Title: The Gravity Dual of Supersymmetric Gauge Theories on a Squashed Five-Sphere

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Abstract: We construct the gravity duals of large N supersymmetric gauge theories on a squashed five-sphere. They are constructed in Euclidean Romans F(4) gauged supergravity in six-dimensions. We find a one- as well as a two-parameter family of solutions and evaluate the renormalised on-shell and fundamental string action for these solutions to find precise agreement with gauge theory.

SUSY Gauge Theories on Squashed 5-sphere and Gravity Dual

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arXiv:1404.1925 with L.F. Alday, MF, P. Richmond and J. Sparks

arXiv:1405.7194 with LFA, MF, C.M. Gregory, PR and JS

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Motivation I

- Localization allows exact computations for rigid supersymmetry on curved backgrounds
- Given observables on such backgrounds, it is natural to ask whether one can construct a gravity dual to it

Example: 3d Gauge theories and their gravity duals

- Backgrounds studied include:
  - 3-manifolds with topology of 3-sphere [*Alday, Martelli, Richmond, Sparks*] and gravity dual [*Farquet, Lorenzen, Martelli, Sparks*]
  - $S^3/\mathbb{Z}_p$ [*Alday, MF, Sparks*] including nontrivial flat connections with gravity dual $AdS_4/\mathbb{Z}_p \times S^7/\mathbb{Z}_k$ [*Martelli, Sparks*]
- Tests include:
  - Supergravity free energy dual to large N gauge theory free energy
  - Gauge theory Wilson loops dual to M2-brane wrapping M-theory cycle

Motivation II: Status in 5d

- In 5d can put SUSY on (e.g.):
  - round and squashed S^5 [*Hosomichi,Seong,Terashima*][*Kim,Kim*][*Imamura*]
  - toric Sasaki-Einstein manifolds, $Y_{p,q}$ [*Qiu,Zabzine*]
- Partition functions computed for S^5, for $Y_{p,q}$ and $S^1 \times S^4$ [*Kallen,Zabzine*][*Kim,Kim*][*Kallen,Qiu,Zabzine*][*Kim,Kim,Kim*] [*Qiu,Zabzine*][*Schmude*][*Qiu,Tizzano,Winding,Zabzine*][*Kim,Kim,Lee*]
- Perturbative part of partition function computed for squashed S^5 [*Imamura*][*Lockhart,Vafa*]

Gravity Duals:

- Large N limit of partition function on S^5 compared to entanglement entropy of dual warped $AdS_6 \times_w S^4$ supergravity solution [*Jafferis,Pufu*]
- Wilson loops on S^5 vs fundamental string/D4-branes action in dual supergravity [*Assel,Estes,Yamazaki*]

Construct (full) supergravity dual for squashed S^5 and check that large-N limit of partition function and Wilson loops of gauge theory coincide with dual supergravity

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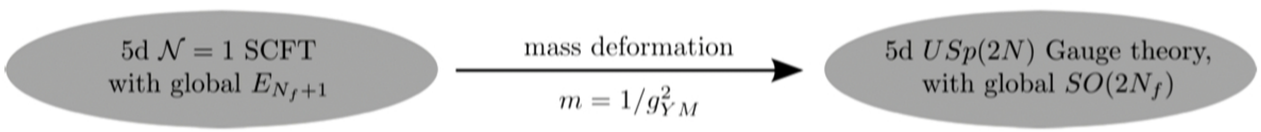
Minimal $\mathcal{N} = 1$ Supersymmetry in 5-dimensions

SUSY gauge theories in $5d$ are *non-renormalizable*

Idea:

- Assuming there is UV (strongly coupled) fixed point, we simply compute quantity in IR description [*Seiberg*]
- As long as the non-renormalizable terms are $\mathcal{Q}$-exact they don't contribute to the localized path integral.
- Has to be made precise!

Example:



Ignore this and compute localised path integral in IR

Setup of $SU(3) \times U(1)$ squashed S^5 [*Imamura*]

- $\mathcal{N} = 1$ gauge theories on a squashed 5-sphere as a fibration over $\mathbb{CP}^2$

$$ds_5^2 = \frac{1}{s^2}(d\tau + \mathcal{C})^2 + ds^2_{\mathbb{CP}^2},$$

where $\mathcal{C}$ is a local 1-form ($\frac{1}{2}d\mathcal{C} = \omega =$ Kähler form on $\mathbb{CP}^2$).

