Title: Physical Hilbert space for the affine group formulation of 4D, gravity of Lorentzian signature.

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Abstract: <span>The authors have revealed a fundamental structure which has been hidden within the Wheeler-DeWitt (WDW) constraint of four dimensional General Relativity (GR) of Lorentzian signature in the Ashtekar self-dual variables. The WDW equation can be written as the commutator of two geometric entities, namely the imaginary part of the Chern-Simons functional Q and the local volume element V(x) of 3-space. Upon quantization with cosmological constant, the WDW equation takes on the form of the Lie algebra of the affine group of transformations of the straight line, with Q and V(x) playing the role of the generators for the Lie algebra. The generators are Hermitian, which addresses the issue of the implementation of the reality conditions of GR at the quantum level. Additionally, the irreducible unitary representations (IUR) implement the positivity of the spectrum of the volume operator V(x) at the quantum level This development has led to the existence of elements of the physical Hilbert space for four dimensional gravity of Lorentzian signature, the full theory, in the form of irreducible, unitary representations of the affine group of transformations of the straight line. The affine Lie algebraic structure of the WDW equation remains intact even in the presence of nongravitational fields. This feature has led to the extension of the affine group formulation to elements of the physical Hilbert space for gravity coupled to the full Standard Model of particle physics, quantized on equal footing. Work on the physical interpretation of the states with respect to gauge-diffeomorphism invariant observables, and spacetime geometries solving the Einstein equations is in progress. The journal reference for these results are as follows: - The first result has been published in CQG 30 (2013) 065013 - The second result has just been published in Annals of Physics Journal Vol.343, pages 153-163, April 2014

Affine Lie algebra generators

- Local volume element of 3-space

$$V(x) = \sqrt{\left|\frac{1}{6}\epsilon_{ijk}\epsilon^{abc}\widetilde{E}^i_a(x)\widetilde{E}^j_b(x)\widetilde{E}^k_c(x)\right|}$$

- Chern Simons functional

$(I_{CS} = Y + iQ)$

Real part *Imaginary part*

$$I_{CS}[A] = I_{CS}[\Gamma] + \gamma\int_\Sigma R^\Gamma_a \wedge k^a + \frac{\gamma^2}{2!}\int_\Sigma k^a \wedge (D^\Gamma k)_a + \frac{\gamma^3}{3!}\int_\Sigma \epsilon^{abc} k_a \wedge k_b \wedge k_c,$$

