Title: Maximum likelihood decoding threshold as a phase transition

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Abstract: In maximum likelihood (ML) decoding, we are trying to find the most likely error given the measured syndrome. While this is hardly ever practical, such a decoder is expected to have the highest threshold.
I will discuss the mapping between the ML threshold for an infinite family of stabilizer codes and a phase transition in an associated family of Ising models with bond disorder [1]. This is a generalization of the map between the toric codes and the square lattice Ising model. Quantum LDPC codes produce generally non-local spin models with few-body interactions. A relatively simple Monte Carlo simulation of such a model can give an upper bound on the decoding threshold for the original code family. This can be used to compare code families irrespectively of decoders, and to establish an absolute measure of decoder performance.

[1] A. A. Kovalev and L. P. Pryadko, "Spin glass reflection of the decoding transition for quantum error correcting codes," unpublished,
arXiv:1311.7688 (2013).

Maximum likelihood decoding threshold as a phase transition

Leonid Pryadko

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July 14, 2014

- Introduction: How it works for the surface codes
- Threshold for Q-LDPC codes with power-law distance
- ML decoding & nonlocal spin models
- Conclusions and open problems

Ilya Dumer (UCR)

Alexey Kovalev (UNL)

arXiv:1208.2317

arXiv:1311.7688

and some new work

Decoding threshold

Decoding threshold p_c: Consider an infinite family of error correcting codes. With probability p for independent errors per (qu)bit, at $p < p_c$, a large enough code can correct all errors with success probability $P \to 1$, but not at $p > p_c$

Example: code family with **finite relative distance** $\delta = d/n$. A code can detect any error involving $w < d$ (qu)bits, and distinguish between any two errors involving $w < d/2$ qubits each. For such a family, $p_c \geq \delta/2$.

The actual value of p_c depends on the decoding algorithm.

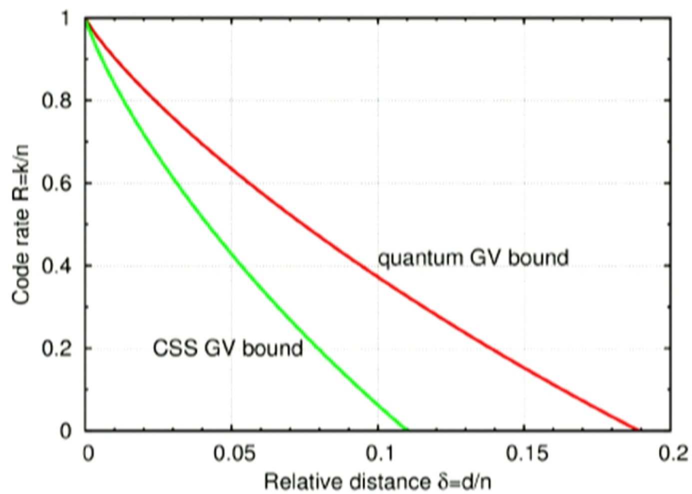
Can we interpret a decoding threshold as a phase transition?

Quantum stabilizer codes

- An $[[n,k,d]]$ stabilizer code $\mathcal{Q}$ is a 2^k-dimensional subspace of the n-qubit Hilbert space $\mathcal{H}_2^{\otimes n}$, a common eigenspace of operators in an Abelian stabilizer group $\mathscr{S} = \langle g_1, \ldots, g_{n-k} \rangle$, $-\mathbf{1} \notin \mathscr{S}$: $\mathcal{Q} \equiv \{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \forall S \in \mathscr{S}\}$.
  - Distance d is minimum weight of a non-trivial operator $E \notin \mathscr{S}$ which commutes with the stabilizer $\mathscr{S}$.
  - Such a code can detect any error affecting up to $d-1$ qubits, and correct any error affecting up to $t \equiv \lfloor d/2 \rfloor$ qubits.

Gilbert-Varshamov (GV) bound: there exist stabilizer codes with rates $R \equiv k/n$, $R > 1 - H_2(\delta) - \delta \log_2 3$, $\delta \equiv d/n$.

Calderbank-Shor-Steane (CSS) codes: stabilizer generators formed by either only X or only Z operators. GV bound: $R > 1 - 2H_2(\delta)$.

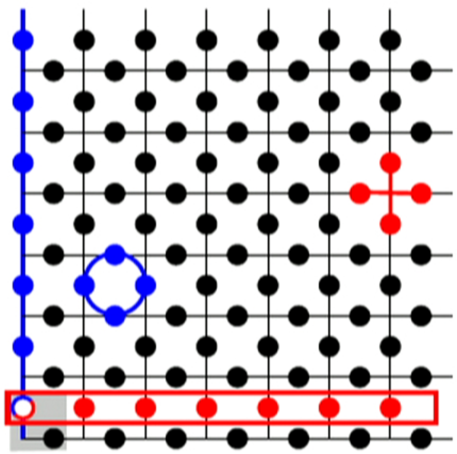


3 **GV bound not of much use for quantum LDPC codes...**

Surface codes

Family of codes invented by Alexey Kitaev (orig: *toric* codes)

Stabilizer generators: plaquette $A_\square = ZZZZ$ and vertex $B_+ = XXXX$ operators (this is a CSS code).



