Title: Spin glass reflection of the decoding transition for space-time codes

Date: Jul 14, 2014 02:30 PM

URL: http://pirsa.org/14070003

Abstract: We introduce space-time quantum code construction which is based on repeating the layers of an arbitrary quantum error correcting code. The error threshold of such space-time construction is shown to be related to the fault tolerant error threshold of the original quantum error correcting code in the presence of errors in syndrome measurements. The decoding transition for space-time codes can be further mapped to random-bond Wegner spin models.
Families of quantum low density parity-check (LDPC) codes with a finite decoding threshold lead to both known models (e.g., random bond Ising and random plaquette Z2 gauge models) as well as unexplored earlier and generally non-local disordered spin models with non-trivial phase diagrams that include the spin glass phase.
 We apply this construction to the simplest examples of recently discovered hypergraph-product codes and numerically find the fault tolerant threshold in excess of 5% by employing Monte-Carlo simulations.

Spin glass reflection of the decoding transition for space-time codes

arXiv:1311.7688 [quant-ph]

Alexey A. Kovalev
University of Nebraska-Lincoln
Leonid Pryadko
University of California, Riverside

Outline

- Introduction: classical and quantum LDPC codes
- *Quantum LDPC* code mapping to Ising spin model of general type suggested by *Wegner*
- *Maximum likelihood decoding* and the partition function
- Local LDPC codes and the corresponding spin models
- Multiplicity of ground states, extended defects and absence of the local order parameter
- Peculiarities of the finite rate LDPC codes and the corresponding spin models
- LDPC code constructions corresponding to *fault tolerant quantum meory* and *space-time codes*
- Mapping of space-time codes to quantum models

Classical error correction

We would like to send some number of bits through a noisy channel randomly flipping bits while being able to recover the message.

The simplest error-correcting code is the repetition code:

0 -> (0,0,0,0,0)
1-> (1,1,1,1,1)

We duplicate information in order to be able to recover it.

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For recovery, it is convenient to use the parity check matrix ***H***.

Suppose one bit got flipped so instead of v=(1,1,1,1,1) we received v1=(1,0,1,1,1).

By multiplying received message with the parity check matrix we obtain syndrome which tells us the position in which the bit was flipped, i.e.:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (1,1,0,0,0)$$. This information is sufficient for recovery.

Stabilizer codes: Binary representation

Pauli operators are mapped to two binary strings, $\mathbf{v}, \mathbf{u} \in \{0,1\}^n$, $U \equiv i^{m'} X^{\mathbf{v}} Z^{\mathbf{u}} \rightarrow (\mathbf{v}, \mathbf{u})$, where $X^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} \ldots X_n^{v_n}$ and $Z^{\mathbf{u}} = Z_1^{u_1} Z_2^{u_2} \ldots Z_n^{u_n}$. A product of two quantum operators corresponds to a sum (mod 2) of the corresponding pairs $(\mathbf{v}_i, \mathbf{u}_i)$.

In this representation, a stabilizer code is represented by parity check matrix written in binary form for X and Z Pauli operators so that, e.g. XIYZYI=-(XIXIXI)x(IIZZZI) -> (101010)|(001110).

$$H = \begin{array}{c} \begin{array}{ccccc|ccccc} & & \text{Ax} & & & & & \text{Az} & & \end{array} \\ \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

Example of a parity check matrix *H* of [[5,1,3]] code written in X-Z form.

$$\boxed{A_X A_Z^T + A_Z A_X^T = 0}$$

Necessary and sufficient condition for existence of stabilizer code with stabilizer commuting operators corresponding to ***H***.

$$\boxed{(z|x) \odot (z'|x')^T = zx'^T + xz'^T}$$

Row orthogonality with respect to symplectic product.

Parity check matrix for a Calderbank-Shor-Steane (CSS) code: $H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right)$, $G_X G_Z^T = 0$ ⟸ commutativity

Classical LDPC codes

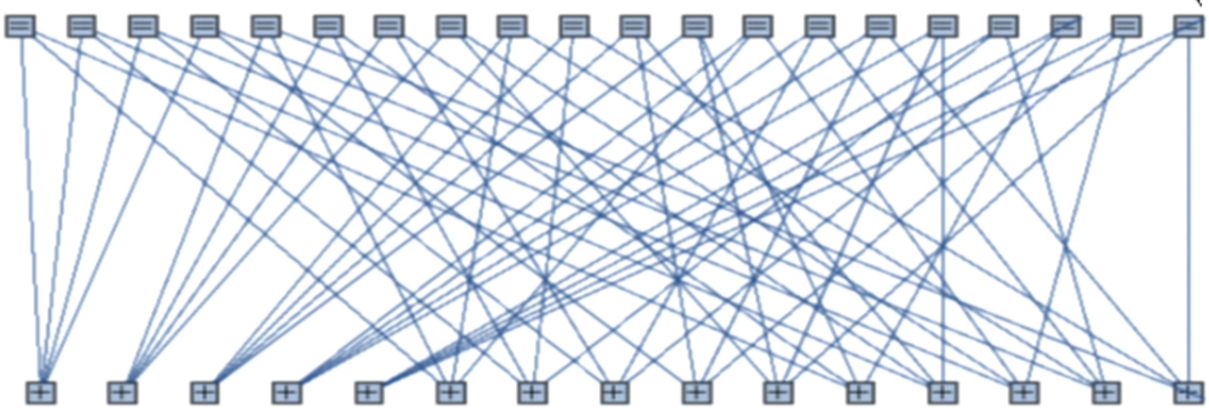
Classical LDPC codes are exceptional for error correction,
e.g. **Gallager codes, IRE Trans. Info. Theory IT-8: 21-28 (1962)**.

Number of 1s in every row (=r), and in every column (=c) for the parity check matrix is fixed.

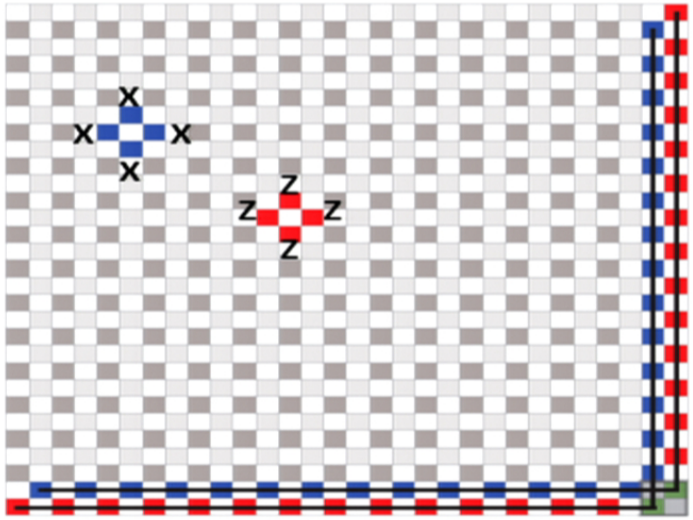
H2 and H3 are formed from H1 by column permutations,
e.g. c= 4, r = 3 for [20,7,6] code

$$\begin{pmatrix} H1 \\ H2 \\ H3 \end{pmatrix} = \begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&= \\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0 \\
0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0 \\
0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0 \\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0 \\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\
=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&=&= \\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0 \\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0 \\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0 \\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{pmatrix}$$

Convenient to represent by bipartite graph:



The simplest LDPC code – toric code

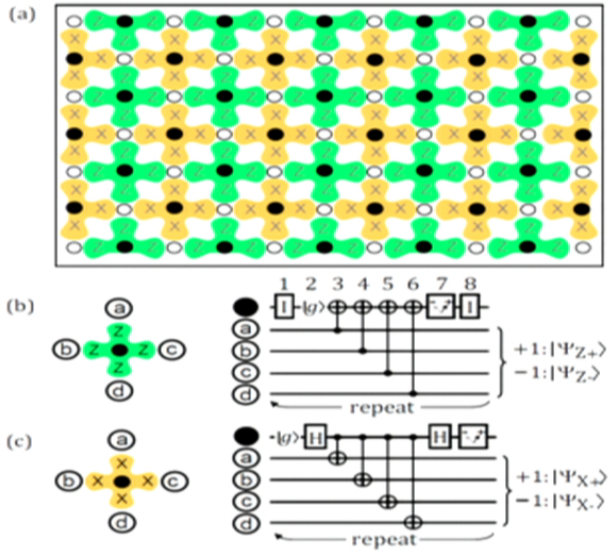


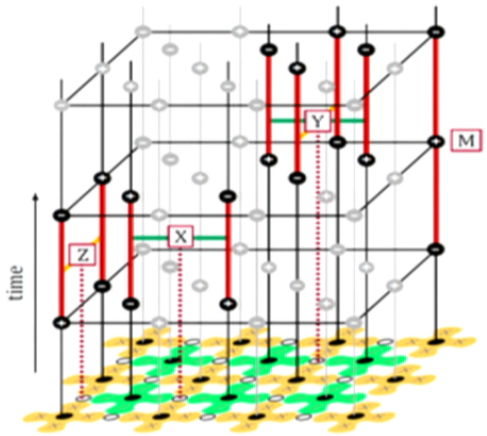
1. **Stabilizer generators commute with line like logical operators.**
2. **Stabilizer generators commute with each other.**
3. **Combinations of stabilizer generators form loops that are topologically different from the logical operators.**
4. **Logical operators can be deformed by stabilizer generators.**

Two stabilizer generators and two pairs of anticommuting logical operators of a toric code (red and blue, respectively, X and Z operators, green - overlap of Z and X operators, dark and light gray - dual sublattices of physical qubits). Other stabilizer generators are obtained by shifts over the same sublattice with periodic boundaries.