- Wheeler-DeWitt equation

$$\{Q, V(x)\} = -\Big(\frac{G\Lambda}{2}\Big)V(x); \quad \{Y, V(x)\} = 0$$

- Physical Hilbert space $(|\psi\rangle \in \mathbf{H}_{Phys})$

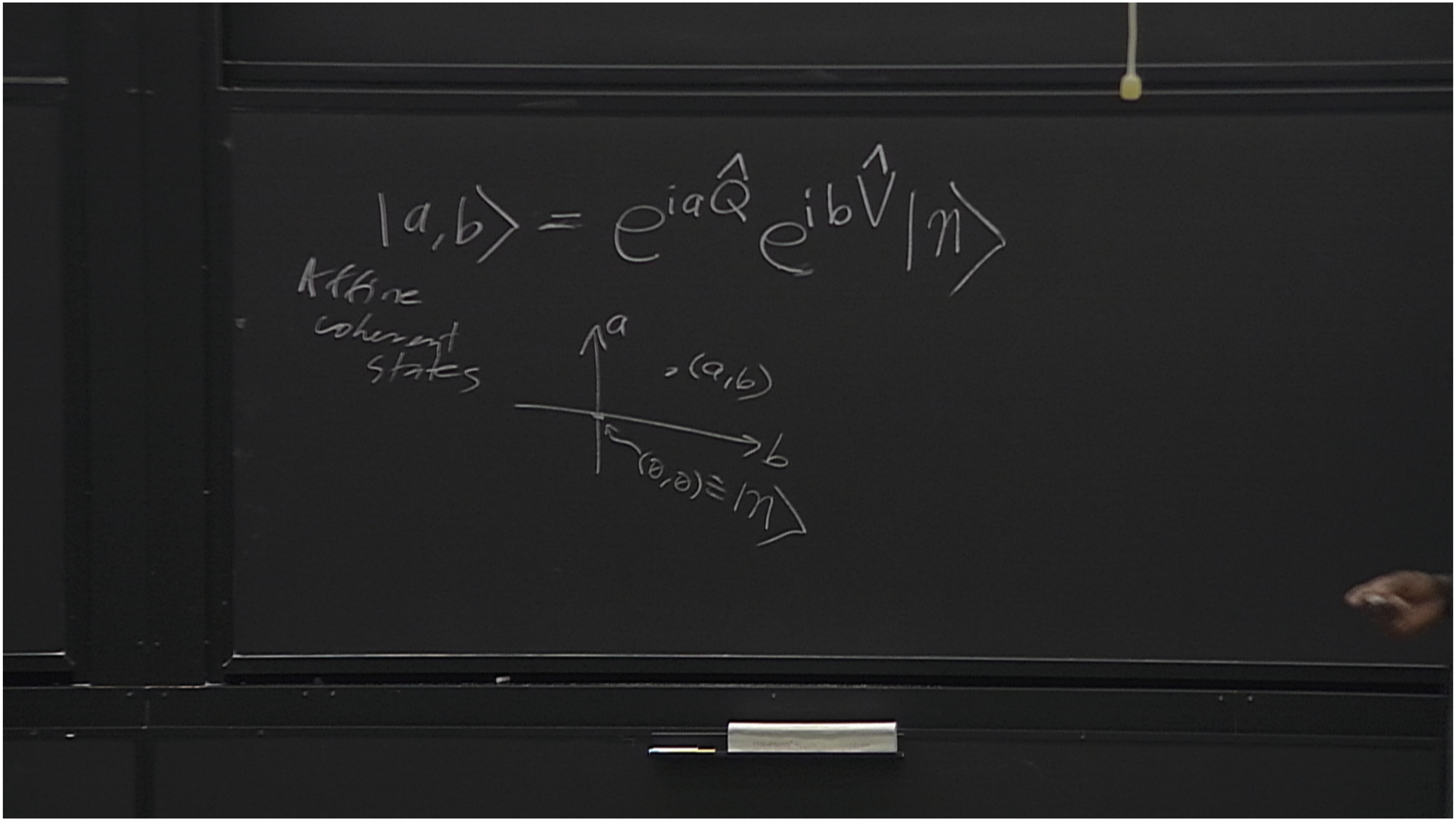
$$[\widehat{Q}, \widehat{V}(x)]|\psi\rangle = -i\Big(\frac{\hbar G\Lambda}{2}\Big)\widehat{V}(x)|\psi\rangle \ \forall x \in \Sigma$$

