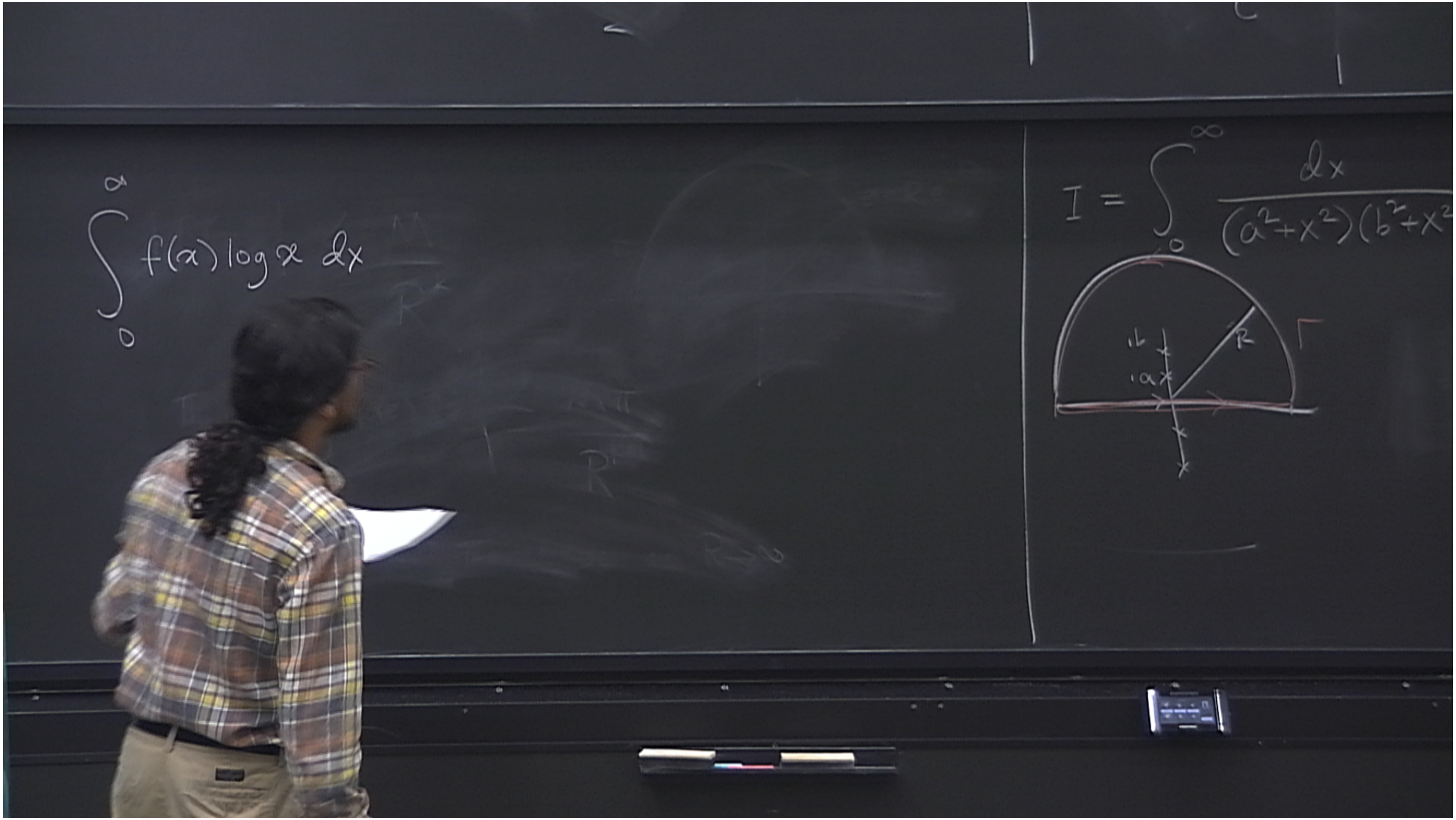
Integrals involving logs

$$\int_0^\infty f(x)\,dx$$

$$\oint_C f(z)\log z\,dz$$

C → Keyhole contours

