Title: Algebraic Geometry in Cosmology

Date: Aug 07, 2014 11:00 AM

URL: http://pirsa.org/14080002

Abstract: The talk is based on joint work with Yuri Manin (arXiv:1402.2158). Using algebro-geometric blowups it is possible to construct a family of models of gluing of aeons across a Big-Bang type singularity, which includes the case of Penrose's conformally cyclic cosmology, as well as inflationary multiverse models generalizing the "eternal symmetree", and BKLL mixmaster type cosmologies. Using the mixmaster dynamics, formulated in terms of elliptic curves and modular curves, we speculate on the geometry of cosmological time near the gluing of aeons. We show also that this type of model allows for phenomena of noncommutativity of spacetime coordinates near the Big-Bang and the crossing of aeons.

Based on:

- Yuri Manin and Matilde Marcolli, *Big Bang, blowup and modular curves: Algebraic geometry in cosmology*, SIGMA 10 (2014), 073, 20 pages, arXiv:1402.2158 [gr-qc]

Complexified Minkowski spacetime and Grassmannian

An algebro-geometric setting for Minkowski space and other 4-dimensional Lorentzian spacetimes (Penrose, Manin)

Basic geometric setup:

- T = 4-dimensional complex vector space
- $G(2,T)$ = Grassmannian of complex 2-planes in T
- $\mathcal{S}$ = tautological plane bundle over $G(2,T)$
- T^* = dual of T
- $\tilde{\mathcal{S}} = \{(x,t) : x \in G(2,T), t \in \mathcal{S}_x^\perp \subset T^*\}$

- trivial vector bundle $\mathcal{T}$ on $G(2,T)$ with fiber T decomposes as

$$0 \to \mathcal{S} \to \mathcal{T} \to \tilde{\mathcal{S}}^* \to 0$$

- this gives an identification

$$\sigma : \Omega^1(G(2,T)) \stackrel{\simeq}{\to} \mathcal{S} \otimes \tilde{\mathcal{S}}$$

- $\Omega^1(G(2,T))$ acts on $\mathcal{S}^* \oplus \tilde{\mathcal{S}}^*$ and maps it to $\mathcal{S} \oplus \tilde{\mathcal{S}}$, via the maps $\Omega^1(G(2,T)) \otimes \mathcal{S}^* \to \tilde{\mathcal{S}}$ and $\Omega^1(G(2,T)) \otimes \tilde{\mathcal{S}}^* \to \mathcal{S}$ induced by σ
- matrices defining this action are the γ-matrices

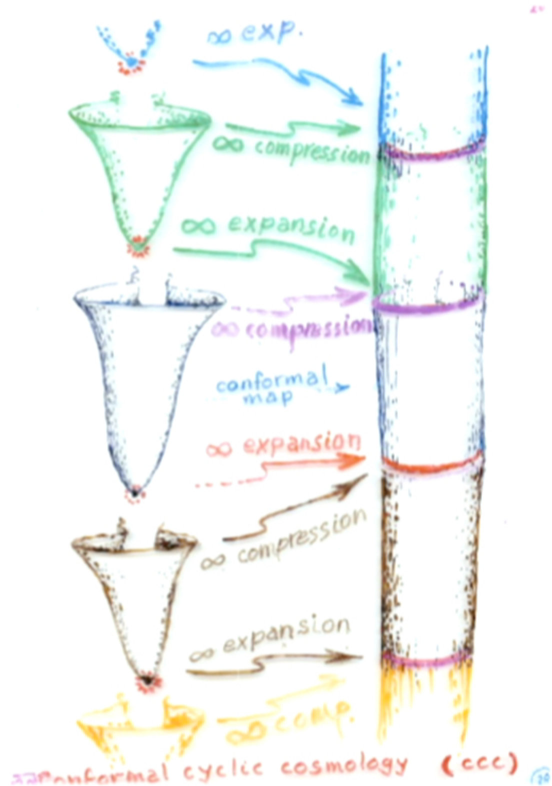
- compactification of $\mathcal{M}^4_{\mathbb{C}}$: boundary cells of $G(2, T)$

$$G(2,T) \smallsetminus U_{\mathbb{C}} = \mathcal{C}(\infty),$$

$\mathcal{C}(\infty)$ = (complex) *light cone* with vertex ∞ and base
$L(\infty) = \mathbb{P}(\mathcal{S}^*_\infty) \times \mathbb{P}(\tilde{\mathcal{S}}^*_\infty)$

