Title: Overview of the theory of spin glasses and its applications to quantum codes

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Abstract: I will review the theory of spin glasses with an emphasis on gauge symmetry. A number of exact results will be shown to be derived, some of which are useful to discuss the properties of quantum LDPC codes. Also will be explained is the combination of gauge symmetry, replica method, and duality argument to predict the precise location of a multicritical point, which is equivalent to the error-tolerance limit of toric code.

Overview of the theory of spin glasses and its applications to quantum codes

Hidetoshi Nishimori
Tokyo Institute of Technology

Overview of the theory of spin glasses

Hidetoshi Nishimori
Tokyo Institute of Technology

My theory of spin glasses

Hidetoshi Nishimori
Tokyo Institute of Technology

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My theory of spin glasses+

Hidetoshi Nishimori
Tokyo Institute of Technology

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Collaborators

Koji Nemoto

Koujin Takeda

Jean-Marie Maillard

Tomohiro Sasamoto

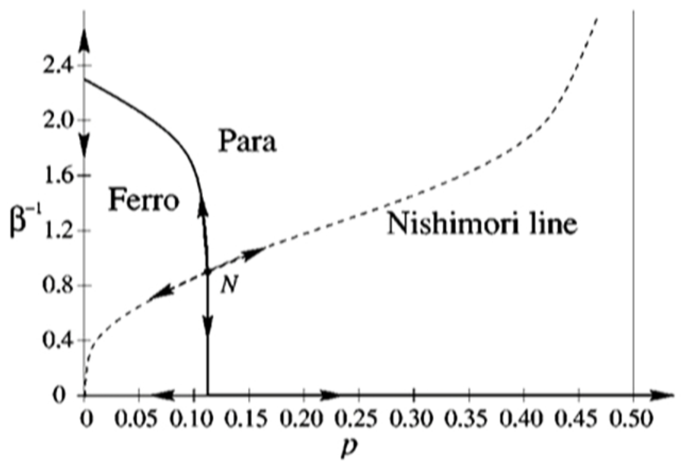
Masayuki Ohzeki

A. Nihat Berker

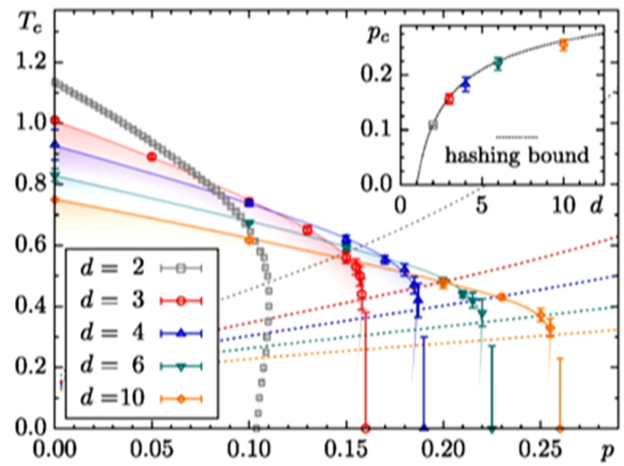
David Sherrington

PART 0
Motivation & goal

Error threshold of surface codes



Dennis, Kitaev, Landahl, Preskill (2002)



Andrist, Wootton, Katzgraber (2014)

cf. Duclos-Cianti and Poulin (2013), Anwar *et al* (2014)