$_4$toric code [[98, 2, 7]]

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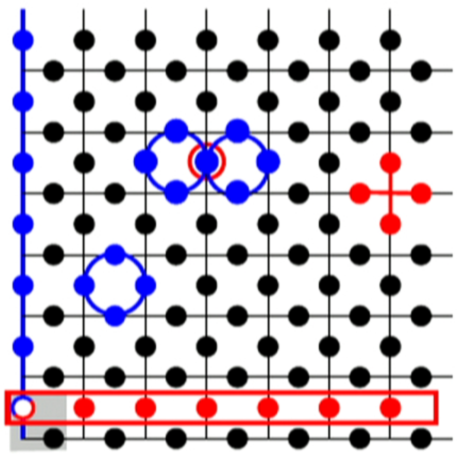

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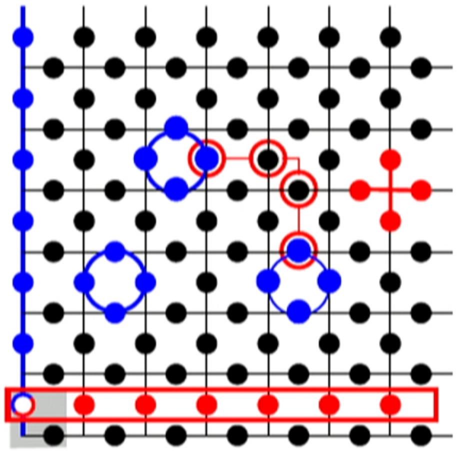

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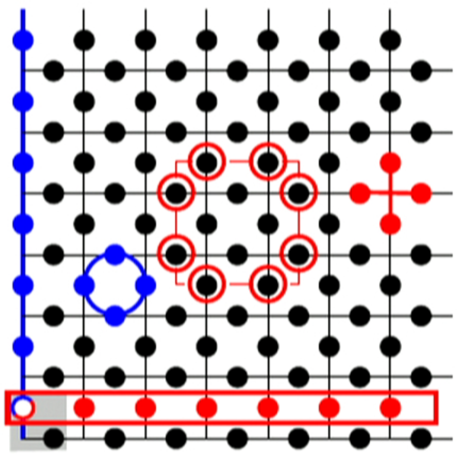


toric code $[[98, 2, 7]]$

Undetectable error: only closed chains

Trivial undetectable error: topologically trivial loops

Bad undetectable error: topologically non-trivial loop $\Rightarrow$ Code distance $d = L \propto \sqrt{n}$.

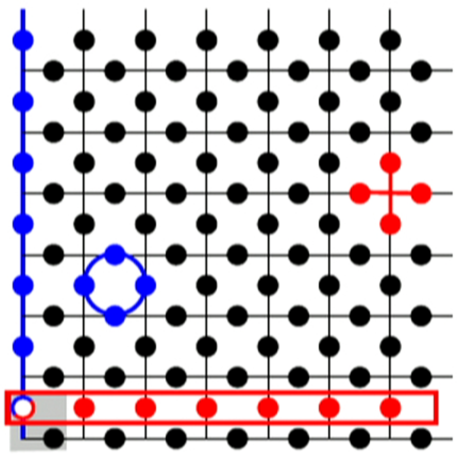
$[[n = 2L^2, k = 2, d = L]]$

Surface codes: finite decoding threshold

Distance scales as $d \propto n^{1/2}$, meaning zero relative distance $\delta \propto n^{-1/2}$, $n \to \infty$. Is there a finite decoding threshold?

Yes! [Dennis, Kitaev, Landahl & Preskill, 2002]

- Counting topologically non-trivial chains
- Mapping to the Ising model with bond disorder



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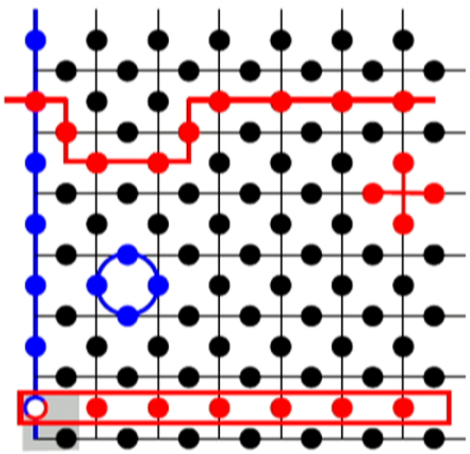

toric code $[[98, 2, 7]]$

Prob. of an undetectable chain of length d:

$$Q_d(x) \leq p^d \#(\mathrm{SAW}_d) \leq (3p)^d$$

Uncorrectable error: such a chain more than half-filled with errors. Probability:

$$P_d \leq \#(\mathrm{SAW}_d) \sum_{m \leq \lfloor d/2 \rfloor} \binom{d}{m} p^m (1-p)^{d-m}$$

$$P_d(x) \leq 3^d \times 2^d [p(1-p)]^{d/2}$$

Neither happens at sufficiently small p!