- conformal metric on $G(2, T)$ induces Minkowski metric on real locus $U_{\mathbb{C}}(\mathbb{R}) = \mathcal{M}^4_{\mathbb{R}}$: real 4-dimensional Minkowski space

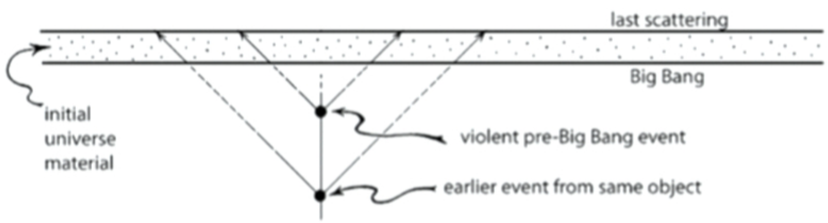
Penrose's Conformally Cyclic Cosmology (CCC)

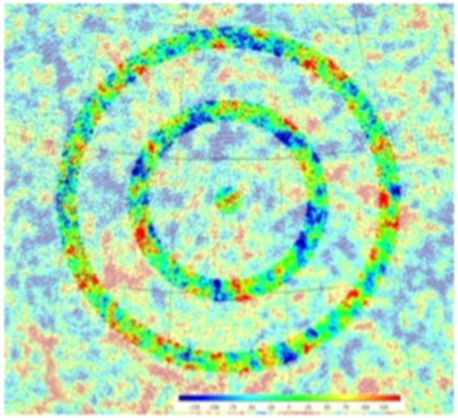


Successive aeons glued by a conformal scaling of the metric at $\pm\infty$
Matching conformal classes, not metrics

Controversial aspects

- Compatibility with inflation?
- Claims of ripples in the CMB
- Relation to other cyclic cosmologies? (mixmaster, Steinhardt-Turok, ...)





A possible approach to some of these issues: a more general (algebro-geometric) setting for such models

General geometric setting

• two complex 4-dimensional manifolds $\mathcal{M}_-$ and $\mathcal{M}_+$ (smooth, not necessarily projective)

• a smooth projective 2-dimensional quadric

$$\mathbb{S}_- \cong \mathbb{P}^1 \times \mathbb{P}^1$$

embedded $\mathbb{S}_- \subset \mathcal{M}_-$ in $\mathcal{M}_-$

• a 3-dimensional complex space $\mathcal{L}$ isomorphic to a neighborhood of the vertex of the complex cone with base $\mathbb{P}^1 \times \mathbb{P}^1$. embedded as a closed submanifold in $\mathcal{M}_+$

• explicit isomorphism of $\mathbb{S}_-$ in $\mathcal{M}_-$ with the "quadrics of null directions" $\mathbb{S}_+$ in $\mathcal{M}_+$

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Conformally Cyclic Cosmology Projectivized

Two cases of blowups

- Quadric hypersurface $Q \subset \mathbb{P}^5$, blown up at a point

$$\mathrm{Bl}_x(Q)$$

Exceptional divisor $\simeq \mathbb{P}^3$

- Projective space $\mathbb{P}^4$ ($\mathbb{P}^4 = \mathbb{A}^4 \cup \mathbb{P}^3$) blown up along a $\mathbb{P}^1$ inside the $\mathbb{P}^3$ at infinity