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill,
J. Math. Phys. 43, 4452 (2002)

Beyond the surface LDPC code





Unfortunately:

$$kd^2 = O(n)$$

S. Bravyi, D. Poulin, and B. Terhal, Phys. Rev. Lett. 104, 050503 (2010)

A. G. Fowler, M. Mariantoni, J. M. Martinis, A. N. Cleland, Phys. Rev. A 86, 032324 (2012)

1. Overhead of ancillary qubits is small.
2. Threshold is very high p=0.57%.
3. All we need to do is to measure ancillas and keep track of errors.

Finite rate LDPC codes allow for linear overhead in the number of encoded qubits!

D. Gottesman, arXiv:1310.2984 [quant-ph]

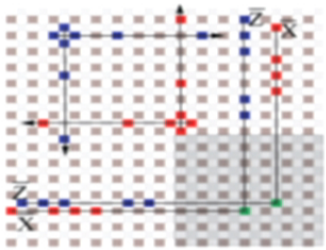
Finite rate LDPC codes can have finite error threshold!

Kovalev&Pryadko, Phys. Rev. A 87, 020304(R) (2013)

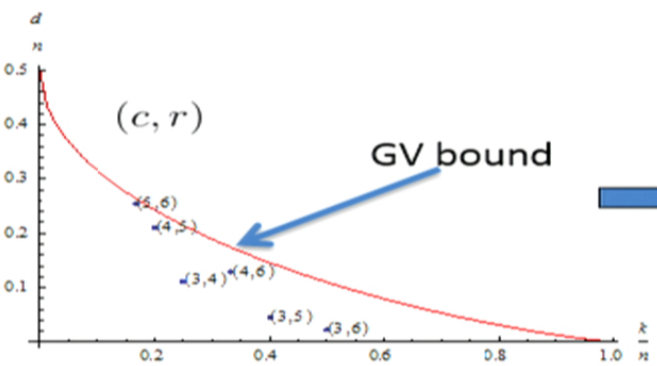
Examples of finite rate LDPC codes:

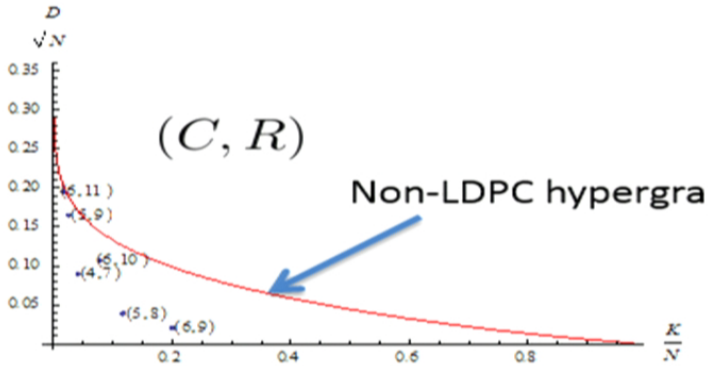
J.P. Tillich and G. Zémor," IEEE Transactions on Information Theory (2014).

Quantum LDPC codes



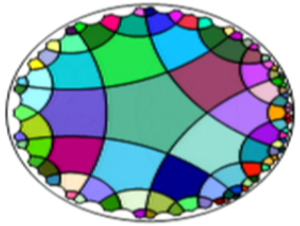
1. Hypergraph-product LDPC codes: $K \propto N,\ D \propto \sqrt{N}, KD^2 \propto N^2$





Tillich & Zemor, In Information Theory, ISIT 2009. IEEE International Symposium on, 799-803. IEEE, 2009

2. Hyperbolic surface codes: $K \propto N,\ D \propto \log N, KD^2 \propto N(\log N)^2$
Zemor, In Workshop on Coding and Cryptology, IWCC 2009 (2009), 259–273

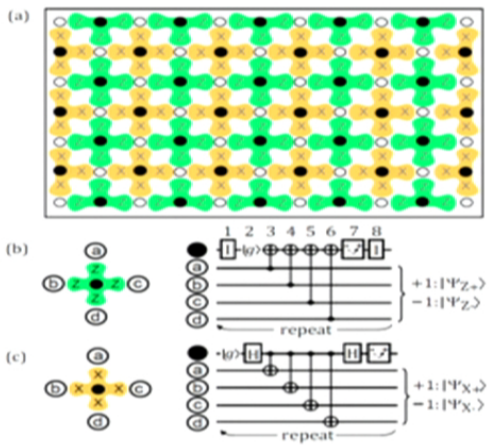


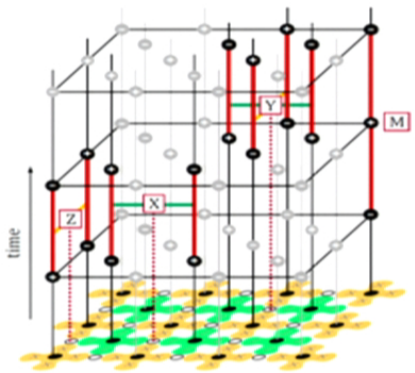
3. 4D hyperbolic code: $K \propto N,\ D \propto N^{\epsilon}, KD^2 \propto N^{1+2\epsilon}$
L. Guth and A. Lubotzky, arXiv:1310.5555

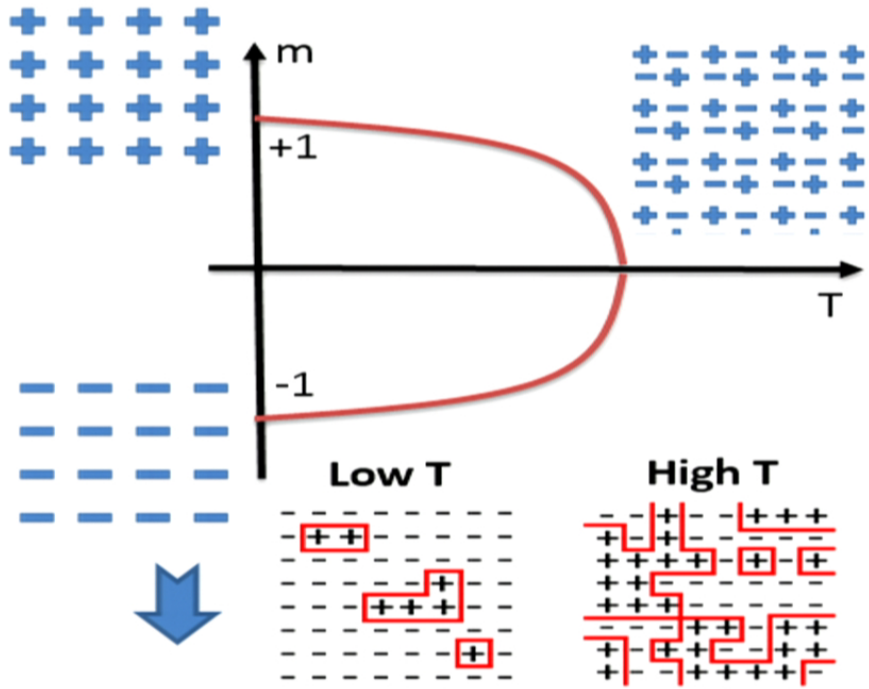
Based on distance scaling these codes should have finite threshold:
Kovalev & Pryadko, Phys. Rev. A 87, 020304(R) (2013)

Decoding: M. B. Hastings, arXiv:1312.2546

Toric code mapping to Ising model



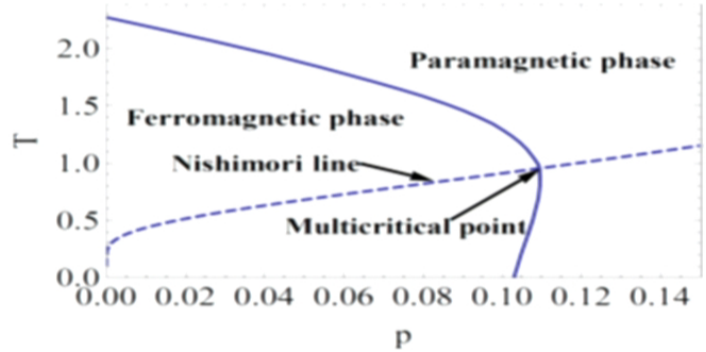




A. G. Fowler, M. Mariantoni, J. M. Martinis, A. N. Cleland, Phys. Rev. A 86, 032324 (2012)

Error model with independent X and Z errors

$$\rho \mapsto p_I \rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$$



For color codes: C. K. Thomas and H. G. Katzgraber, Phys. Rev. E 84, 040101 (2011)

General code mapping to Wegner models

$$S = -\sum_b I_b R_b \qquad R_b = \prod_r S_r^{\Theta_{r,b}} \qquad I_b = 1$$

F. Wegner, J. Math. Phys. 2259, 12 (1971)
Example: 3D plaquette model without local order parameter.

1. Represent code via binary matrices by **mapping stabilizer generators to binary strings** corresponding to X and Z Pauli operators, e.g. XIYZYI=-(XIXIXI)x(IIZZZI).