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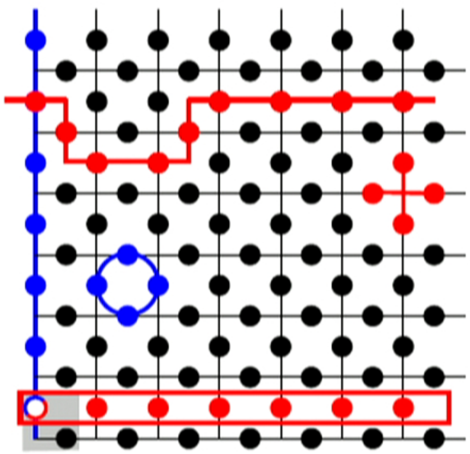

${}_5$toric code $[[98, 2, 7]]$

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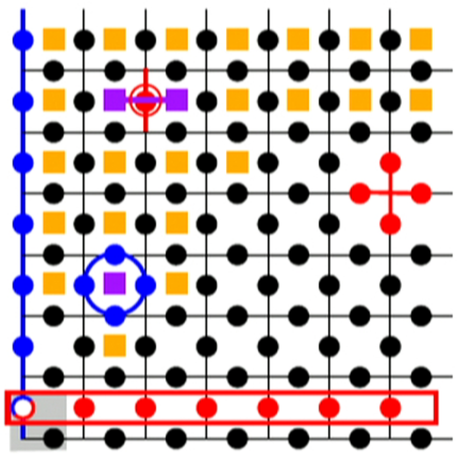


toric code $[[98, 2, 7]]$

Conditional probability of an error in a given sector given the syndrome **s** = sum over loop configurations on top of a representative error **e**.

$\to$ partition function of a **bond disordered Ising model** on dual lattice

ML decoding transition $\Leftrightarrow$ phase transition along the Nishimori line

More general quantum LDPC codes

- All codes with local stabilizer generators in 2D satisfy $kd^2 = \mathcal{O}(n)$ [Bravyi et al, 2010]
- General quantum LDPC codes: remove requirement of locality

Construction (CSS): encode generators g_i in binary matrices.
Commutativity condition: $G_x G_z^T = 0 \bmod 2$

Example

Ansatz [Tillich & Zemor, 2009]: use two binary matrices H_1, H_2

$$G_x = (E^{(r_2 \times r_2)} \otimes H_1^{(r_1 \times n_1)}, H_2^{(r_2 \times n_2)} \otimes E^{(r_1 \times r_1)})$$
$$G_z = (H_2^T \otimes E^{(n_1 \times n_1)}, \qquad E^{(n_2 \times n_2)} \otimes H_1^T)$$

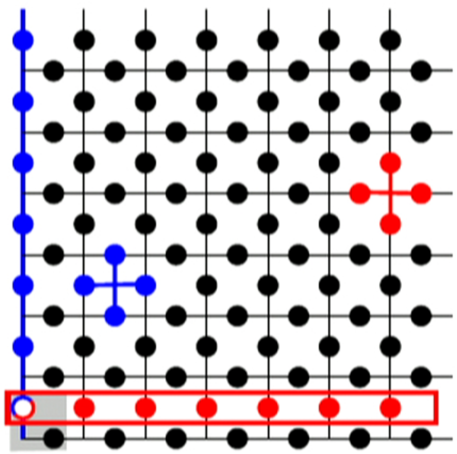
If the binary codes with check matrices H_i, H_i^T, $i = 1, 2$ have parameters $[n_i, k_i, d_i]$ and $[\tilde{n}_i, \tilde{k}_i, \tilde{d}_i]$ respectively, the CSS code has $[[n = n_1 r_2 + n_2 r_1, k = k_1 \tilde{k}_2 + \tilde{k}_1 k_2, d]]$, where $\min(d_i, \tilde{d}_i) \leq d \leq \max(d_i, \tilde{d}_i) \quad \rightarrow \quad$ finite k/n, $d \propto n^{1/2}$

Quantum hypergraph-product codes

- Finite-rate generalization of surface codes; distances $d \propto n^{1/2}$
  - Small-weight generators g_i, can be measured
  - Can be measured in parallel