$$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^4)$$

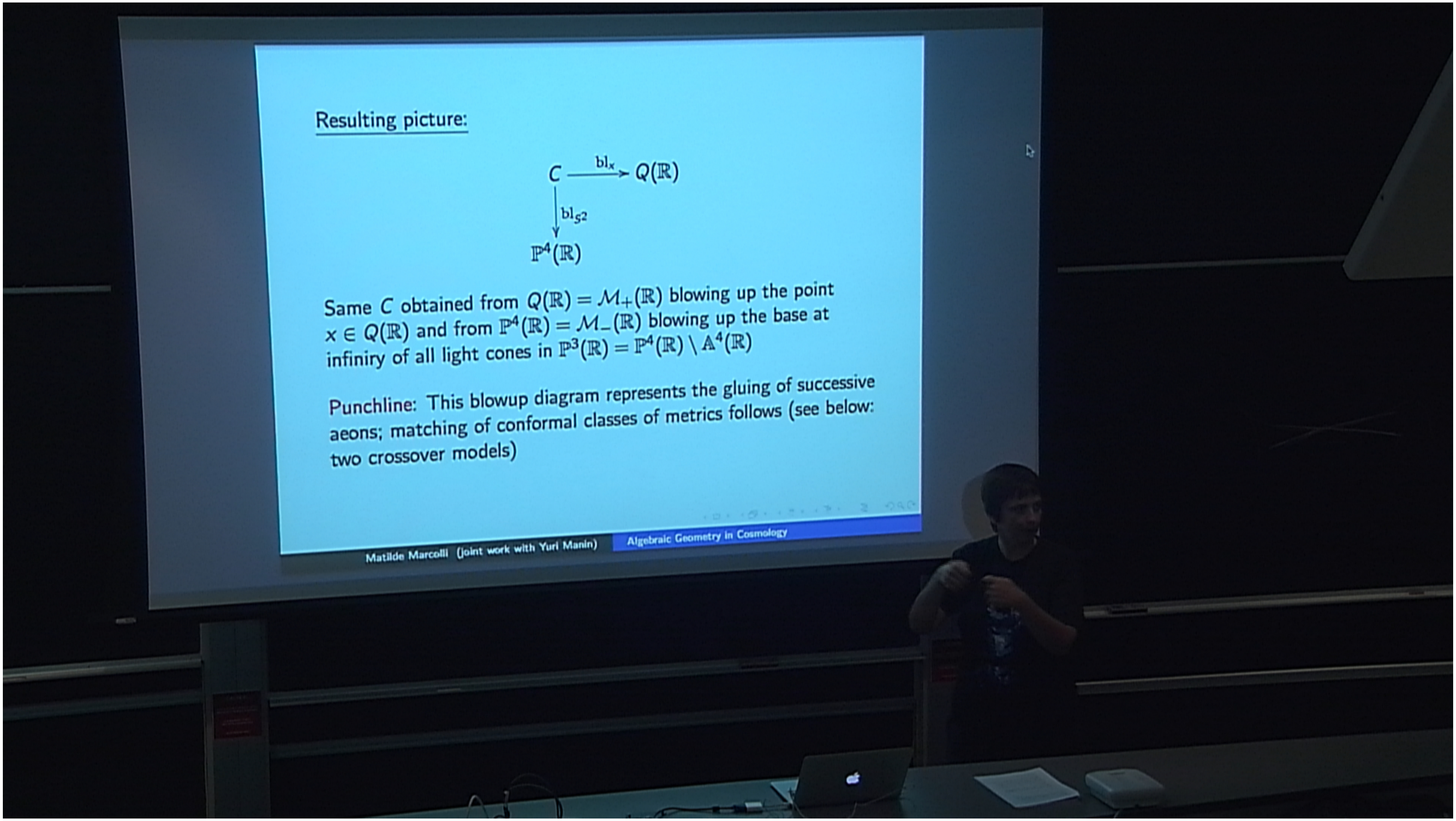
Exceptional divisor $\simeq \mathbb{P}^1 \times \mathbb{P}^1$

Real structures

• real involution on $\mathbb{S}$ comes from exchanging the two spinor bundles $\mathcal{S}$ and $\tilde{\mathcal{S}}$, so that $\mathbb{S}(\mathbb{R}) = S^2$: the diagonal $\mathbb{P}^1(\mathbb{C})$ interpreted as the "sky"

• $\mathcal{M}_- = \mathbb{P}^4$ with real locus $\mathcal{M}_-(\mathbb{R}) = \mathbb{P}^4(\mathbb{R})$, 4-dimensional real projective space

• the boundary at infinity $\mathbb{P}^3(\mathbb{R})$ contains an embedded 2-sphere S^2: the common base at infinity of all light cones in the real Minkowski 4-space $\mathcal{M}_-(\mathbb{R}) \smallsetminus \mathbb{P}^3(\mathbb{R})$


Resulting picture:
$C \xrightarrow{\mathrm{bl}_x} Q(\mathbb{R})$
bl_{S^2}
$\mathbb{P}^4(\mathbb{R})$
Same C obtained from $Q(\mathbb{R}) = \mathcal{M}_+(\mathbb{R})$ blowing up the point
$x \in Q(\mathbb{R})$ and from $\mathbb{P}^4(\mathbb{R}) = \mathcal{M}_-(\mathbb{R})$ blowing up the base at
infiniry of all light cones in $\mathbb{P}^3(\mathbb{R}) = \mathbb{P}^4(\mathbb{R}) \setminus \mathbb{A}^4(\mathbb{R})$
Punchline: This blowup diagram represents the gluing of successive
aeons; matching of conformal classes of metrics follows (see below:
two crossover models)
Matilde Marcolli (joint work with Yuri Manin)
Algebraic Geometry in Cosmology