PART 1
Introduction

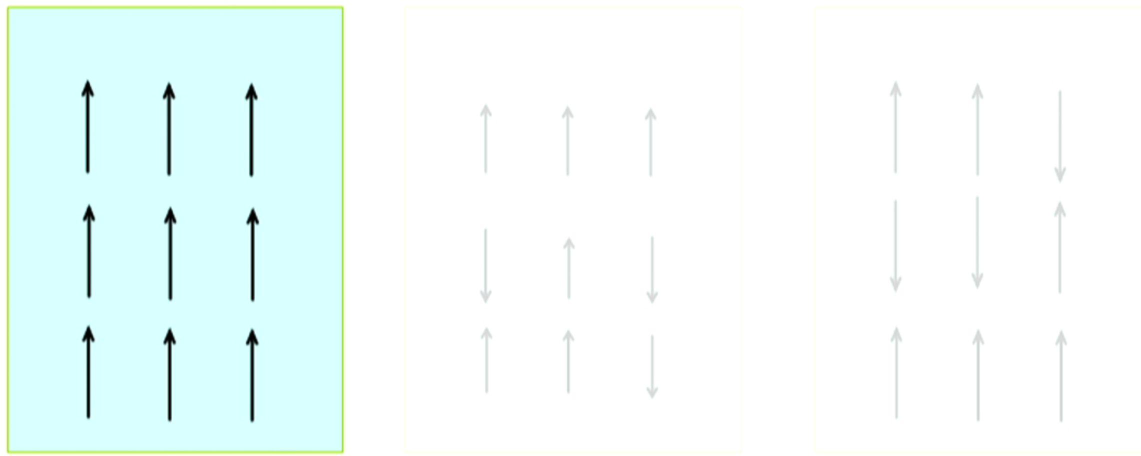
	Ferromagnetic	Paramagnetic	Spin glass

Time :	ordered	random	ordered
Space :	ordered	random	random

PART 1

Introduction

	Ferromagnetic	Paramagnetic	Spin glass



	Ferromagnetic	Paramagnetic	Spin glass
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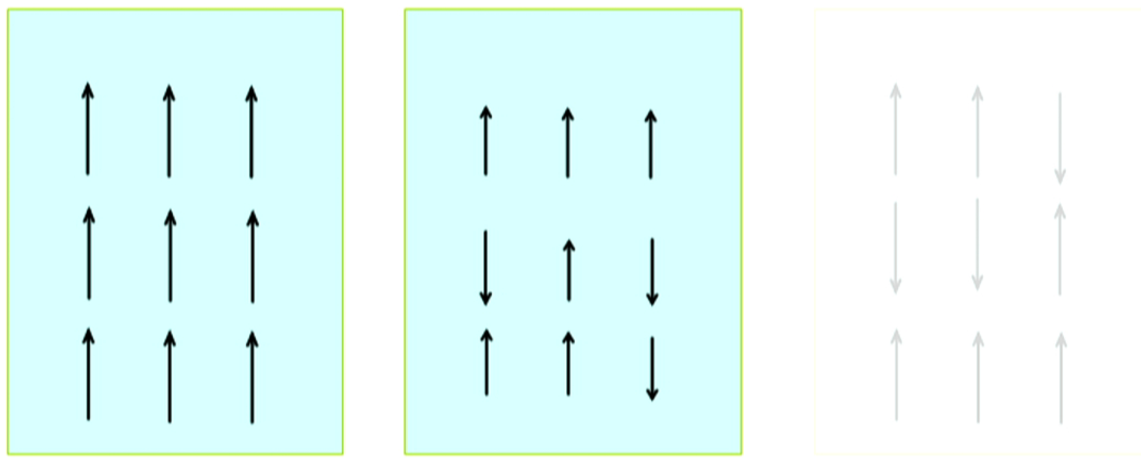
PART 1

Introduction

Ferromagnetic | Paramagnetic | Spin glass



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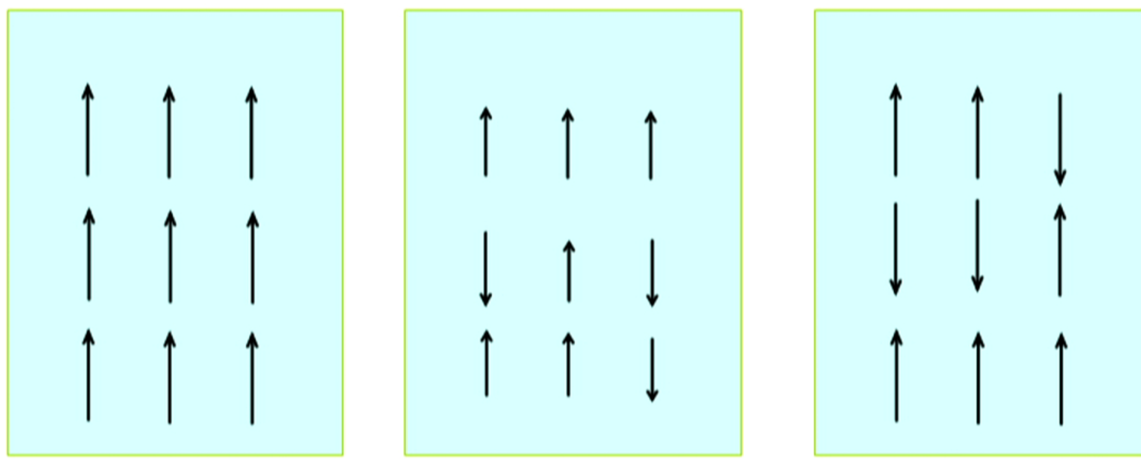
PART 1

