Title: Spatially coupled quantum LDPC codes

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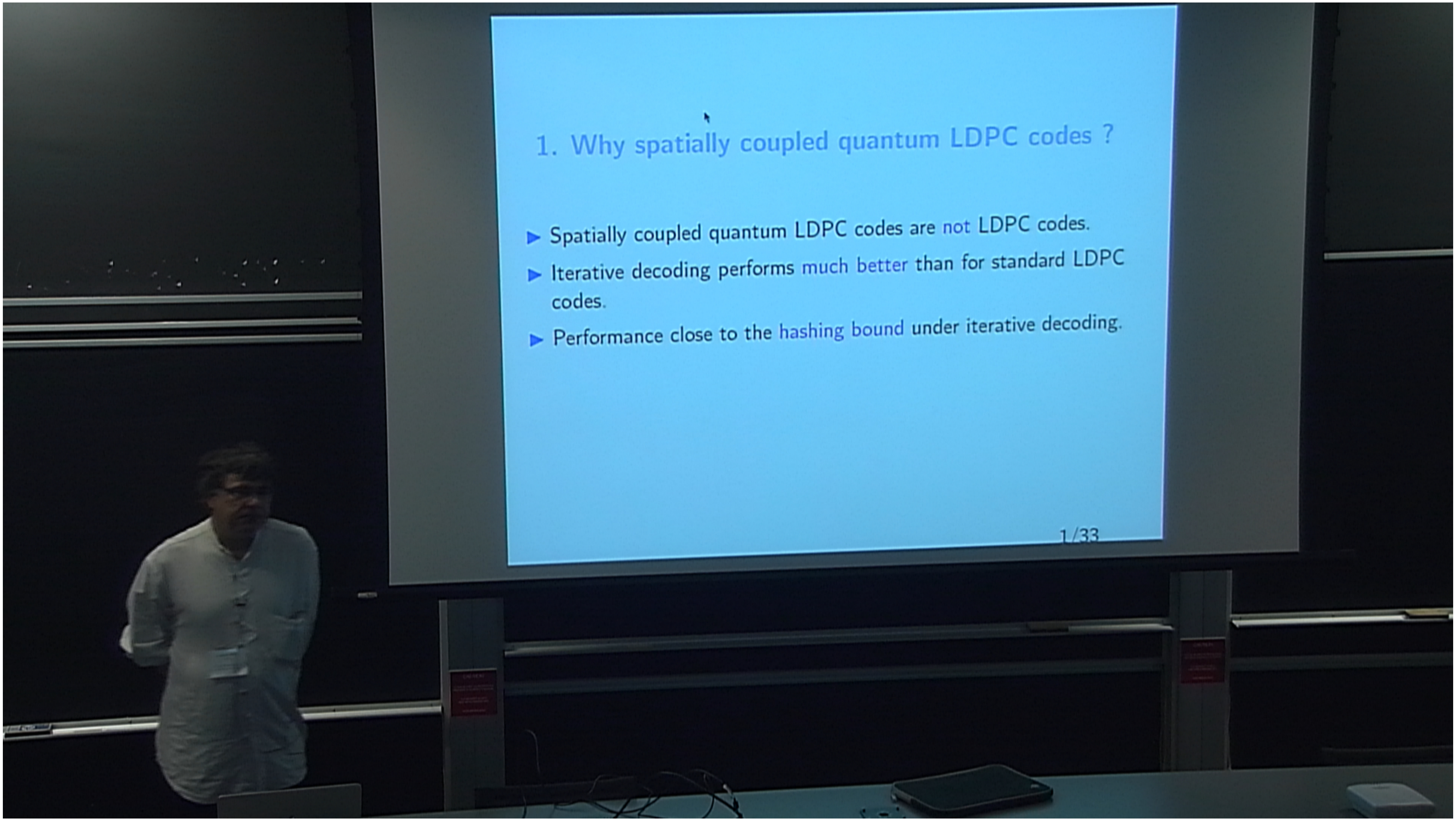
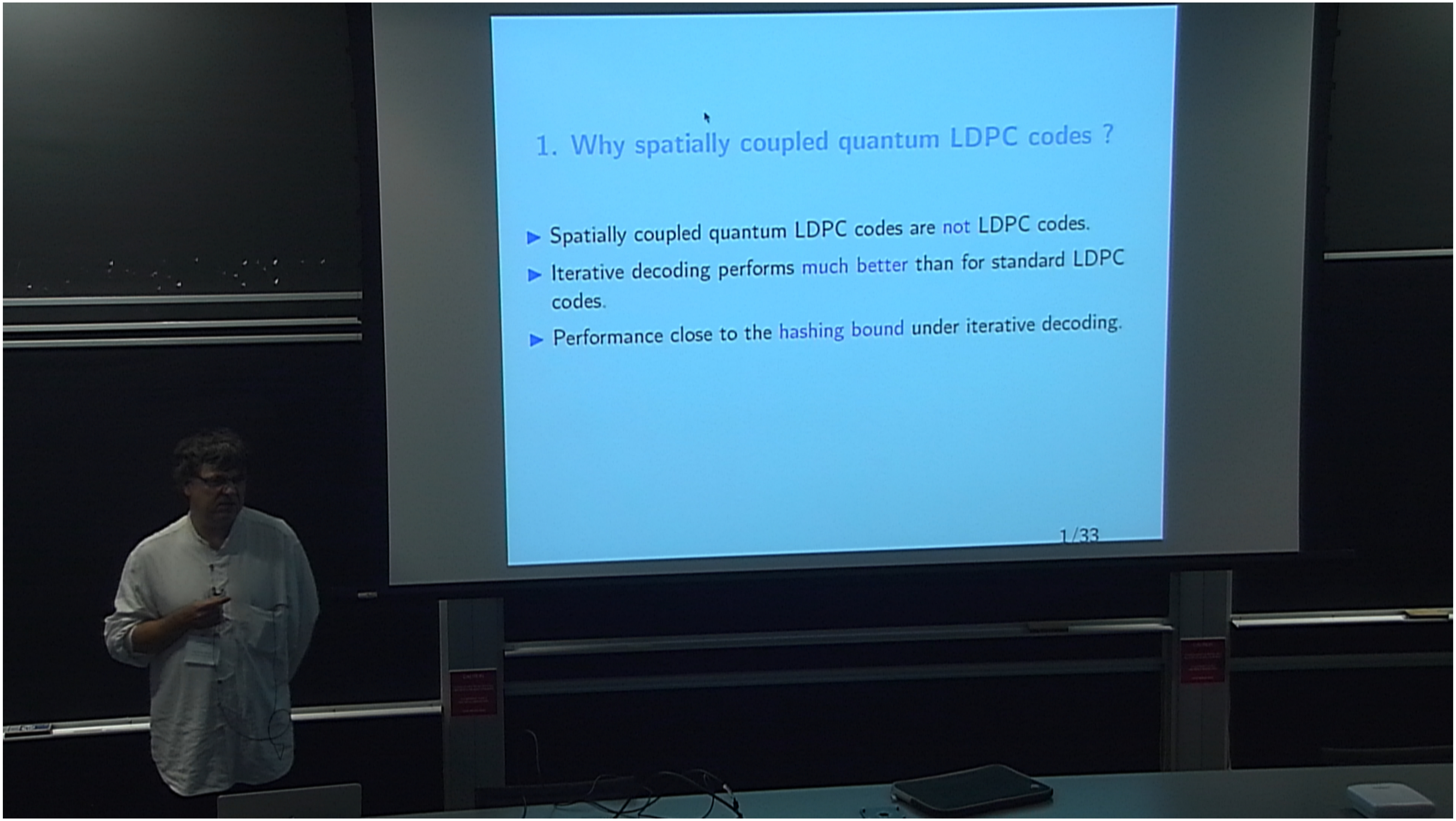
Abstract: Spatially coupled LDPC were introduced by FelstrÃ¶m and Zigangirov in 1999. They might be viewed in the following way, take several several instances of a certain LDPC code family, arrange them in a row and then mix the edges of the codes randomly among neighboring layers. Moreover fix the bits of the first and last layers to zero. It has soon been found out that iterative decoding behaves much better for this code than for the original LDPC code. A breakthrough occurred when it was proved by Kudekar, Richardson and Urbanke that these codes attain the capacity of all binary input memoryless output-symmetric channels.

All these nice features of classical spatially coupled LDPC codes suggest to study whether they have a quantum analogue. The fact that spatially coupled LDPC codes may afford to have large degrees and still perform well under iterative decoding would be quite interesting in the quantum setting, since by the very nature of the quantum construction of stabilizer codes the rows of the parity-check matrix of the quantum code have to belong to the code which is decoded by the iterative decoder. This implies that we should have rather large row weights to avoid severe error-floor phenomena and/or oscillatory behavior of iterative decoding which degrades significantly its performance.

With Andriyanova and Maurice, I showed last year that it is possible to come up with coupled versions of quantum LDPC codes that perform excellently under iterative decoding. For instance we have constructed a spatially coupled LDPC code family of rate $\approx \frac{1}{4}$ which performs well under iterative decoding even for noise values close to the hashing bound $p \approx 0.127$.
This represents a tremendous improvement over all previous known families of quantum LDPC codes of the same rate.
I will discuss in this talk what can be expected from this approach when these spatially coupled LDPC codes are used for performing fault tolerant computation.


1. Why spatially coupled quantum LDPC codes ?
Spatially coupled quantum LDPC codes are not LDPC codes.
Iterative decoding performs much better than for standard LDPC codes.
Performance close to the hashing bound under iterative decoding.
1/33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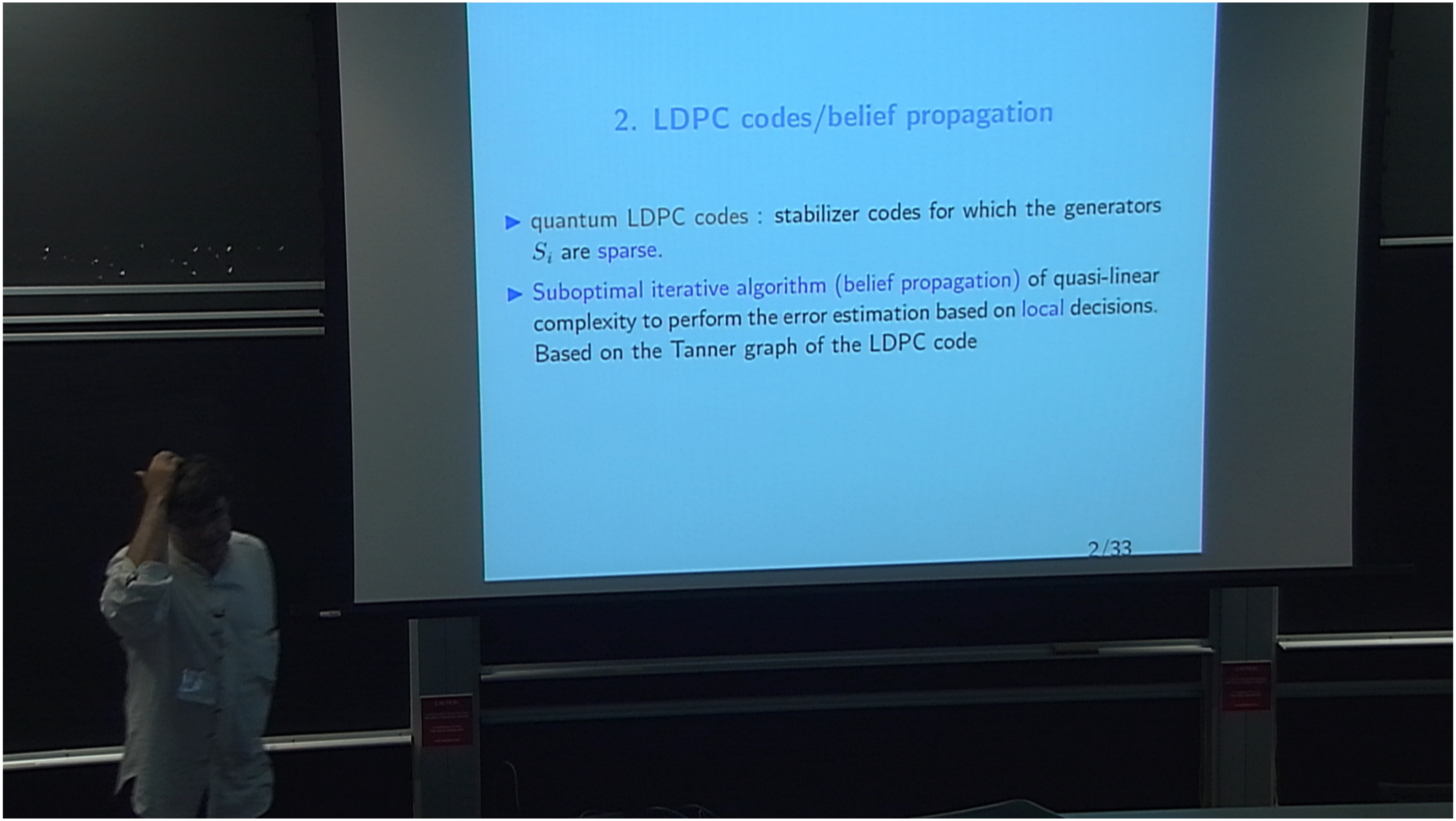
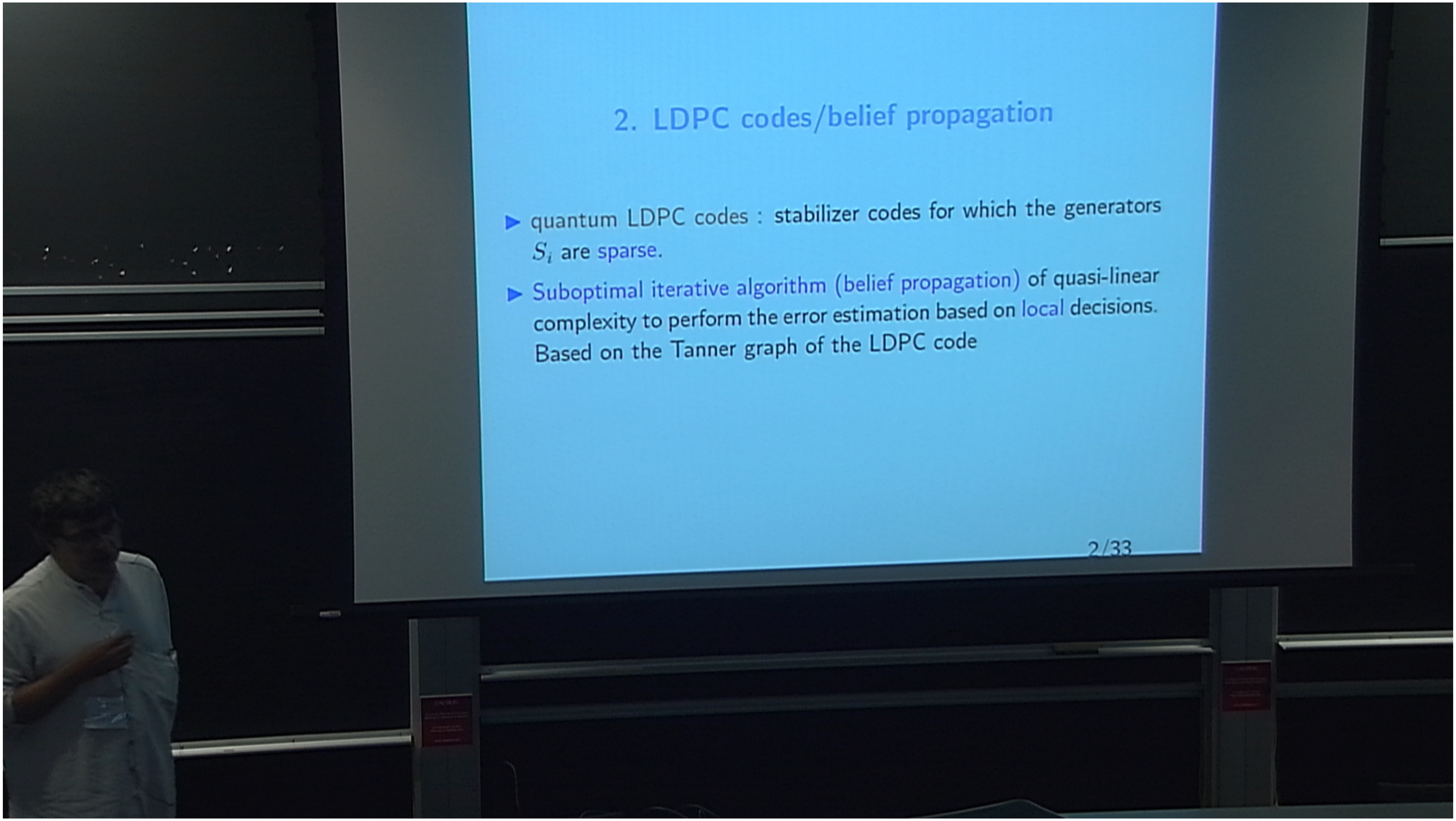