Real structures and orientations

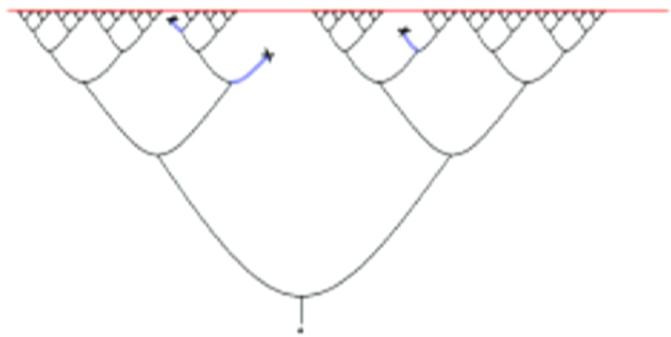
- for $n \geq 2$ have $\pi_1(\mathbb{P}^n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ with universal cover $S^n \to \mathbb{P}^n(\mathbb{R})$, sphere S^n parameterizing *oriented* lines in $\mathbb{R}^{n+1}$
- similarly for Grassmannians: real Grassmannian of *oriented* d-spaces is universal cover (double cover) of real points of the complex Grassmannian (unoriented spaces)
- blowup diagram lifts to these coverings with future and past boundaries
- can glue future boundary of $\mathcal{M}_-$ to past boundary of $\mathcal{M}_+$

Two crossover models

1. Identify 3–dim projective space $\mathrm{bl}_x^{-1}(x)$ with 3-dim projective space at infinity of $\mathbb{P}^4(\mathbb{R})$ so that sphere of null–directions in $\mathrm{bl}_x^{-1}(x)$ identified with common base at infinity of all light cones
2. Identify divisor $\mathrm{bl}_{S^2}^{-1}(\mathbb{P}^3(\mathbb{R})) \subset \mathrm{Bl}_{S^2}(\mathbb{P}^4(\mathbb{R}))$ (=infinity of the first aeon) with divisor $\mathrm{bl}_x^{-1}(L_x) \subset \mathrm{Bl}_x(Q(\mathbb{R}))$ (=Big Bang of the second aeon)

Second model reflects Penrose's CCC, first leads to an idea for inflation/multiverse

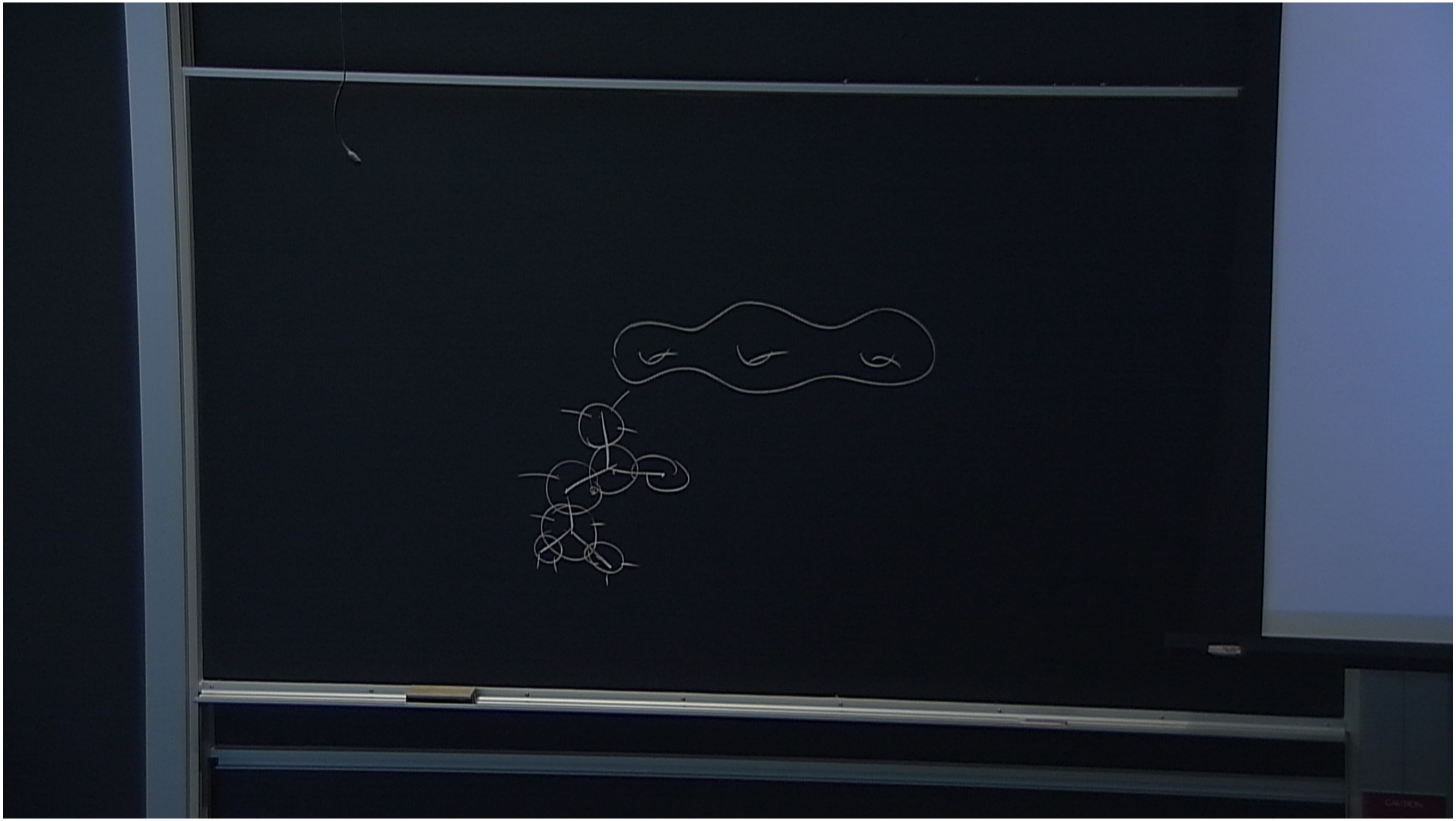
Eternal Symmetree: a discretized model of eternal inflation