- **Price to pay: non-local stabilizer generators**

$H_1 = H_2^T$: 7×7 circulant $h(x) = 1 + x$.



toric code $[[98, 2, 7]]$

Tillich & Zémor '09

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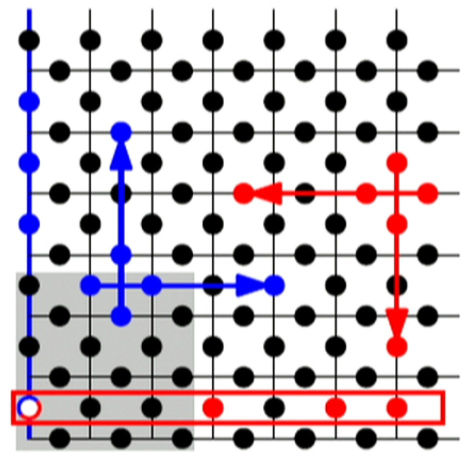

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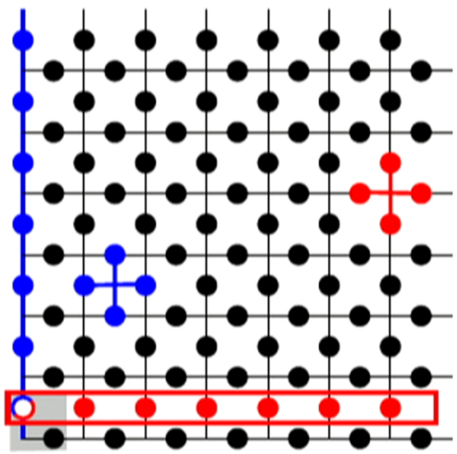

$_7$toric code $[[98, 2, 7]]$

Tillich & Zémor '09

Given a layout, it is easy to come up with a spin model map. . . **but does it have a transition?**

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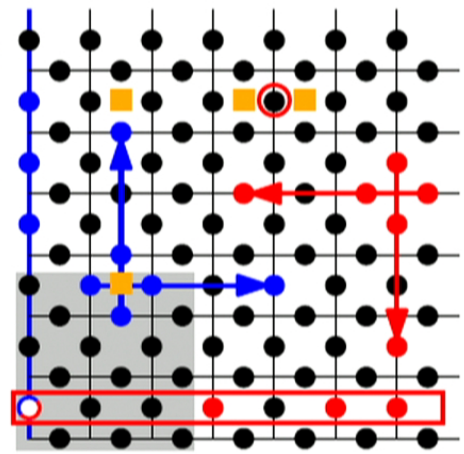

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Threshold theorem and sparse-graph codes

A general code of distance d can detect all errors of weight $w < d$ and correct all errors of weight up to $t = \lfloor d/2 \rfloor$ (this guarantees that any two errors will take $\mathcal{Q}$ to mutually orthogonal spaces.)

A finite per-qubit error probability p typically generates errors of weight $\sim pn$. "Good-distance" code families have finite d/n and thus can correct errors with high likelihood up to a threshold p_c.

Problem: no (limited-weight) quantum LDPC codes are known with good asymptotic distance. At best, $d \propto n^{1/2}$.

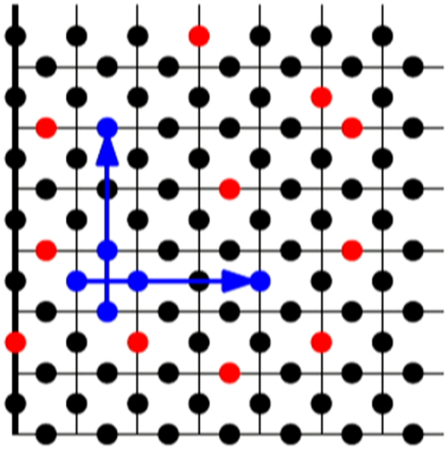
Tillich & Zémor 2009
Andriyanova et al. 2012
...

Can such codes correct errors in a finite fraction of qubits?

8

Threshold theorem and sparse-graph codes (cont'd)

- Start with a small per-qubit error probability $p \ll 1$.
- Connect errors affecting common generators. For small p and a sparse code these form small disconnected clusters



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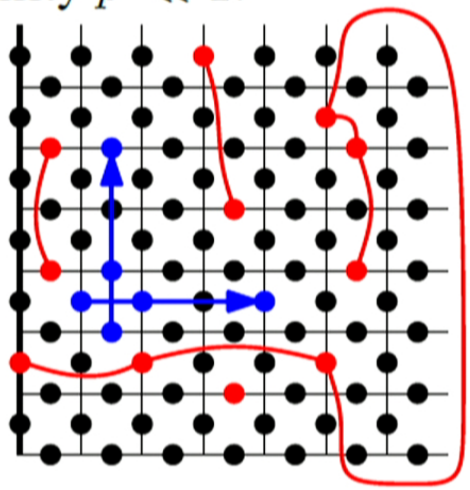