- Squashing breaks round-sphere isometry ($s \neq 1$)

$$SO(6) \sim SU(4) \longrightarrow SU(3) \times U(1),$$

and consequently the supercharges transforming in $\mathbf{4} + \overline{\mathbf{4}}$ break to

(i) $\mathbf{3}_{+1} + \overline{\mathbf{3}}_{-1}$ for 3/4 BPS
(ii) $\mathbf{1}_{-3} + \mathbf{1}_{+3}$ for 1/4 BPS

- Turn on background R-symmetry gauge field

$$\mathcal{A} = \frac{1}{s^2}\left(1 + Q\sqrt{1 - s^2}\right)\sqrt{1 - s^2}\,(d\tau + \mathcal{C})$$

where $U(1)_R \subset SU(2)_R$, and $Q = 1$, $Q = -3$ give rise to 3/4 BPS, 1/4 BPS respectively.

SUSY on Squashed S^5 from $6d$

Perturbative part of partition function is obtained by a twisted reduction of the index of a corresponding theory on $\mathbb{R} \times S^5$ [Imamura]

1. Theory on $\mathbb{R} \times S^5$ with metric $\mathrm{d}s^2 = \mathrm{d}t^2 + \sum_{i=1}^{3} |\mathrm{d}w_i|^2$, $w_i \in \mathbb{C}$
2. Cpf t-direction with twist: $(t, w_i) \sim (t + \beta, e^{i\mu_i\beta} w_i)$ for $\beta > 0$
3. Change of coordinates and identifying $-\mu_1 = \mu_2 = \mu_3 = i\sqrt{1 - s^2}$ for 3/4 BPS and $\mu_j = i\sqrt{1 - s^2}$ for 1/4 BPS gives squashed 5-sphere metric.

Hence:

Start with $6d$ $\mathcal{N} = (1, 0)$-theory and compute Index and reduce to squashed S^5 partition function by taking limit $\beta \to 0$.

SUSY on Squashed S^5

Applying this procedure, one obtains:

For generic theory with gauge group G, prepotential F and matter in $\mathbf{R} \oplus \bar{\mathbf{R}}$ representation the perturbative partition function is given by

$$Z_{\text{pert}} \propto \prod_{a=1}^{\text{rk } G} \int_{-\infty}^{\infty} \mathrm{d}\sigma_a \, \mathrm{e}^{-\frac{(2\pi)^3}{b_1 b_2 b_3} F(\sigma)} \frac{\prod_\alpha S_3\left(-\mathrm{i}\alpha(\sigma) \mid \mathbf{b}\right)}{\prod_\rho S_3\left(-\mathrm{i}\rho(\sigma) + \frac{1}{2}(b_1 + b_2 + b_3) \mid \mathbf{b}\right)}$$

- $\mathbf{b} = (b_1, b_2, b_3)$, where $b_i = 1 + \mathrm{i}\mu_i$, and in terms of squashing s

$$\mathbf{b} = \begin{cases} \left(1 + \sqrt{1-s^2}, 1 - \sqrt{1-s^2}, 1 - \sqrt{1-s^2}\right) \ , & 3/4 \text{ BPS} \ , \\ \left(1 + \sqrt{1-s^2}, 1 + \sqrt{1-s^2}, 1 + \sqrt{1-s^2}\right) \ , & 1/4 \text{ BPS} \ . \end{cases}$$

- $S_3\left(z \mid \mathbf{b}\right) \equiv \Gamma_3(z \mid \mathbf{b})^{-1} \, \Gamma_3(b_{\text{tot}} - z \mid \mathbf{b})^{-1}$ the triple sine function, where $\Gamma_K(z \mid \mathbf{b})$ is the so-called Barnes' multiple gamma function
- For 1/4 BPS dependence on s can be explicitly removed from Z_{pert}.