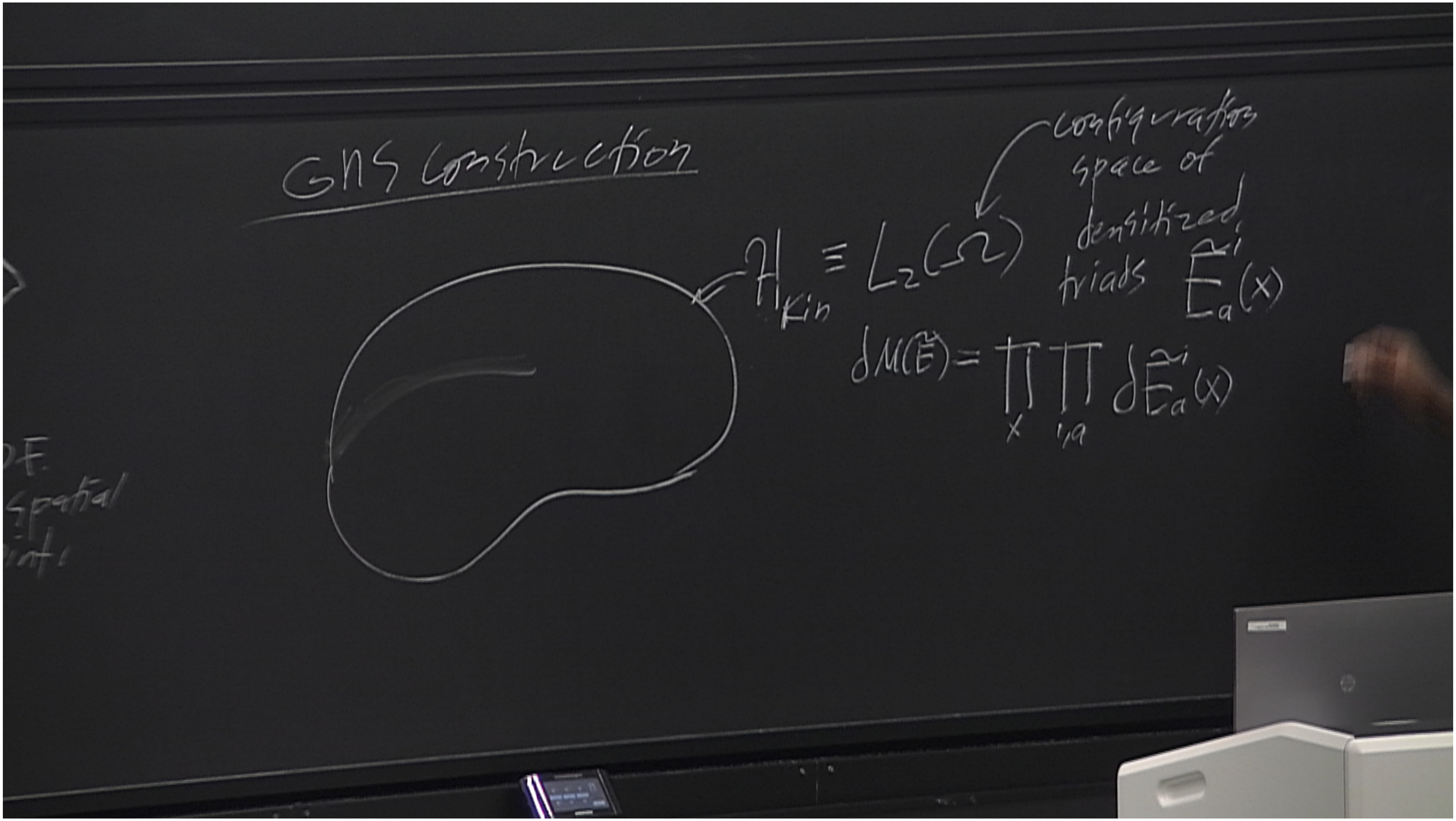
Affine Group Quantization Results

- Physical Hilbert space via Unitary, Irreducible Representations of the affine group
  - Affine gravitational coherent states
- Preserves positivity of spectrum of volume operator
- Reality conditions implemented at a quantum level
- Analogous construction extends to matter coupled to gravity
  - Gravity plus Standard Model

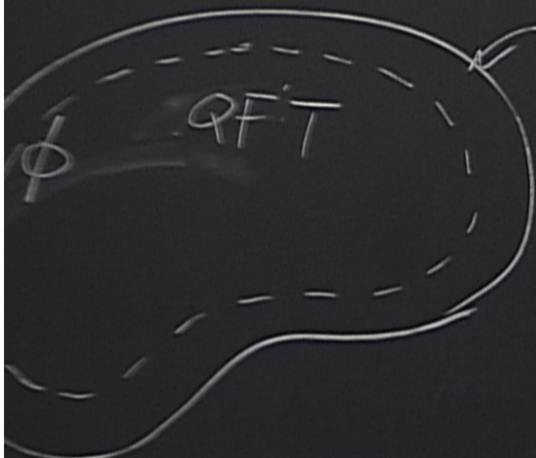
Issues in progress

- Only three Lie algebra generators
  – Do not separate the points of the full classical phase space of GR: Need to extend algebra
- Provide an explicit example of a fiducial vector to solidify correspondence
- Interpretation of the semiclassical limit of states in relation to spacetime geometries
- Verification of the Dirac algebra


$|a,b\rangle = e^{ia\hat{Q}} e^{ib\hat{V}} |\eta\rangle$
Affine coherent states
a
$\cdot (a,b)$
b
$(0,0) \equiv |\eta\rangle$