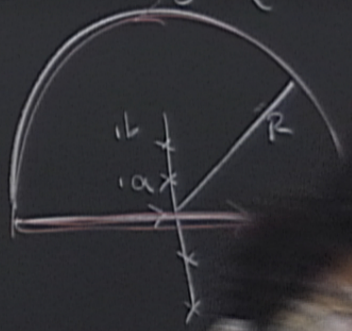


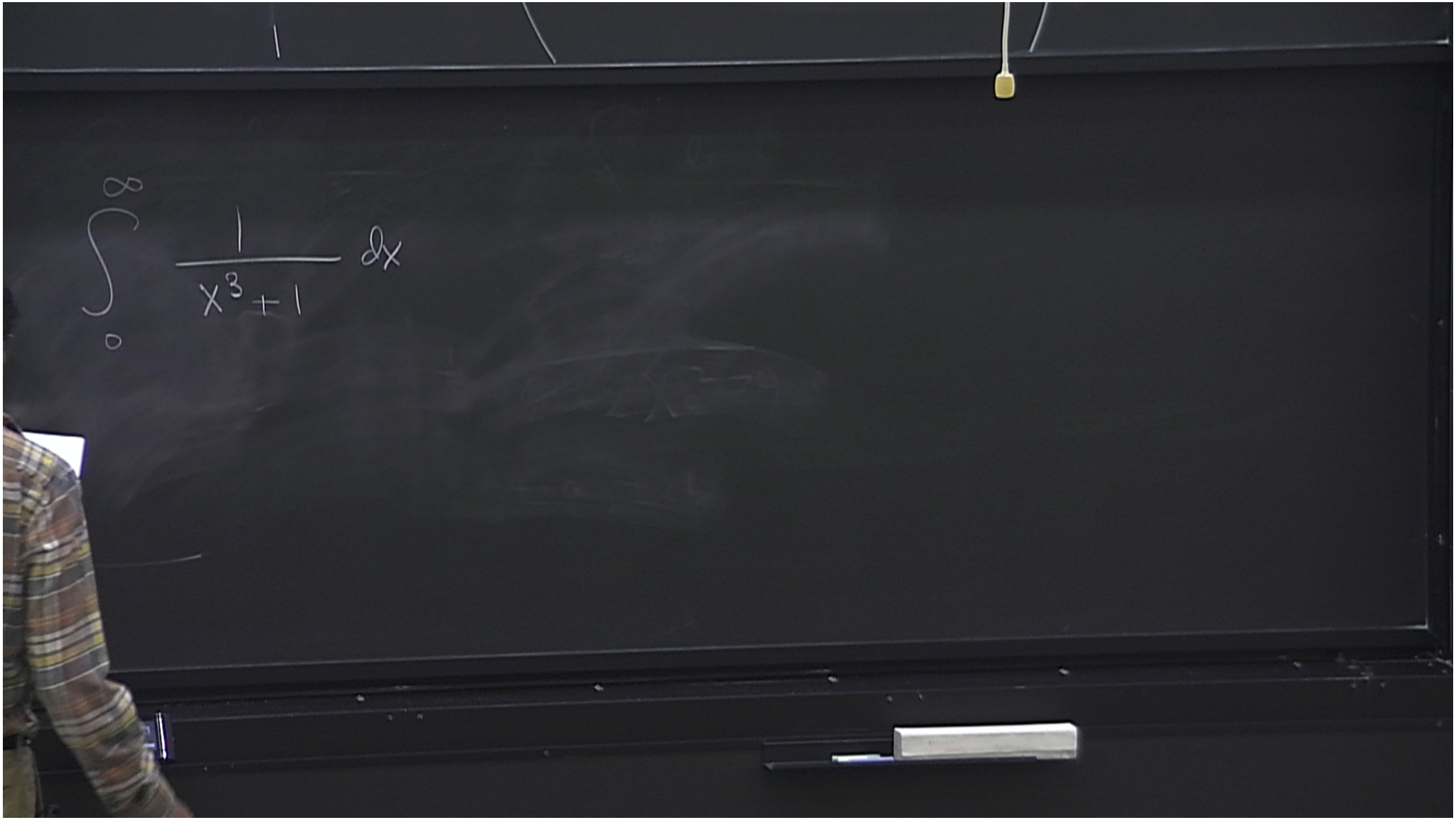

$\int_0^\infty f(x) \log x \, dx$
$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2}$
R
ib
ia