Introduction

	Ferromagnetic	Paramagnetic	Spin glass



Time :	ordered	random	ordered
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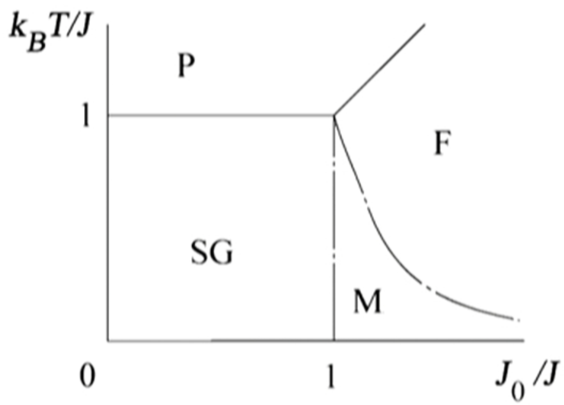
Mean-field theory

Sherrington-Kirkpatrick model

$$H = -\sum_{i>j} J_{ij} S_i S_j \,, \quad P(J_{ij}) \propto e^{-\frac{(J_{ij}-J_0)^2}{2J^2}}$$

Parisi Solution with replica symmetry breaking (RSB)

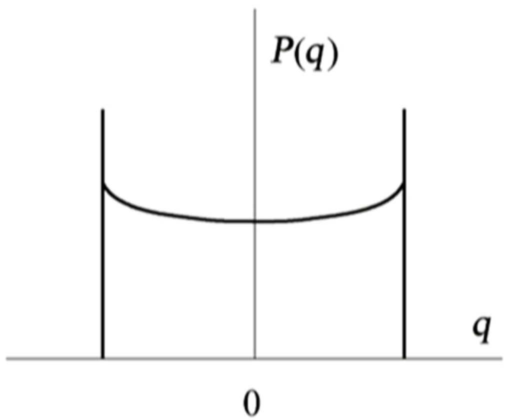


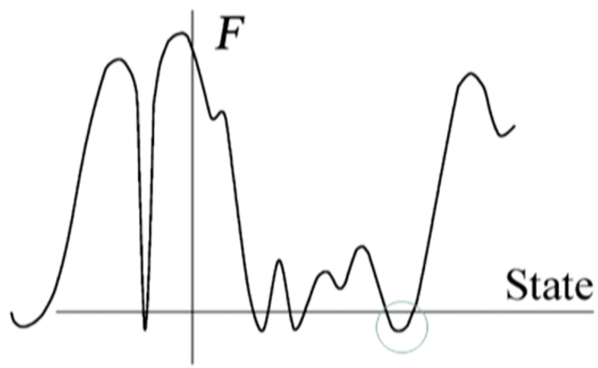
Replica symmetry breaking

Order parameter

$$q^{\alpha\beta} = \left[\left\langle S_i^{\alpha} S_i^{\beta} \right\rangle\right]$$

Continuous distribution of *q* and rugged energy landscape





What about finite dimensions?

Questions

1. Existence (or not) of the spin-glass phase
2. Properties of the spin-glass phase: Mean-field like or not?

What about finite dimensions?

Questions

1. Existence (or not) of the spin-glass phase
2. Properties of the spin-glass phase: Mean-field like or not?

Suggestion (mainly from simulations)