2. LDPC codes/belief propagation
quantum LDPC codes : stabilizer codes for which the generators S_i are sparse.
Suboptimal iterative algorithm (belief propagation) of quasi-linear complexity to perform the error estimation based on local decisions. Based on the Tanner graph of the LDPC code
2/33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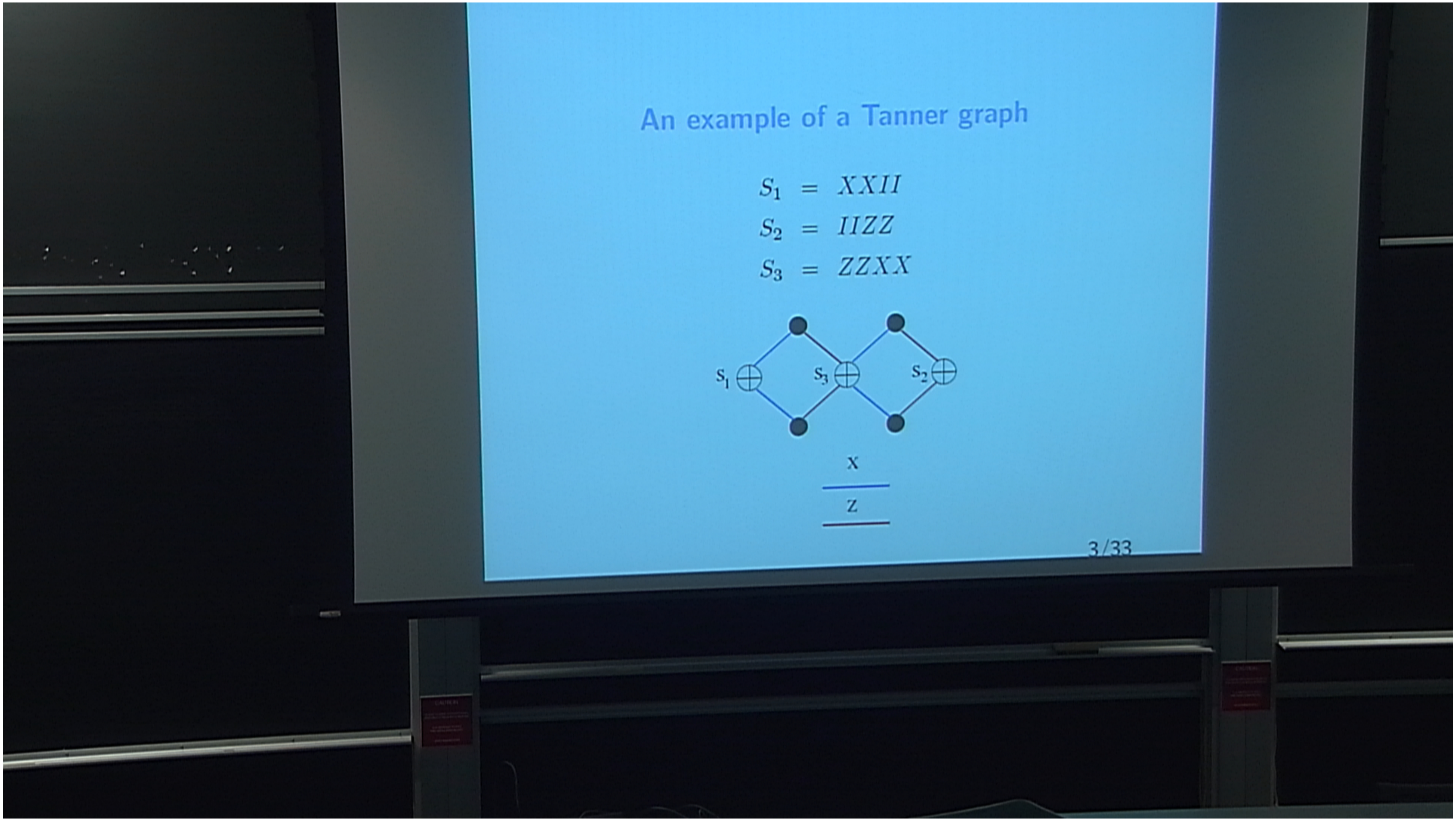
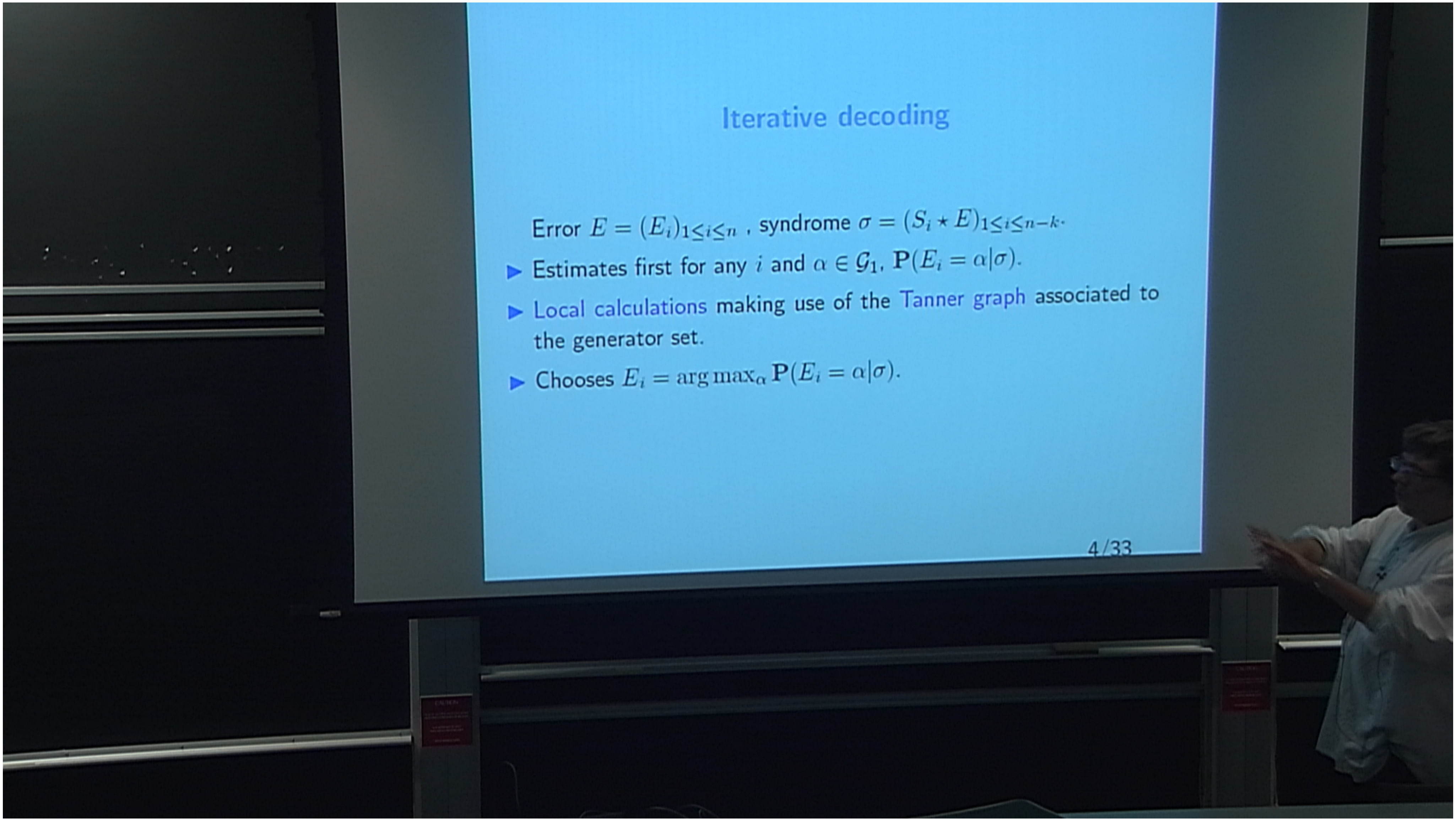
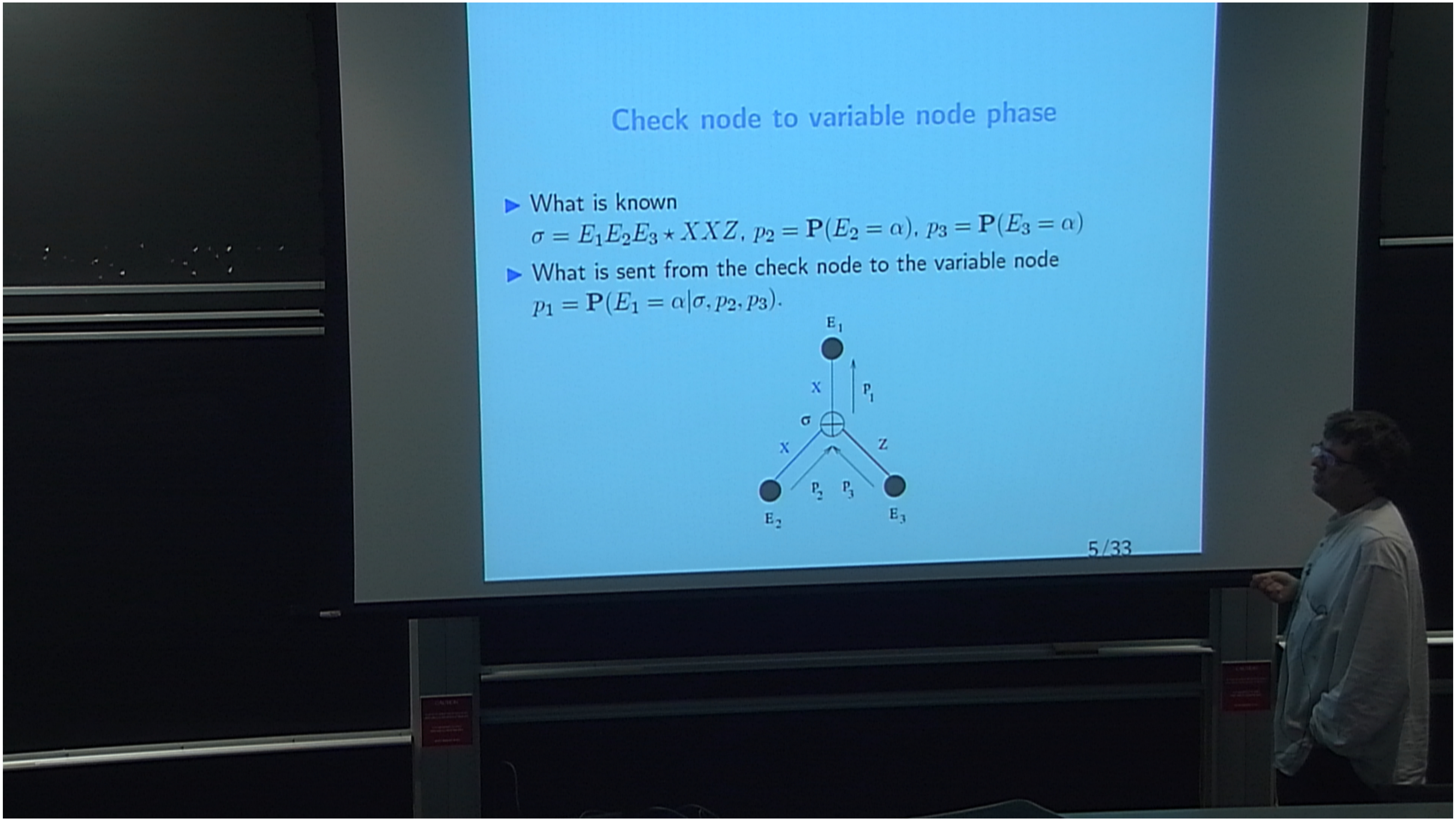
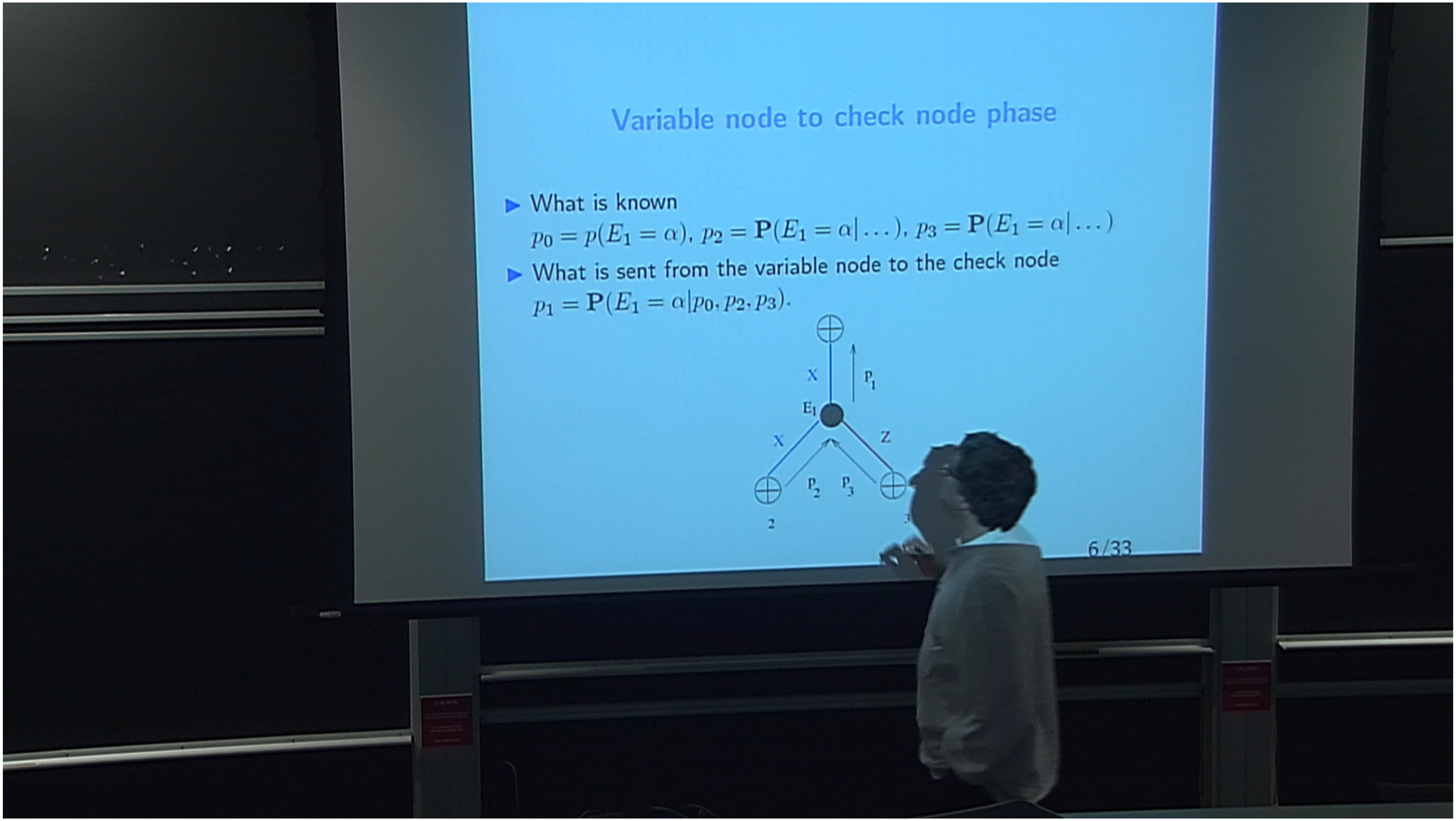
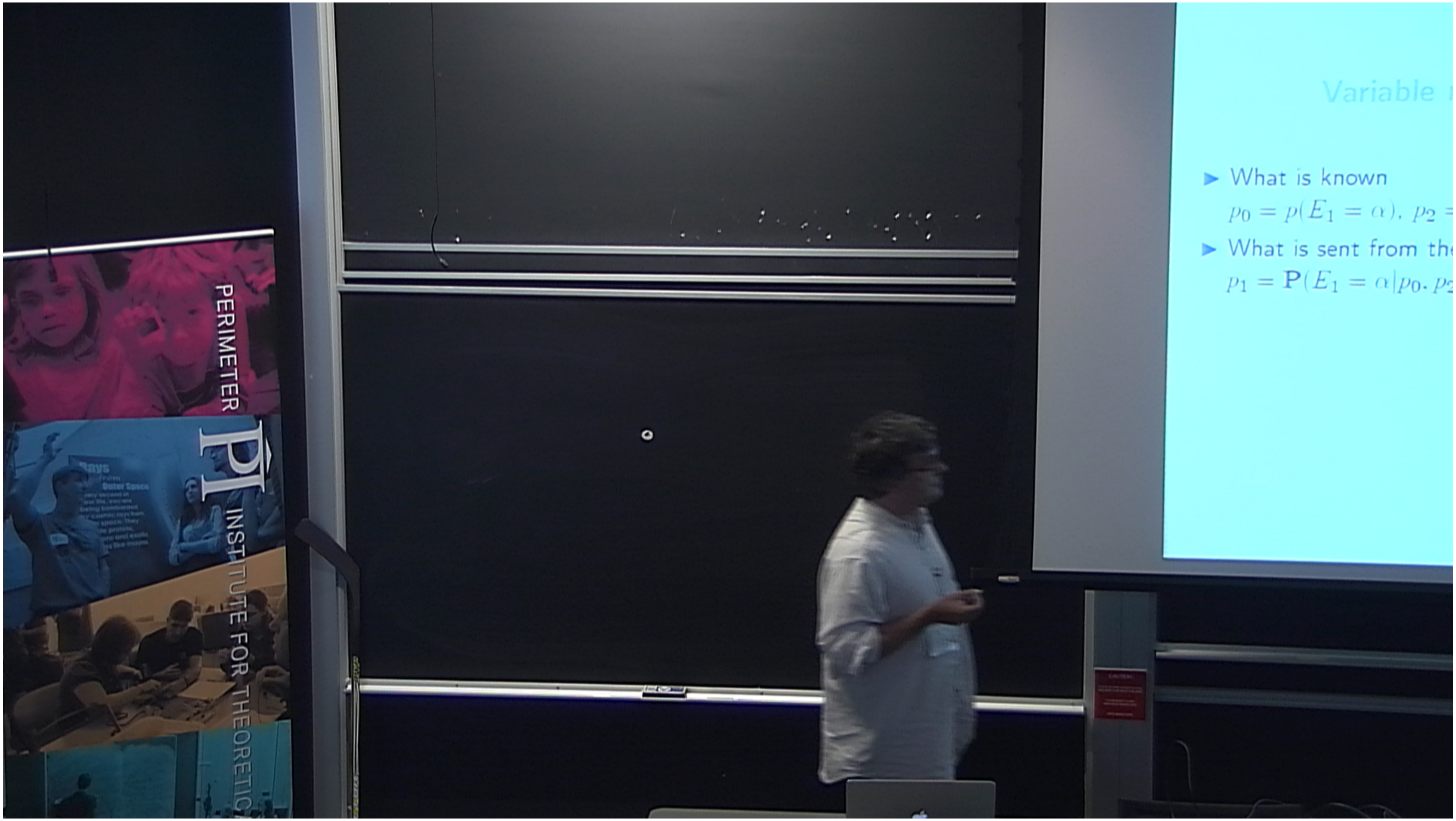
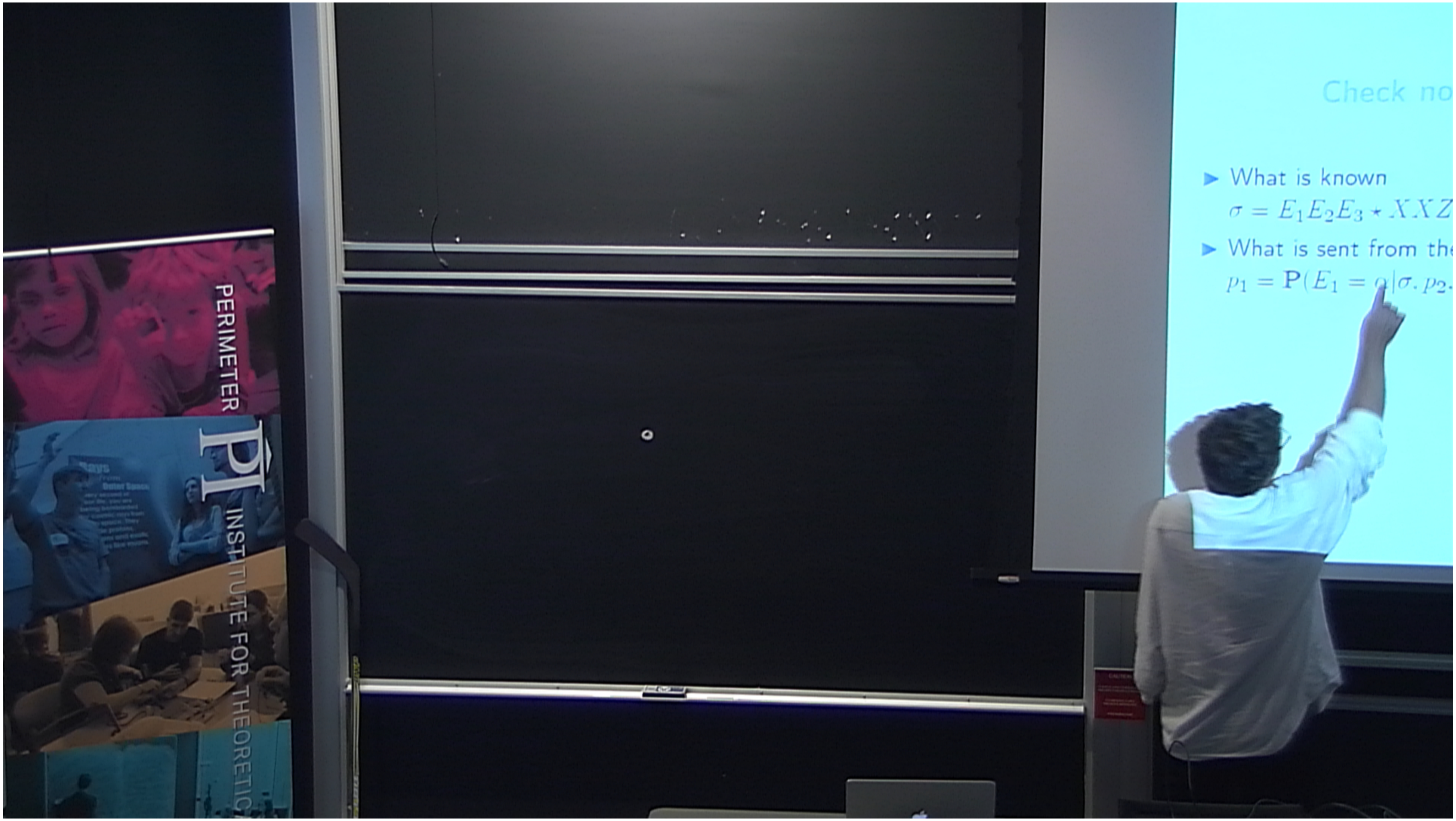

An example of a Tanner graph
$S_1 = XXII$
$S_2 = IIZZ$
$S_3 = ZZXX$
S_1
S_3
S_2
X
Z
3/33

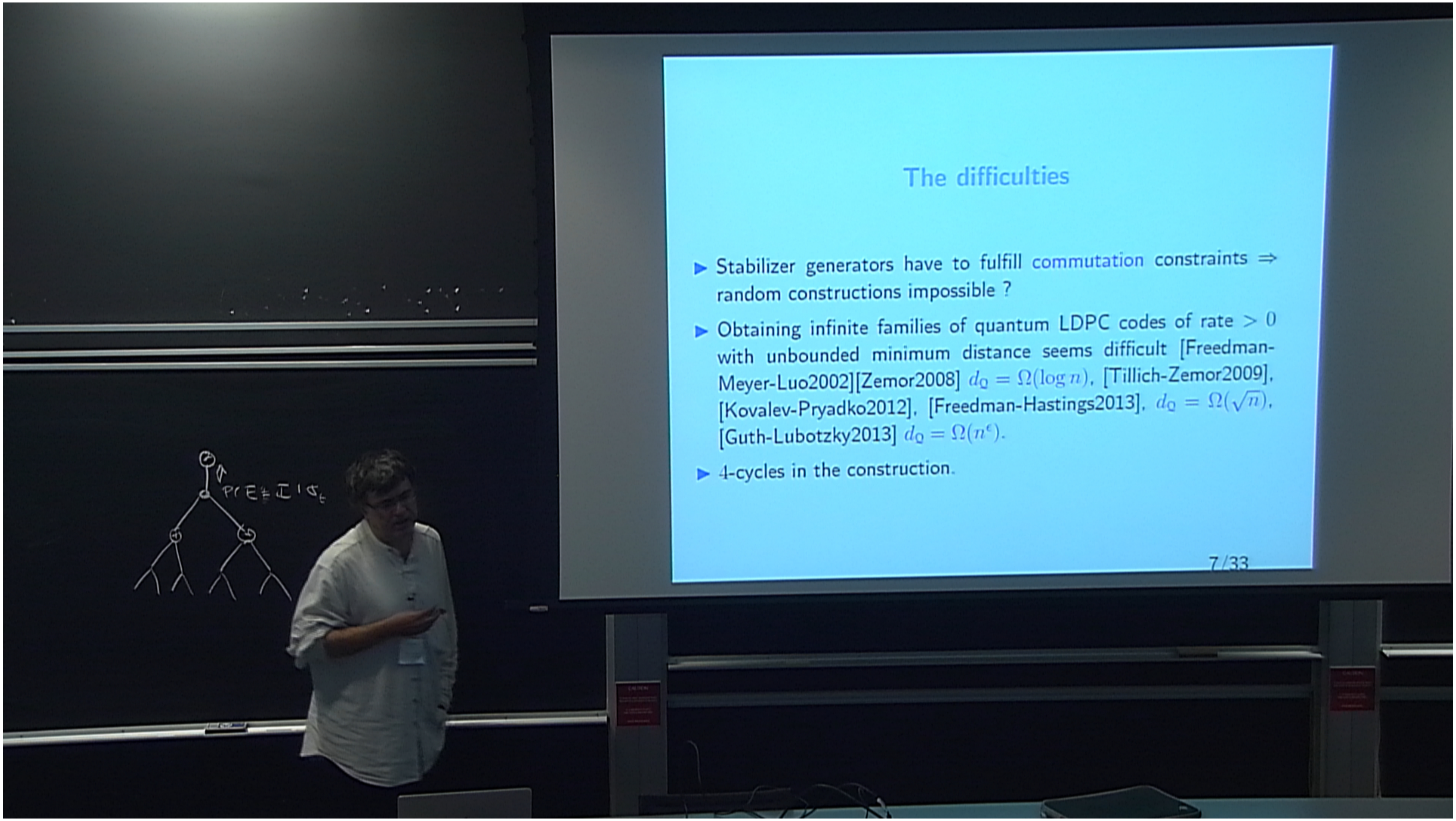

Iterative decoding
Error $E = (E_i)_{1 \leq i \leq n}$, syndrome $\sigma = (S_i \star E)_{1 \leq i \leq n-k}$.
Estimates first for any i and $\alpha \in \mathcal{G}_1$, $\mathbf{P}(E_i = \alpha | \sigma)$.
Local calculations making use of the Tanner graph associated to the generator set.
Chooses $E_i = \arg\max_\alpha \mathbf{P}(E_i = \alpha | \sigma)$.
4/33