Example for [[5,1,3]] code: **H=**

$$\begin{array}{c} \text{Ax} \qquad\qquad\qquad\qquad \text{Az} \\ \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

2. Error model corresponds to independent X and Z errors happening with probability p.

$$\rho \mapsto p_I \rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$$

3. Calculate probability of an error and probability of equivalent errors:

$$P(\mathbf{e}) = \prod_{i=1}^{N_b} p^{e_i}(1-p)^{1-e_i} = p^w (1-p)^{N_b - w} \Rightarrow P_c(\mathbf{e}) = \frac{1}{2^{N_g}} \sum_{\boldsymbol{\sigma}} p^w (1-p)^{N_b - w}, \quad w \equiv \mathrm{wgt}(\mathbf{e} + \mathbf{c} + \boldsymbol{\sigma} H)$$

$$\boxed{Z_{\mathbf{e},\mathbf{c}}(\Theta = H; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r = \pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b (-1)^{e_b + c_b} R_b\right]}{2\cosh\beta}}$$

$$K_b = \beta I_b \qquad \beta = \beta_p \qquad e^{-2\beta_p} = p/(1-p)$$

General code mapping to Wegner models

$$S = -\sum_b I_b R_b \qquad R_b = \prod_r S_r^{\Theta_{r,b}} \qquad I_b = 1$$

F. Wegner, J. Math. Phys. 2259, 12 (1971)

Example: 3D plaquette model without local order parameter.

1. Represent code via binary matrices by **mapping stabilizer generators to binary strings** corresponding to X and Z Pauli operators, e.g. XIYZYI=-(XIXIXI)x(IIZZZI).

Example for [[5,1,3]] code:

$$\mathbf{H}= \begin{array}{c} \begin{array}{cccccccccc} \text{Ax} & & & & & \text{Az} & & & & \end{array} \\ \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

2. Error model corresponds to independent X and Z errors happening with probability p.

$$\rho \mapsto p_I \rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$$

3. Calculate probability of an error and probability of equivalent errors:

$$P(\mathbf{e}) = \prod_{i=1}^{N_b} p^{e_i}(1-p)^{1-e_i} = p^w (1-p)^{N_b - w} \Rightarrow P_c(\mathbf{e}) = \frac{1}{2^{N_g}} \sum_{\boldsymbol{\sigma}} p^w (1-p)^{N_b - w}, \quad w \equiv \mathrm{wgt}(\mathbf{e} + \mathbf{c} + \boldsymbol{\sigma} H)$$

$$\boxed{Z_{\mathbf{e},\mathbf{c}}(\Theta = H; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r = \pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b (-1)^{e_b + c_b} R_b\right]}{2\cosh\beta}}$$

$$K_b = \beta I_b \qquad \beta = \beta_p \qquad e^{-2\beta_p} = p/(1-p)$$

Maximum likelihood decoding

1. **Maximum likelihood decoding** corresponds to the equivalence class with maximum probability (partition function) on the Nishimori line: $e^{-2\beta_p} = p/(1-p)$

$P_{\max}(\mathbf{e})$ -- the maximum probability among equivalence classes.

$$P_{\text{tot}}(\mathbf{e}) = \sum_c P_c(\mathbf{e}) \Rightarrow \mathcal{P}_{\text{dec}} = [P(\mathbf{e})/P_{\max}(\mathbf{e})]_e$$ -- probability of success

2. **Decoding transition** is characterized by: $\boxed{\mathcal{P}_{\text{dec}} \to 1}$

$$\boxed{\kappa = \left[Z_0(\mathbf{e};\beta) / \sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta) \right]_e}$$

ML decoder $\left[Z_{\max}(\mathbf{e};\beta) / \sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta) \right]_e$

Order parameter

3. **Decodable phase** corresponds to divergent Free energy cost of extended defect:

$$\boxed{\Delta F = \beta^{-1} \left[\log \frac{Z_{\mathbf{e}}}{Z_{\mathbf{e}+\mathbf{c}}} \right]_e}$$

By studying the phase diagram of the quantum-code Wegner model we can gain information about the error threshold of the code!

Code mapping to Wegner models: CSS codes

1. A CSS code results in two **Wegner models**, i.e. $\Theta = G_X$ and $\Theta = G_Z$

For CSS code: $H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right)$

$$Z_{\mathbf{e}}(\Theta; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r = \pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b (-1)^{e_b} R_b\right]}{2 \cosh \beta}$$

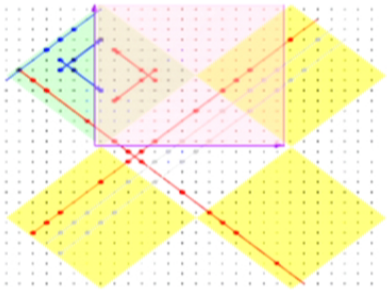
2. With this mapping toric code maps to Ising models

3. Hyperbolic surface codes will lead to Ising models on hyperbolic surfaces

4. **Hypergraph-product codes** map to generalized Ising models where $\otimes$-Kronecker product, E-unit matrix

$$G_X = (E_2 \otimes \mathcal{H}_1 \;\; \mathcal{H}_2 \otimes E_1)$$
$$G_Z = (\mathcal{H}_2^T \otimes \widetilde{E}_1 \;\; \widetilde{E}_2 \otimes \mathcal{H}_1^T)$$

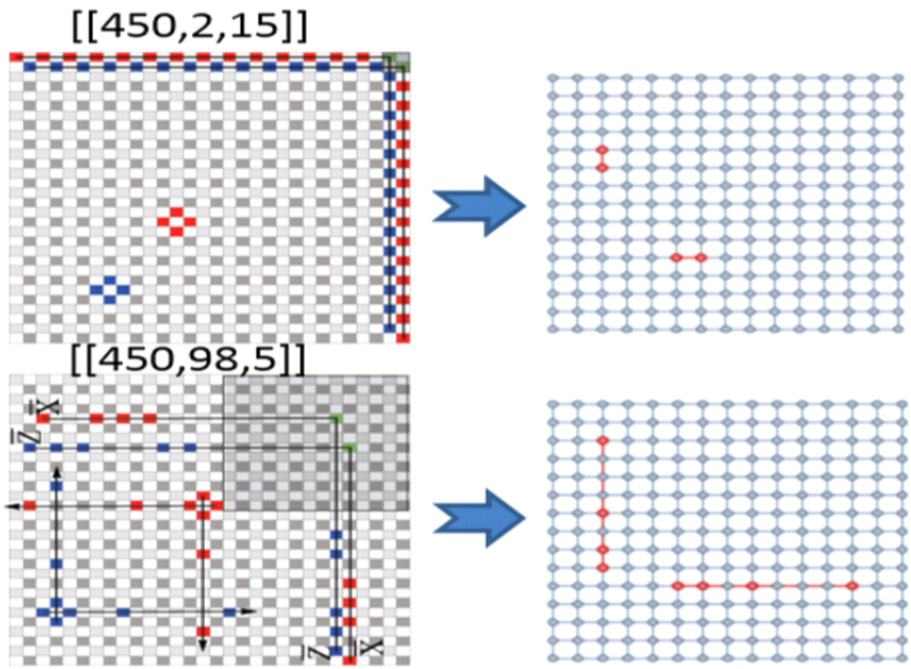
5. Rotated hypergraph-product (hyperbicycle) codes lead to generalized Ising models with rotated boundaries.
A.A. Kovalev and L. P. Pryadko, Phys. Rev. A 88, 012311 (2013)

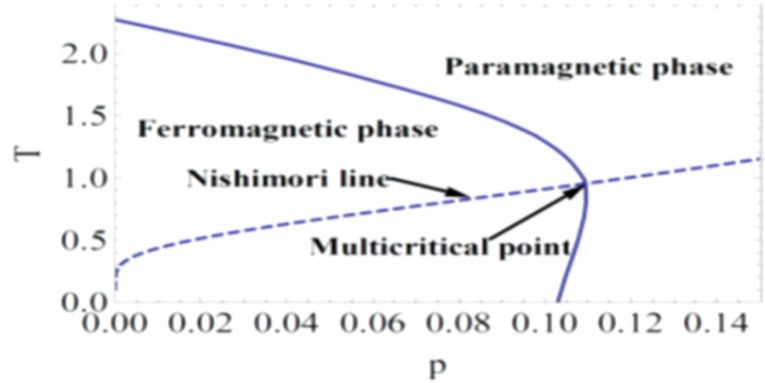


Local LDPC codes

Still bounded by $kd^2 = O(n)$ but can improve number of encoded qubits by a factor compared to toric code for not too large code dimensions.

Monte-Carlo analysis suggests that both models have similar phase diagram:





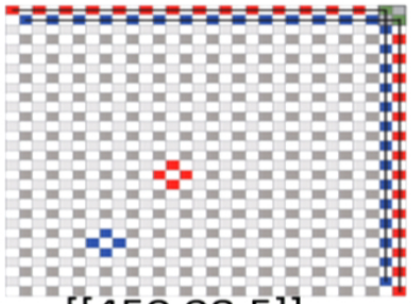
Consistent with Nishimori's strong disorder **self-duality conjecture**: $H_2(p_c) = 1/2$

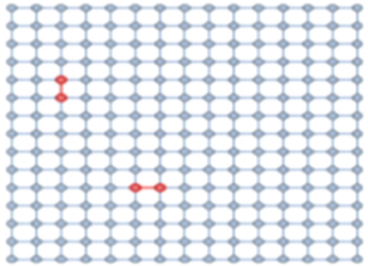
H. Nishimori, Journal of Physics C: Solid State Physics 12, L905 (1979)

Local LDPC codes

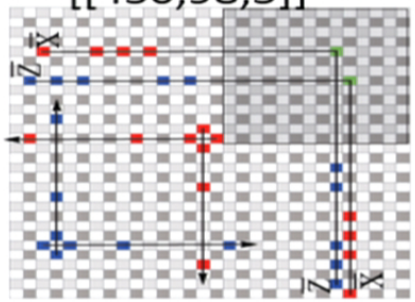
Still bounded by $kd^2 = O(n)$ but can improve number of encoded qubits by a factor compared to toric code for not too large code dimensions.