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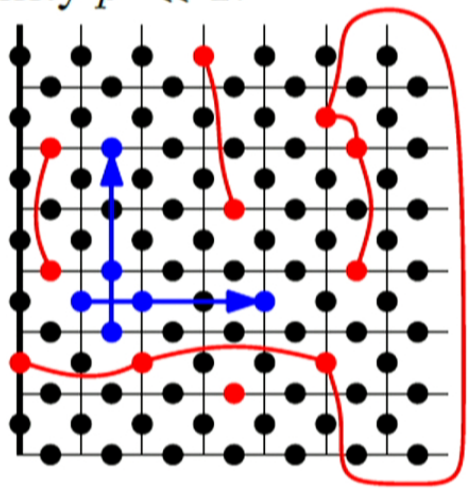

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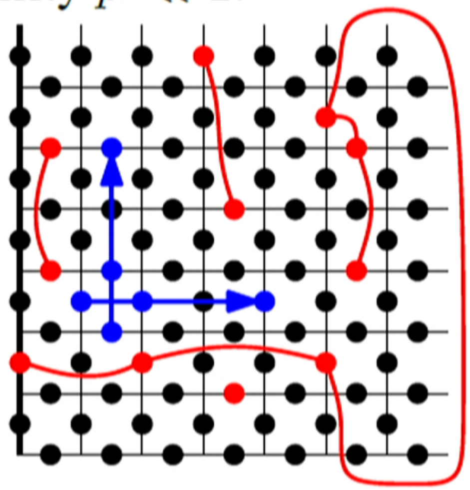

This implies that errors formed by clusters of weight $w < d$ are all detectable

- Below percolation limit p_c, probability to have a cluster of large weight w is exponentially small with w.
- Maximum cluster size grows logarithmically with n (for small enough p this is also true for confusing half-filled clusters)

Conclusion: as long as $d \propto n^{\alpha}$, $\alpha > 0$ (or even logarithmic), a sparse-graph code can correct errors at finite p. [Kovalev & LPP, '13]

9

Threshold theorem for quantum LDPC codes

Kovalev & LPP '13

Theorem: For an infinite family of (j, ℓ)-limited LDPC codes, quantum or classical, where the distance d scales as a power law at large n, asymptotically certain recovery is possible for (qu)bit *depolarizing* probabilities $p < p_d$, where $p_d \geq [2e(z-1)]^{-2}$, and e is the base of the natural logarithm. A threshold $p_d > 0$ also exists for code families with distance scaling logarithmically at large n.

$z \equiv j(\ell - 1)$.

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Recent progress: **stronger CSS bound**
$p_d \geq [2\gamma(\ell - 1)]^{-2}$, $\gamma \approx 1.2$

Proof steps:

- A **codeword** should satisfy all associated checks. To make a particular check happy, we need to choose from $\ell - 1$ positions, thus $\# \leq (\ell - 1)^w$ optimal clusters of size w.
- Configurations where decoding could fail: codewords of size $w \geq d$, where errors occupy half or more qubits.

Decoding transition

Consider an infinite family of stabilizer LDPC codes with rate $R = k/n$ and decoding probability $\overset{n\to\infty}{\to} 1$ for $p < p_c$ but not above p_c.

Do we have a phase transition at p_c?

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Consider maximum-likelihood (ML) decoding, given error syndrome $\mathbf{s} = G_x \mathbf{e} \bmod 2$, $\mathbf{e} \equiv \mathbf{e}_z$

Trivial errors: combination of rows of G_z, $\boldsymbol{\alpha} G_z$
Degeneracy: $\mathbf{e}_1 \simeq \mathbf{e}_2$ iff $\mathbf{e}_1 = \mathbf{e}_2 + \boldsymbol{\alpha} G_z \bmod 2$.

Non-trivial codewords $\mathbf{c} \not\simeq \mathbf{0}$: satisfy $G_x \mathbf{c} = 0 \bmod 2$

Degeneracy: need to find error up to degeneracy class.
Net probability: $P_0(\mathbf{e}) = \sum_{\boldsymbol{\alpha}} \mathrm{Prob}(\mathbf{e} + \boldsymbol{\alpha} G_z)$

Choose among competing $P_{\mathbf{c}}(\mathbf{e}) \equiv P_0(\mathbf{e} + \mathbf{c})$ for all 2^k inequivalent codewords $\mathbf{c}$.

Spin model for the ML decoding threshold

Take $\mathrm{Prob}(\mathbf{e}) = p^w (1-p)^{n-w}$, $w \equiv \mathrm{wt}(\mathbf{e})$

Nishimori temperature $T = 1/\beta_p$, $e^{-2\beta_p} = p/(1-p)$.