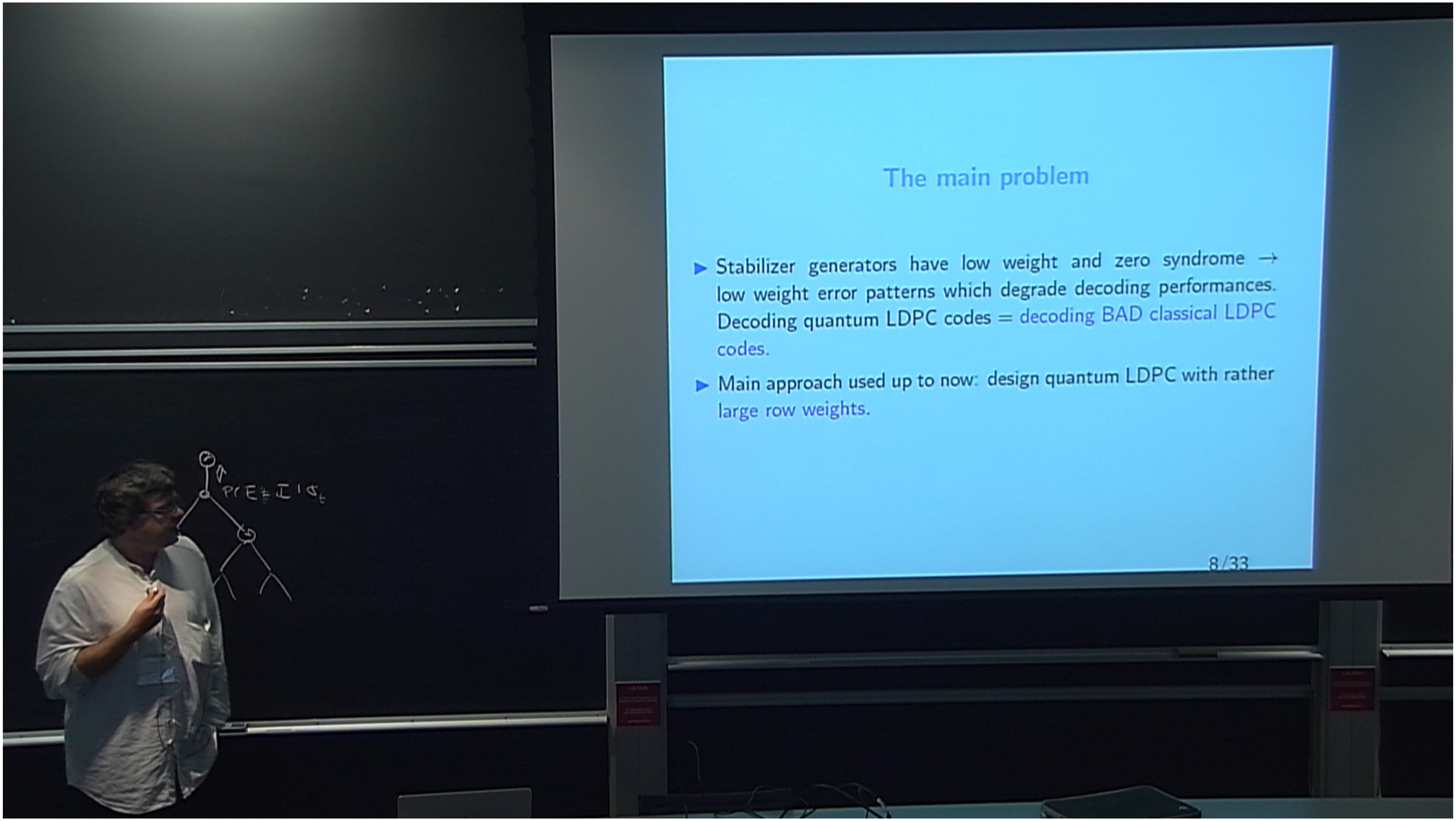

Check node to variable node phase
What is known
$\sigma = E_1 E_2 E_3 \star XXZ$, $p_2 = \mathbf{P}(E_2 = \alpha)$, $p_3 = \mathbf{P}(E_3 = \alpha)$
What is sent from the check node to the variable node
$p_1 = \mathbf{P}(E_1 = \alpha | \sigma, p_2, p_3)$.
E_1
X
p_1
σ
X
Z
p_2
p_3
E_2
E_3
5/33

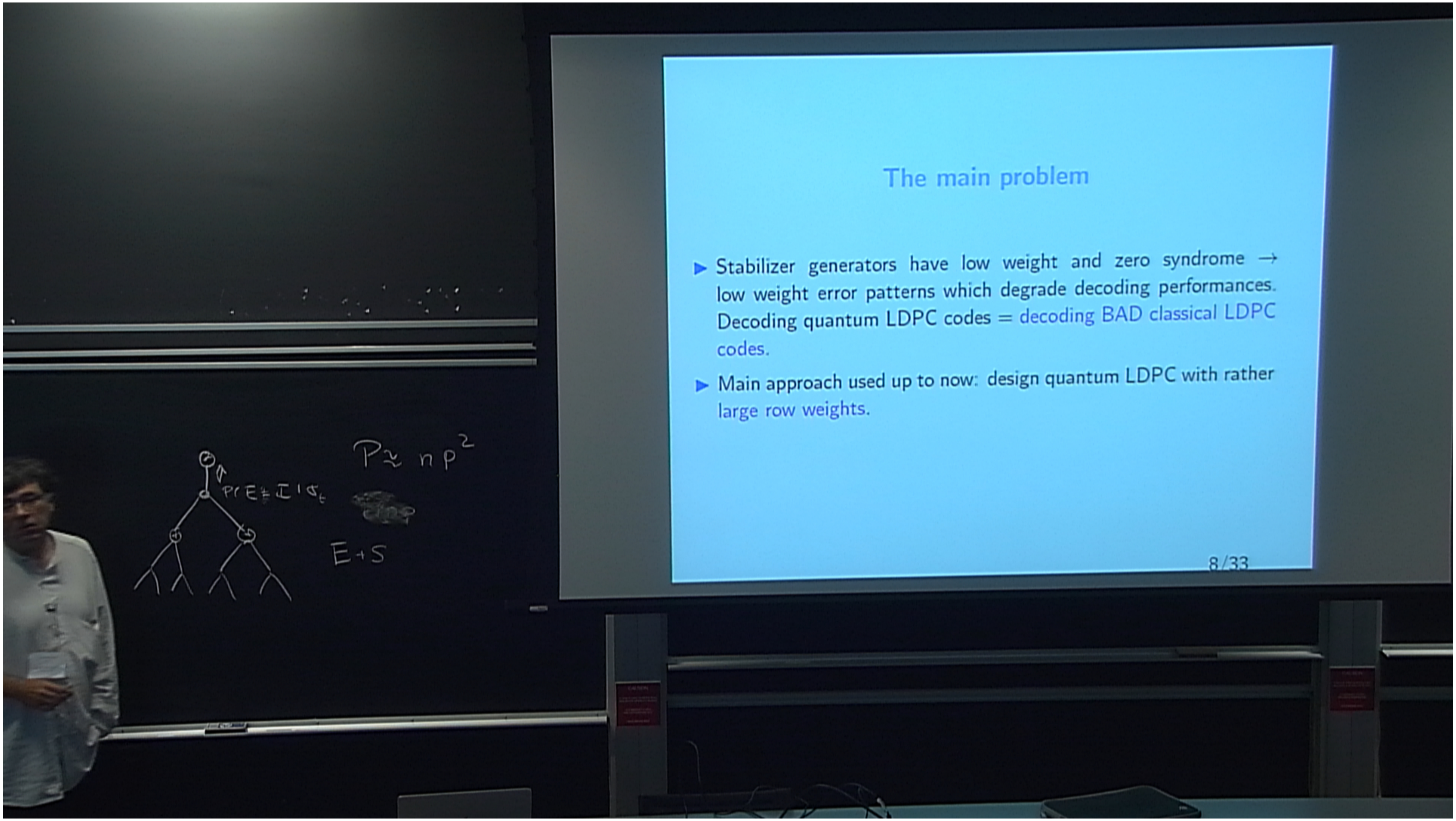

Variable node to check node phase
What is known
$p_0 = p(E_1 = \alpha)$, $p_2 = \mathbf{P}(E_1 = \alpha | \ldots)$, $p_3 = \mathbf{P}(E_1 = \alpha | \ldots)$
What is sent from the variable node to the check node
$p_1 = \mathbf{P}(E_1 = \alpha | p_0, p_2, p_3)$.
X
P_1
E_1
X
Z
P_2
P_3
2
6/33


PERIMETER
PI
What is known
What is sent from


PERIMETER
What is known


The difficulties
Stabilizer generators have to fulfill commutation constraints ⇒ random constructions impossible ?
Obtaining infinite families of quantum LDPC codes of rate > 0 with unbounded minimum distance seems difficult [Freedman-Meyer-Luo2002][Zemor2008] $d_Q = \Omega(\log n)$, [Tillich-Zemor2009], [Kovalev-Pryadko2012], [Freedman-Hastings2013], $d_Q = \Omega(\sqrt{n})$, [Guth-Lubotzky2013] $d_Q = \Omega(n^\epsilon)$.
4-cycles in the construction.
7/33


The main problem
Stabilizer generators have low weight and zero syndrome → low weight error patterns which degrade decoding performances. Decoding quantum LDPC codes = decoding BAD classical LDPC codes.
Main approach used up to now: design quantum LDPC with rather large row weights.
8/33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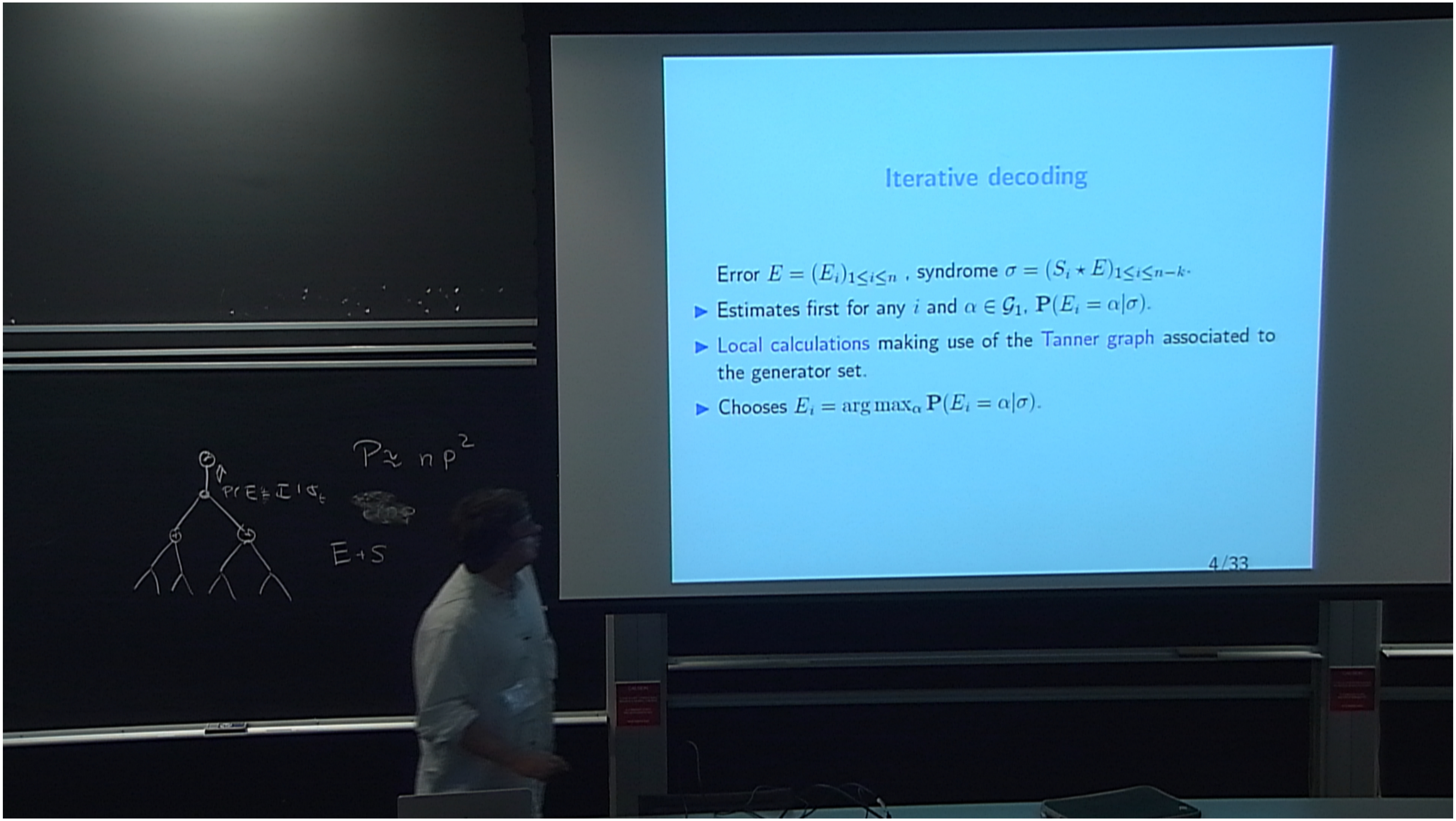
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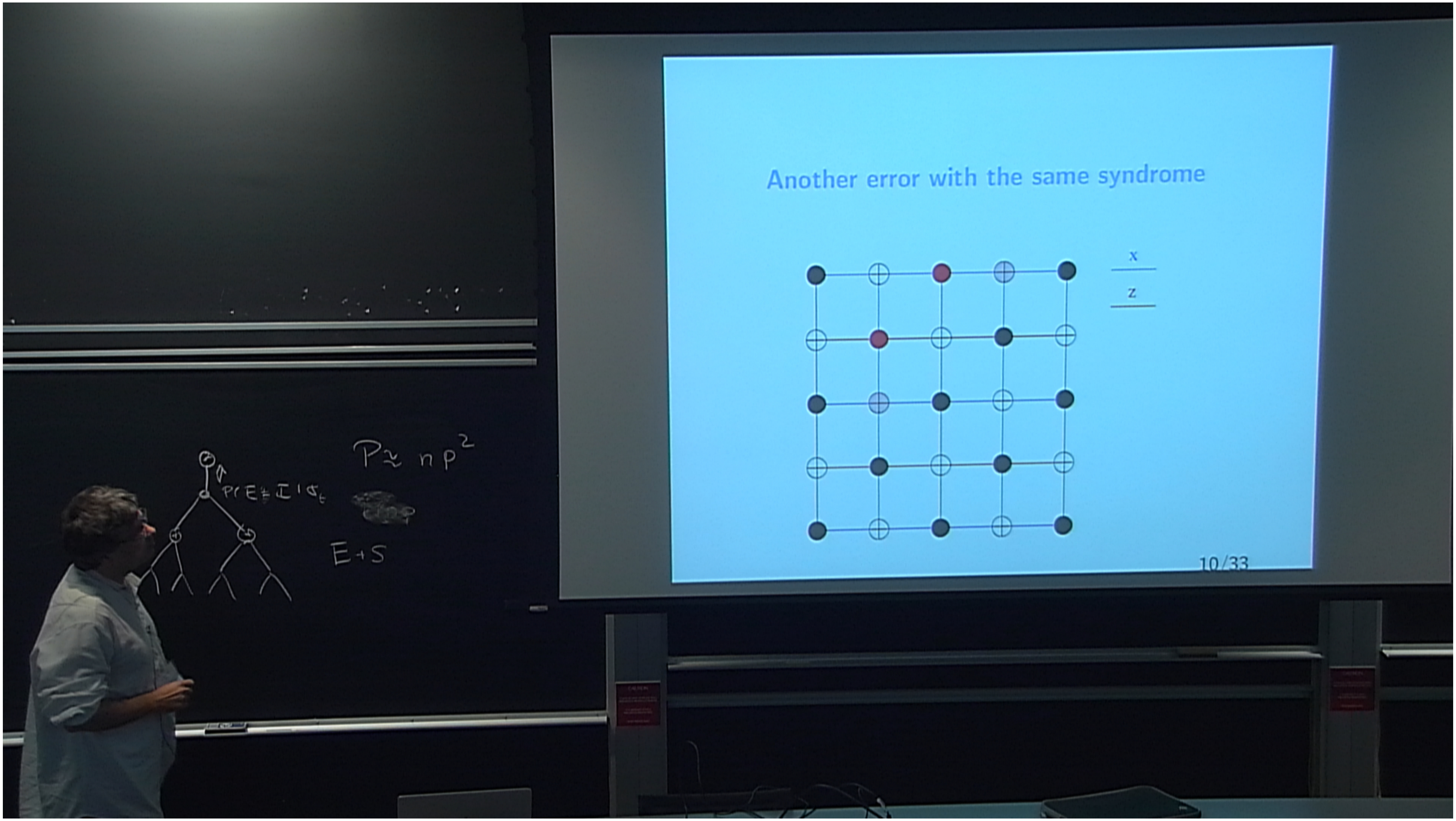
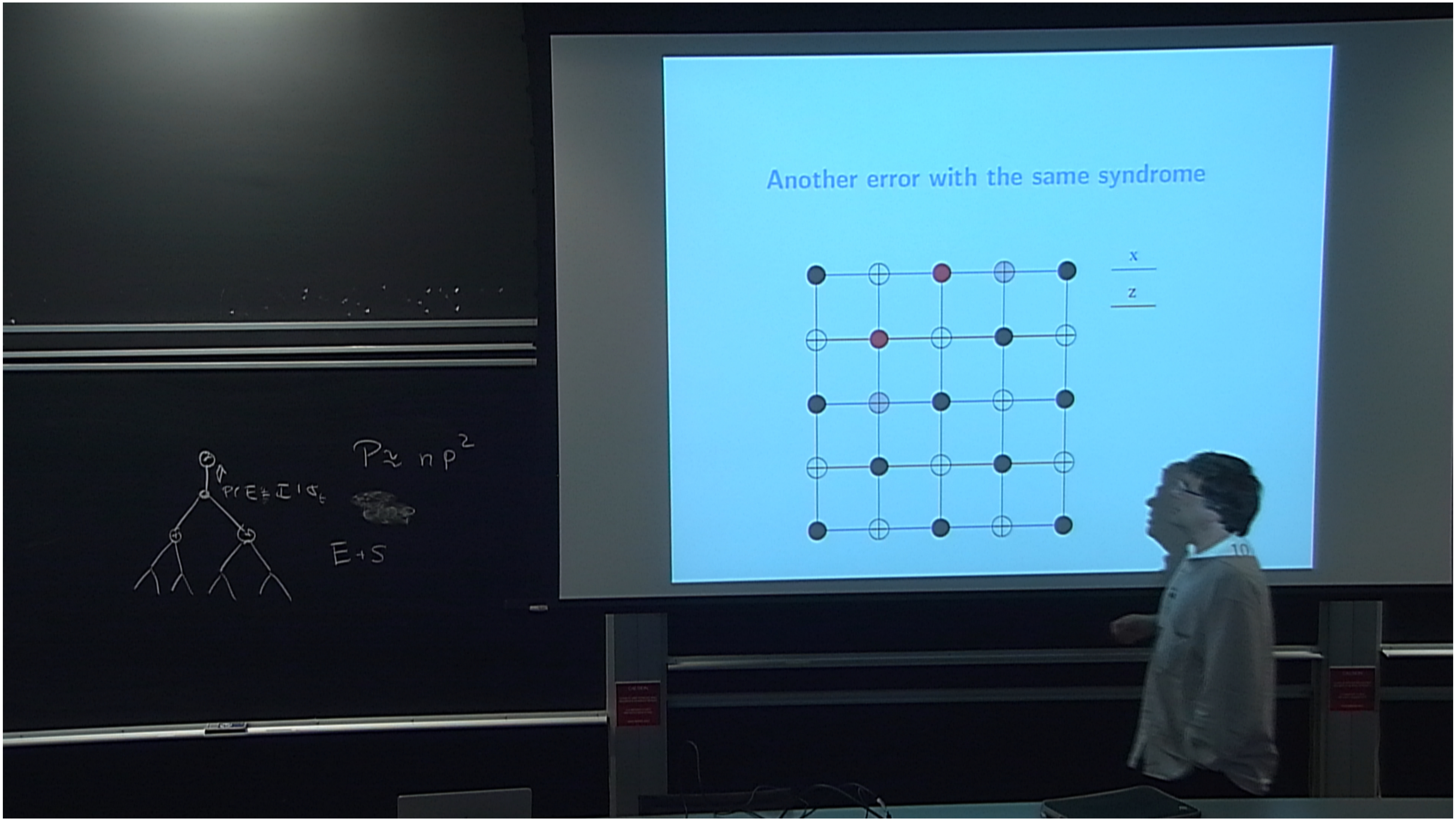