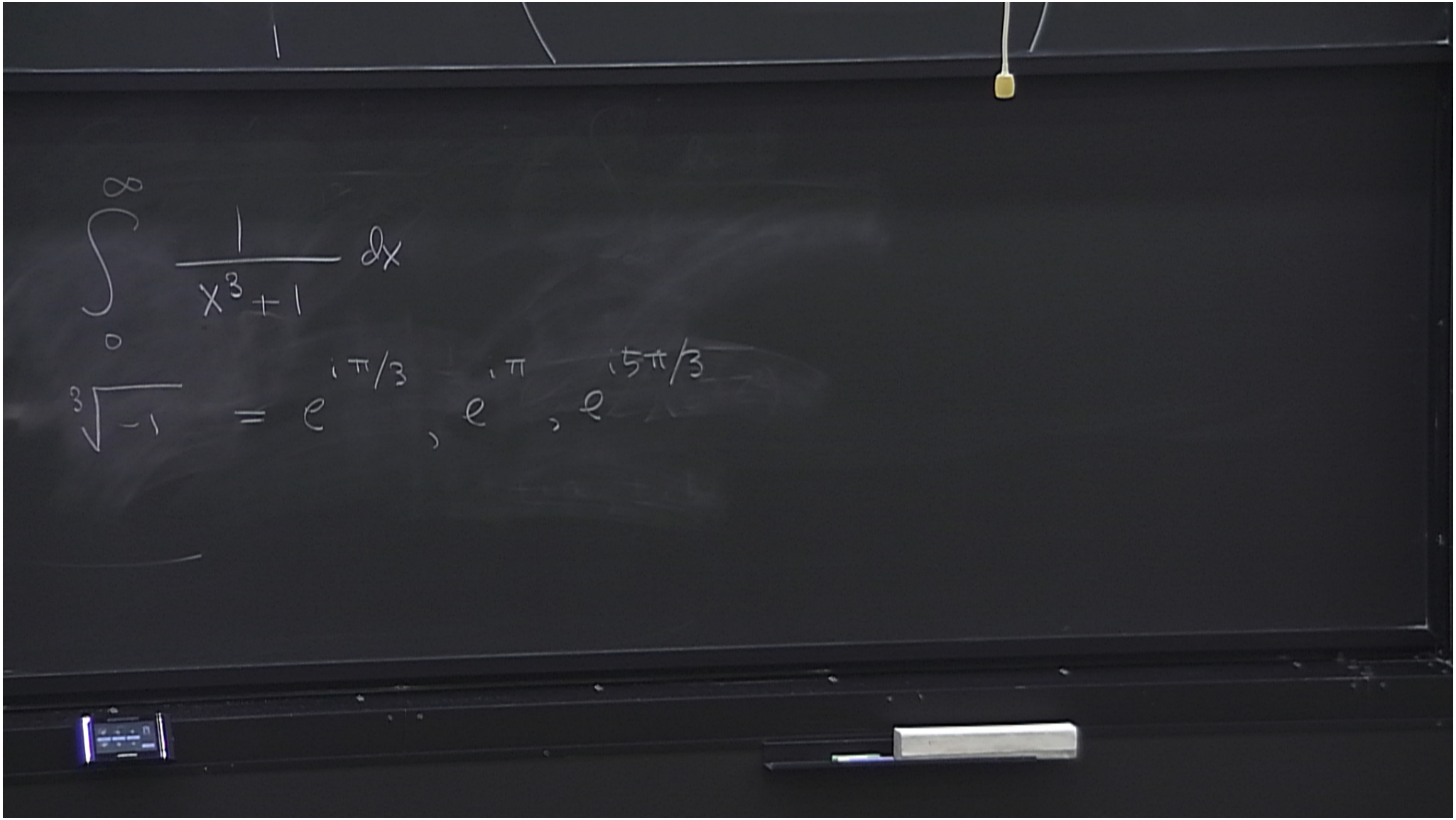
$$\int_0^\infty f(x)\log x\,dx + \int_\infty^0 f(x)(\log x + 2\pi i)\,dx$$