Define Ising partition function [Wegner '71]:
$\mathcal{Z}_0(\mathbf{e}, \Theta; \beta) = A\ \mathrm{Tr}_{\mathbf{S}} \exp(-\beta \sum_b (-1)^{e_b} R_b)$,

$$R_b \equiv \prod_j S_j^{\Theta_{jb}},\ S_j = \pm 1$$

We have $P_0(\mathbf{e}) = \mathcal{Z}_0(\mathbf{e}, G_z; \beta = \beta_p)$, $P_{\mathbf{c}}(\mathbf{e}) = P_0(\mathbf{e} + \mathbf{c})$

- Correspondence:
  - Multi-spin bond per col of G_x
  - Each non-zero bit of $\mathbf{e}$: flipped bond.
  - Decoding transition along the Nishimori line, $\beta = \beta_p$.
  - Codeword $\mathbf{c}$: extended post-topological defect

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Features of the obtained models

- Applicability:
  - Mapping works for any stabilizer code
  - Could be defined for any error model
  - Useful for LDPC codes (limited bond size)
- Typical features
  - Non-locality
  - No local order parameter
  - Wegner self-duality at $p = 0$ [$\equiv$ MacWilliams identities]
  - Non-degenerate ground state
- Yet if the original code has a decoding threshold $p_c > 0$, there is a phase transition from an "ordered" (decodable) phase at p, T small, to a "disordered" phase at p and/or T large
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13

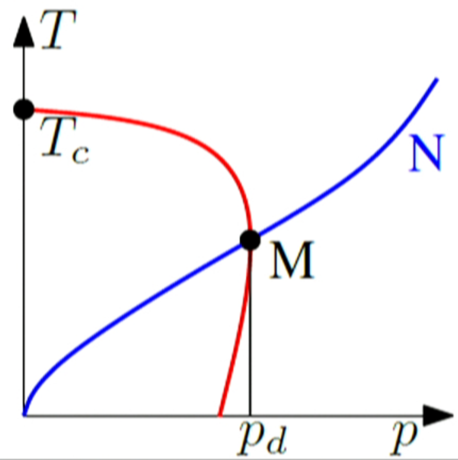
Results summary

Ordered phase: $\dfrac{P_0(\mathbf{e})}{P_{\text{tot}}(\mathbf{e})} \equiv \dfrac{P_0}{P_0 + P_{\mathbf{c}_1} + \ldots + P_{\mathbf{c}_{2^k-1}}} \overset{n\to\infty}{\to} 1$ a.s.

Analog of line tension $\lambda_{\mathbf{c}} = \dfrac{1}{d_{\mathbf{c}}} \left\langle \ln \dfrac{P_0(\mathbf{e})}{P_{\mathbf{c}}(\mathbf{e})} \right\rangle_{\mathbf{e}}$

- In ordered phase $\langle \lambda_{\mathbf{c}} \rangle_{\mathbf{c}\neq\mathbf{0}} \geq RT \ln 2$. Transition mechanisms:
  - $R = 0$: $\lambda_{\mathbf{c}} \to 0$, defect proliferation by **vanishing tension**.
  - $R \neq 0$: either $\lambda_{\mathbf{c}} \to 0$, or with all $\lambda_{\mathbf{c}}$ finite, defect proliferation driven by the **entropy of defect types**.

- Nishimori gauge theory of spin glass:
  - Energy known exactly along the Nishimori line (**N**).
  - Specific heat finite along **N**.
  - Multicritical point at $p_d(\mathbf{N})$
  - No ordered phase at $p > p_d(\mathbf{N})$.



Indicator correlation functions

Define the spin correlation functions:

$$Q_{\mathbf{c}}^{\mathbf{m}}(\mathbf{e}) \equiv [Z_{\mathbf{c}}(\mathbf{e})]^{-1} \operatorname{Tr}_{\mathbf{S}} \prod_b R_b^{m_b} \exp\Big(-\beta(-1)^{e_b} R_b\Big),$$

$$Q_{\text{tot}}^{\mathbf{m}}(\mathbf{e}) \equiv \sum_{\mathbf{c}} (-1)^{\mathbf{c}\cdot\mathbf{m}} \frac{Z_{\mathbf{c}}(\mathbf{e}) Q_{\mathbf{c}}^{\mathbf{m}}(\mathbf{e})}{Z_{\text{tot}}(\mathbf{e})},$$

where $R_b = \prod_j S_j^{\Theta_{jb}}$, $S_j = \pm 1$, and $\Theta = G_z$

- These satisfy $|Q^{\mathbf{m}}| \le 1$.
- Invariant under $\mathbf{m} \to \mathbf{m} + \boldsymbol{\gamma} G_x$, $\mathbf{c} \to \mathbf{c} + \boldsymbol{\alpha} G_z$

Take $\mathbf{m} = \mathbf{b}$ one of the dual codewords $G_z \mathbf{b} = 0$ [cf. $G_x \mathbf{c} = 0$]

- Then $Q_{\mathbf{c}}^{\mathbf{m}}(\mathbf{e}) = 1$ for any codeword $\mathbf{c}$.
- If only one of $Z_{\mathbf{c}}(\mathbf{e})$ is dominant, $Q_{\text{tot}}^{\mathbf{m}} = (-1)^{\mathbf{c}\cdot\mathbf{m}}$
- Use these to identify defect-free phase with dominant $\mathbf{c} = \mathbf{0}$

Can we get a simpler expression for $Q_{\text{tot}}^{\mathbf{m}}$?