GNS construction
configuration space of densitized triads $\tilde{E}^i_a(x)$
$\mathcal{H}_{Kin} \equiv L_2(\Omega)$
$d\mu(\tilde{E}) = \prod_x \prod_{i,a} d\tilde{E}^i_a(x)$

s construction



$\mathcal{H}_{Kin} \equiv L_2(\Omega)$ configuration space of densitized triads $\tilde{E}^i_a(x)$

$$d\mu(\tilde{E}) = \prod_x \prod_{i,a} d\tilde{E}^i_a(x)$$

$$\tilde{f}_\epsilon(\emptyset) \longrightarrow \delta^{(3)}(\emptyset)$$

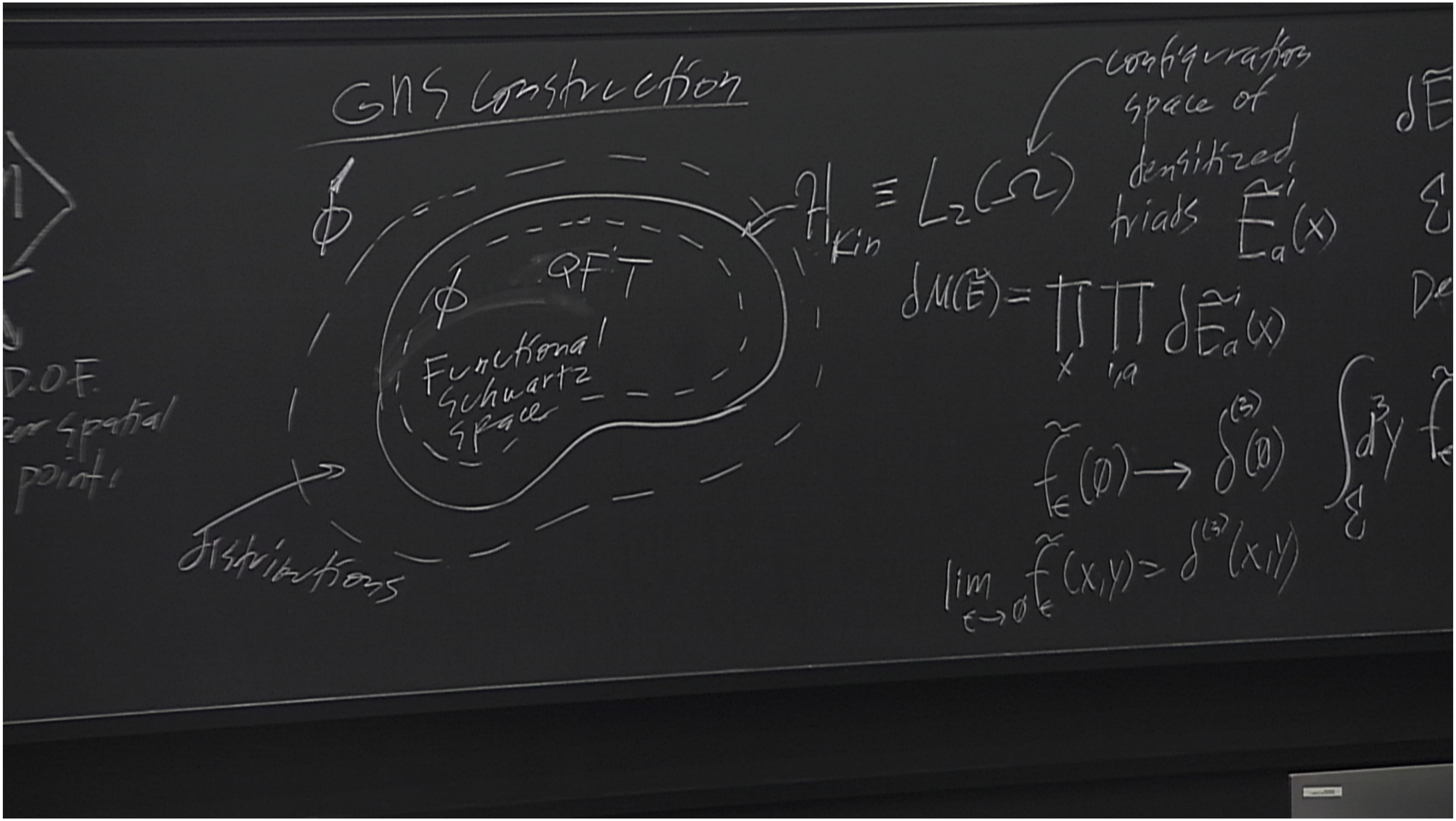
$$\lim_{\epsilon \to 0} \tilde{f}_\epsilon(x,y) = \delta^{(3)}(x,y)$$