Another error with the same syndrome
x
z
10/33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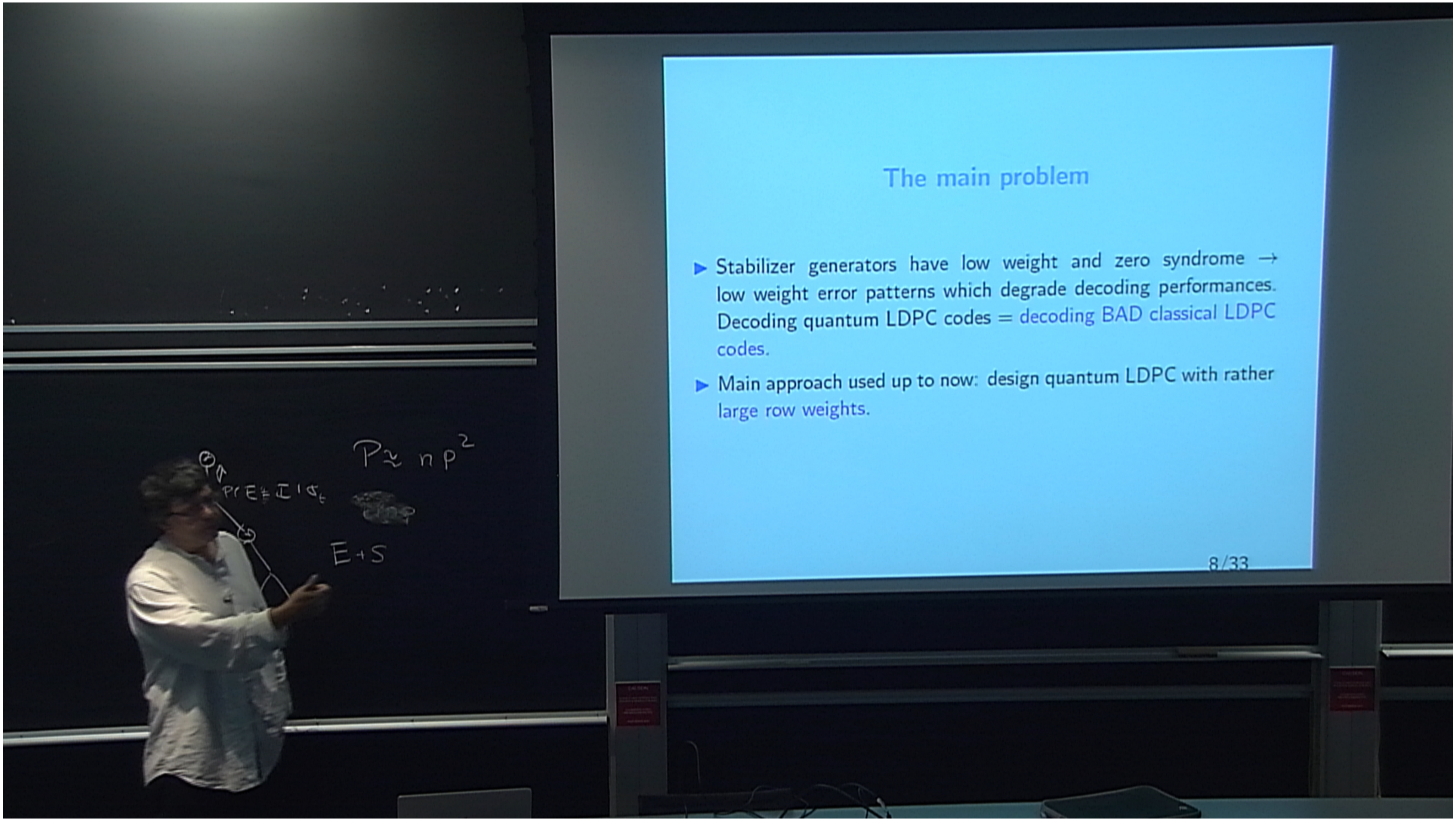
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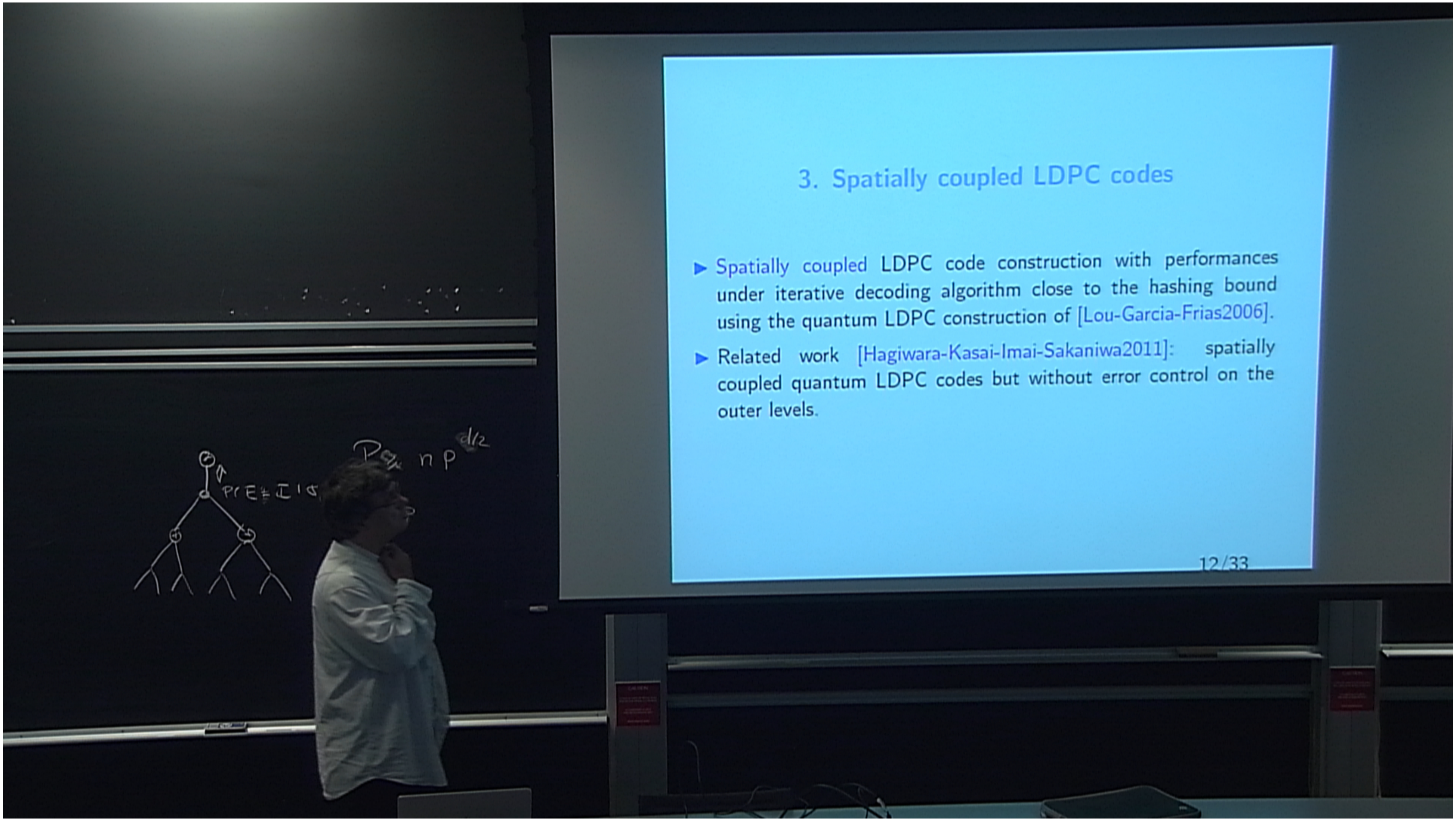
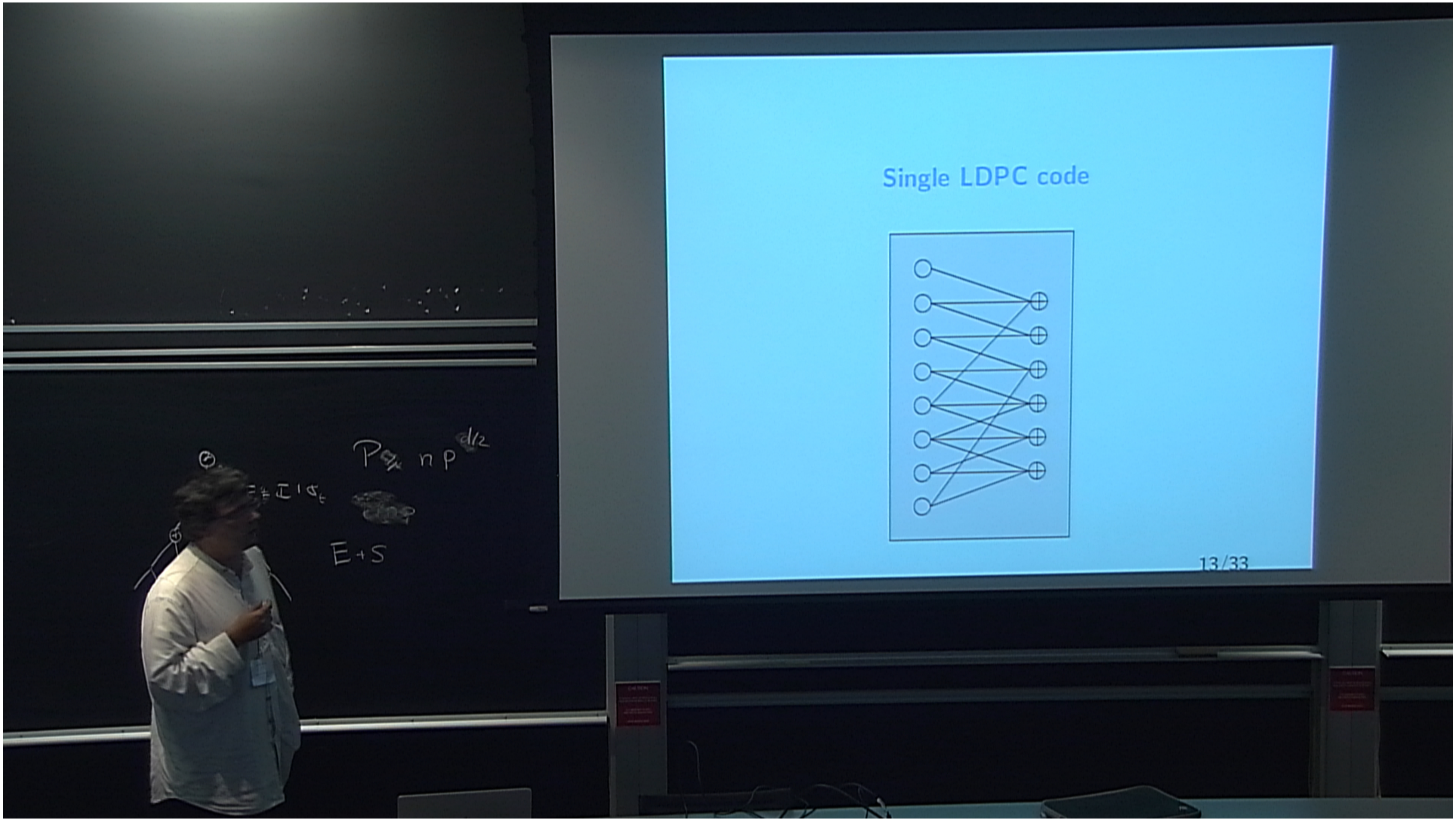
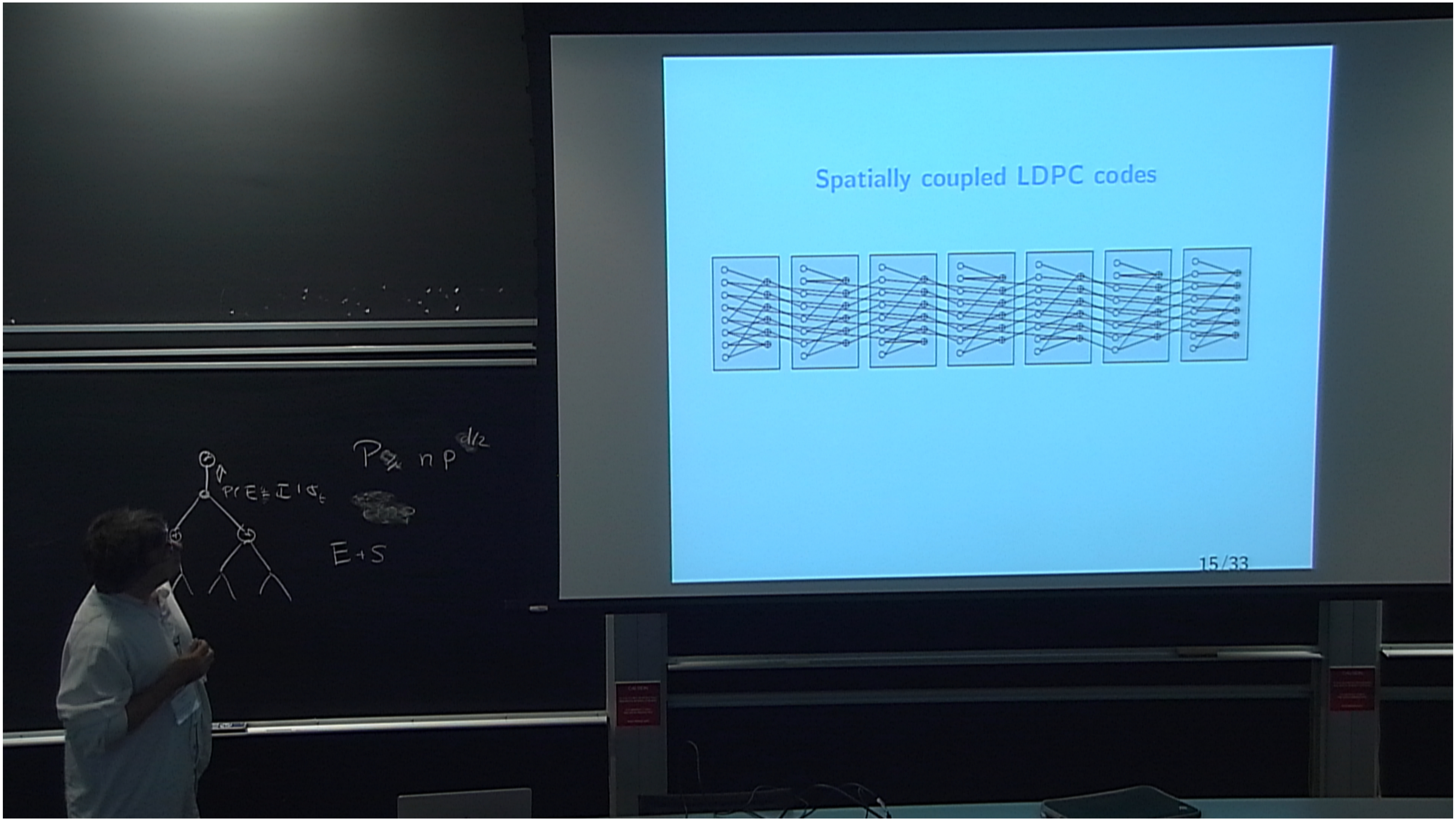
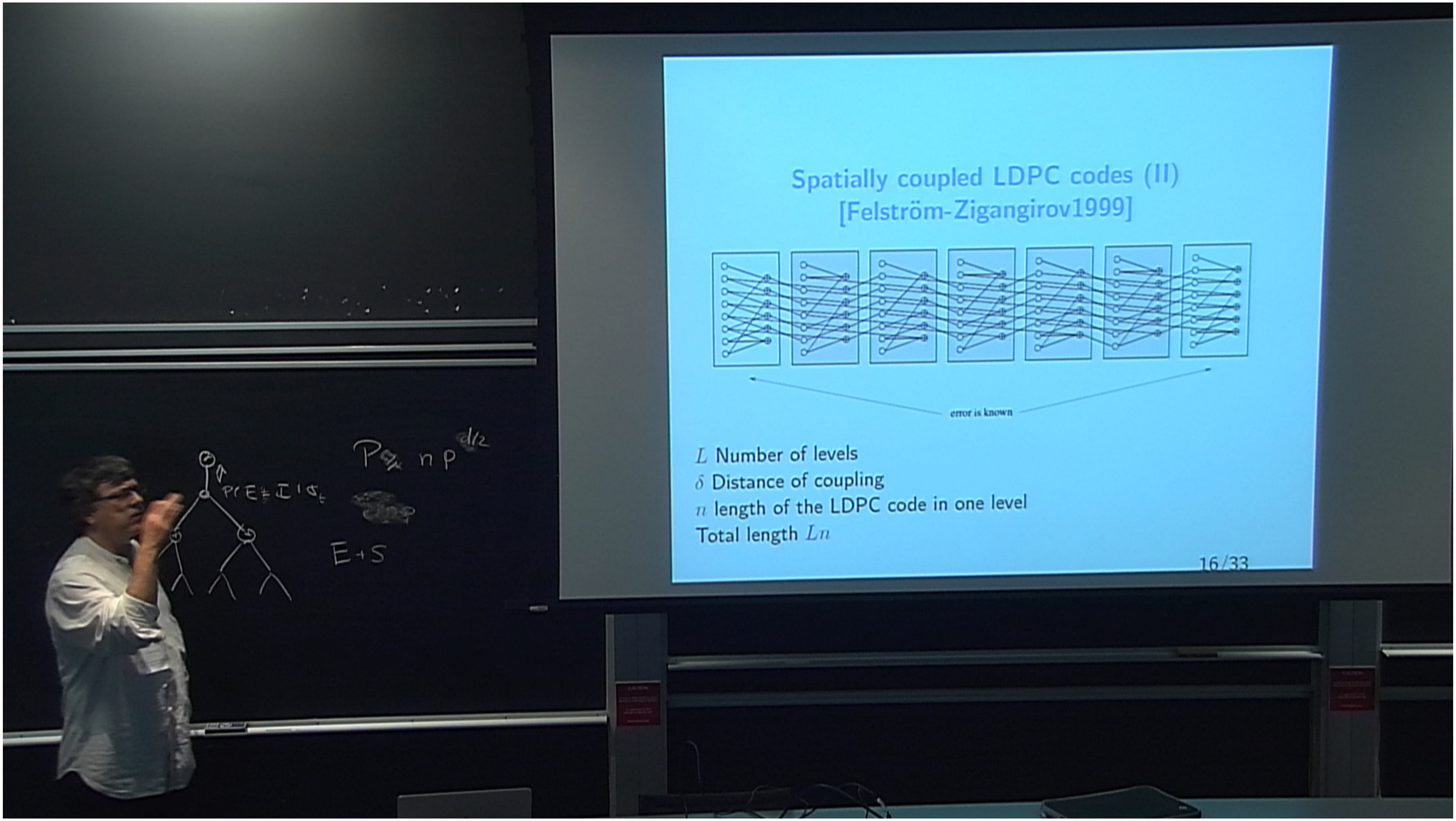
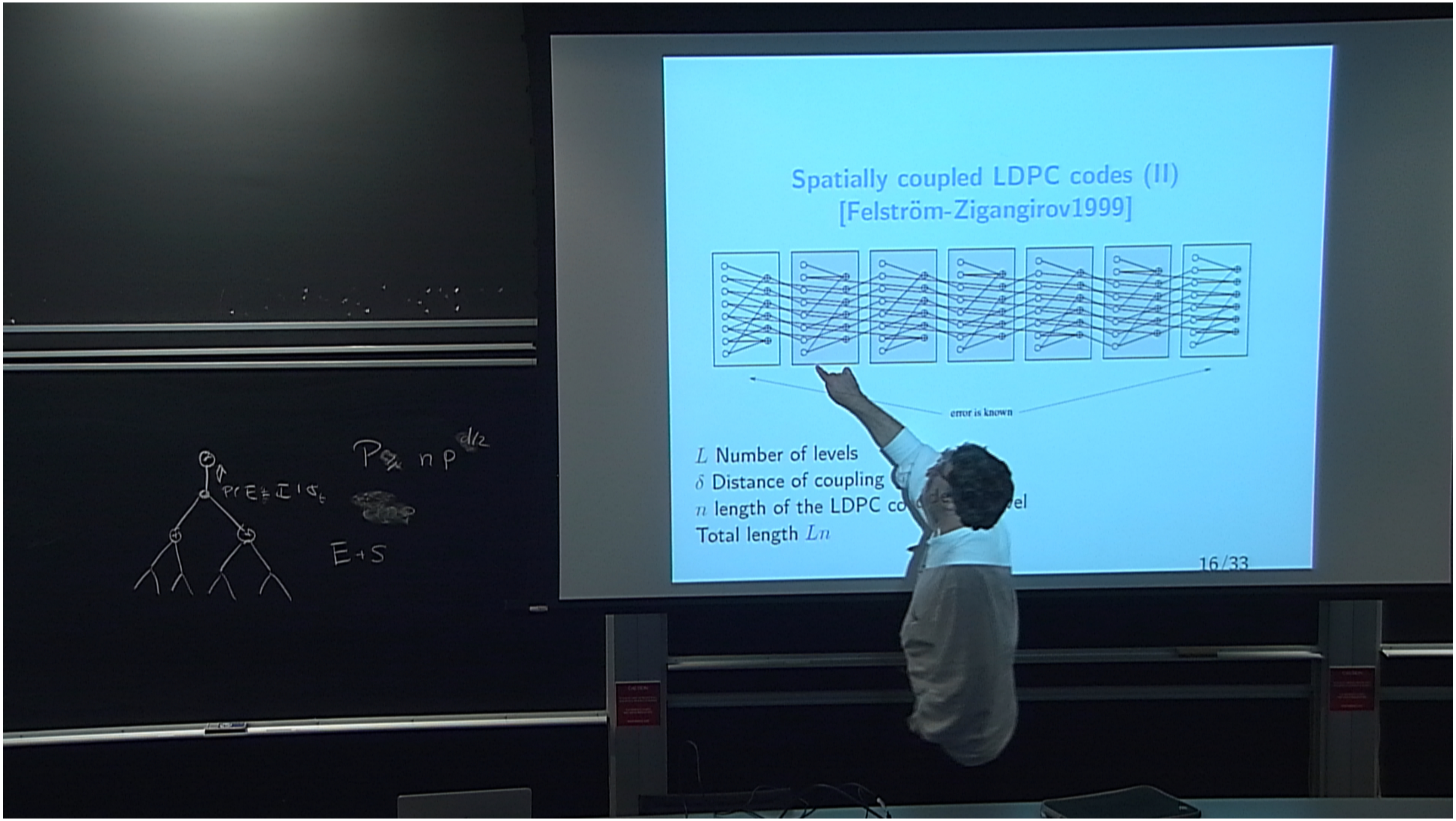
P ≃ n p²
E+S


3. Spatially coupled LDPC codes
▶ Spatially coupled LDPC code construction with performances under iterative decoding algorithm close to the hashing bound using the quantum LDPC construction of [Lou-Garcia-Frias2006].
▶ Related work [Hagiwara-Kasai-Imai-Sakaniwa2011]: spatially coupled quantum LDPC codes but without error control on the outer levels.
12/33