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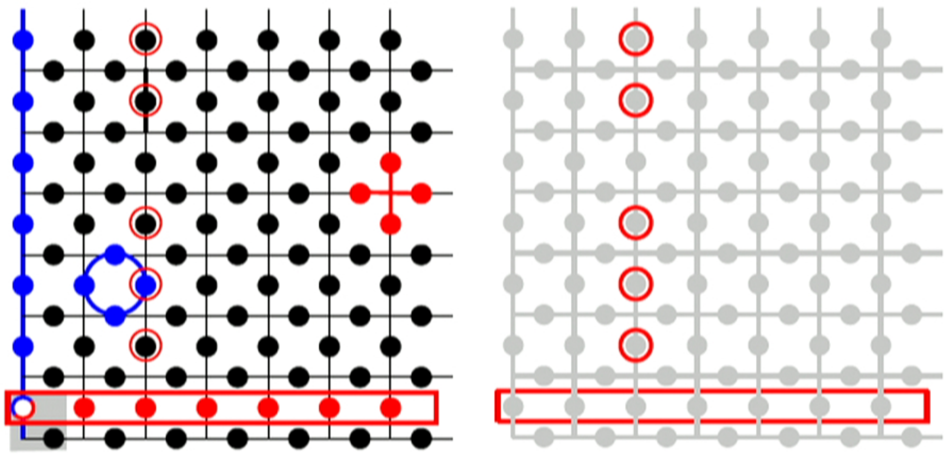
Exact Wegner-style duality

$$Q^{\mathbf{m}}_{\text{tot}}(\Theta = G_z, \beta; \mathbf{e}) = (-1)^{\mathbf{e}\cdot\mathbf{m}} Q^{\mathbf{e}}_{\mathbf{0}}(\Theta = G_x, \beta^*; \mathbf{m}),$$

where $\tanh \beta = e^{-2\beta^*}$ $\Rightarrow$ the indicator functions can be computed by a MC simulation at a higher temperature...

This is potentially doable for LDPC codes, except for the minus sign problem...



16 toric code $[[98, 2, 7]]$

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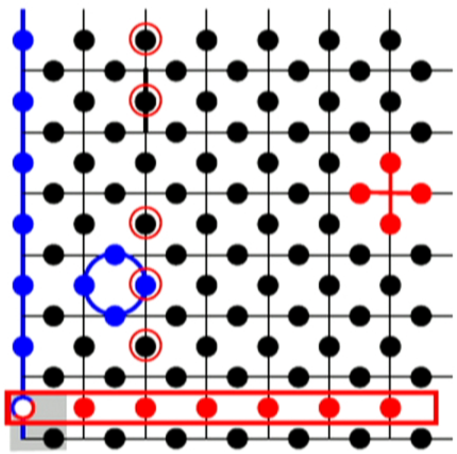