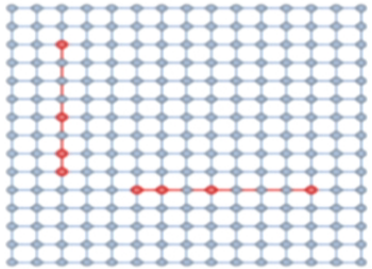
[[450,2,15]]



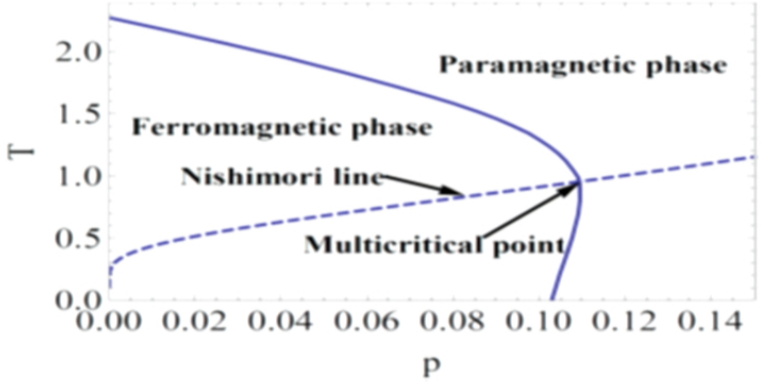


[[450,98,5]]





Monte-Carlo analysis suggests that both models have similar phase diagram:



Consistent with Nishimori's strong disorder **self-duality conjecture**: $H_2(p_c) = 1/2$

H. Nishimori, Journal of Physics C: Solid State Physics 12, L905 (1979)

Domain walls and phase transition

1. Ordered phase is associated with the divergent Free energy cost of an extended defect (domain wall):

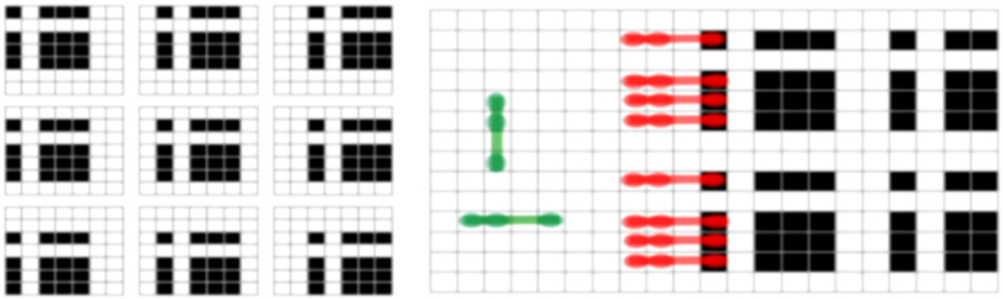
$$\left[\Delta F_{\mathbf{c}}^{(0)}(\mathbf{e};\beta)\right]_e \equiv \beta^{-1}\left[\log\frac{Z_0(\mathbf{e};\beta)}{Z_{\mathbf{c}}(\mathbf{e};\beta)}\right]_e$$

2. We introduce the order parameter which on Nishimori line exactly corresponds to the probability of successful decoding:

$$\kappa = \left[Z_0(\mathbf{e};\beta)/\sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta)\right]_e$$

3. The ground state is degenerate where the degeneracy is related to the number of encoded qubits in the originating code:

4. The last example leads to the first order phase transition by proliferations of extended defects



J. M. Debierre and L. Turban, Journal of Physics A: Mathematical and General 16, 3571 (1983)

Finite rate LDPC codes

1. **Hypergraph-product code** construction: $H = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right)$ $\begin{array}{l} G_X = (E_2 \otimes \mathcal{H}_1 \;\; \mathcal{H}_1^T \otimes E_1) \\ G_Z = (\mathcal{H}_1 \otimes \widetilde{E}_1 \;\; \widetilde{E}_2 \otimes \mathcal{H}_1^T) \end{array}$

   E – unit matrices and a random LDPC matrix $\mathcal{H}_1 =$

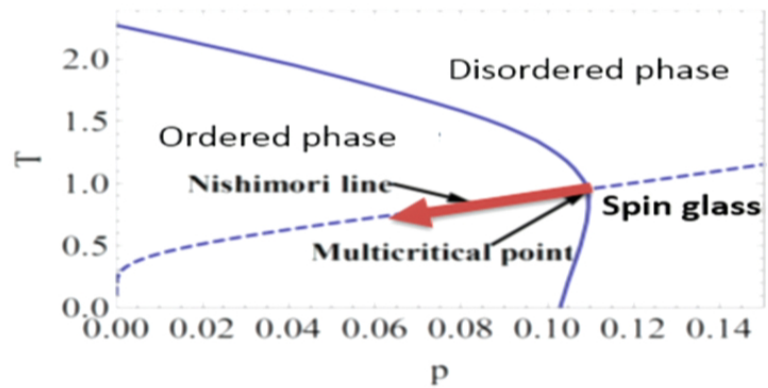
$$\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
\\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{pmatrix}$$

   Gallager codes, IRE Trans. Info. Theory IT-8: 21-28 (1962).

2. The dimension of encoded space and the number of **extended (post-topological) defects** grows exponentially with code size.

3. **Self duality conjecture** should not hold for finite rate LDPC codes:

4. There can be additional phase transitions due to large number of **extended defects** driven by entropy associated with large number of defects.



5. Existence of the **decoding transition**;
   Kovalev & Pryadko,
   Phys. Rev. A 87, 020304(R) (2013)

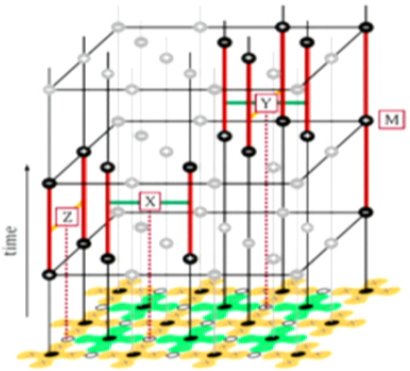
Generalizations: Errors in syndrome measurements

1. Consider repeated measurements of syndromes with the purpose of recovering syndrome errors

2. Simple error model with the same probability of syndrome error

$$H(\ldots + \mathbf{e}_{-1}^T + \mathbf{e}_0^T + \mathbf{e}_1^T) + \mathbf{e}_1^{(s)} = \mathbf{s}_1$$

$$H(\ldots + \mathbf{e}_0^T + \mathbf{e}_1^T + \mathbf{e}_2^T) + \mathbf{e}_2^{(s)} = \mathbf{s}_2, \text{ etc.}$$



A.G. Fowler et al., Phys. Rev. A 86, 032324 (2012)

3. By considering differences we obtain:

$$\begin{aligned}
H\mathbf{e}_1^T + \quad & \mathbf{e}_1^{(s)} && = \mathbf{s}_1, \\
H\mathbf{e}_2^T + \quad & \mathbf{e}_2^{(s)} - \mathbf{e}_1^{(s)} && = \mathbf{s}_2 - \mathbf{s}_1, \\
& \ldots \\
H\mathbf{e}_{r_c}^T + \quad & \mathbf{e}_{r_c}^{(s)} - \mathbf{e}_{r_c-1}^{(s)} && = \mathbf{s}_{r_c} - \mathbf{s}_{r_c-1},
\end{aligned}$$

Will work with any stabilizer code!

This can be interpreted as enlarged classical code with syndromes on the right!

For color codes: A. J. Landahl, J. T. Anderson, P. R. Rice, arXiv:1108.5738

Gauge freedom of the enlarged code

1. Not all errors need to be corrected: errors that are cycles can be ignored.

2. This leads to a code with a gauge freedom, we call it space time codes.

$$
\begin{aligned}
H\mathbf{e}_1^T + \quad & \mathbf{e}_1^{(s)} && = \mathbf{s}_1, \\
H\mathbf{e}_2^T + \quad & \mathbf{e}_2^{(s)} - \mathbf{e}_1^{(s)} && = \mathbf{s}_2 - \mathbf{s}_1, \\
& \cdots \\
H\mathbf{e}_{r_c}^T + \quad & \mathbf{e}_{r_c}^{(s)} - \mathbf{e}_{r_c-1}^{(s)} && = \mathbf{s}_{r_c} - \mathbf{s}_{r_c-1},
\end{aligned}
\quad \Rightarrow \quad
\mathcal{G} = (E_1 \otimes H, \quad R \otimes E_2)
$$

$$
R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
$$

3. The gauge freedom can be readily constructed leading to a CSS code where only threshold with respect to Z errors is relevant for the fault tolerant analysis.