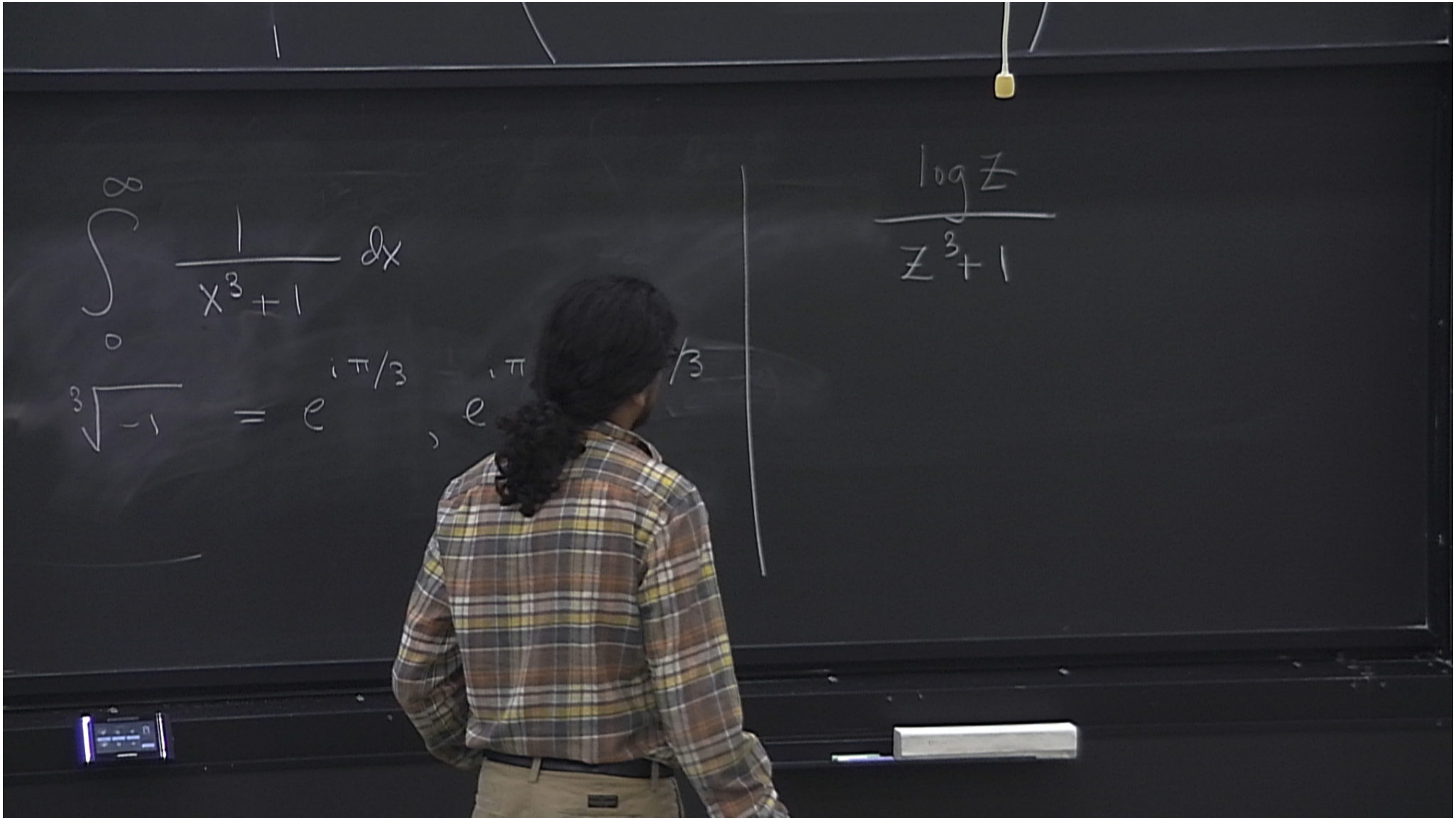
$$= -2\pi i \int_0^\infty f(x)\,dx$$