Single LDPC code
13/33


Spatially coupled LDPC codes
15/33
E+S


Spatially coupled LDPC codes (II)
[Felström-Zigangirov1999]
error is known
L Number of levels
δ Distance of coupling
n length of the LDPC code in one level
Total length Ln
16/33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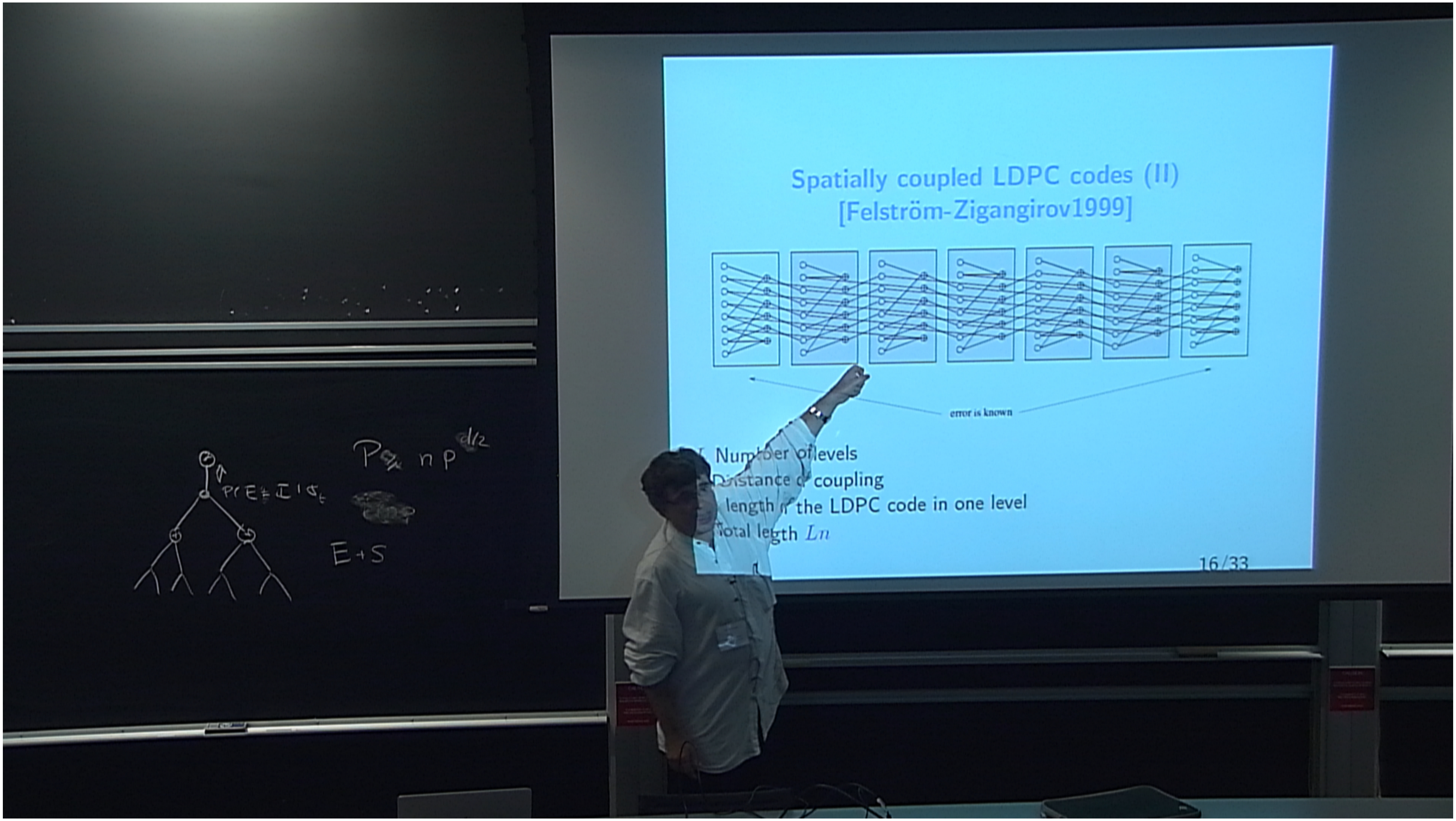
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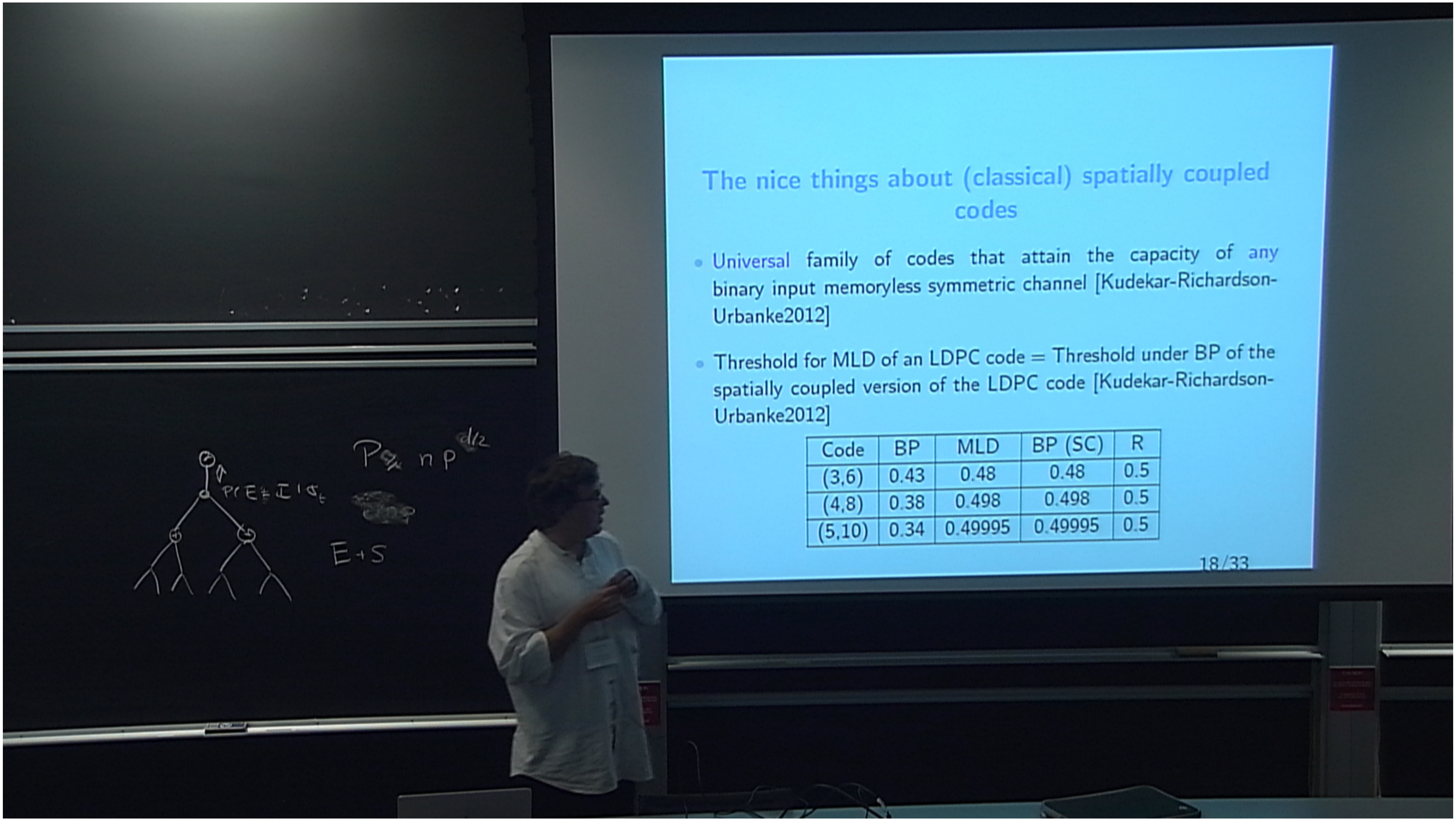
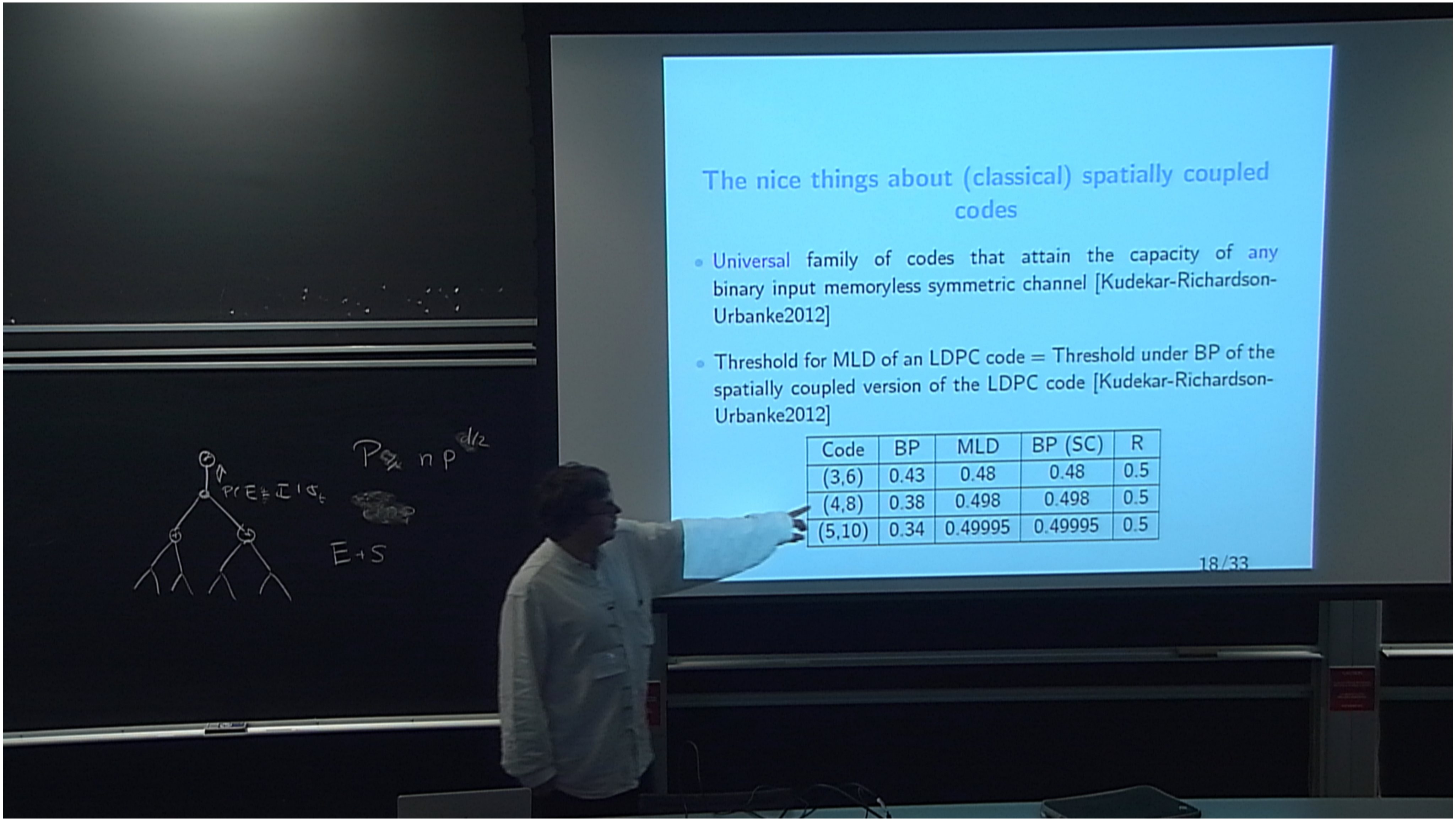

The nice things about (classical) spatially coupled codes
Universal family of codes that attain the capacity of any binary input memoryless symmetric channel [Kudekar-Richardson-Urbanke2012]
Threshold for MLD of an LDPC code = Threshold under BP of the spatially coupled version of the LDPC code [Kudekar-Richardson-Urbanke2012]

Code	BP	MLD	BP (SC)	R
(3,6)	0.43	0.48	0.48	0.5
(4,8)	0.38	0.498	0.498	0.5
(5,10)	0.34	0.49995	0.49995	0.5

18/33
E+S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18/33
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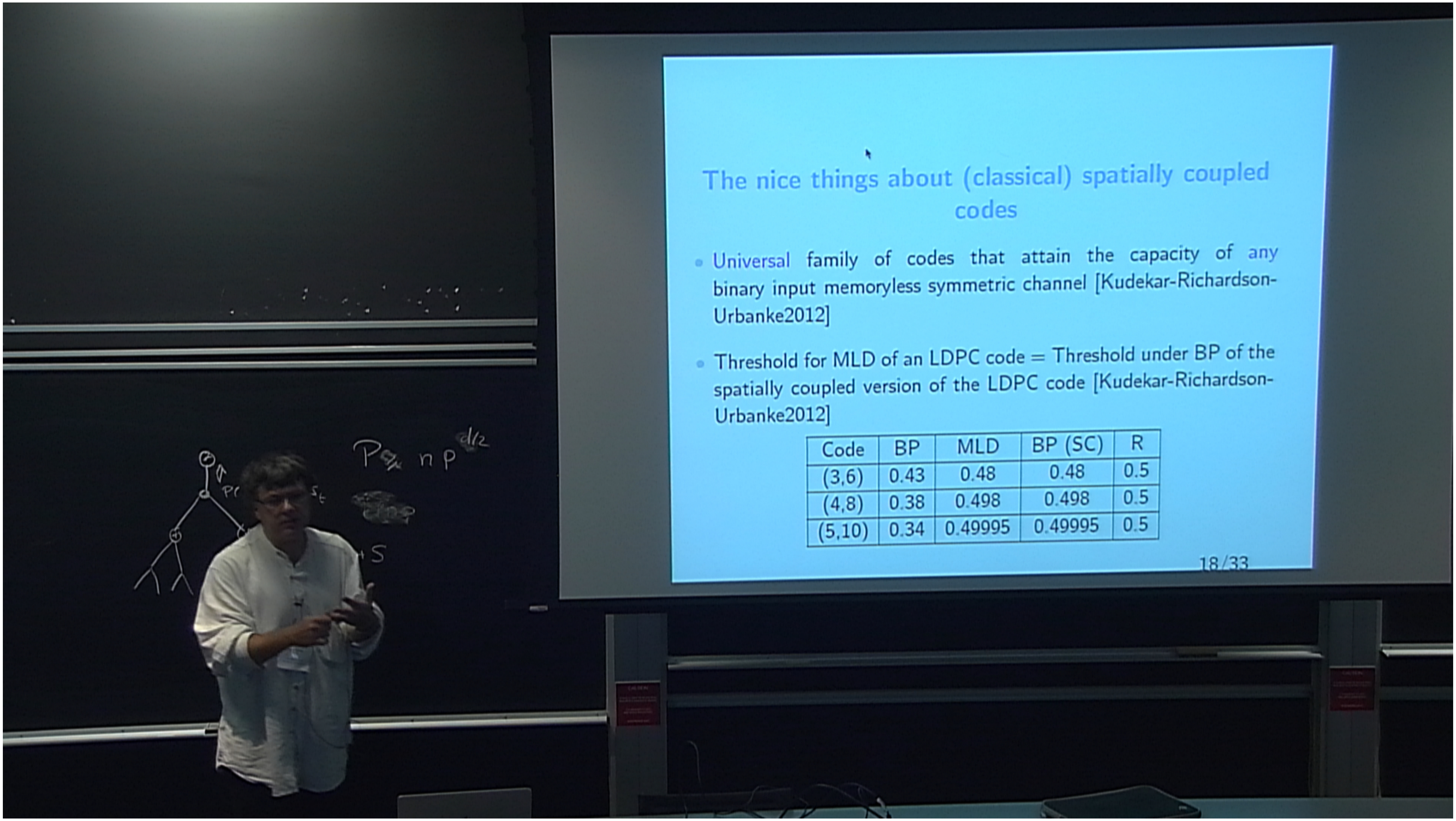
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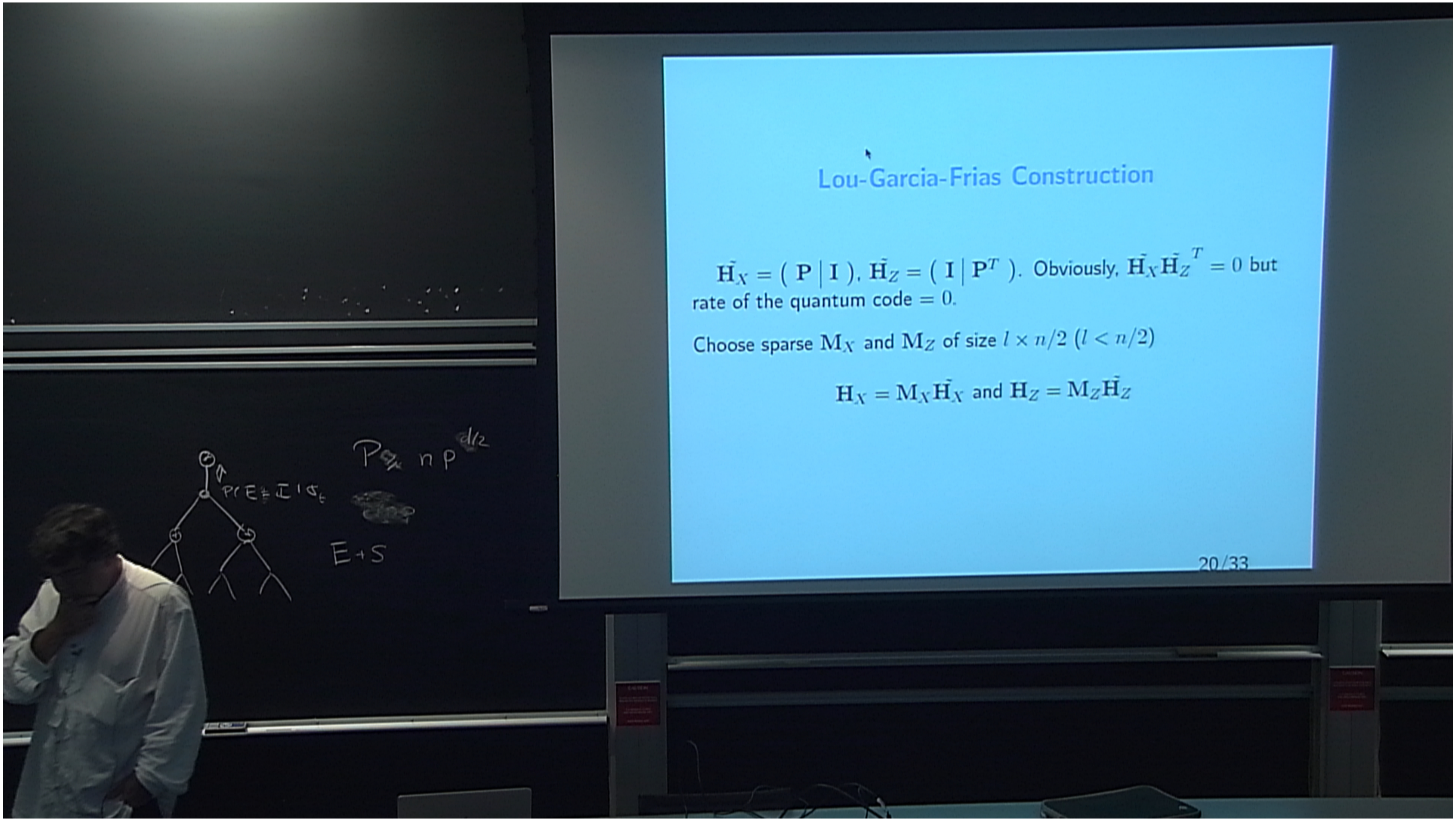
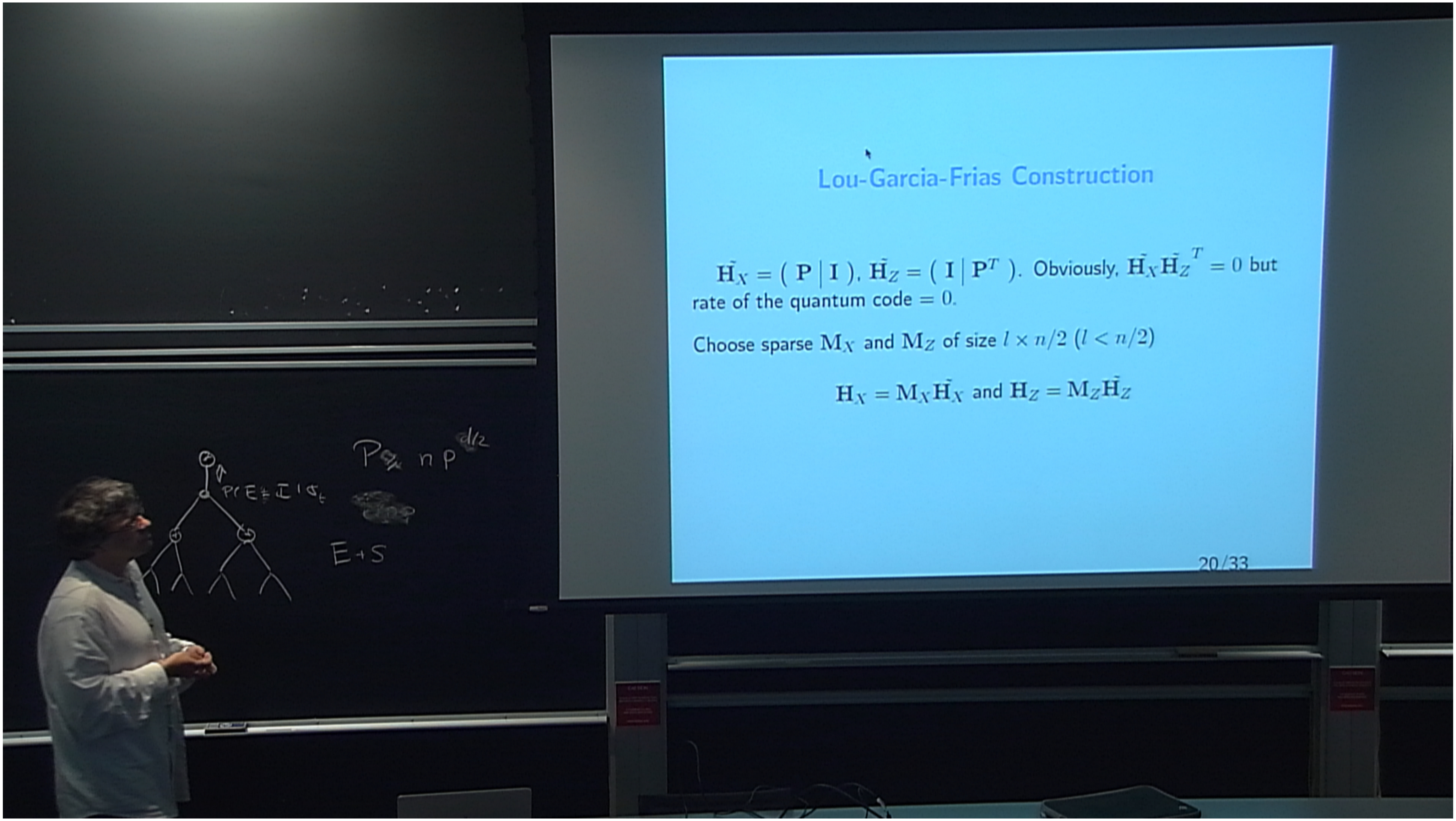