$$
\mathcal{G}_X = (E_1 \otimes H, \quad R \otimes E_2)
$$

$$
\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix}
$$

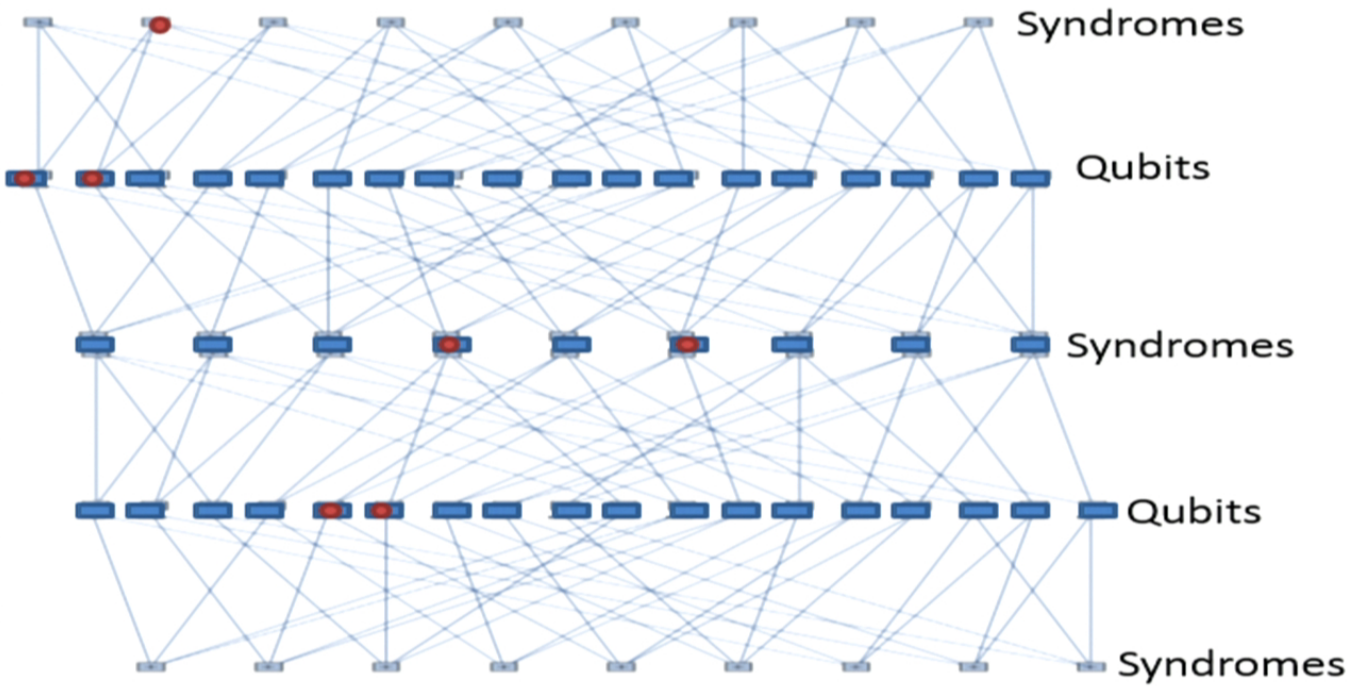
$$
H = (H_x, H_z)
$$

$$
\widetilde{H} = (H_z, H_x)
$$

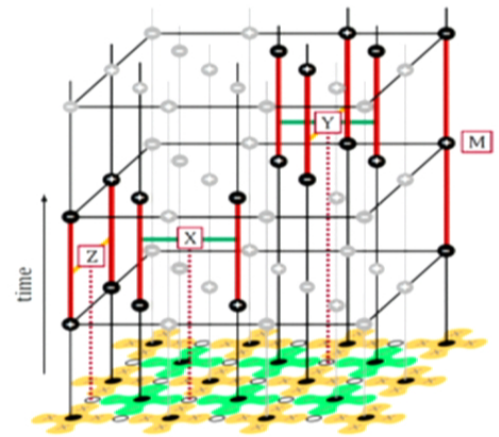
4. The decoding transition in such a model corresponds to decoding transition of the original quantum code when **errors in syndrome measurements** are present.

5. Such codes are also LDPC codes for which the existence of the **decoding transition** has been shown; **Kovalev&Pryadko, Phys. Rev. A 87, 020304(R) (2013)**.

Space-time LDPC codes



Time



Compare to toric code case.

$$\widetilde{H} = (H_z, H_x)$$

$$H = (H_x, H_z)$$

In the presence of syndrome errors we can still represent errors on a graph with a different connectivity.

Maximum degree of the graph:

$$z = (R-1)C$$
$$z -> z' = (R+1)C$$

$$\mathcal{G}_X = (E_1 \otimes H, \quad R \otimes E_2)$$

$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix}$$

Kovalev&Pryadko, Phys. Rev. A 87, 020304(R) (2013)

Gauge freedom of the enlarged code

1. Not all errors need to be corrected: errors that are cycles can be ignored.

2. This leads to a code with a gauge freedom, we call it space time codes.

$$
\begin{aligned}
H\mathbf{e}_1^T + \quad & \mathbf{e}_1^{(s)} && = \mathbf{s}_1, \\
H\mathbf{e}_2^T + \quad & \mathbf{e}_2^{(s)} - \mathbf{e}_1^{(s)} && = \mathbf{s}_2 - \mathbf{s}_1, \\
& \cdots \\
H\mathbf{e}_{r_c}^T + \quad & \mathbf{e}_{r_c}^{(s)} - \mathbf{e}_{r_c-1}^{(s)} && = \mathbf{s}_{r_c} - \mathbf{s}_{r_c-1},
\end{aligned}
\quad \Rightarrow \quad
\mathcal{G} = (E_1 \otimes H, \quad R \otimes E_2)
$$

$$
R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
$$

3. The gauge freedom can be readily constructed leading to a CSS code where only threshold with respect to Z errors is relevant for the fault tolerant analysis.

$$
\begin{aligned}
\mathcal{G}_X &= (E_1 \otimes H, \quad R \otimes E_2) \\
\mathcal{G}_Z &= \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix}
\end{aligned}
\qquad
\begin{aligned}
H &= (H_x, H_z) \\
\widetilde{H} &= (H_z, H_x)
\end{aligned}
$$

4. The decoding transition in such a model corresponds to decoding transition of the original quantum code when **errors in syndrome measurements** are present.

5. Such codes are also LDPC codes for which the existence of the **decoding transition** has been shown; **Kovalev&Pryadko, Phys. Rev. A 87, 020304(R) (2013)**.

Mapping space-time codes to Wegner models

1. The mapping to space time codes describes gauge shifts thus the matrix $\mathcal{G}_Z$ is used in the construction.

$$S = -\sum_b I_b R_b \qquad R_b = \prod_r S_r^{\Theta_{r,b}}$$

$$Z_{\mathbf{e}}(\Theta = \mathcal{G}_Z; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r=\pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b(-1)^{e_b} R_b\right]}{2\cosh\beta}$$

$$I_b = 1$$
$$K_b = \beta I_b$$
$$\beta = \beta_p$$
$$e^{-2\beta_p} = p/(1-p)$$

$$\mathcal{G}_X = (E_1 \otimes H, \quad R \otimes E_2)$$
$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix}$$

2. Formally, we can also consider a Wegner model constructed from the matrix $\mathcal{G}_X$

$$Z_{\mathbf{e}}(\Theta = \mathcal{G}_X; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r=\pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b(-1)^{e_b} R_b\right]}{2\cosh\beta}$$

3. By studying the phase diagram of the Wegner models corresponding to space-time codes we can learn useful information about the fault tolerant threshold of the original quantum code

Space-time codes from CSS codes

1. CSS code can be mapped to two space time codes, each dealing with X or Z type errors:

$$\mathcal{G}_X = (E_1 \otimes G_X, \quad R \otimes E_2) \qquad \mathcal{G}_X = (E_1 \otimes G_Z, \quad R \otimes E_2)$$

$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes G_X^T \\ E_1 \otimes G_Z, & 0 \end{pmatrix} \qquad \mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes G_Z^T \\ E_1 \otimes G_X, & 0 \end{pmatrix}$$

To correct Z errors **To correct X errors**

2. For the toric code we recover two **3D random bond Ising models** from $\mathcal{G}_X$ and two **3D random plaquette gauge models** from $\mathcal{G}_Z$

3. For the toric code of relevance is the random plaquette gauge model

$$Z_{\mathbf{e}}(\Theta = \mathcal{G}_Z; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r = \pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b (-1)^{e_b} R_b\right]}{2\cosh\beta}$$

4. The error threshold (fault tolerant threshold) of 3.3% has been calculated for the toric code for qubit flip and syndrome errors (random plaquette gauge model) **E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002).**

Space-time codes from CSS codes

1. CSS code can be mapped to two space time codes, each dealing with X or Z type errors:

$$\mathcal{G}_X = (E_1 \otimes G_X, \quad R \otimes E_2) \qquad \mathcal{G}_X = (E_1 \otimes G_Z, \quad R \otimes E_2)$$

$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes G_X^T \\ E_1 \otimes G_Z, & 0 \end{pmatrix} \qquad \mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes G_Z^T \\ E_1 \otimes G_X, & 0 \end{pmatrix}$$

To correct Z errors **To correct X errors**

2. For the toric code we recover two **3D random bond Ising models** from $\mathcal{G}_X$ and two **3D random plaquette gauge models** from $\mathcal{G}_Z$

3. For the toric code of relevance is the random plaquette gauge model

$$Z_{\mathbf{e}}(\Theta = \mathcal{G}_Z; \{K\}) \equiv \frac{1}{2^{N_g}} \sum_{\{S_r = \pm 1\}} \prod_{b=1}^{N_b} \frac{\exp\left[K_b (-1)^{e_b} R_b\right]}{2\cosh\beta}$$