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Specify to Theory with Gravity Dual

- Class of 5*d* SCFT arising from a N D4 and N_f D8-branes in presence of orientifold planes, is expected to have a large N description in terms of massive type IIA supergravity [Seiberg]

- Near horizon geometry given by warped product of M_6 (deformation of AdS_6) with S^4-hemisphere [Ferrara,Kehagias,Partouche,Zafferoni] [Brandhuber,Oz][Bergman,Rodriguez-Gomez]

5*d* Gauge Theory with Gravity Dual:

Theory living on D4-branes and their Mirrors: $USp(2N)$ gauge group with $N_f \leq 7$ matter fields in the fundamental and single hypermultiplet in the antisymmetric representation.

Large N Limit:

Remarks:

1. Set CS-level to zero for $USp(2N)$
2. **Q:** $Z = Z_{\text{pert}} + \sum_{n\neq 0} e^{-16\pi^3 nr/g^2_{YM}}\, Z_{\text{inst},n}$ and $g_{YM} \to \infty$ at UV fixed point. We should include instantons?
   A: No, instantons go with $\frac{r}{g^2_{\text{eff}}} = \frac{r}{g^2_{YM}} + \frac{8-N_f}{12\pi^2}|\sigma_i| \sim \mathcal{O}\left(N^{1/2}\right)$ and hence exponentially suppressed.
   However: Explicit computation is required.

Computing asymptotic expansions of $\log S_3(z \mid \mathbf{b})$ and applying *saddle point approach* yields:

$$\mathcal{F}\left(b_1, b_2, b_3\right) \sim \begin{cases} \frac{(b_1+b_2+b_3)^3}{27\, b_1 b_2 b_3}\, \mathcal{F}_{S^5_{\text{round}}}\,, & 3/4 \text{ BPS}\,, \\ \mathcal{F}_{S^5_{\text{round}}}\,, & 1/4 \text{ BPS}\,, \end{cases}$$

where the round sphere case $\mathcal{F}_{S^5_{\text{round}}} = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}} + \mathcal{O}\left(N^{3/2}\right)$ [Jafferis,Pufu].

Romans $F(4)$ Gauged Supergravity

- $USp(2N)$ SCFTs on S^5 conjectured to be dual to $AdS_6 \times_w S^4$ solutions of IIA massive supergravity
- Romans $F(4)$ supergravity solutions uplift to massive IIA supergravity on S^4 [Cvetic,Lu,Pope]

Thus:

Possible supergravity duals are described in the Euclidean version of Romans $F(4)$ gauged supergravity in $6d$.

- Bosonic field content:
  - **metric**,
  - **dilaton** ϕ, which we shall write $X = \exp(-\phi/2\sqrt{2})$
  - **one-form** potential A, with field strength $F = \mathrm{d}A + \frac{2}{3}gB$
  - **two-form** potential B, with field strength $H = \mathrm{d}B$
  - $SO(3) \sim SU(2)$ R-symmetry **gauge field** A^i, $i = 1, 2, 3$, with $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$
- Set the gauge coupling constant $g \equiv 1$.

Euclidean Version of Romans $F(4)$ Supergravity

One-form A is a Stueckelberg field, which can be set to $A = 0$ by a gauge transformation. The B-field becomes massive and the Euclidean action is:

$$\begin{aligned} I_E \; = \; -\frac{1}{16\pi G_N} \int \Big[& R * 1 - 4X^{-2}\mathrm{d}X \wedge *\mathrm{d}X - \left(\tfrac{2}{9}X^{-6} - \tfrac{8}{3}X^{-2} - 2X^2\right) * 1 \\ & - \tfrac{1}{2}X^{-2}\left(\tfrac{4}{9}B \wedge *B + F^i \wedge *F^i\right) - \tfrac{1}{2}X^4 H \wedge *H \\ & - \mathrm{i}B \wedge \left(\tfrac{2}{27}B \wedge B + \tfrac{1}{2}F^i \wedge F^i\right)\Big] \, . \end{aligned}$$