Lou-Garcia-Frias Construction
$\tilde{\mathbf{H}}_X = (\mathbf{P} \mid \mathbf{I})$, $\tilde{\mathbf{H}}_Z = (\mathbf{I} \mid \mathbf{P}^T)$. Obviously, $\tilde{\mathbf{H}}_X \tilde{\mathbf{H}}_Z^T = 0$ but rate of the quantum code $= 0$.
Choose sparse $\mathbf{M}_X$ and $\mathbf{M}_Z$ of size $l \times n/2$ $(l < n/2)$
$\mathbf{H}_X = \mathbf{M}_X \tilde{\mathbf{H}}_X$ and $\mathbf{H}_Z = \mathbf{M}_Z \tilde{\mathbf{H}}_Z$
20/33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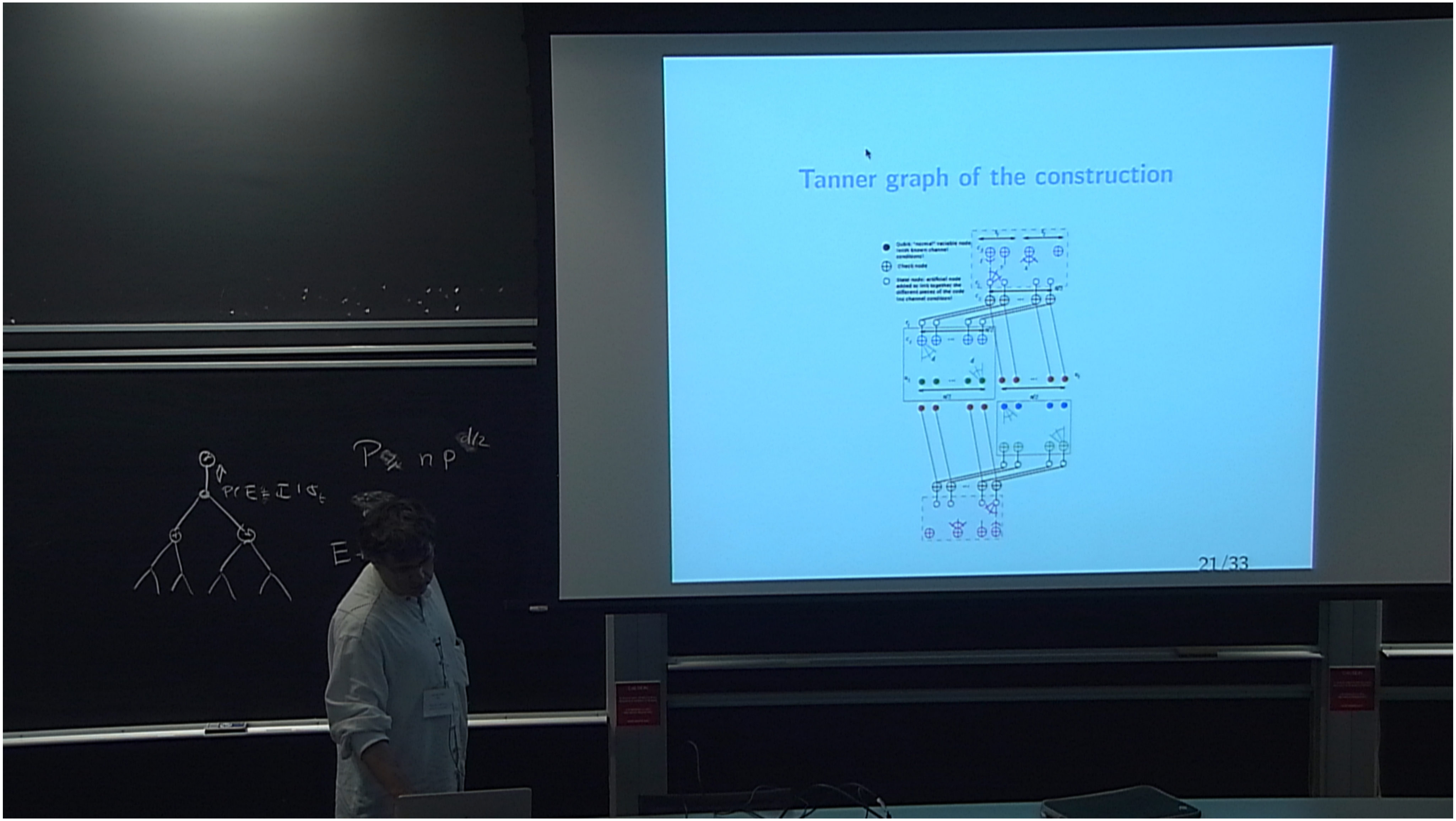
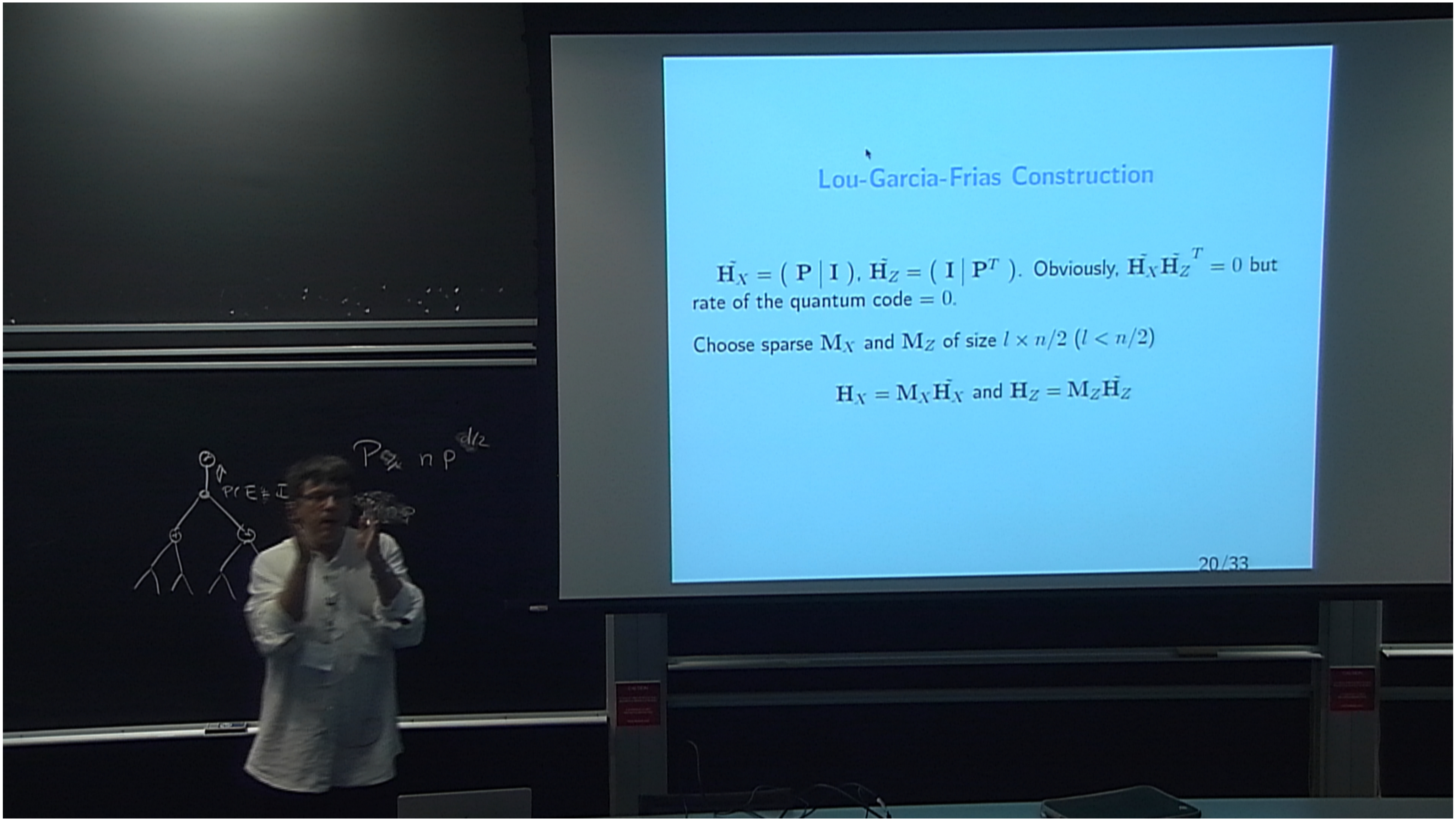
20/33


Tanner graph of the construction
21/33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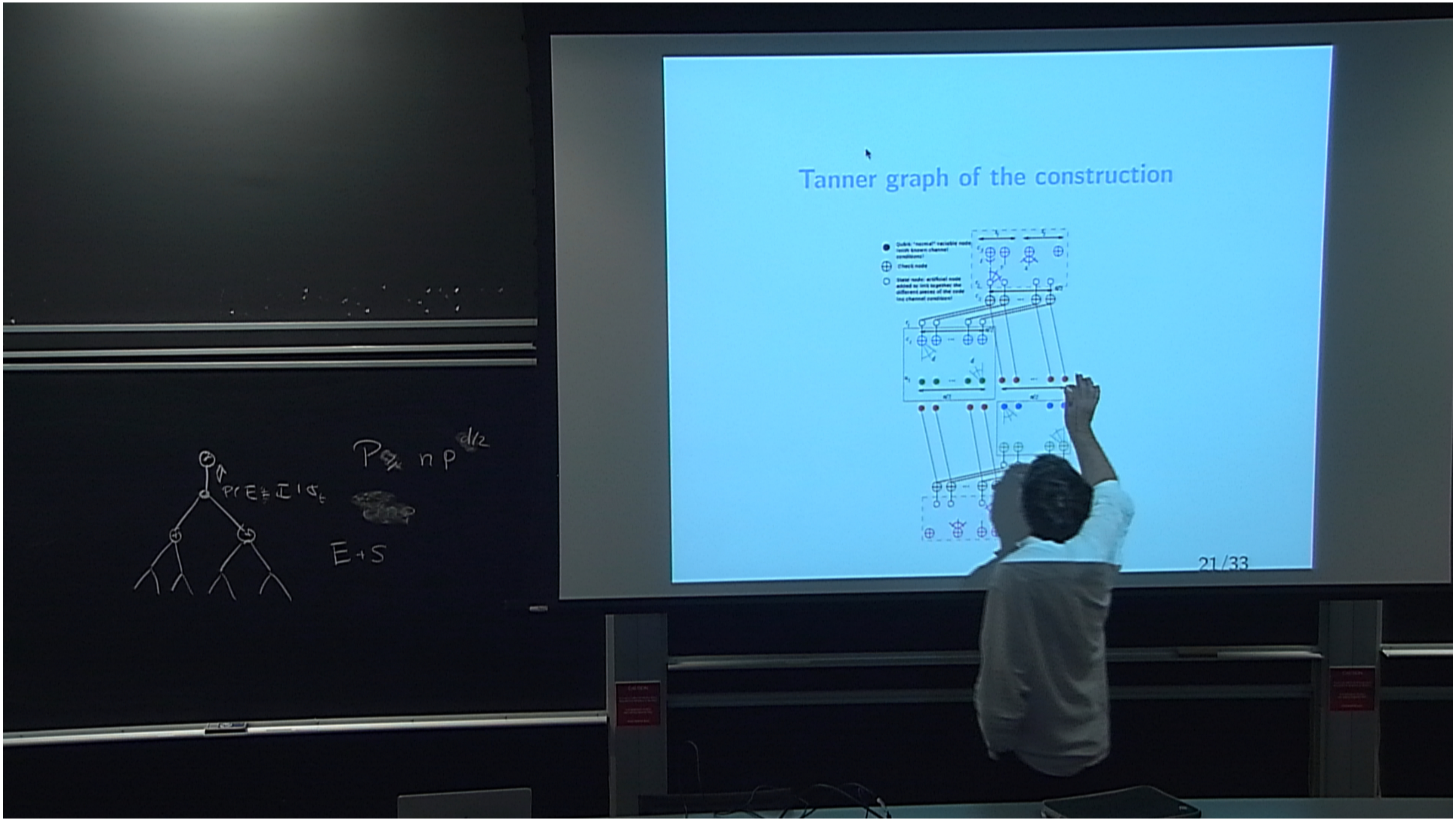
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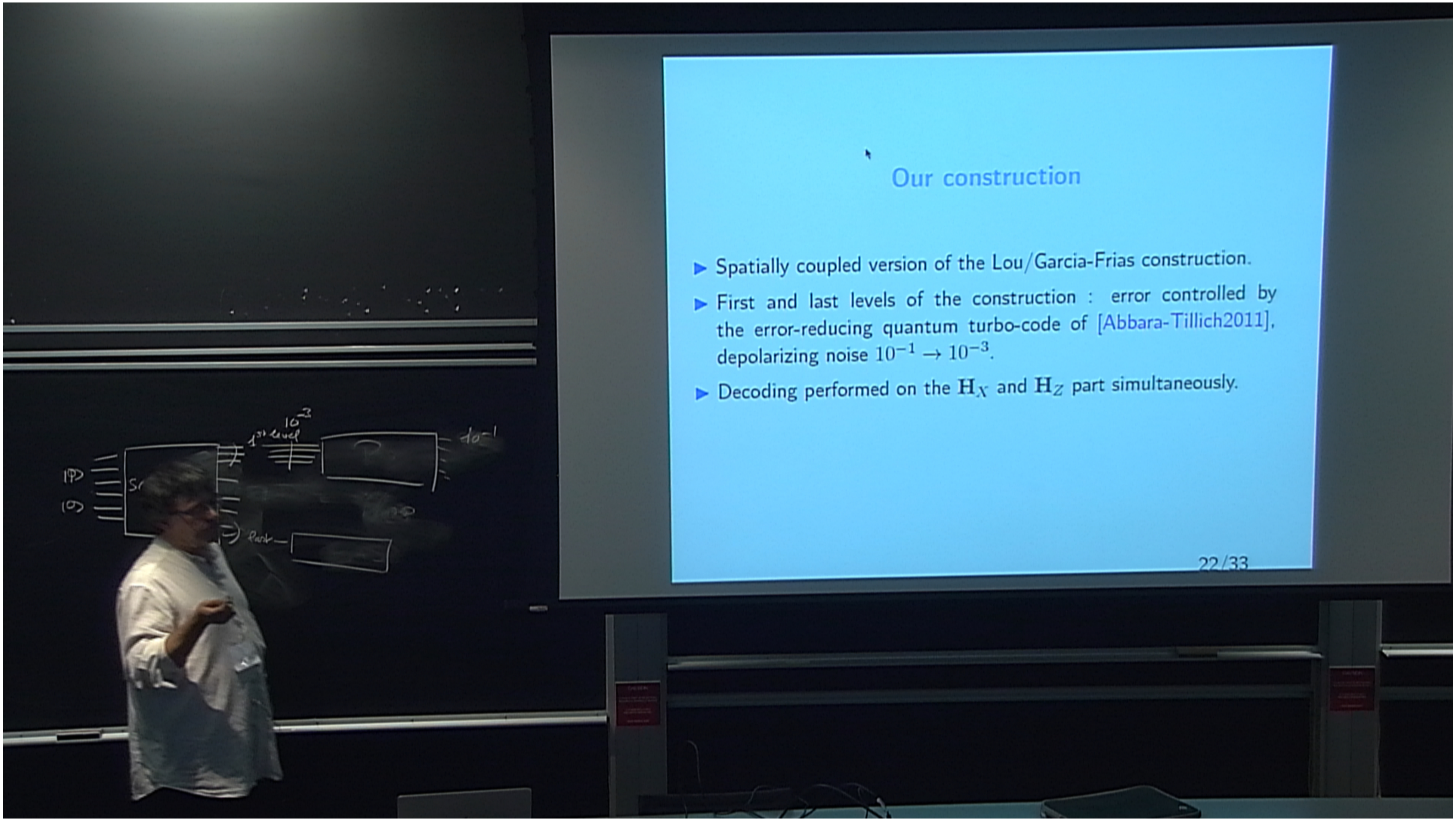
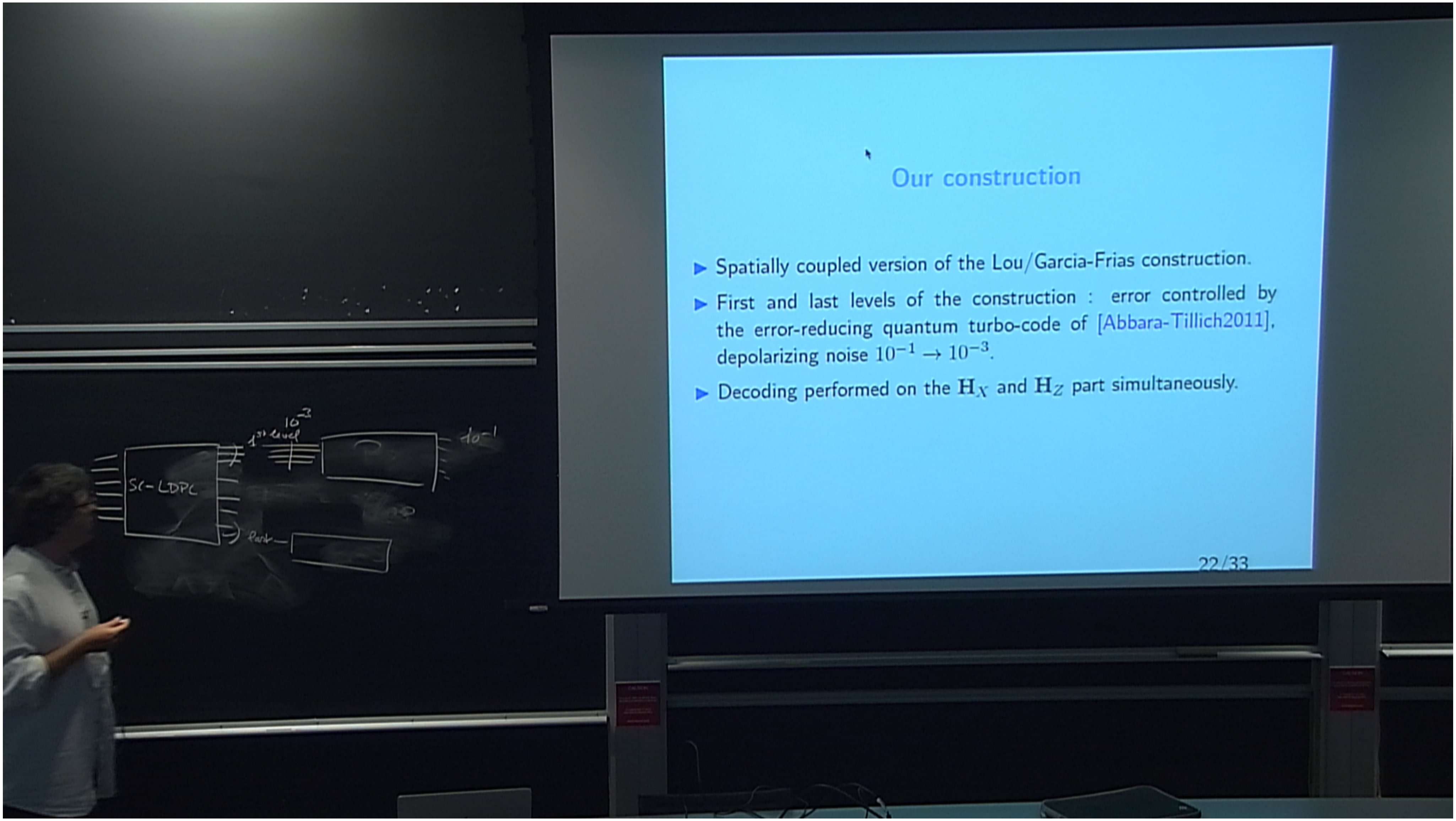

Our construction
▶ Spatially coupled version of the Lou/Garcia-Frias construction.
▶ First and last levels of the construction : error controlled by the error-reducing quantum turbo-code of [Abbara-Tillich2011], depolarizing noise $10^{-1} \to 10^{-3}$.
▶ Decoding performed on the $\mathbf{H}_X$ and $\mathbf{H}_Z$ part simultaneously.
22/33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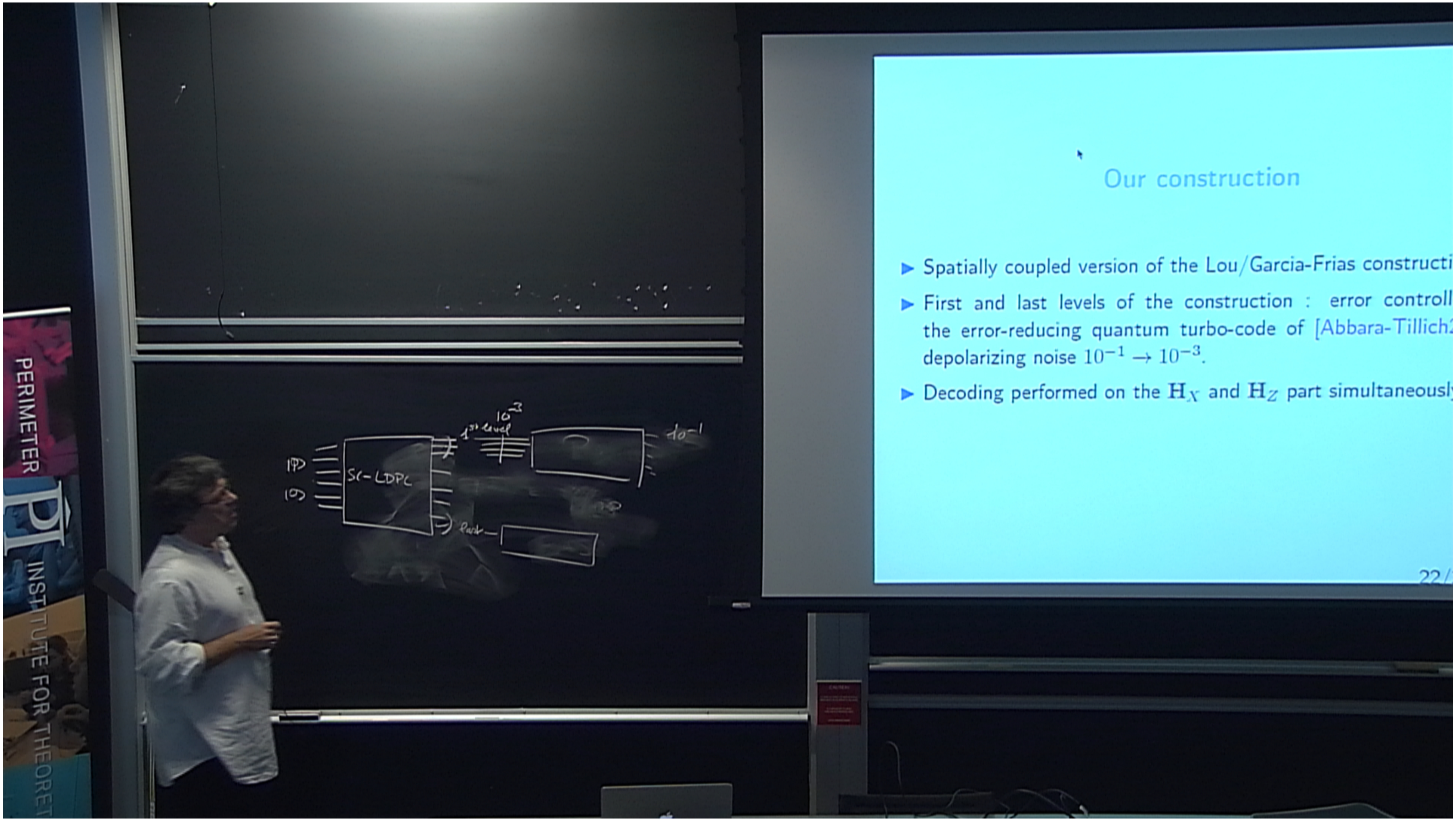
22/33
SC-LDPC
10^{-3}
10^{-1}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SC-LDPC
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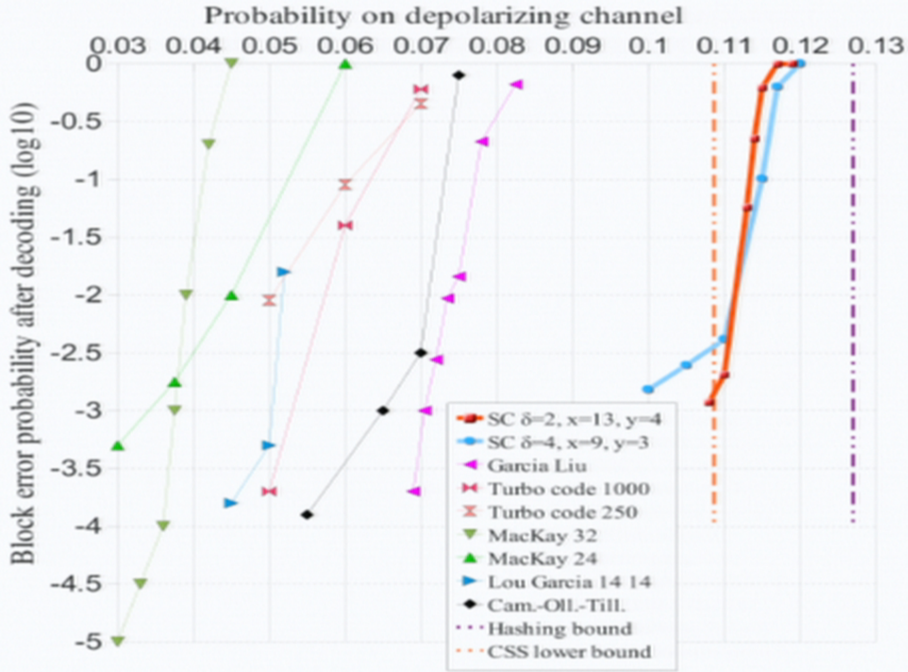
Performances for $L = 150$, $n = 3840$ and rate=$\frac{1}{4}$
Probability on depolarizing channel
0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1 0.11 0.12 0.13
Block error probability after decoding (log10)
0 -0.5 -1 -1.5 -2 -2.5 -3 -3.5 -4 -4.5 -5
Turbo code 1000
Turbo code 250
MacKay 32
MacKay 24
Hashing bound
23/33
SC-LDPC
1st level
10^{-3}
10^{-1}