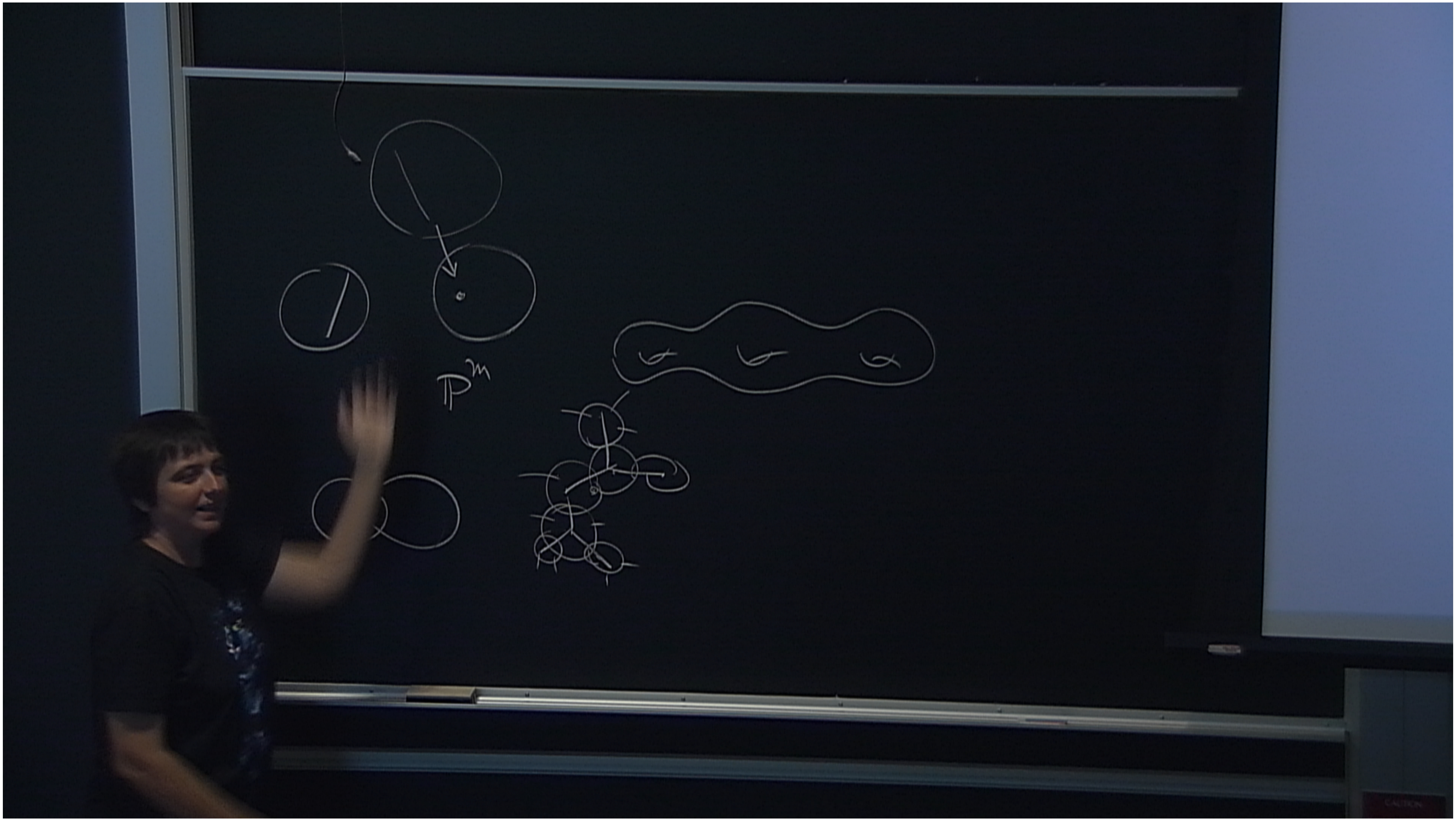


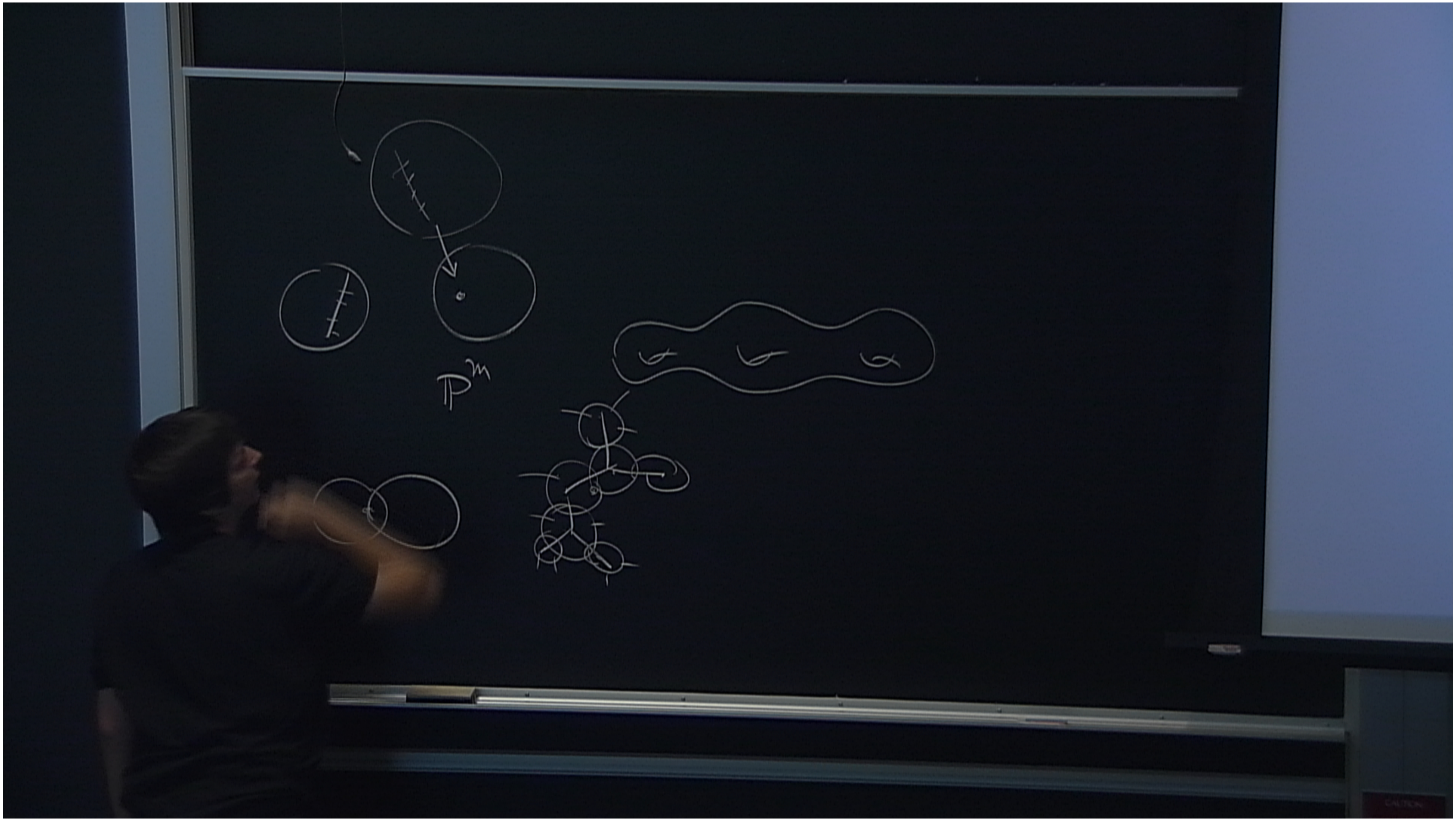
branching off according to a tree (pruning, terminal vacua)