Solutions to EOM are supersymmetric provided the Killing spinor and dilatino equations hold

$$\begin{aligned} D_\mu \epsilon_I \; &= \; \tfrac{\mathrm{i}}{4\sqrt{2}}(X + \tfrac{1}{3}X^{-3})\Gamma_\mu \Gamma_7 \epsilon_I - \tfrac{\mathrm{i}}{24\sqrt{2}}X^{-1}B_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu \Gamma^\rho)\epsilon_I \\ & \quad - \tfrac{1}{48}X^2 H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu \Gamma_7 \epsilon_I + \tfrac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J \, , \\ 0 \; &= \; -\mathrm{i}X^{-1}\partial_\mu X \Gamma^\mu \epsilon_I + \tfrac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7 \epsilon_I + \tfrac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7 \epsilon_I \\ & \quad - \tfrac{1}{18\sqrt{2}}X^{-1}B_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \, , \end{aligned}$$

where ϵ_I, $I = 1, 2$ two Dirac spinors and $D_\mu \epsilon_I = \nabla_\mu \epsilon_I + \frac{\mathrm{i}}{2}A^i_\mu (\sigma^i)_I{}^J \epsilon_J$.

Ansatz:

- Squashed S^5 has $SU(3) \times U(1)$ symmetry; expect to be preserved by bulk filling. This leads to Ansatz:

$$
\begin{aligned}
\mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\tau + \mathcal{C})^2 + \beta^2(r)\mathrm{d}s^2_{\mathbb{CP}^2} \, , \\
A^i &= f_i(r)(\mathrm{d}\tau + \mathcal{C}) \, , \\
B &= p(r)\mathrm{d}r \wedge (\mathrm{d}\tau + \mathcal{C}) + \frac{1}{2} q(r)\mathrm{d}\mathcal{C} \, .
\end{aligned}
$$

  together with $X = X(r)$.
- Want to construct smooth, SUSY, asymptotically locally Euclidean *AdS* solution with topology $M_6 \sim B_6$, with conformal boundary the squashed S^5 [*Imamura*].
- Reparametrization invariance allows us to fix $\beta(r) = 3\sqrt{6r^2 - 1}/\sqrt{2}$ to its *AdS*-value.
- An $SO(3)$ rotation sets $f_3(r) = f(r)$ and $f_1(r) = f_2(r) = 0$.

Solutions as Expansions:

Plug Ansatz into Romans EOM to get 7 coupled ODEs for 6 unknown functions. Very hard to solve analytically!

Found solutions as *expansions* by solving EOM order-by-order:

1) around AdS_6 in terms of a parameter δ

$$\mathcal{X} = \mathcal{X}^{AdS}(r) + \delta \cdot \mathcal{X}^{(1)}(r) + \delta^2 \cdot \mathcal{X}^{(2)}(r) + \cdots \ ,$$

where $\mathcal{X} = \alpha, \gamma, \beta, f, p, q, X$ and δ is related to the squashing parameter s via

(i) $s^{-1} = 1 + \delta^2$ for 3/4 BPS

(ii) $s^{-1} = 1 + \delta$ for 1/4 BPS.

2) around the conformal boundary, $r \to \infty$, e.g.

$$\alpha(r) = \frac{3}{\sqrt{2}} \frac{1}{r} + \frac{1}{r^2} \alpha_{(1)} + \frac{1}{r^2} \alpha_{(2)} + \cdots$$

Supersymmetry constraints

- In order to get solutions preserving SUSY (or at least parts of it), we require solutions ϵ_1 and ϵ_2 to Killing spinor and dilatino equations
- To avoid further differential equations, we consider instead the integrability condition:

$$[D_\mu, D_\nu]\epsilon_I - \tfrac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\epsilon_I - \tfrac{i}{2}F^i_{\mu\nu}\left(\sigma^i\right)_I{}^J\epsilon_J \equiv \mathcal{I}_{\mu\nu I}{}^J\epsilon_J = 0$$

- After careful analysis this can be rewritten as:

$$\begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0 \ ,$$

- Necessary condition for SUSY preserving solutions $\det(A \pm B) = 0$
- Determinant factorises into

$$\det(A+B) = F_1\, F_2\, F_3\, F_4 = 0$$

and we find:

(i) 3/4 BPS: $F_1 = F_2 = F_3 = 0$ and $F_4 \neq 0$
(ii) 1/4 BPS: $F_1,\ F_2,\ F_3 \neq 0$ and $F_4 = 0$

(i) 3/4 BPS solutions I

By employing expansions 1) and 2), we find a **one-parameter family** of solutions for metric and fields. For example the solution in the expansion 2) around $r \to \infty$ is given by:

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots\,, \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots\,, \\
p(r) &= -\frac{\mathrm{i}\sqrt{2/3}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots\,, \\
q(r) &= -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{2/3}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots\,, \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots\,,
\end{aligned}
$$

where the parameter κ is uniquely determined by requiring this to extend to a smooth solution on the ball $M_6 \sim B_6$, i.e. $\kappa = \kappa^{(0)} + \kappa^{(1)}\delta + \cdots$

(i) 3/4 BPS solutions II

- Compute the $SU(2)_R$-doublet of Killing spinors based on solutions 1) and 2) $\Longrightarrow$ 3/4 of SUSY preserved $\Longrightarrow$ 3/4 BPS.
- Killing vector bilinear $K_\mu = \varepsilon^{IJ}\epsilon_I^T \mathscr{C}\Gamma_\mu \epsilon_J$ and impose symplectic Majorana $\mathscr{C}\epsilon_I^* = \varepsilon_I{}^J \epsilon_J$.
- Requiring K to sit in the Lie algebra of $U(1)^3 \subset SU(3) \times U(1)$ fixes constants of integration and we obtain

$$K = b_1 \cdot \partial_{\varphi_1} + b_2 \cdot \partial_{\varphi_2} + b_3 \cdot \partial_{\varphi_3} \ ,$$

$$b_1 = 1 + \sqrt{1-s^2}, \quad b_2 = 1 - \sqrt{1-s^2}, \quad b_3 = 1 - \sqrt{1-s^2} \ .$$

with $\varphi_1 = -\tau$, $\varphi_2 = \tau - \frac{1}{2}(\psi + \varphi)$, $\varphi_3 = \tau - \frac{1}{2}(\psi - \varphi)$, the standard azimuthal variables of the embedding $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$.

- Very much analogous to $3d$ case!

(ii) 1/4 BPS solutions

By employing expansions 1) and 2), we find a **two-parameter family** of solutions for metric and fields

- The family of solutions is parametrised by the squashing s and another parameter f_0, which we have to tune in order to match the gauge field $\mathcal{A}$ at the conformal boundary.
- The Killing vector is automatically in the Lie algebra of the maximal torus $U(1)^3 \subset SU(3) \times U(1)$, namely

$$K = \partial_\tau = b_1 \cdot \partial_{\varphi_1} + b_2 \cdot \partial_{\varphi_2} + b_3 \cdot \partial_{\varphi_3} \ ,$$
$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1 \ .$$

- This should imply the existence of other results in $5d$:
  - (1) $f_0 = 0$ and $s \neq 1$ gives $1/2$ BPS solution ($F_1 = F_3$ and $F_2 = F_4$)
  - (2) $f_0 \neq 0$ and $s = 1$ gives round sphere with nontrivial, different BG-field $\mathcal{A}$

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Holographic Free Energy

Regularized action is

$$I_{\text{ren}} = I_{\text{bulk}} + I_{\text{Gibbons-Hawking}} + I_{\text{ct}} \, .$$

Had to compute I_{ct}: Straightforward, but very long (B is massive and has cubic CS interaction, which leads to much more complicated analysis). We find:

$$\begin{aligned} I_{\text{ct}} = \frac{1}{8\pi G_N} \int_{\partial M_6} \Big\{ \Big[& \frac{4\sqrt{2}}{3} + \frac{1}{2\sqrt{2}} R(h) - \frac{1}{6\sqrt{2}} \|B\|_h^2 + \frac{3}{4\sqrt{2}} R(h)_{mn} R(h)^{mn} - \frac{15}{64\sqrt{2}} R(h)^2 - \frac{3}{4\sqrt{2}} \|F^i\|_h^2 + \frac{1}{12\sqrt{2}} \mathrm{Tr}_h B^4 - \\ & - \frac{13}{192\sqrt{2}} \|B\|_h^4 - \frac{1}{\sqrt{2}} \|\mathrm{d}B\|_h^2 + \frac{5}{8\sqrt{2}} \|\mathrm{d} *_h B + \frac{\mathrm{i}\sqrt{2}}{3} B \wedge B\|_h^2 - \frac{1}{4\sqrt{2}} \langle B, \mathrm{d}\delta_h B + \frac{\mathrm{i}\sqrt{2}}{3} \mathrm{d} *_h B \wedge B \rangle_h + \frac{4\sqrt{2}}{3} (1 - X)^2 - \\ & - \frac{1}{\sqrt{2}} \langle \mathrm{Ric}(h) \circ B, B \rangle_h + \frac{9}{32\sqrt{2}} R(h) \|B\|_h^2 \Big] \sqrt{\det h} \, \mathrm{d}^5 x - \frac{1}{4\sqrt{2}} B \wedge \Big[\mathrm{d} *_h \mathrm{d}B + \frac{\sqrt{2}\mathrm{i}}{3} B \wedge \delta_h B - \frac{2}{9} B \wedge *_h (B \wedge B) \Big] \Big\} \, , \end{aligned}$$