Performances for $L = 150$, $n = 3840$ and rate$=\frac{1}{4}$



23 / 33

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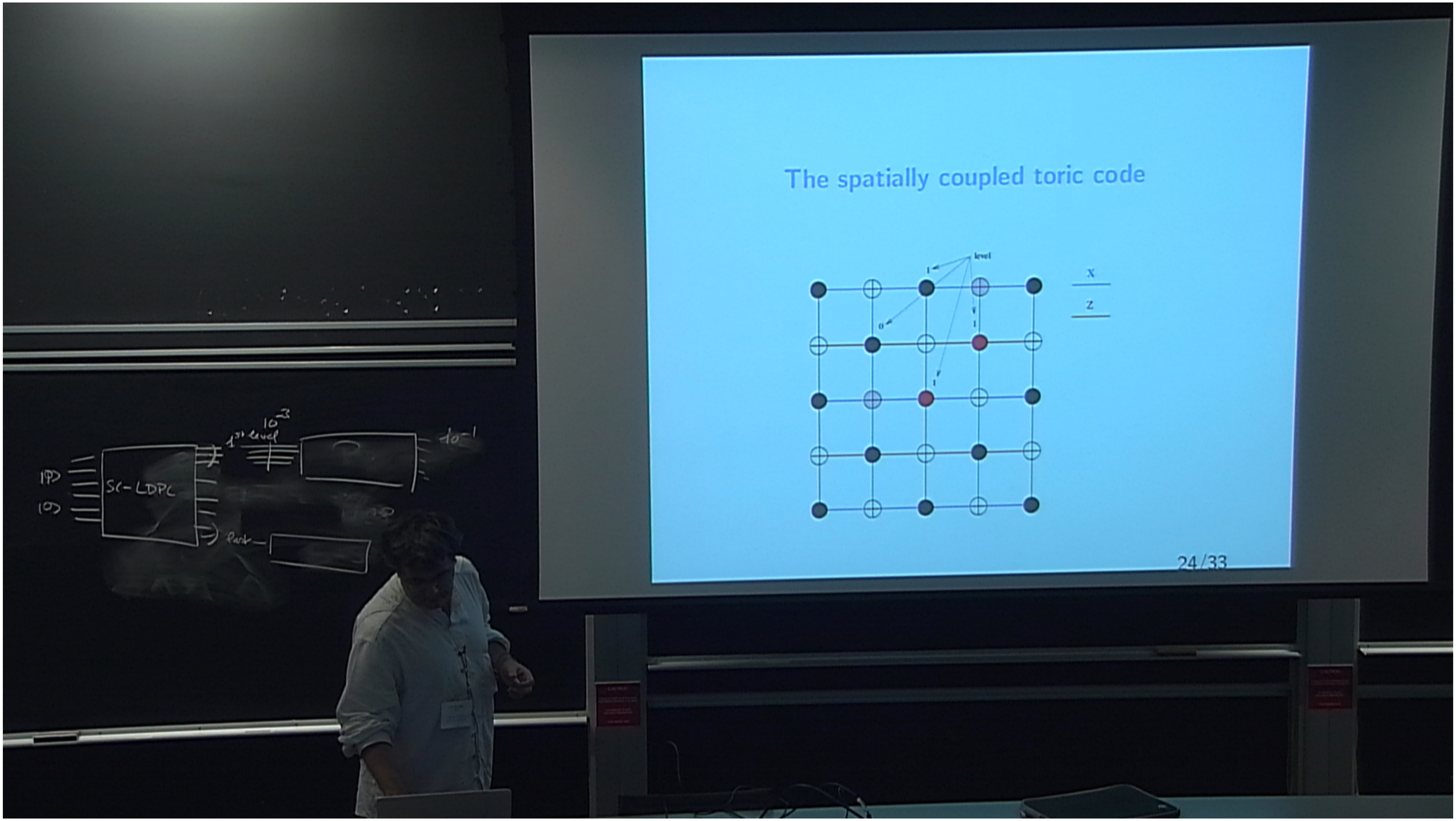
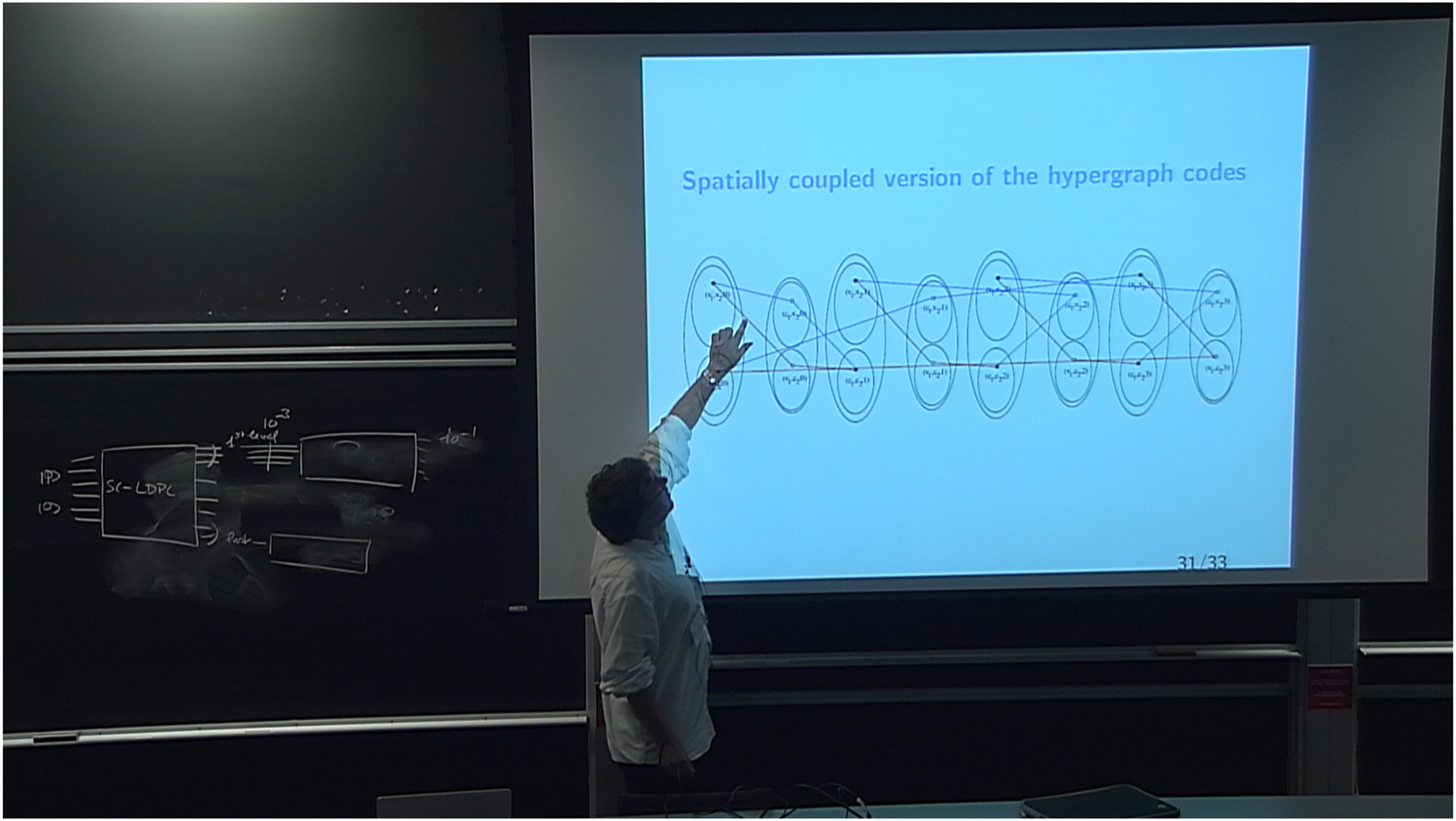
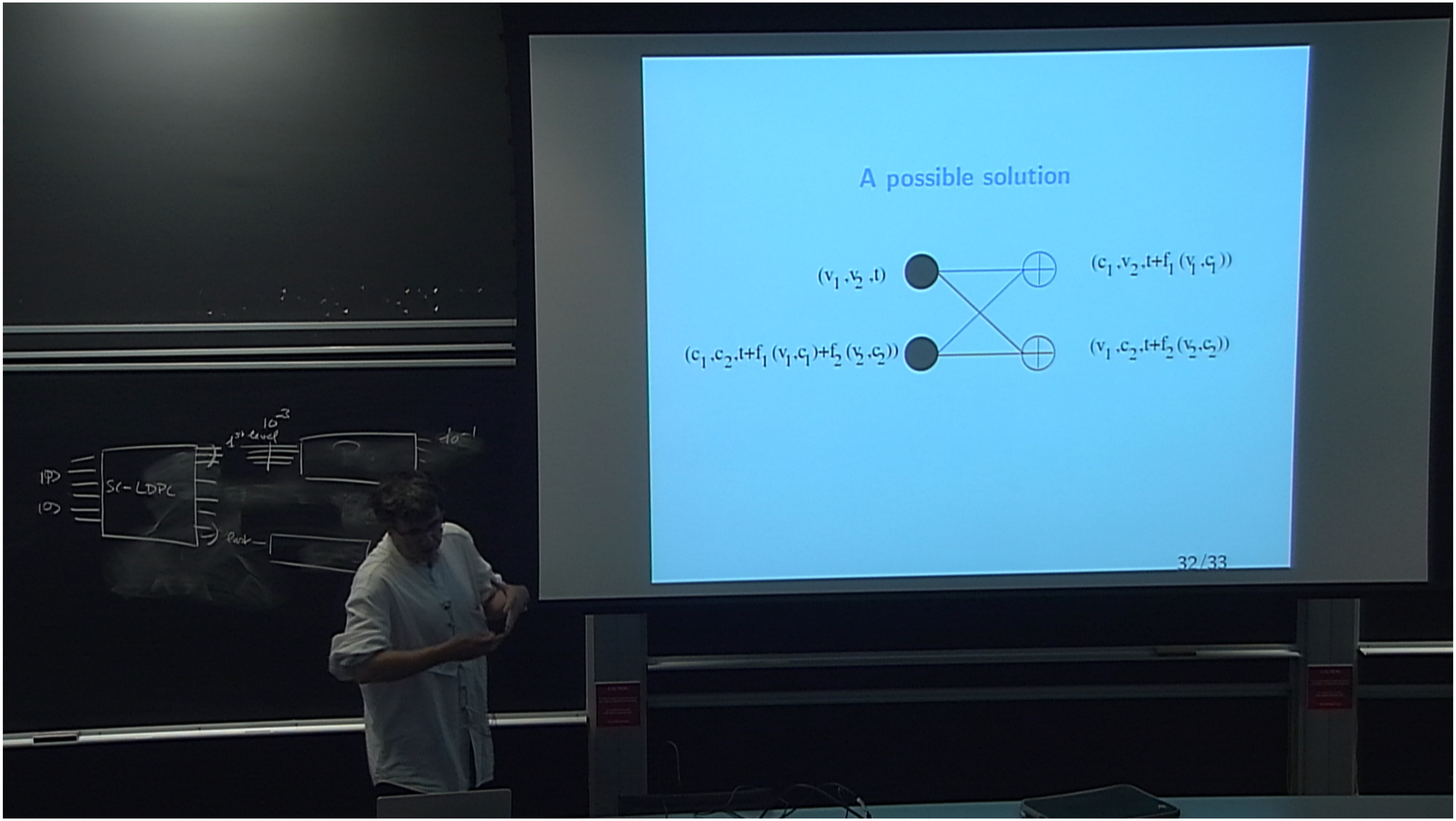
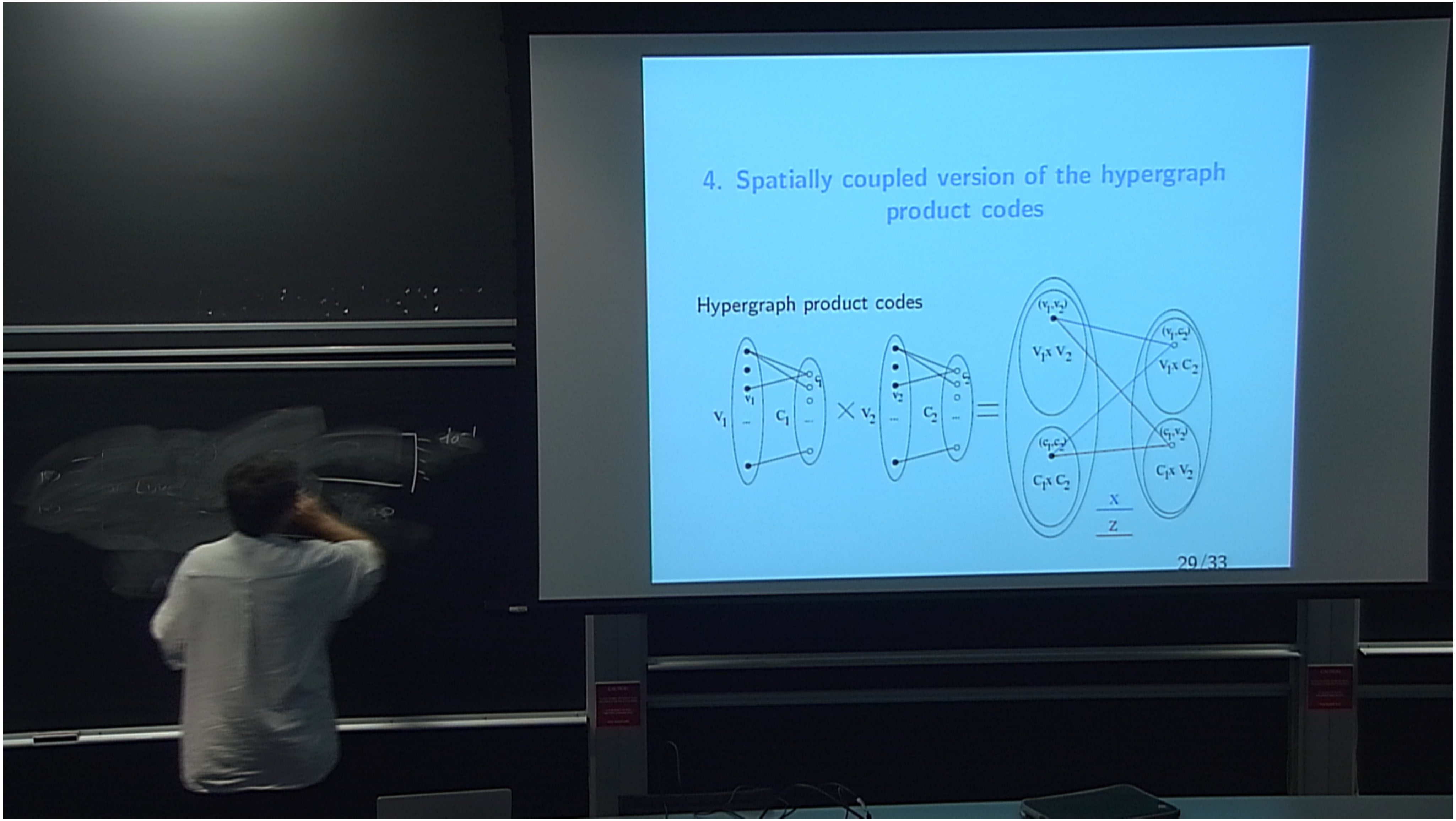
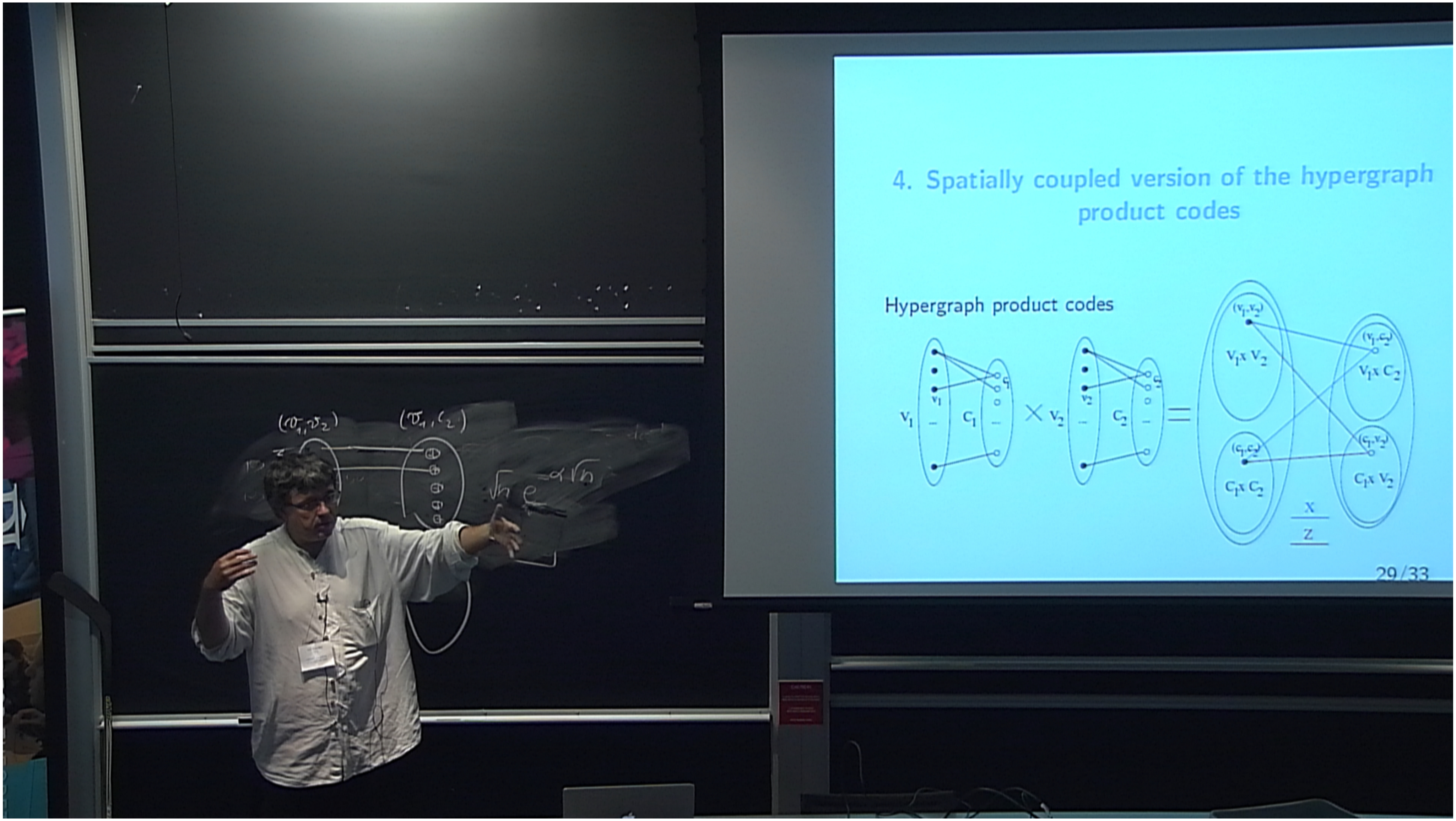
Hashing bound
CSS lower bound
23/33
SC-LDPC
1st level
10^{-3}
10^{-1}


The spatially coupled toric code
level
1
0
1
1
X
Z
24/33
SC-LDPC
1st level
10^{-3}
10^{-1}