1. Yes for $d > 2$.
2. No for $2 < d < 6$; Yes for $d > 6$.

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PART 2
Gauge symmetry

Problem setting

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$$

$\pm J$ model

$$P(J_{ij}) = p\delta(J_{ij} - 1) + (1 - p)\delta(J_{ij} + 1)$$

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PART 2

Gauge symmetry

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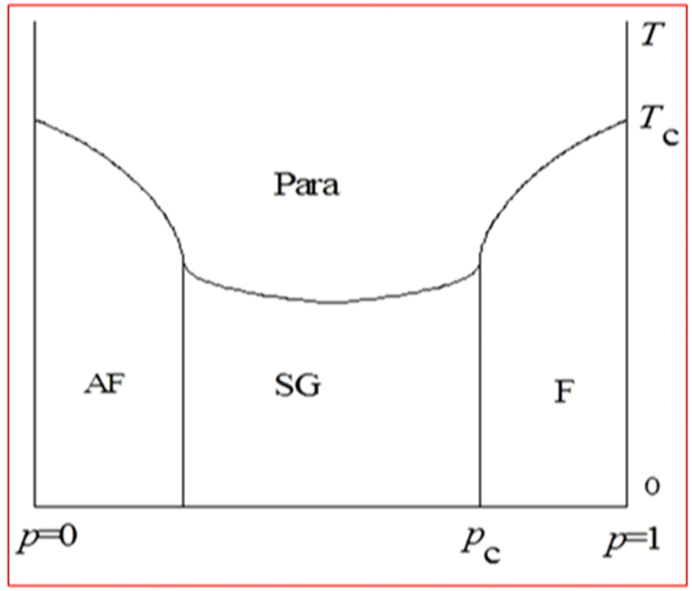

$\pm J$ model

$$P(J_{ij}) = p\delta(J_{ij} - 1) + (1 - p)\delta(J_{ij} + 1)$$

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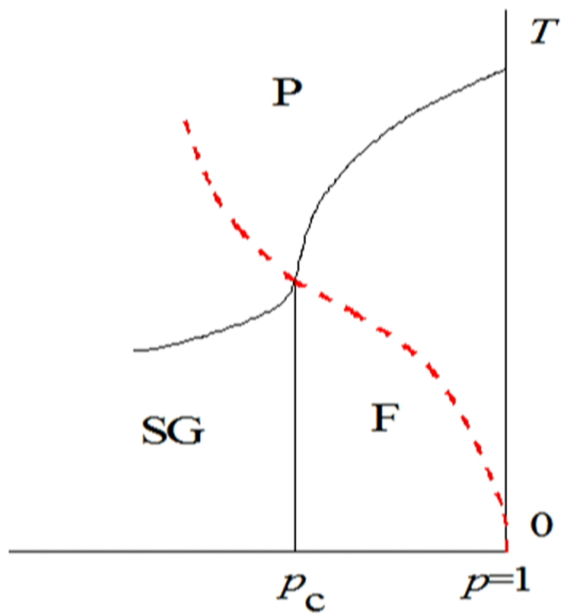
Results : condition



$$\exp(-2\beta) = \frac{1-p}{p}$$

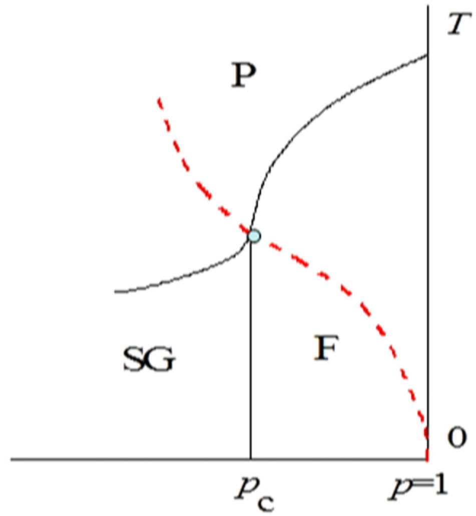
$$\beta = \frac{1}{T}$$

Results

H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981)

Exact energy $$[\langle E \rangle] = -N_B \tanh \beta$$

No singularity across the MCP



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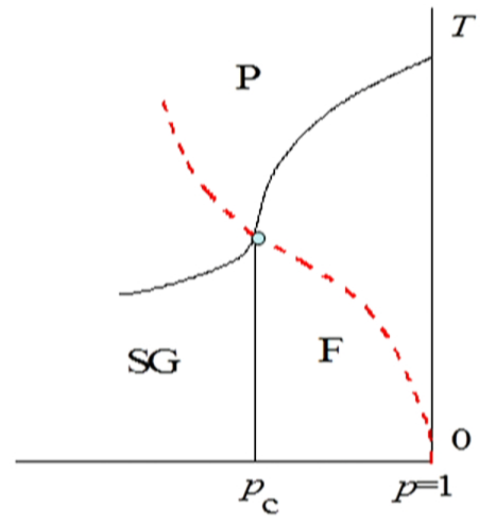
No singularity across the MCP



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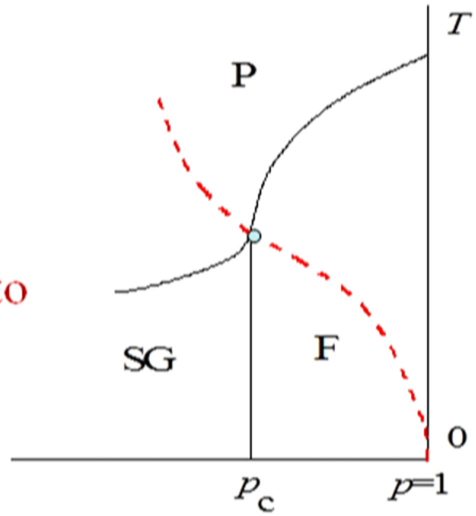
Referee comment:

"The model is artificial and the result is thin to warrant publication."