with $R(h)_{ij}$, $R(h)$ the Ricci tensor, scalar of the bdry metric h_{ij}, $\langle \nu_1, \nu_2 \rangle_h \sqrt{\det h} \, \mathrm{d}^5 x = \nu_1 \wedge *_h \nu_2$, adjoint δ_h of d wrt h_{ij} is $\delta_h B = *_h \mathrm{d} *_h B$ and $\mathrm{Tr}_h B^4 = B_i^j B_j^k B_k^l B_l^i$ and $(S \circ \nu)_{i_1 \ldots i_p} = S_{[i_1}^j \nu_{|j| i_2 \ldots i_p]}$ for S_{ij} any symmetric 2-tensor and ν a p-form.

Holographic Free energy for 3/4 and 1/4 BPS solutions

(i) For the 3/4 BPS case we obtain the following expansion

$$I_{\text{ren}}^{(3/4\text{ BPS})} = -\frac{27\pi^2}{4G_N}\left(1 + \frac{8}{3}\delta^2 + \frac{16\sqrt{2}}{27}\delta^3 + \dots\right), \quad G_N = \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}},$$

which precisely agrees with the large N limit of the free energy

$$\mathcal{F}(b_1, b_2, b_3) \sim -\frac{(b_1+b_2+b_3)^3}{27\, b_1 b_2 b_3} \cdot \frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}} + \mathcal{O}\left(N^{3/2}\right) .$$

(ii) Similarly for the 1/4 BPS, we obtain

$$I_{\text{ren}}^{(1/4\text{ BPS})} = -\frac{27\pi^2}{4G_N}\left(1 + \mathcal{O}(\delta^5)\right), \quad G_N = \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}},$$

precisely agreeing with $\mathcal{F}(b_1, b_2, b_3) \sim -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}} + \mathcal{O}\left(N^{3/2}\right)$.

Fundamental Wilson loop in the large N limit

- VEV of Wilson loop in representation $\mathbf{R}$ of gauge group G:

$$\langle W_{\mathbf{R}} \rangle = \frac{1}{\dim \mathbf{R}} \left\langle \mathrm{Tr}_{\mathbf{R}} \mathcal{P} \exp \int (\mathscr{A}_m \dot{x}^m + \sigma |\dot{x}|) \, \mathrm{d}t \right\rangle .$$

- SUSY $\implies$ along orbit of Killing bilinear: $K_m = \varepsilon^{IJ} {\chi_I}^T \mathscr{C}_{(5)} \gamma_m \chi_J$.
- Insertion of the Wilson loop into the path integral does not affect the leading order saddle point configuration. Thus the VEV for $\mathbf{R} = \mathbf{fund}$ of $USp(2N)$, is computed in the large N matrix model as

$$\langle W_{\mathrm{fund}} \rangle = \int_0^{x_*} \mathrm{e}^{2\pi \mathscr{L} \lambda(x)} \rho(x) \mathrm{d}x ,$$

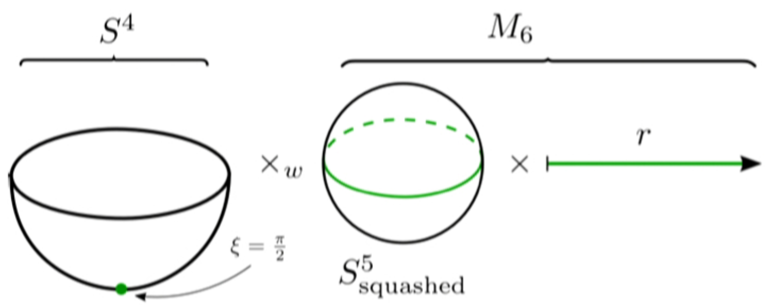
where $2\pi\mathscr{L} = \int |\dot{x}| \, \mathrm{d}t$ and $\rho(x)$ is SP eigenvalue distribution of $\mathcal{F}$