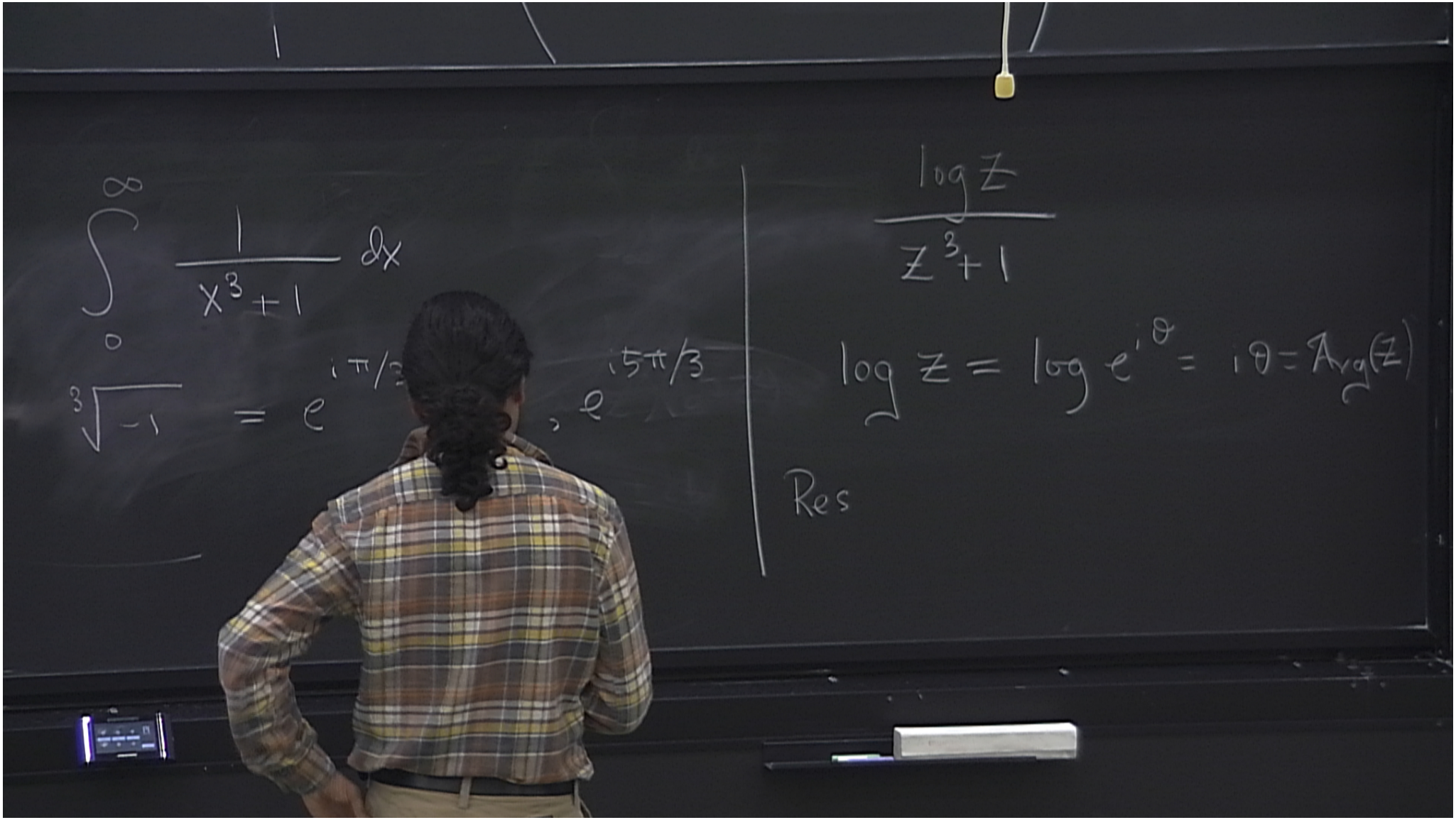
$$I = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2}$$