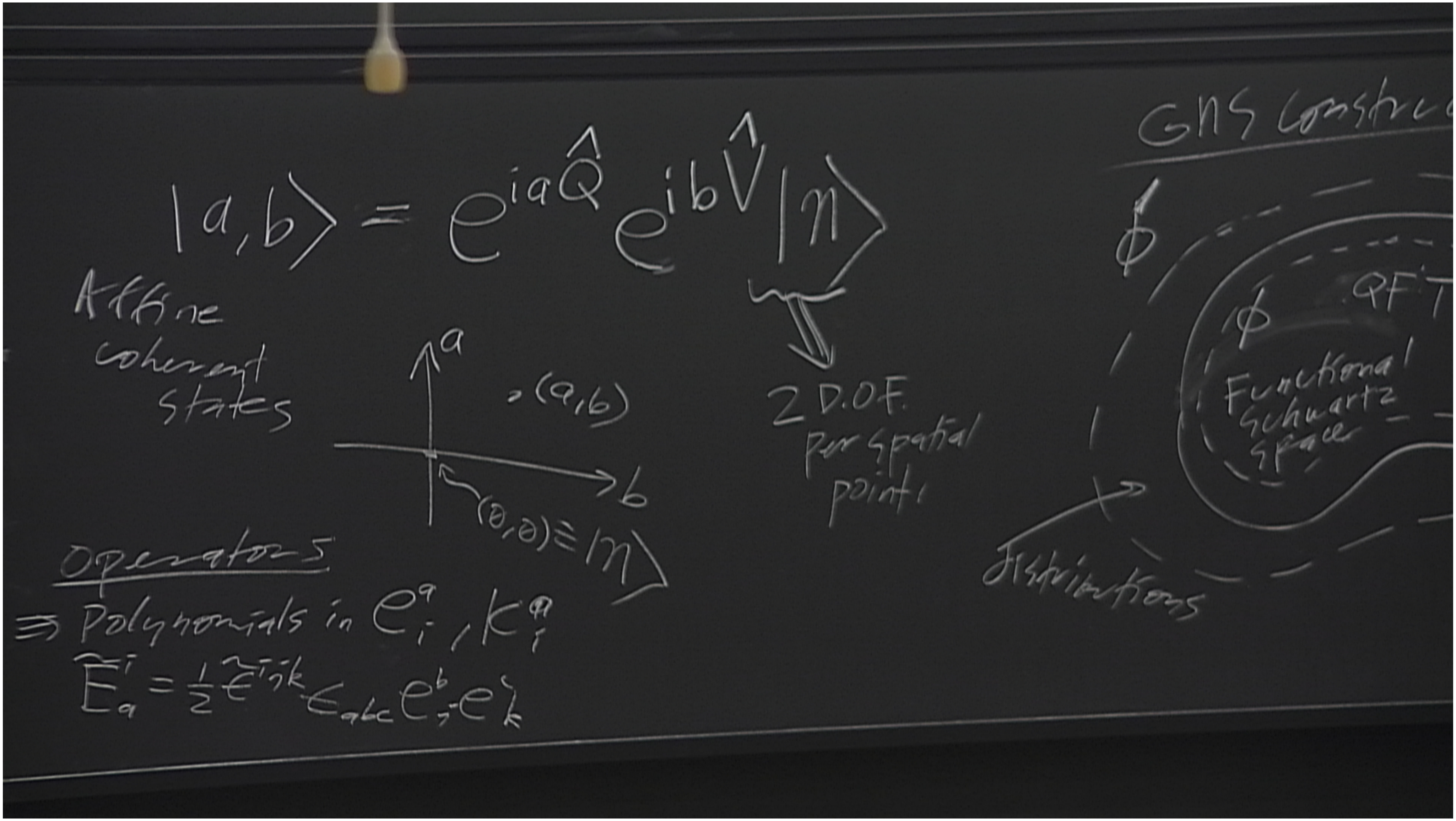
$$d\tilde{E}^i_a(x) \in T^*_{\Sigma}(\Omega)$$