Reference:

1. D. Harlow, S. Shenker, D. Stanford, L. Susskind, *Eternal Symmetree*, arXiv:1110.0496




$\mathbb{P}^m$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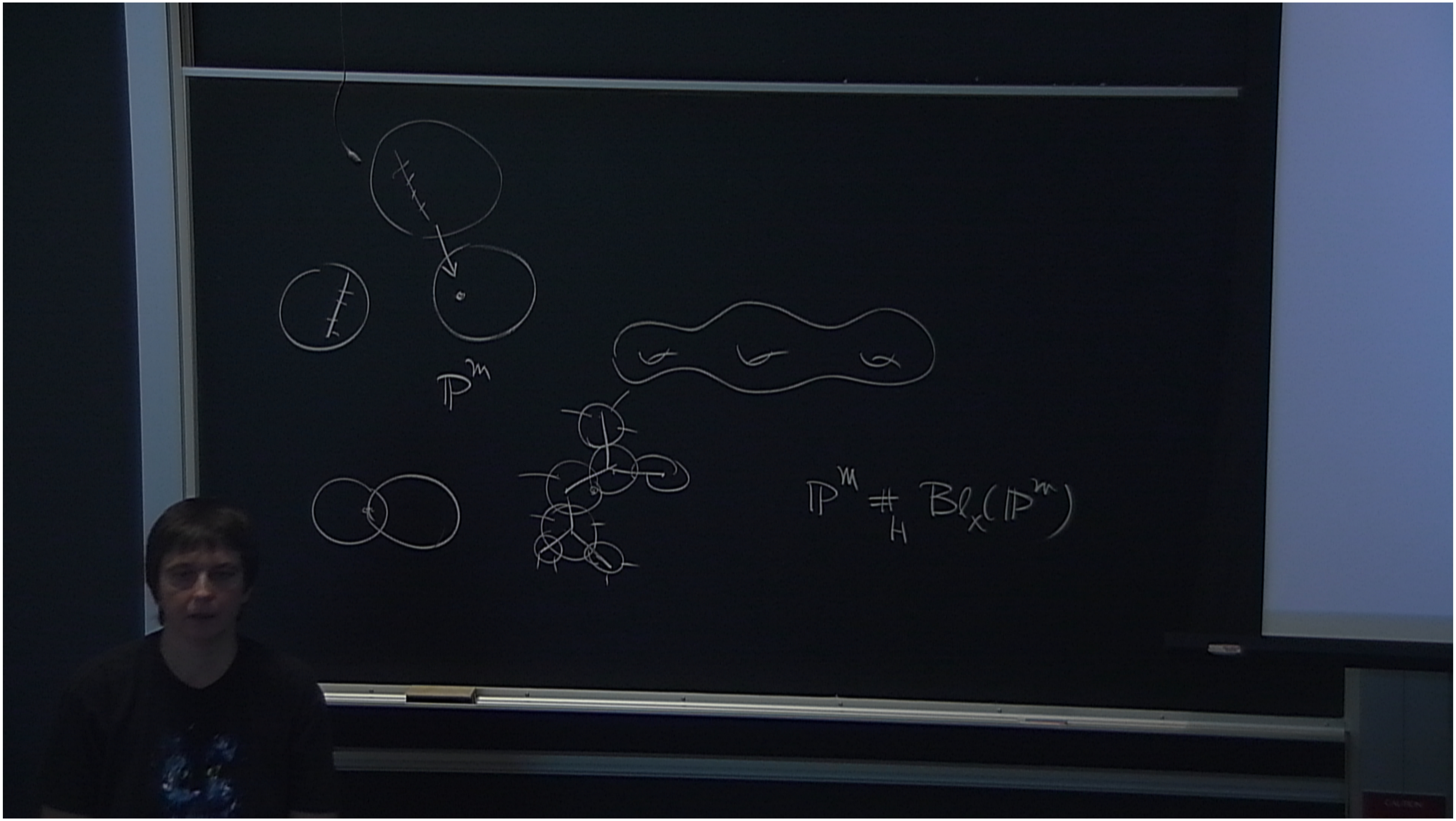
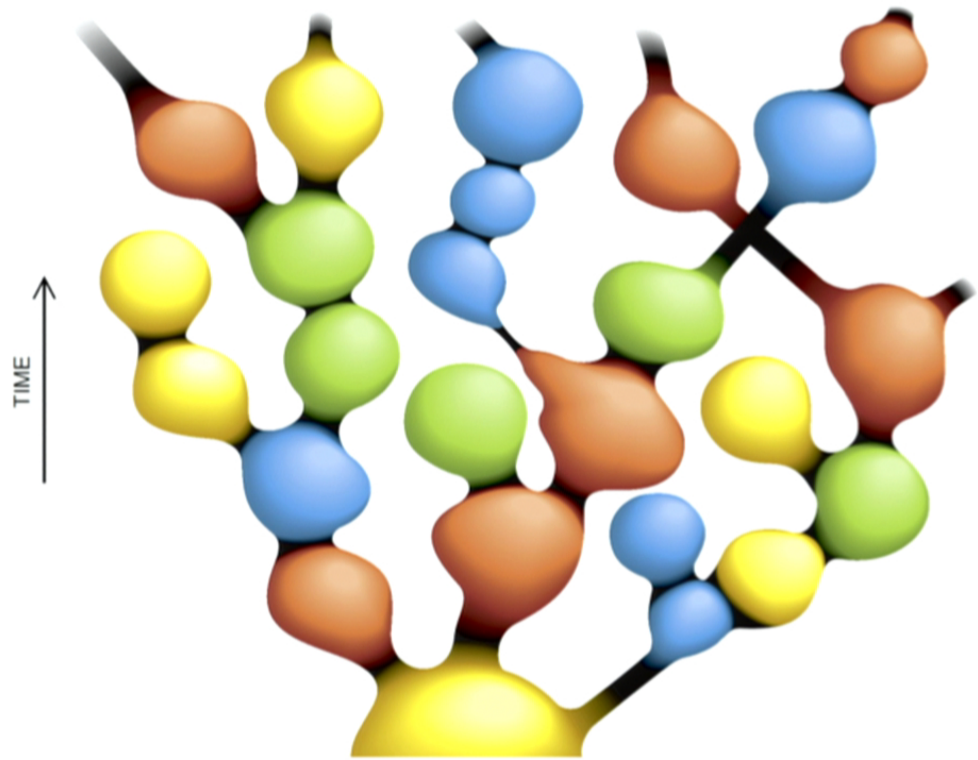