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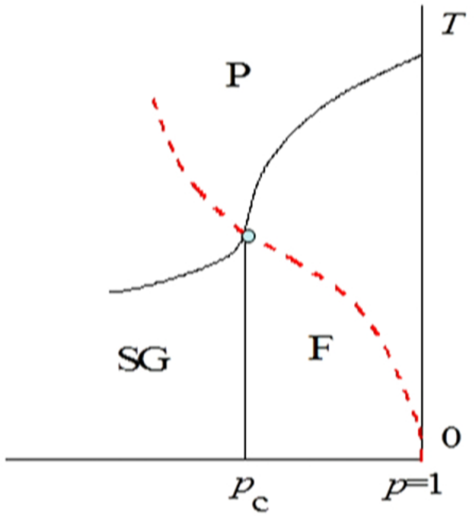
Specific heat $$T^2[\langle C\rangle] \le N_B \mathrm{sech}^2 \beta < \infty$$

Weak singularity at MCP



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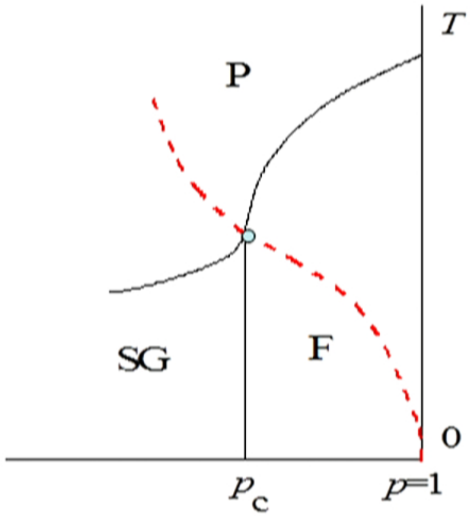


Specific heat $T^2\left[\langle C \rangle\right] \leq N_B \mathrm{sech}^2 \beta < \infty$

Weak singularity at MCP

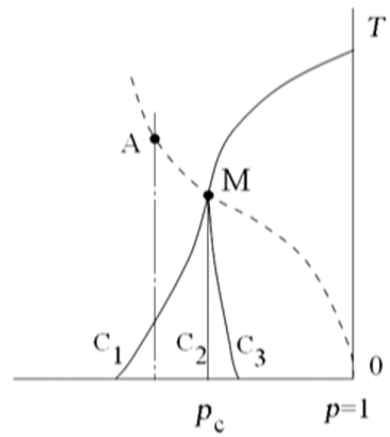
Order parameter $m = q$ $\quad m = \left[\langle S_i \rangle\right] \; q = \left[\langle S_i \rangle^2\right]$

No SG (m=0, q>0) on NL

Implications

Correlation inequality

$$\left|\left[\langle S_i S_j \rangle_\beta\right]_{K_p}\right| \le \left[\left|\langle S_i S_j \rangle_{K_p}\right|\right]_{K_p}$$



C_1 forbidden

Proof: $\left[\langle E\rangle\right] = -N_B \tanh\beta$

$$\left[\langle E\rangle\right] = \left[\frac{\sum_S He^{-\beta H}}{\sum_S e^{-\beta H}}\right] = \sum_{\{J_{ij}=\pm 1\}}\left(\prod_{\langle ij\rangle}\frac{e^{K_p J_{ij}}}{e^{K_p}+e^{-K_p}}\right)\frac{-\partial_\beta Z}{Z}$$

$$J_{ij} = 1: \ \frac{e^{K_p}}{e^{K_p}+e^{-K_p}} = p$$

$$J_{ij} = -1: \ \frac{e^{-K_p}}{e^{K_p}+e^{-K_p}} = 1-p$$

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$$\sum_{\{J_{ij}=\pm 1\}} \left(\prod_{\langle ij \rangle} \frac{e^{K_p J_{ij}}}{e^{K_p} + e^{-K_p}} \right) \frac{-\partial_\beta Z}{Z}$$

$$= \sum_{\{J_{ij}=\pm 1\}} \frac{e^{K_p \sum_{<ij>} J_{ij}}}{\left(e^{K_p} + e^{-K_p}\right)^{N_B}} \frac{-\partial_\beta Z}{\sum_S e^{\beta \sum_{\langle ij \rangle} J_{ij} S_i S_j}}$$