Σ 3-space

Define $\tilde{f}_\epsilon(x,y)$

$$\int_\Sigma d^3y\, \tilde{f}_\epsilon(x,y)\varphi(y) = \varphi(x)$$


GNS construction
D.O.F. per spatial point
ϕ
QFT
Functional Schwartz space
Distributions
$\mathcal{H}_{kin} \equiv L_2(\Omega)$
configuration space of densitized triads $\tilde{E}^i_a(x)$
$d\mu(\tilde{E}) = \prod_x \prod_{i,a} d\tilde{E}^i_a(x)$
$\tilde{f}_\epsilon(0) \to \delta^{(3)}(0)$
$\lim_{\epsilon \to 0} \tilde{f}_\epsilon(x,y) = \delta^{(3)}(x,y)$
$\int_\Sigma d^3y$



|a,b> = e^{iaQ̂} e^{ibV̂} |η>
Affine coherent states
(a,b)
(0,0) ≅ |η>
2 D.O.F. per spatial point
GNS construc
QFT
Functional Schwartz Space
Distributions
Operators
⇒ Polynomials in e^a_i , K^a_i

$$e = (\det e_i^a) \qquad A_i^a = \Gamma_i^a + i K_i^a \qquad \hat{K}_i^a \psi = i(\hbar G)\frac{\delta \psi}{\delta E_a^i}$$

$$[iQ, V(x)] = \left[\int \tfrac{1}{2}\left(R_a^i[\Gamma]\, K_i^a + K_i^a R_a^i\right) + \int K \wedge K \wedge K,\, V(x)\right]$$

$$= e\left(\tfrac{1}{2} R[\Gamma] + \Lambda\right)$$

3D spatial curvature scalar on Σ

$$\left[\hat{K}^a_i(x), \hat{V}(y)\right] = \frac{i}{2}(\hbar G)\, e^a_i(x)\, \delta^{(3)}(x,y)$$

$e = (\det e_i^a)$ $\quad A_i^a = \Gamma_i^a + i K_i^a$ $\quad \hat{K}_i^a \psi = i(\hbar G)\dfrac{\delta \psi}{\delta \tilde{E}^i_a}$

$$[iQ, V(x)] = \left[\int \tfrac{1}{2}\left(\tilde{R}^i_a[\Gamma]\, K^a_i + K^a_i \tilde{R}^i_a\right) + \int K\wedge K\wedge K,\ V(x)\right]$$

$KKe + KeK + eKK$

$$= e\Big(\underbrace{\tfrac{1}{2}R[\Gamma]}_{\text{3D spatial curvature scalar on } \Sigma} + \Lambda\Big)$$

$$\left[\hat{K}^a_i(x), \hat{V}(y)\right] = \frac{i}{2}(\hbar G)\, e^a_i(x)\, \delta^{(3)}(x,y)$$

$$\delta^{(3)}(0) \Rightarrow f_\epsilon(0)$$

$$\hat{H}(x)\Psi[E] = \Big[e\Big(\frac{R}{2} + \Lambda + \frac{9}{2}\left(\hbar G f_\epsilon(0)\right)^2\Big) + \frac{1}{2}(\hbar G)^2\, \epsilon_{ijk}\, \epsilon^{abc} \frac{\delta}{\delta E^i_a}$$