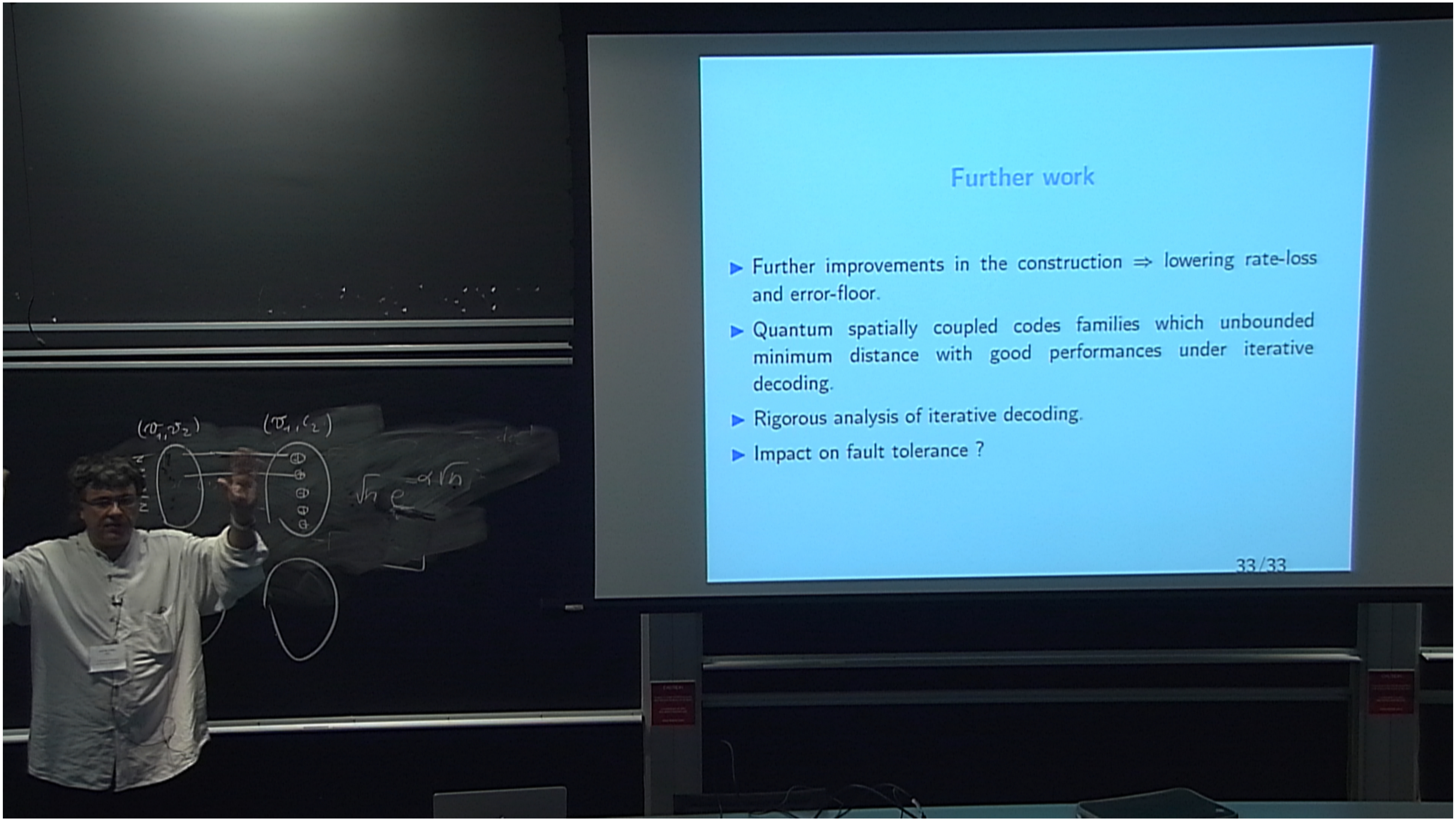

Spatially coupled version of the hypergraph codes
31/33
SC-LDPC

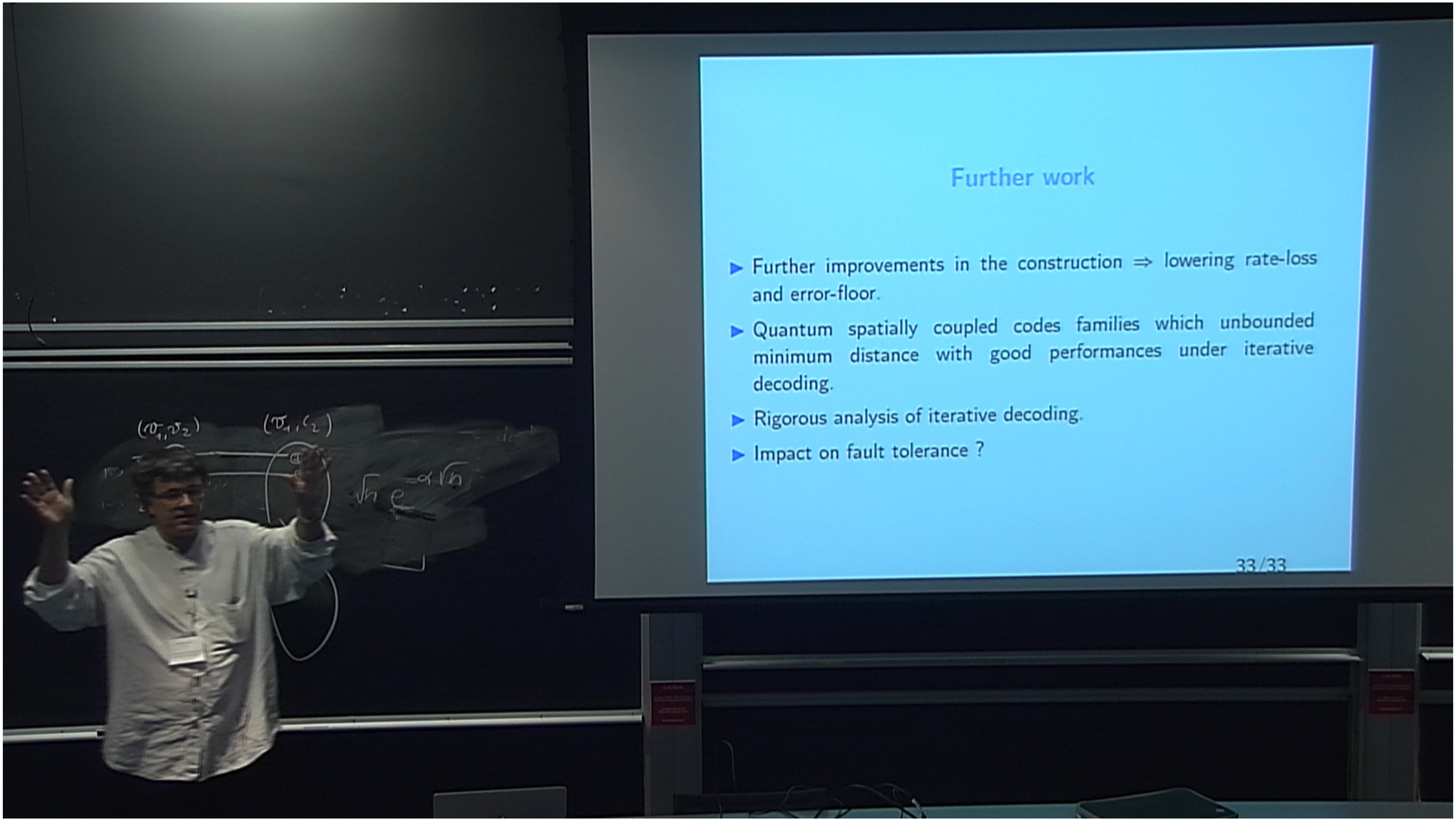

A possible solution
(v_1, v_2, t)
$(c_1, c_2, t+f_1(v_1, c_1)+f_2(v_2, c_2))$
$(c_1, v_2, t+f_1(v_1, c_1))$
$(v_1, c_2, t+f_2(v_2, c_2))$
32/33
SC-LDPC


4. Spatially coupled version of the hypergraph product codes
Hypergraph product codes
V_1
C_1
V_2
C_2
$V_1 x V_2$
$V_1 x C_2$
$C_1 x C_2$
$C_1 x V_2$
X
Z
29/33



4. Spatially coupled version of the hypergraph product codes
Hypergraph product codes
V1
C1
V2
C2
V1x V2
V1x C2
C1x C2
C1x V2
X
Z
29/33


Further work
▶ Further improvements in the construction ⇒ lowering rate-loss and error-floor.
▶ Quantum spatially coupled codes families which unbounded minimum distance with good performances under iterative decoding.
▶ Rigorous analysis of iterative decoding.
▶ Impact on fault tolerance ?
33/33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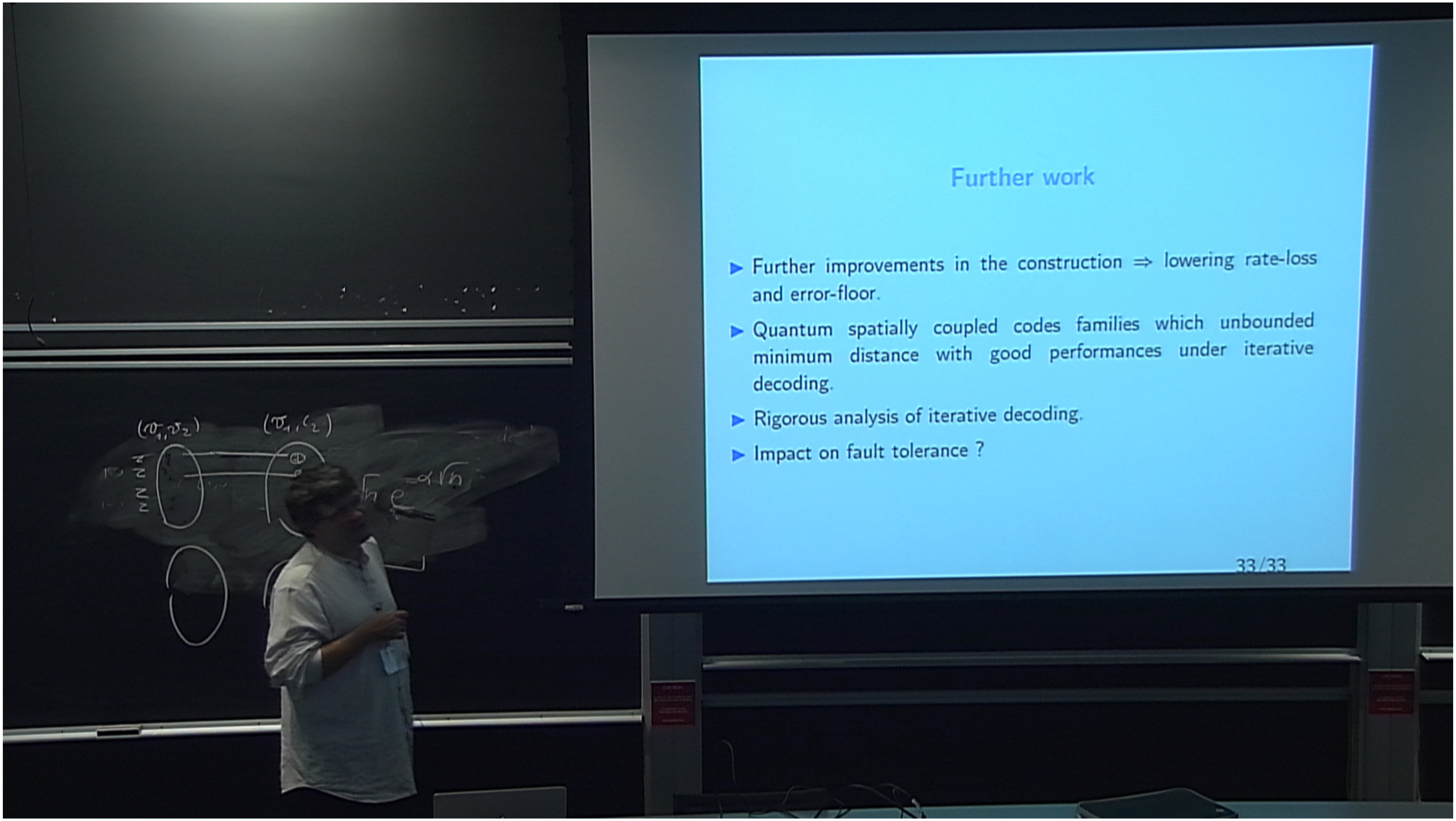
Rigorous analysis of iterative decoding.
Impact on fault tolerance ?
33/33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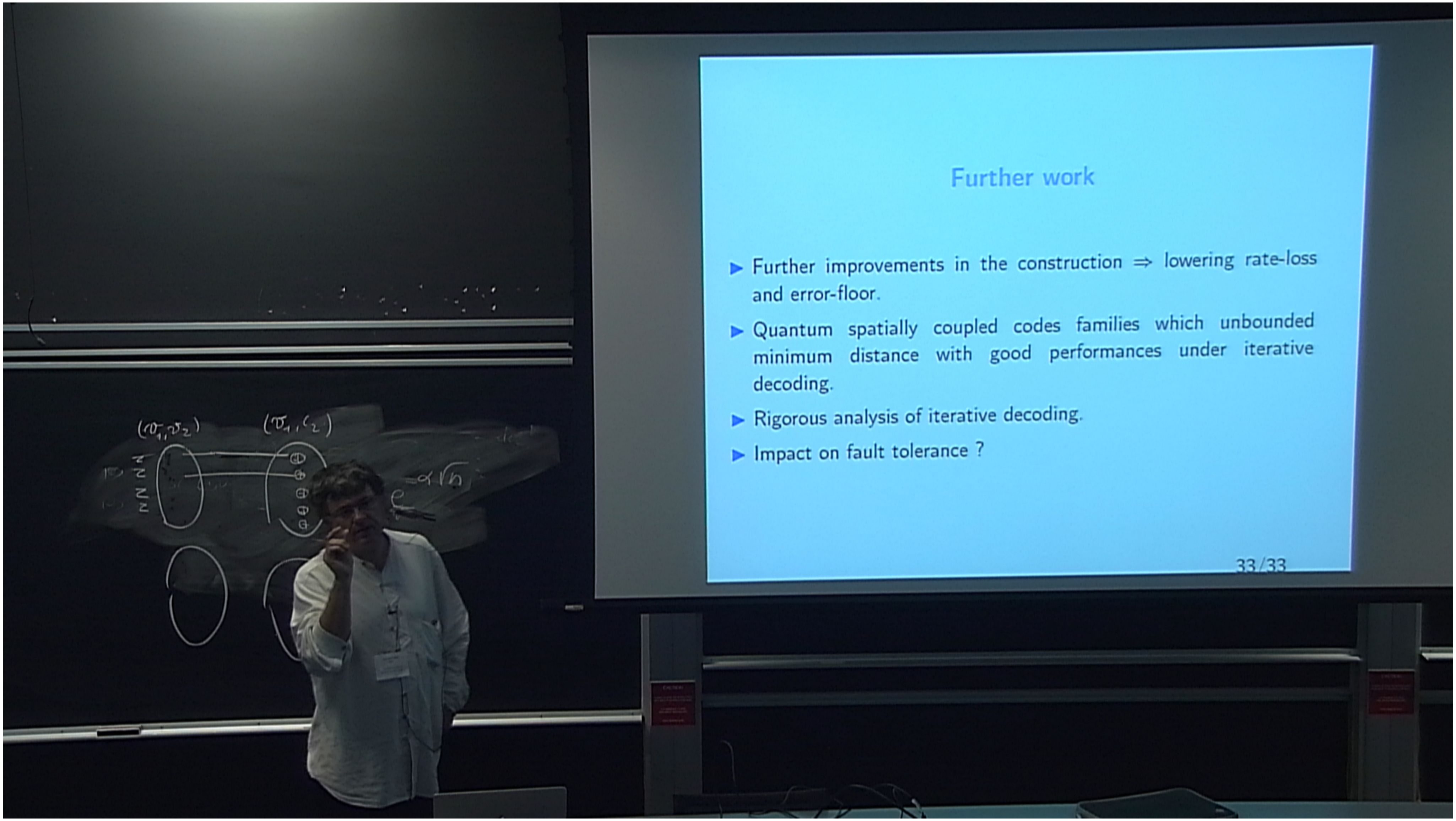
33/33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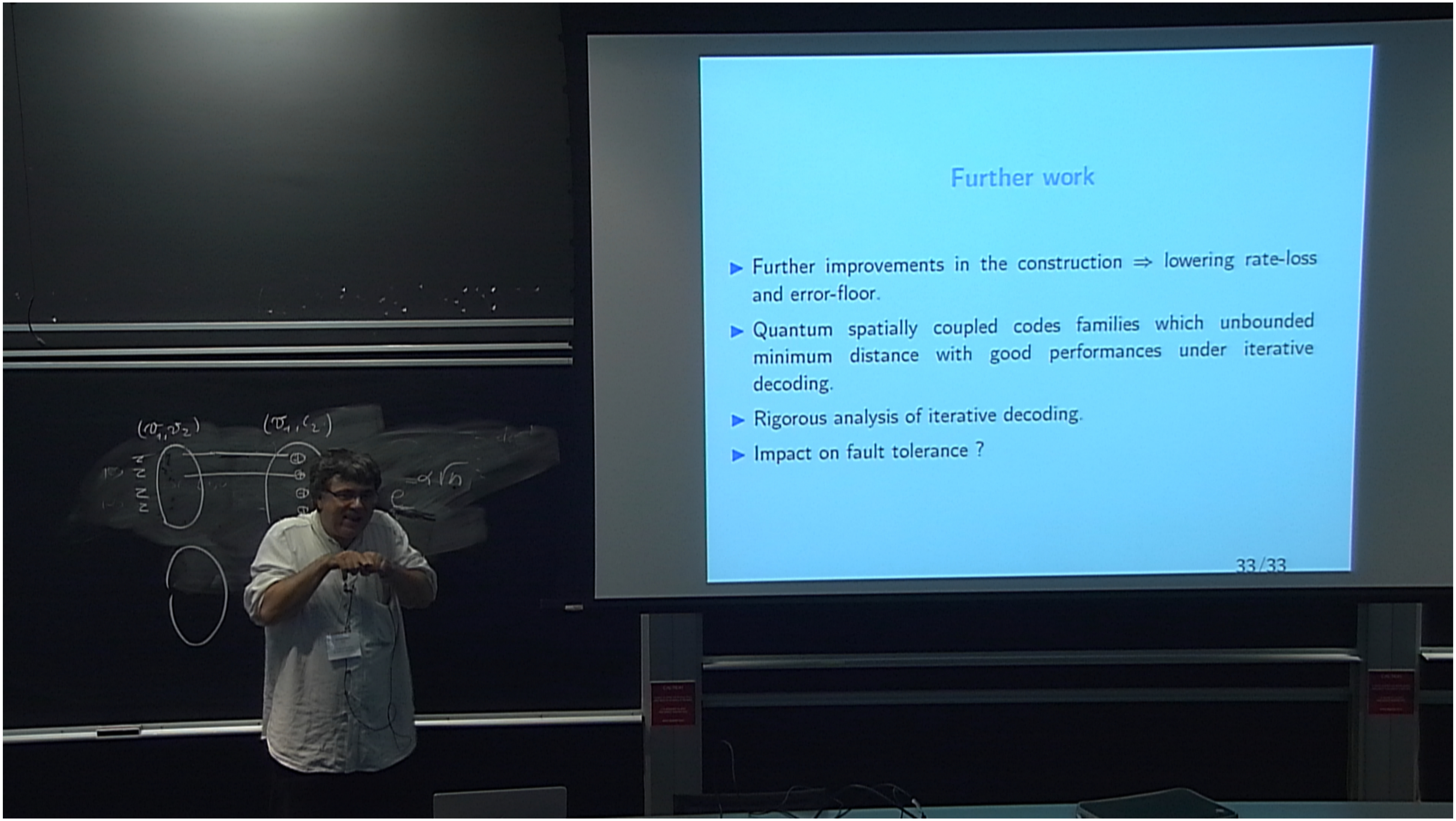
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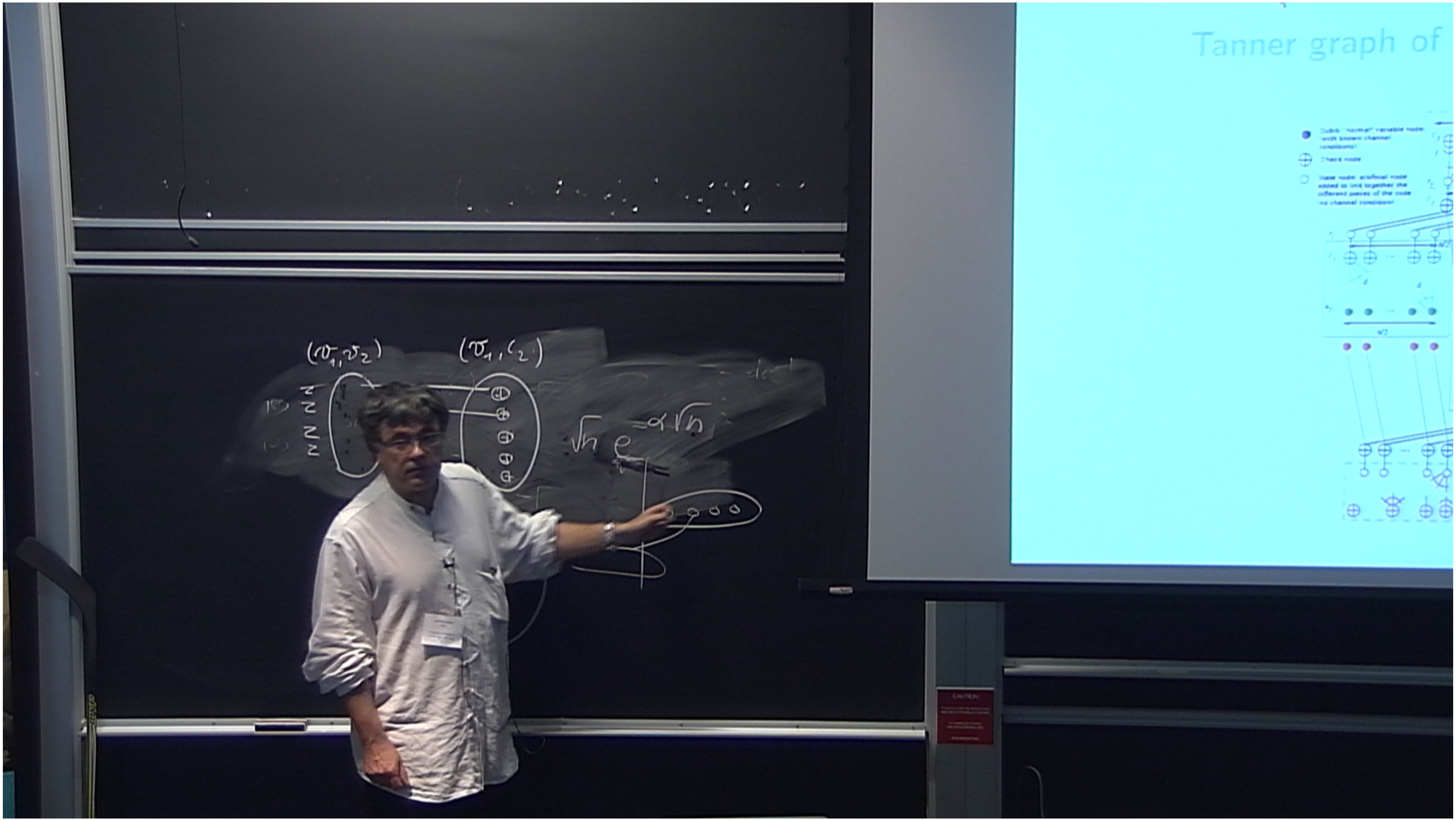
33/33


Tanner graph of