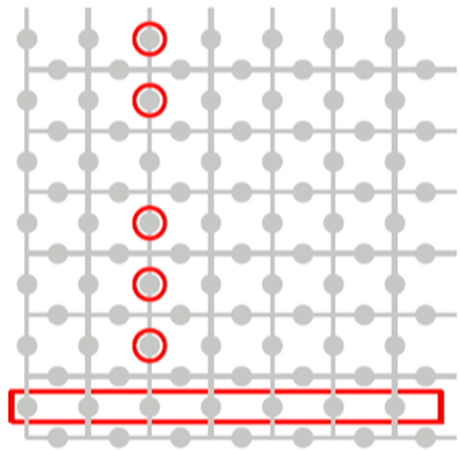

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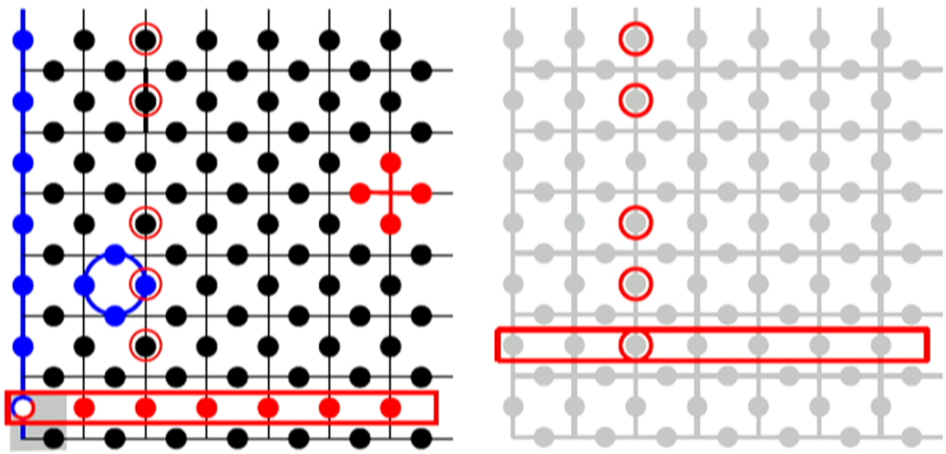

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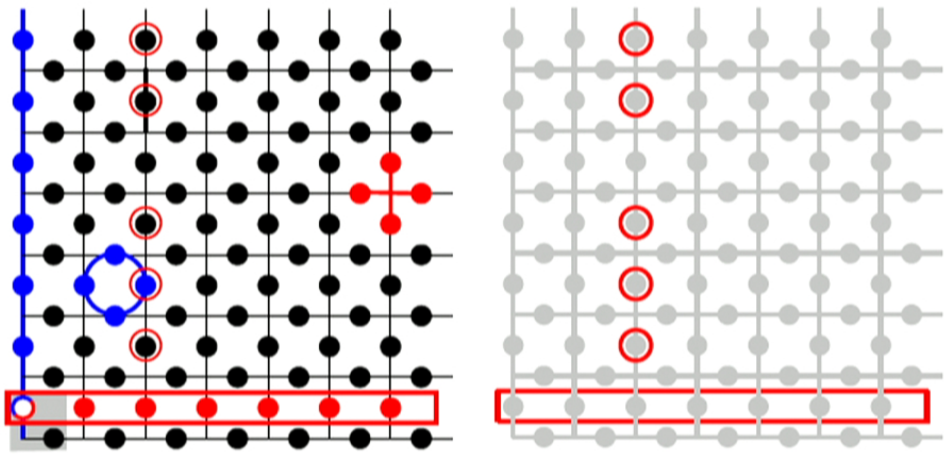

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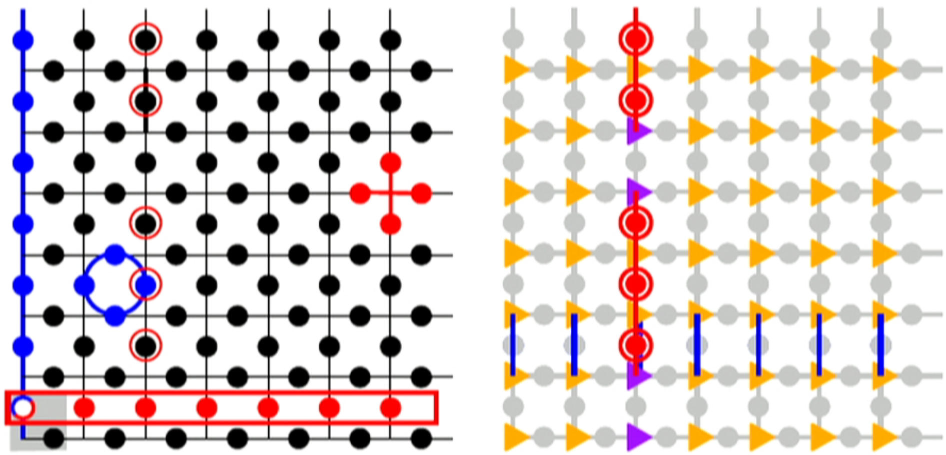

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Lattice: rows of G_x

Flip bonds from $\mathbf{m}$

Average product of bonds from $\mathbf{e}$

Summary

- LDPC stabilizer codes with **limited-weight** generators have a finite decoding threshold (also in FT setting, see next talk)
- ML decoding threshold of **any stabilizer code** corresponds to a multicritical point in an associated bond-disordered spin model
  - These models are interesting in their own right . . .
  - They also seem to suggest that decoding should not be so hard since defects' free energies are large. . .
- A relatively inexpensive Monte-Carlo simulation can be used to:
  - Establish threshold for a given code family, independent of decoder
  - Give an absolute measure of decoder performance
  - And (possibly) even help with decoding. . .
- **Many open questions in theory of quantum LDPC codes, inluding:**
  - Is there a fast general-purpose decoder approaching ML threshold?
  - Any families of limited-weight stabilizer codes with finite $\delta \equiv d/n$?
  - Any tight bounds on parameters of quantum LDPC codes?
  - . . .