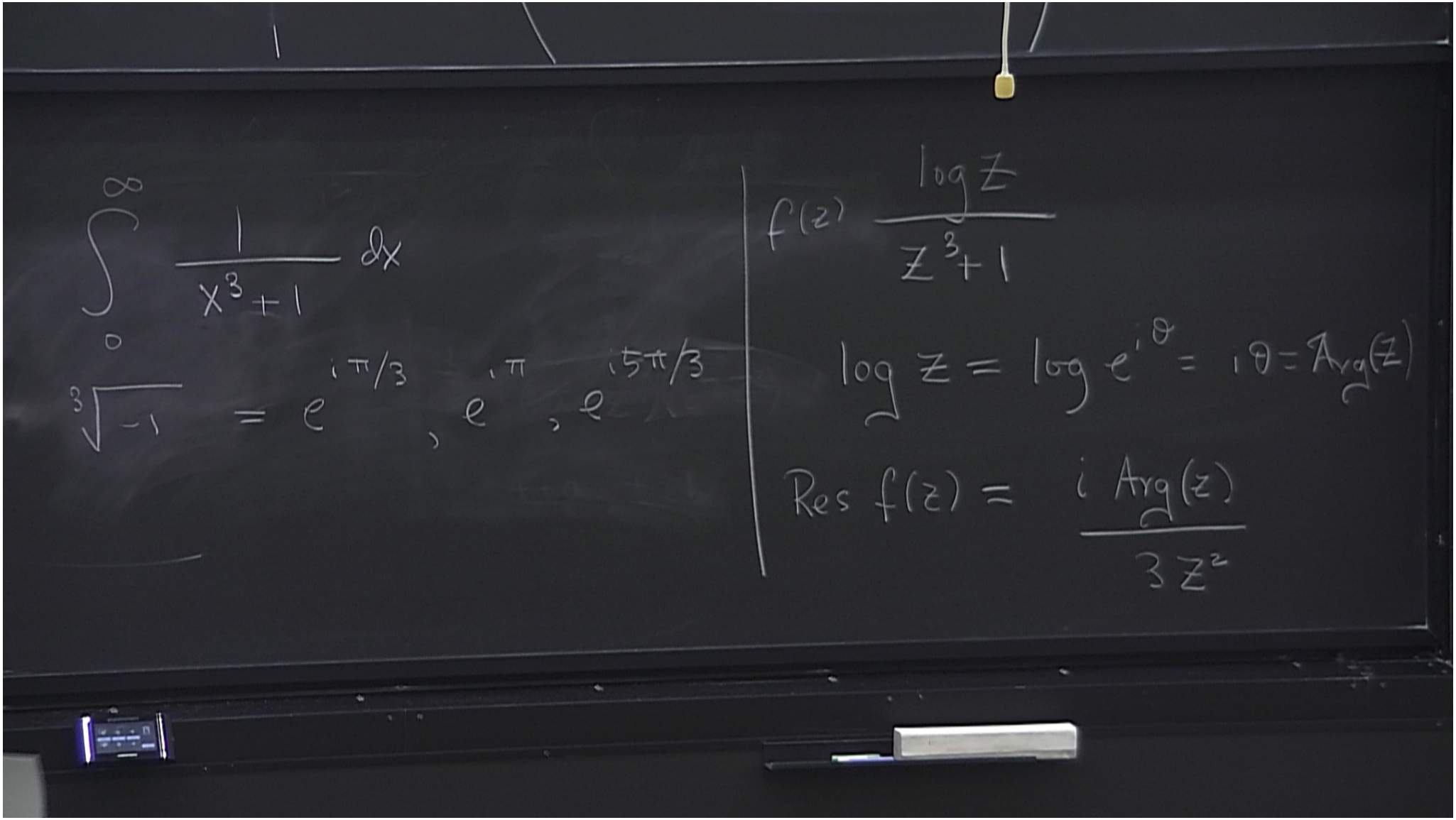


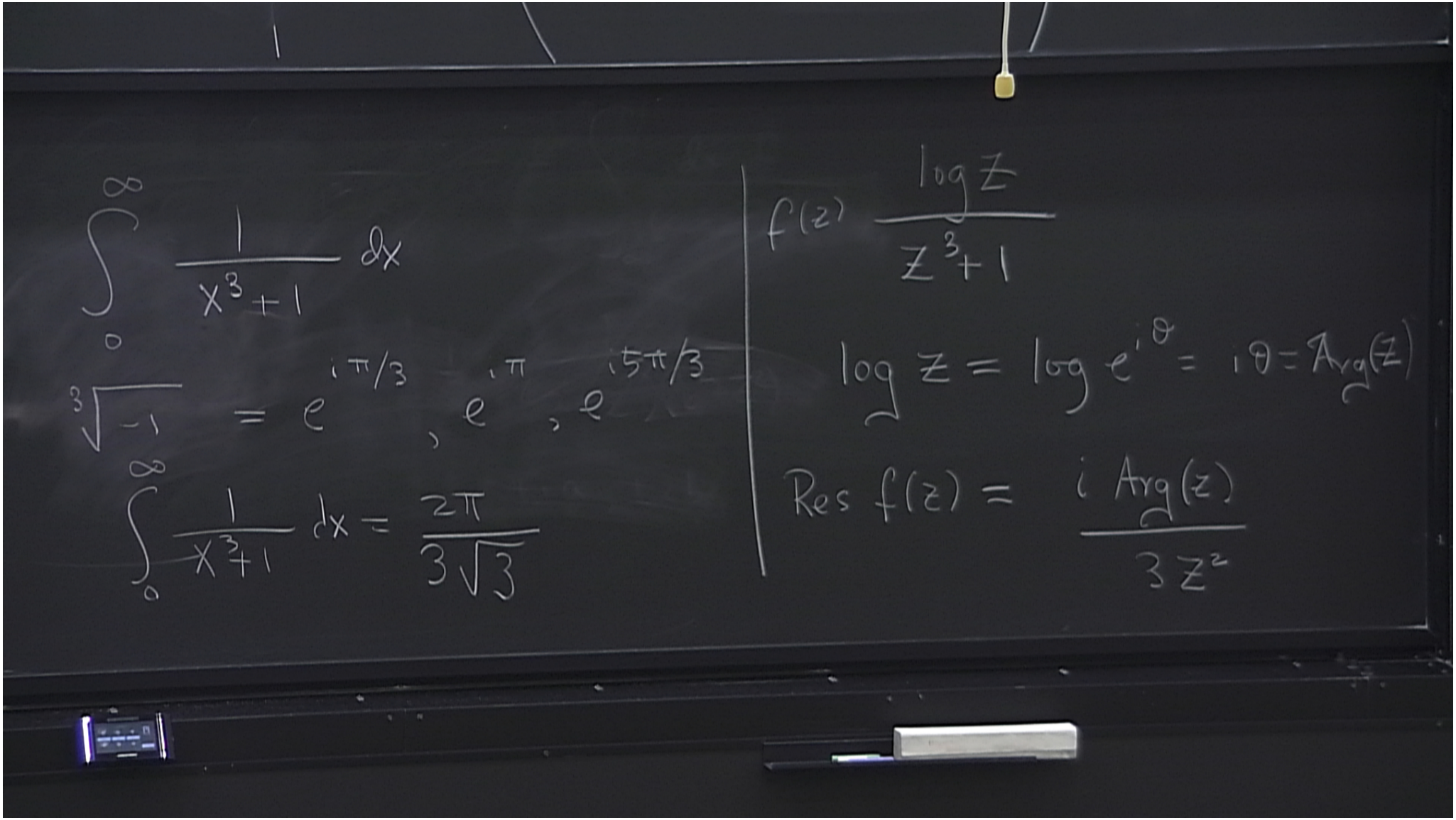

$\int_0^\infty \frac{1}{x^3+1}\,dx$


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$\sqrt[3]{-1} = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$\sqrt[3]{-1} = e^{i\pi/3}, e^{i\pi}$
$\frac{\log z}{z^3+1}$


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$\sqrt[3]{-1} = e^{i\pi/3}$
$e^{i5\pi/3}$
$\frac{\log z}{z^3+1}$
$\log z = \log e^{i\theta} = i\theta = \mathrm{Arg}(z)$
Res


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$\sqrt[3]{-1} = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$
$f(z) \quad \frac{\log z}{z^3+1}$
$\log z = \log e^{i\theta} = i\theta = i\,\mathrm{Arg}(z)$
$\mathrm{Res}\, f(z) = \frac{i\,\mathrm{Arg}(z)}{3z^2}$


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$\sqrt[3]{-1} = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$
$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3\sqrt{3}}$
$f(z)\ \frac{\log z}{z^3+1}$
$\log z = \log e^{i\theta} = i\theta = i\,\mathrm{Arg}(z)$
$\mathrm{Res}\, f(z) = \frac{i\,\mathrm{Arg}(z)}{3z^2}$