$\Rightarrow$ Polynomials in e^a_i, K^a_i

$\tilde{E}^i_a = \frac{1}{2}\tilde{\epsilon}^{ijk}\epsilon_{abc}e^b_j e^c_k$

$\hat{H}(x)$

distributions

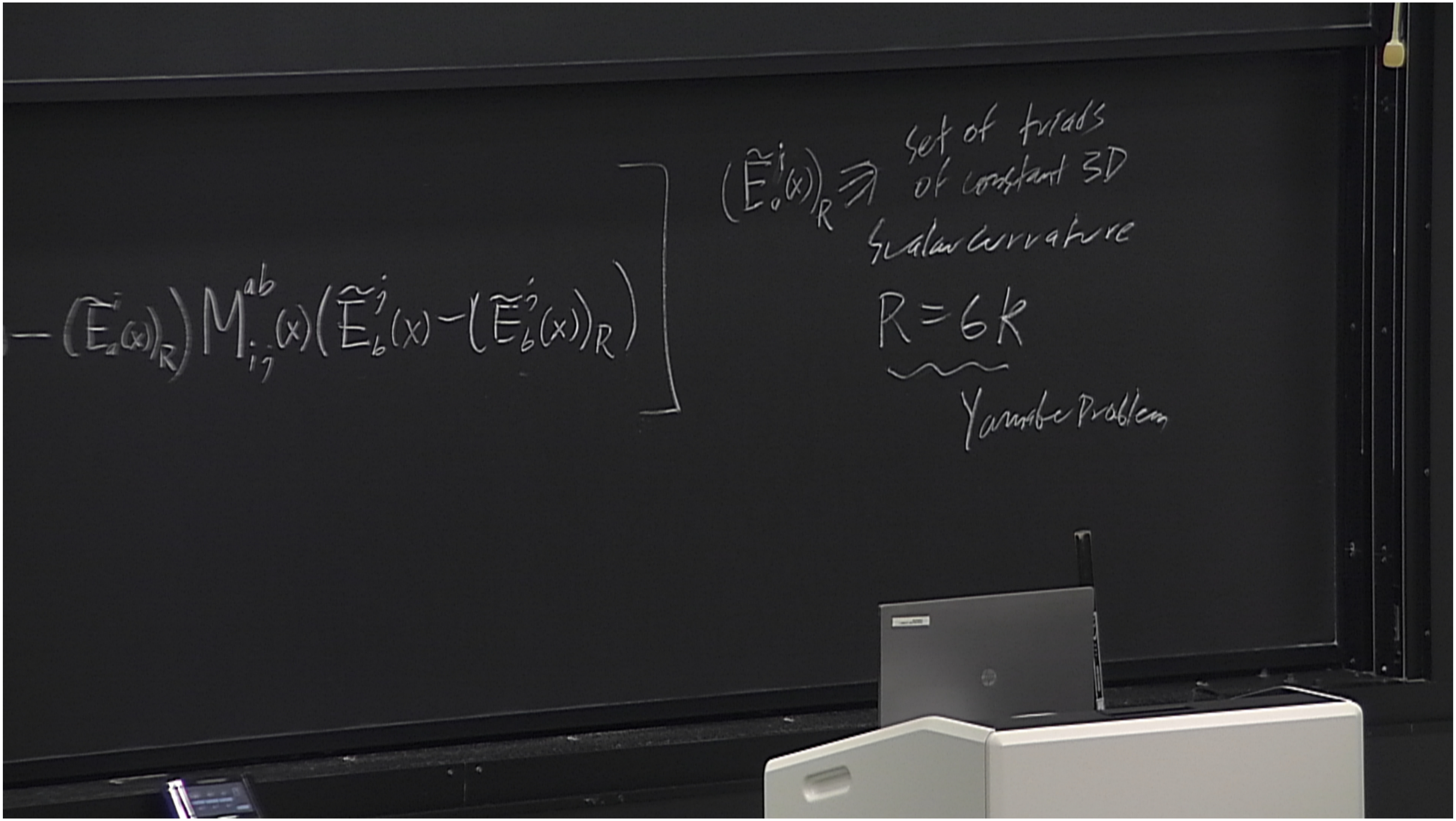
$\lim_{\epsilon\to 0} \tilde{f}_\epsilon(x,y) =$

$$\left[\hat{K}^a_i(x), \hat{V}(y)\right] = \frac{i}{2}(\hbar G)\, e^a_i(x)\, \delta^{(3)}(x,y)$$