4. The error threshold (fault tolerant threshold) of 3.3% has been calculated for the toric code for qubit flip and syndrome errors (random plaquette gauge model) **E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002).**

Parameters of space-time codes

1. The most general code is constructed from an arbitrary quantum code and a binary matrix *R*

$$\mathcal{G}_X = (E_1 \otimes H, \quad R \otimes E_2) \qquad \widetilde{H} = (H_z, H_x)$$

$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix} \qquad H = (H_x, H_z)$$

2. The parameters of such CSS code are given in terms of (1) $[n_c, k_c, d_c]$ classical code with *R* as the parity check; (2) $[\widetilde{n}_c, \widetilde{k}_c, \widetilde{d}_c]$ classical code with a transposed parity check; (3) $[[n_q, k_q, d_q]]$ quantum CSS code with $H \oplus \widetilde{H}$ parity check (4) $[[\widetilde{n}_q, \widetilde{k}_q, \widetilde{d}_q]]$ quantum CSS code with $H^T \oplus \widetilde{H}^T$ parity check

The parameters of the space-time code are given by $n = r_q n_c + r_c n_q$, $k = k_q \widetilde{k}_c + k_c \widetilde{k}_q$, while the distance d satisfies the conditions $d \geq \min(d_q, d_c, \widetilde{d}_q, \widetilde{d}_c)$, and two upper bounds: if $k_q > 0$ and $\widetilde{k}_c > 0$, then $d \leq d_q$, $d \leq \widetilde{d}_c$; if $k_c > 0$ and $\widetilde{k}_q > 0$, then $d \leq d_c$, $d \leq \widetilde{d}_q$.

Formally these expressions are similar to those for a hypergraph-product (HP) code. The difference is that in this construction we use a quantum and a classical codes while in hypergraph-product construction we use two classical codes.

Space-time codes – code dimension

We obtain tensor product of subspaces divided by degeneracy:

$$\begin{aligned} \operatorname{rank} \mathcal{G}_X &= r_q r_c - \widetilde{k}_q \widetilde{k}_c \\ \operatorname{rank} \mathcal{G}_Z &= n_q n_c + (r_q - \widetilde{k}_q)(\widetilde{k}_c - k_c) - k_q k_c \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} (a^T \otimes b^T) \cdot \mathcal{G}_X &= 0 \\ (c^T \otimes d^T) \cdot \mathcal{G}_Z &= 0 \end{aligned}$$

For CSS code we know:

$$K = N - \operatorname{rank} \mathcal{G}_X - \operatorname{rank} \mathcal{G}_Z$$
$$N = (n_q r_c + n_c r_q)$$

By relating subspaces $\mathcal{C}_{\mathcal{H}_i}$ **and** $\mathcal{C}_{\mathcal{H}_i^T}$

$$\boxed{K = k_q \widetilde{k}_c + k_c \widetilde{k}_q}$$

$$k_c = 0,\ k_q = 0 \quad \Longrightarrow \quad K = 0$$

Space-time codes – code dimension

We obtain tensor product of subspaces divided by degeneracy:

$$\begin{aligned} \operatorname{rank} \mathcal{G}_X &= r_q r_c - \widetilde{k}_q \widetilde{k}_c \\ \operatorname{rank} \mathcal{G}_Z &= n_q n_c + (r_q - \widetilde{k}_q)(\widetilde{k}_c - k_c) - k_q k_c \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} (a^T \otimes b^T) \cdot \mathcal{G}_X &= 0 \\ (c^T \otimes d^T) \cdot \mathcal{G}_Z &= 0 \end{aligned}$$

For CSS code we know:

$$\begin{aligned} K &= N - \operatorname{rank} \mathcal{G}_X - \operatorname{rank} \mathcal{G}_Z \\ N &= (n_q r_c + n_c r_q) \end{aligned}$$

By relating subspaces $\mathcal{C}_{\mathcal{H}_i}$ **and** $\mathcal{C}_{\mathcal{H}_i^T}$

$$\boxed{K = k_q \widetilde{k}_c + k_c \widetilde{k}_q}$$

$$k_c = 0,\ k_q = 0 \quad \Longrightarrow \quad K = 0$$

Space-time codes – code distance

Suppose we found vector such that $\mathcal{G}_X v = 0$ and $\text{weight}[v] < d$

$$d = \min(d_q, d_c, \widetilde{d}_q, \widetilde{d}_c)$$

$$\mathcal{G}_X = (E_1 \otimes H, \quad R \otimes E_2)$$

$$\mathcal{G}_Z = \begin{pmatrix} R^T \otimes \widetilde{E}_2, & E_1 \otimes H^T \\ E_1 \otimes \widetilde{H}, & 0 \end{pmatrix}$$

$$v \quad \begin{array}{ccc} \downarrow\downarrow & \downarrow & \downarrow\downarrow \\ H & o & o \\ o & H & o \\ o & o & H \\ & \dots & \end{array} \quad \dots$$

We form a reduced code from columns of H corresponding to non-zero positions of vector v

The reduced code will have $k_c = 0, \; k_q = 0$

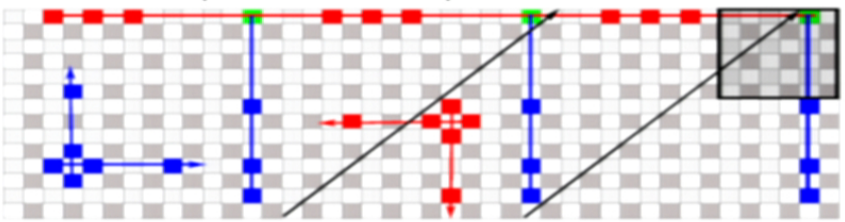
$$K = 0$$

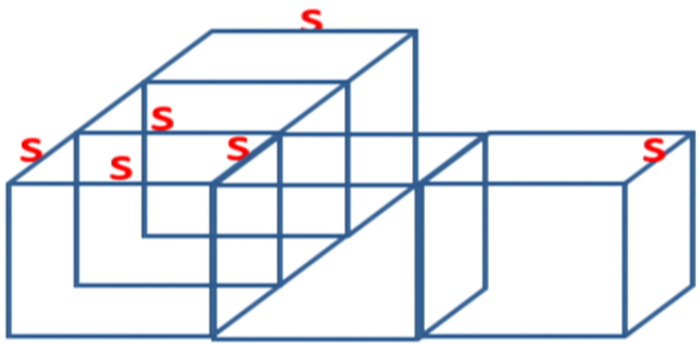
$$\boxed{D \geq d}$$

Example of a space-time code (1,1,0,1)

$$\mathcal{G} \equiv \begin{pmatrix} R^T \otimes \widetilde{\mathcal{E}}_q, & \widetilde{\mathcal{E}}_r \otimes H^T \\ \mathcal{E}_r \otimes \widetilde{H}, & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$$

temporal spatial





1. The local structure of stabilizer generators corresponds to the polynomial $g = 1 + x + x^3$

2. By employing the Monte-Carlo analysis we can find the transition point as a function of the flip probability for random bonds

3. The transition point on the Nishimori line will reveal the position of the decoding transition

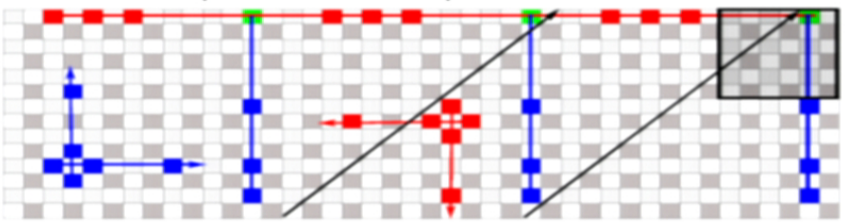
4. As local order parameter does not necessarily exist we could use the order parameter derived from the decoding transition:

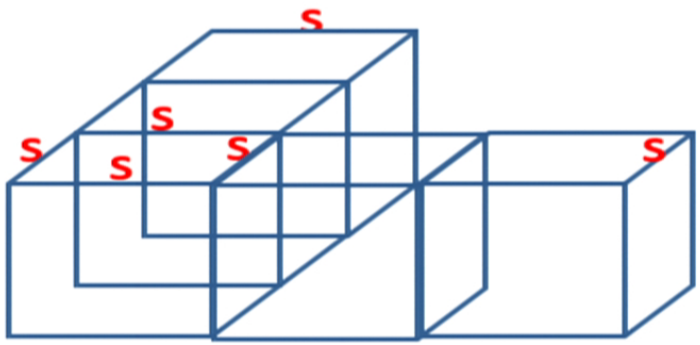
$$\kappa = \left[Z_0(\mathbf{e};\beta) / \sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta) \right]_e$$