$\mathbb{P}^m$
$\mathbb{P}^m \#_H Bl_x(\mathbb{P}^m)$

Eternal inflation



Multiverse landscape: can it fit into the general framework above?

Multiverse models:

Spacetime picture

• take each universe to be a $\mathbb{P}^4(\mathbb{R})$ and use crossover model 2, with exceptional divisor of blowup at a point (=Big-Bang of second aeon) glued to hyperplane at infinity of previous aeon

• eternal inflation type of multiverse picture: multiverse fields live on moduli spaces $T_{4,n}$

Twistor picture

• use twistor spaces $\mathbb{P}^3(\mathbb{C})$ as twistor transforms of complex spacetimes $G(2, T)$

• result of sequence of blowups and gluing for tree of $\mathbb{P}^3(\mathbb{C})$'s is not necessarily the twistor space of another smooth 4–dimensional spacetime, but can be deformed to one (Donaldson–Friedman)

• deformation happens in moduli space $T_{3,n}$

Mixmaster dynamics and modular curve

- continued fraction expansion of $x > 1$

$$x = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \dots}} := [k_0, k_1, k_2, \dots]$$

discrete dynamics: one-sided shift operator

$$[k_0, k_1, k_2, \dots] \mapsto [0, k_0, k_1, k_2, \dots], \quad x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right]$$

- information about both sequences (u, Ω): invertible (two-sided) shift

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]$$

or else

$$(x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y + [1/x]} \right).$$

with $x = [k_0, k_1, k_2, \dots]$ and $y = [0, k_0, k_{-1}, k_{-2}, \dots]$