$$\delta^{(3)}(0) \equiv f_\epsilon(0)$$

$$\hat{H}(x)\Psi[\tilde{E}] = \left[e\left(\frac{R}{2} + \Lambda + \frac{\rho}{2}(\hbar G f_\epsilon(0))^2\right) + \frac{1}{2}(\hbar G)^2 \underset{\sim}{\epsilon}_{ijk}\epsilon^{abc}\frac{\delta}{\delta \tilde{E}^i_a(x)} e^b_j(x) \frac{\delta}{\delta \tilde{E}^k_c(x)}\right]\Psi[\tilde{E}] = 0$$

$$\Psi[\tilde{E}] = \left(\prod_x \sqrt{\frac{\alpha f_\epsilon(\theta)}{\pi}}\right) \exp\Big[-\alpha f_\epsilon(\theta) \int_{\mathcal{E}} d^3x \left(\tilde{E}^i_a(x) - (\tilde{E}^i_a(x))_R\right)$$


$-\left(\tilde{E}^i_a(x)\right)_R\Big) M^{ab}_{ij}(x)\Big(\tilde{E}^j_b(x)-\left(\tilde{E}^j_b(x)\right)_R\Big)\Big]$
$\left(\tilde{E}^i_a(x)\right)_R \ni$ set of triads of constant 3D scalar curvature
$\underbrace{R = 6K}$
Yamabe Problem

$$= \left[e\left(\frac{R}{2} + \Lambda + \frac{\epsilon}{2}\left(\hbar G f_\epsilon(\emptyset)\right)^2\right) + \frac{1}{2}(\hbar G)\, \epsilon_{ijk}\epsilon^{abc} \frac{\delta}{\delta E^i_a(x)}\, e^{j}(x)\, \frac{\delta}{\delta E^k_c(x)}\right]$$

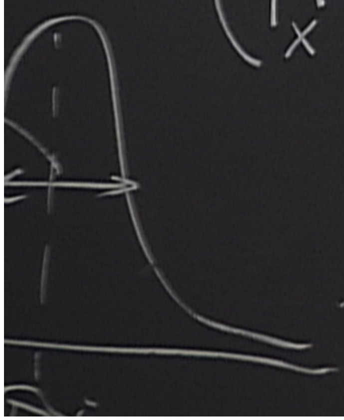
$$\Rightarrow \quad R - 6k = 0 \qquad (R - 6k)\Psi[E]\Big|_{E = E_R} = 0$$

Hessian

$$\epsilon_{ijk}\epsilon^{abc}\left(E^k_c(x)\right)_R$$

$$\Psi[\tilde E] = \left(\prod_x \sqrt{\frac{\alpha f_\epsilon(\emptyset)}{\pi}}\right) \exp\left[-\alpha f_\epsilon(\emptyset) \int_\Sigma d^3x \left(\tilde E^i_a(x) - \left(E^i_a(x)\right)_R\right) M^{ab}_{ij}(x) \left(\tilde E^j_b(x) - (\tilde E\right.\right.$$

$$\alpha f_\epsilon(\emptyset) + \frac{(3k - \Lambda)}{f_\epsilon(\emptyset)}$$



$\tilde E^i_a(x)$

$\forall x \in \Sigma$

ϕ

$$\Psi[\tilde{E}] \to \delta(\hat{\tilde{E}} - \hat{\tilde{E}}_R)$$

Gauge-Invariant, Diffeo-Invariant measure.

$\delta(\hat{\tilde{E}} - \hat{\tilde{E}}_R) \Rightarrow$ Lebesgue measure on unimodular D.O.F.

$$= \prod_{x\in\Sigma} \prod_{i,a} \delta\left(\tilde{E}^i_a(x) - (\tilde{E}^i_a(x))_R\right)$$

$$\sqrt{g}\, d^n x$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$e^{ds^2}$$

$$M^{ab}_{ij}$$

$$\int_{\Sigma^t} d^3x\, \delta s^2(x)$$

$$= \int_{\Sigma^t} d^3x\, \delta \tilde{\hat{E}}^i_a(x) \left(G^{ab}_{ij}(x)\right)_R \delta \tilde{\hat{E}}^j_b(x)$$