Example of a space-time code (1,1,0,1)

$$\mathcal{G} \equiv \begin{pmatrix} R^T \otimes \widetilde{\mathcal{E}}_q, & \widetilde{\mathcal{E}}_r \otimes H^T \\ \mathcal{E}_r \otimes \widetilde{H}, & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$$

temporal spatial





1. The local structure of stabilizer generators corresponds to the polynomial $g = 1 + x + x^3$

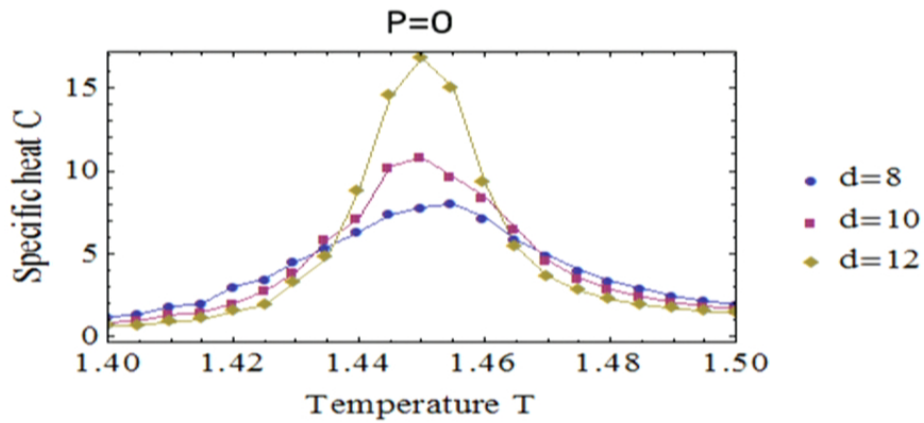
2. By employing the Monte-Carlo analysis we can find the transition point as a function of the flip probability for random bonds

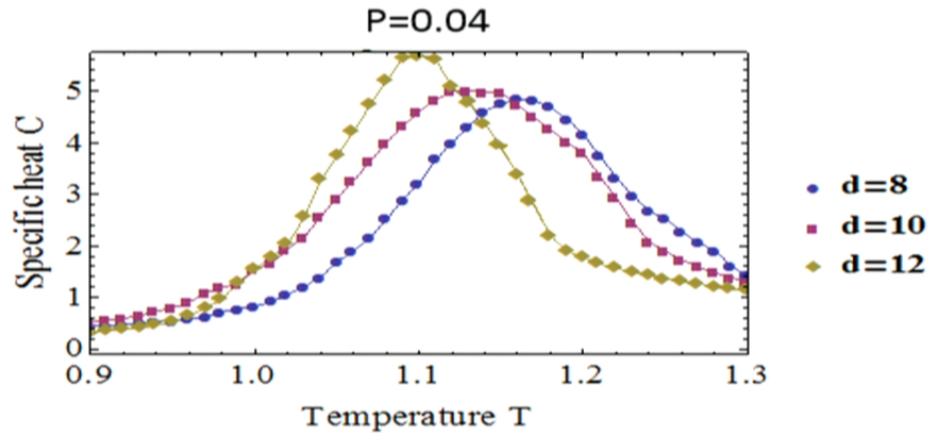
3. The transition point on the Nishimori line will reveal the position of the decoding transition

4. As local order parameter does not necessarily exist we could use the order parameter derived from the decoding transition:

$$\kappa = \left[Z_0(\mathbf{e};\beta) / \sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta) \right]_e$$

Results of Monte-Carlo numerics

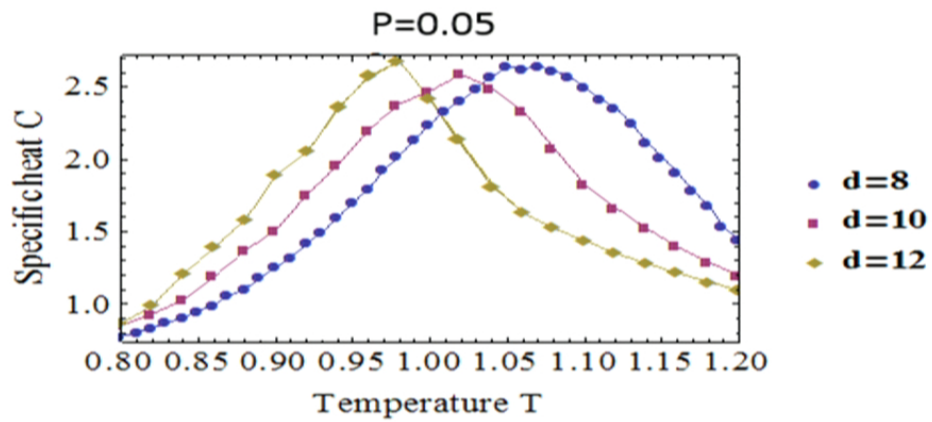




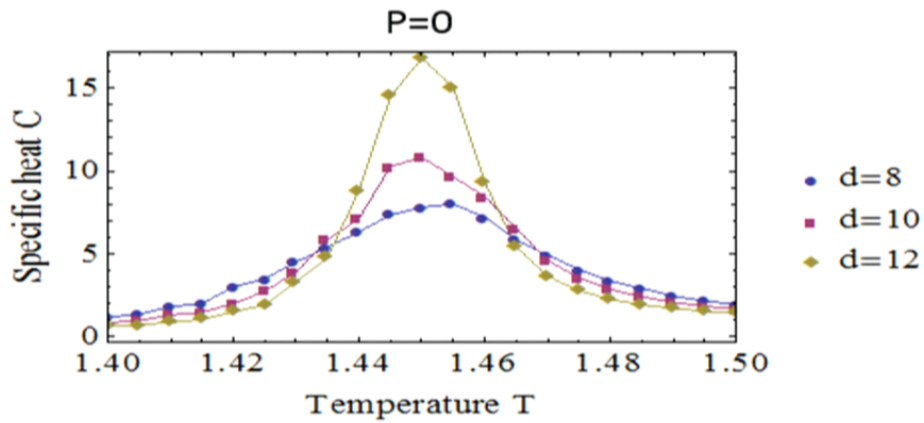
1. The phase transition seems to be of the first order

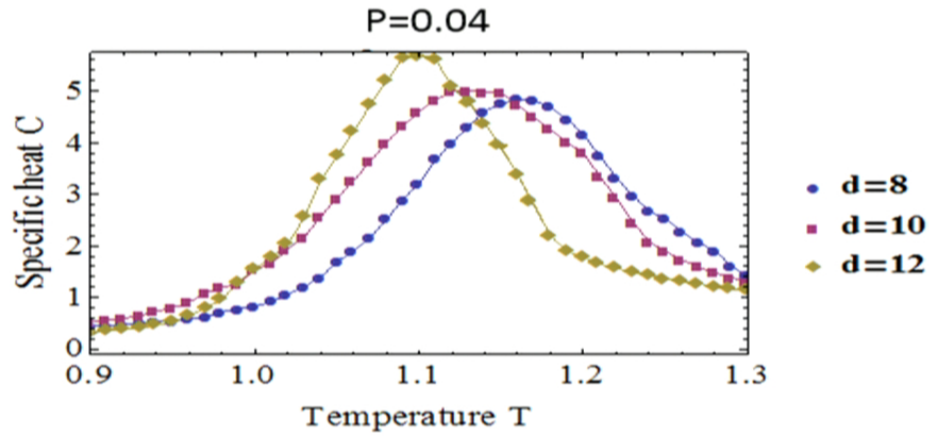
2. Close to the multicritical point the manifestation of the first order phase transition is suppressed

3. Simulations of codes of larger size are necessary for conclusive results



Results of Monte-Carlo numerics

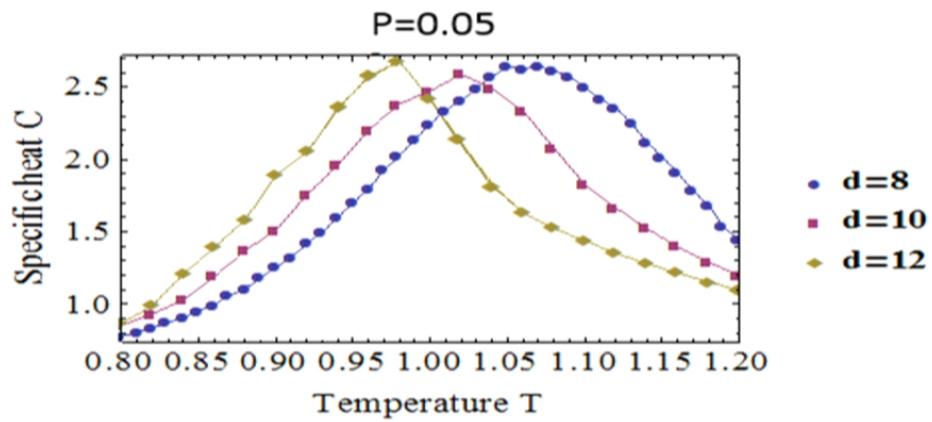




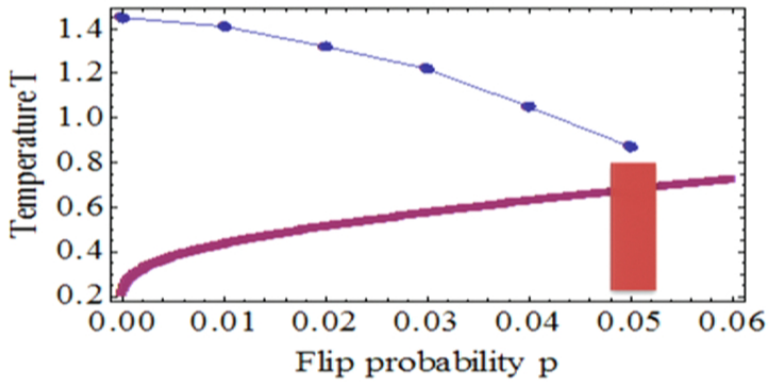
1. The phase transition seems to be of the first order