- Call $U(1)^3$ invariant circles S^1_i, $i = 1, 2, 3$, then $\mathscr{L} = 1/b_i$ and we may write

$$\log \langle W_{\mathrm{fund}, S^1_i} \rangle = \frac{(b_1 + b_2 + b_3)}{3 b_i} \log \langle W_{\mathrm{fund}} \rangle_{\mathrm{round}} .$$

Dual Fundamental Strings: General Analysis

- Type IIA background is a warped and fibred product $M_6 \times S^4$, together with various non-trivial background fluxes.
- Dual of fund WL is a fundamental string tracing out $\Sigma \sim \mathbb{R}^2 \subset M_6$ similar to round sphere case [Assel, Estes, Yamazaki]



- compute the regularized action of a FS wrapping this submanifold.
- Uplift to $10d$ supergravity and including counter term yields the regularised string action:

$$S_{\text{string}} = \frac{N^{1/2}\sqrt{2}}{3\sqrt{(8-N_f)}}\left[\int_\Sigma \left(X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\right) - \frac{3}{\sqrt{2}}\mathrm{length}(\partial\Sigma)\right] .$$

Dual Fundamental Strings: 3/4 and 1/4 BPS Solutions

(i) 3/4 BPS:

For S^1_i the fundamental string wraps the circle φ_i together with the r direction. One can show (e.g.)

$$\frac{S_{\mathrm{string},S^1_1}}{S_{\mathrm{string}}\,|_{\delta=0}} = 1-\frac{4\sqrt{2}}{3}\delta+\frac{8}{3}\delta^2-\frac{5\sqrt{2}}{3}\delta^3+\frac{4}{3}\delta^4-\frac{7}{12\sqrt{2}}\delta^5+\ldots\,,$$

and similarly for S^1_2 and S^1_3. These results precisely agree with the series expansions computed in field theory.

(ii) 1/4 BPS:

Evaluating this for the two-parameter family of 1/4 BPS solutions, as a series in the parameter δ, we find

$$S_{\mathrm{string},S^1_i} = -\frac{3\sqrt{2}\pi}{\sqrt{8-N_f}}N^{1/2}+\mathcal{O}(\delta^5)\,,$$

precisely agreeing with the large N field theory result, since $b_i=1$.

1. SUSY on the Squashed S^5
2. Large N limit
3. Euclidean Romans $F(4)$ Gauged Supergravity
4. Constructing Our Solutions
5. Holographic Free Energy
6. Wilson Loops
7. **Conjectures, Summary and Outlook**

Conjectures

Based on these results and very similar statements in $3d/4d$, we can conjecture:

(A) For any supersymmetric supergravity solution with the topology of the six-ball, with at least $U(1) \times U(1) \times U(1)$ isometry, and for which the Killing vector takes the form

$$K = b_1 \cdot \partial_{\varphi_1} + b_2 \cdot \partial_{\varphi_2} + b_3 \cdot \partial_{\varphi_3} \ ,$$

the holographic free energy is equal to

$$\mathcal{F}(b_1, b_2, b_3) \ = \ -\frac{(b_1 \ + \ b_2 \ + \ b_3)^3}{27\, b_1 b_2 b_3} \cdot \mathcal{F}_{S^5_{\mathrm{round}}} \ .$$

(B) If we define a supersymmetric gauge theory on the conformal boundary of the background in point (A), the finite N partition function depends only on b_1, b_2, b_3.

Summary:

- Performed highly nontrivial check of AdS/CFT in $5d$ by comparing free energy and Wilson loop
- Construct all the necessary ingredients to analyse supergravity dual in detail
- Find agreement on conformal boundary of $6d/5d$ KSE

Outlook:

- Sufficiency conditions for SUSY in Romans theory - look at bilinears, G-structures
- General asymptotic KSE is curious - generalised charged conformal Killing spinor equation
- Can we prove conjecture in $5d$? Start from appropriate rigid limit or alternatively from KSE at conformal boundary