2. Close to the multicritical point the manifestation of the first order phase transition is suppressed

3. Simulations of codes of larger size are necessary for conclusive results



Preliminary phase diagram for code (1,1,0,1)



1. We approximately located the decoding transition point

2. One could also try to directly calculate the decoding order parameter

$$\kappa = \left[Z_0(\mathbf{e};\beta) / \sum_{\mathbf{c}} Z_{\mathbf{c}}(\mathbf{e};\beta) \right]_e$$

Analytical results valid for any code

1. By invoking the gauge transformation argument on spin correlators one can obtain that the phase diagram for the generalized decoding transition (characterized by κ) is vertical or reentrant

2. On the Nishimori line: below the decoding transition $\kappa = 1$, above the decoding transition the order parameter is separated from 1 by a finite number

Quantum models from space-time codes

1. In general error probabilities of syndrome errors and qubit errors are asymmetric

2. We can study such asymmetries directly through statistical model while additional insight can be achieved by mapping statistical models to quantum model

3. To this end we employ Wick rotation:

The amplitude of a transition for quantum model: $Z = \sum_{\text{paths}} \exp[\frac{i}{\hbar} S_m]$

$$S_m = \int \mathcal{L} dt$$

For the corresponding statistical model: $t = -i\tau \quad Z = \sum_{\text{stat}} \exp[-\frac{1}{\hbar} S]$

Transfer matrix method: $Z = \text{tr}[\hat{T}^N] \qquad \frac{\epsilon}{\hbar} \hat{H} = -\log \hat{T}$

4. The role of temperature will be played by $\hbar \leftrightarrow T$

Mapping for anisotropic Ising model

$$S = -\beta \sum_{m,\tau} S_{m,\tau} S_{m+1,\tau} - \beta_\tau \sum_{m,\tau} S_{m,\tau+1} S_{m,\tau}$$

1. We consider lowest order processes of only one spin flip between different time layers

2. By comparing transfer matrix with the action of the Hamiltonian we recover the quantum model:

$$L(\tau) = -\frac{1}{2}\beta \sum_m S_{m,\tau} S_{m+1,\tau} - \frac{1}{2}\beta \sum_m S_{m,\tau+1} S_{m+1,\tau+1} - \beta_\tau \sum_m S_{m,\tau+1} S_{m,\tau}$$

$$S = \sum_\tau L(\tau)$$

$$\hat{T} = e^{-i\frac{\tau}{\hbar}\hat{H}} = 1 - i\frac{\tau}{\hbar}\hat{H}$$

$$\boxed{\hat{H} = -\sum_m X_m - \lambda \sum_m Z_m Z_{m+1}}$$

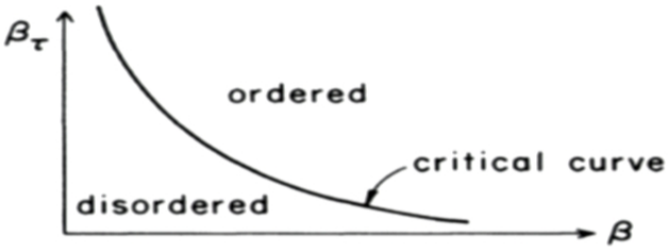
3. The coupling constants have to be adjusted properly:

$$\tau/\hbar = \exp(-2\beta_\tau)$$

$$\beta = \lambda\tau/\hbar$$

The phase diagram of Ising model is known:

$$\sinh 2\beta_\tau \sinh 2\beta = 1$$

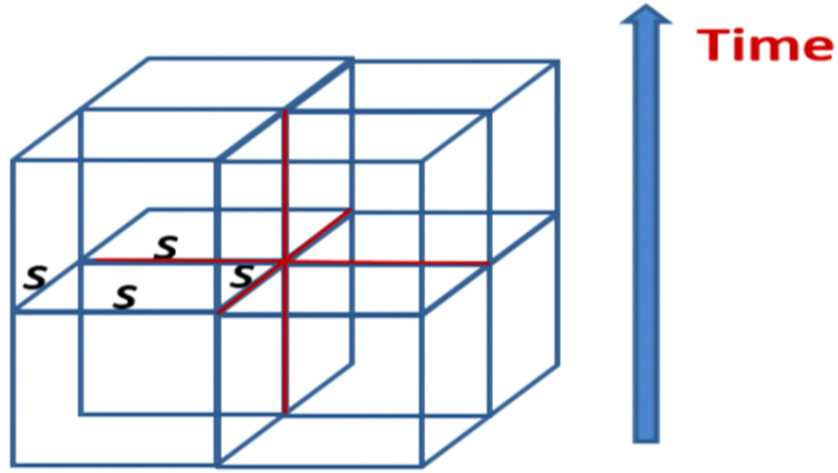


Anisotropic plaquette model

1. Space time code for the toric code is a plaquette model

$$S = -\beta \sum_{plaq} SSSS - \beta_\tau \sum_{plaq} SSSS$$

$$S = -\beta \sum_{plaq} SSSS - \beta_\tau \sum_{m,\tau} S_{m,\tau+1} S_{m,\tau}$$



2. By fixing the gauge we arrive at simplified model

3. The transfer matrix method results in the Hamiltonian (sum of toric code Z stabilizers):

$$\hat{H} = -\sum_m X_m - \lambda \sum_{plaq} ZZZZ$$

4. Limitations on coupling constants: $\tau/\hbar = \exp(-2\beta_\tau)$

$$\beta = \lambda\tau/\hbar$$

Mapping for space-time codes

1. Anisotropic coupling for temporal and spatial bonds:

$$\mathcal{G} \equiv \begin{pmatrix} R^T \otimes \widetilde{\mathcal{E}}_q, & \widetilde{\mathcal{E}}_r \otimes H^T \\ \mathcal{E}_r \otimes \widetilde{H}, & 0 \end{pmatrix}, \qquad R=\begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$$

temporal $I_b = \beta_\tau$ spatial $I_b = \beta$

$$S = -\sum_b I_b \prod_r S_r^{\mathcal{G}_{r,b}}$$

2. To fix the gauge we find the gauge matrix: $G = \left(\widetilde{\mathcal{E}}_r \otimes \widetilde{H}, R^T \otimes \mathcal{E}_q\right)$

$$G \cdot \mathcal{G} = 0$$

3. By this gauge we can fix additional spins corresponding to syndromes arriving at

$$S = -\beta \sum_{b,\tau} \prod_x S_{x,\tau}^{G_{b,x}} - \beta_\tau \sum_{x,\tau} S_{x,\tau+1} S_{x,\tau}$$

4. The quantum Hamiltonian can be readily written:

$$\boxed{\hat{H} = -\sum_m X_m - \lambda \sum_b \prod_m Z_m^{G_{b,m}}}$$

$$\tau/\hbar = \exp(-2\beta_\tau)$$
$$\beta = \lambda\tau/\hbar$$

Mapping for space-time codes

1. Anisotropic coupling for temporal and spatial bonds:

$$\mathcal{G} \equiv \begin{pmatrix} R^T \otimes \widetilde{\mathcal{E}}_q, & \widetilde{\mathcal{E}}_r \otimes H^T \\ \mathcal{E}_r \otimes \widetilde{H}, & 0 \end{pmatrix}, \qquad R=\begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$$

temporal $I_b = \beta_\tau$ spatial $I_b = \beta$

$$S = -\sum_b I_b \prod_r S_r^{\mathcal{G}_{r,b}}$$

2. To fix the gauge we find the gauge matrix: $G = \left(\widetilde{\mathcal{E}}_r \otimes \widetilde{H}, R^T \otimes \mathcal{E}_q\right)$

$$G \cdot \mathcal{G} = 0$$

3. By this gauge we can fix additional spins corresponding to syndromes arriving at

$$S = -\beta \sum_{b,\tau} \prod_x S_{x,\tau}^{G_{b,x}} - \beta_\tau \sum_{x,\tau} S_{x,\tau+1} S_{x,\tau}$$

4. The quantum Hamiltonian can be readily written:

$$\boxed{\hat{H} = -\sum_m X_m - \lambda \sum_b \prod_m Z_m^{G_{b,m}}}$$

$$\tau/\hbar = \exp(-2\beta_\tau)$$
$$\beta = \lambda\tau/\hbar$$

Conclusions

- We suggest to study the decoding transition for quantum LDPC codes by employing the mapping to random bond Wegner models
- This mapping also works in the case when errors in syndrome measurements are present (e.g. 3D random plaquette gauge model for toric code)
- Obtained spin models can be a source of new physics associated with phase transitions in non-local models and spin glass phase
- In the context of QECCs, this map allows to define an absolute upper limit for error threshold for a given code and for a given decoder thus one can talk about decoder efficiency
- In progress:

  1) Numerical calculations of error thresholds in the presence of errors in syndrome measurements

  2) Detailed phase diagrams by employing generalizations of cluster Wolff algorithm

Conclusions

- We suggest to study the decoding transition for quantum LDPC codes by employing the mapping to random bond Wegner models
- This mapping also works in the case when errors in syndrome measurements are present (e.g. 3D random plaquette gauge model for toric code)
- Obtained spin models can be a source of new physics associated with phase transitions in non-local models and spin glass phase
- In the context of QECCs, this map allows to define an absolute upper limit for error threshold for a given code and for a given decoder thus one can talk about decoder efficiency
- In progress:

  1) Numerical calculations of error thresholds in the presence of errors in syndrome measurements

  2) Detailed phase diagrams by employing generalizations of cluster Wolff algorithm