Gauge transformation (of running variables)

$$S_i \to S_i \sigma_i \quad \forall i \qquad \sigma_i = \pm 1$$

$$J_{ij} \to J_{ij} \sigma_i \sigma_j \quad \forall ij$$

Invariance by $S_i \to S_i\sigma_i,\quad J_{ij} \to J_{ij}\sigma_i\sigma_j$

Sums

$$\sum_{\{J_{ij}=\pm 1\}} \to \sum_{\{J_{ij}=\pm 1\}} \qquad \sum_{\{S_i=\pm 1\}} \to \sum_{\{S_i=\pm 1\}}$$

Hamiltonian

$$-\sum_{\langle ij\rangle} J_{ij}S_iS_j \to -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \cdot S_i\sigma_i \cdot S_j\sigma_j = -\sum_{\langle ij\rangle} J_{ij}S_iS_j$$

Energy

$$[\langle E\rangle] = \sum_{\{J_{ij}=\pm 1\}} \frac{e^{K_p \sum_{<ij>} J_{ij}}}{\left(e^{K_p}+e^{-K_p}\right)^{N_B}} \frac{-\partial_\beta Z}{\sum_S e^{\beta\sum_{\langle ij\rangle} J_{ij}S_iS_j}}$$

$$= \sum_{\{J_{ij}=\pm 1\}} \frac{e^{K_p \sum_{<ij>} J_{ij}\sigma_i\sigma_j}}{\left(e^{K_p}+e^{-K_p}\right)^{N_B}} \frac{-\partial_\beta Z}{\sum_S e^{\beta\sum_{\langle ij\rangle} J_{ij}S_iS_j}}$$

Indep of σ_i

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$$\left[\langle E\rangle\right]=\frac{1}{2^N}\sum_{\{J_{ij}=\pm1\}}\frac{\sum_\sigma e^{K_p\sum_{<ij>}J_{ij}\sigma_i\sigma_j}}{\left(e^{K_p}+e^{-K_p}\right)^{N_B}}\frac{-\partial_\beta Z}{\sum_S e^{\beta\sum_{\langle ij\rangle}J_{ij}S_iS_j}}$$

$K_p=\beta$

$$=\frac{1}{2^N}\sum_{\{J_{ij}=\pm1\}}\frac{-\partial_\beta Z}{\left(e^{\beta}+e^{-\beta}\right)^{N_B}}=\frac{-1}{2^N\left(e^{\beta}+e^{-\beta}\right)^{N_B}}\partial_\beta\sum_{\{S_i=\pm1\}}\sum_{\{J_{ij}=\pm1\}}e^{\beta\sum_{<ij>}J_{ij}S_iS_j}$$

$$=\frac{-1}{2^N\left(e^{\beta}+e^{-\beta}\right)^{N_B}}\partial_\beta\sum_{\{S_i=\pm1\}}\prod_{<ij>}(e^{\beta}+e^{-\beta})$$

$$=-N_B\tanh\beta$$

$$\left[\langle E\rangle\right] = \frac{1}{2^N}\sum_{\{J_{ij}=\pm1\}} \frac{\sum_\sigma e^{K_p\sum_{<ij>} J_{ij}\sigma_i\sigma_j}}{\left(e^{K_p}+e^{-K_p}\right)^{N_B}} \frac{-\partial_\beta Z}{\sum_S e^{\beta\sum_{\langle ij\rangle} J_{ij}S_iS_j}}$$

$K_p=\beta$

$$= \frac{1}{2^N}\sum_{\{J_{ij}=\pm1\}} \frac{-\partial_\beta Z}{\left(e^\beta+e^{-\beta}\right)^{N_B}} = \frac{-1}{2^N\left(e^\beta+e^{-\beta}\right)^{N_B}}\partial_\beta \sum_{\{S_i=\pm1\}}\sum_{\{J_{ij}=\pm1\}} e^{\beta\sum_{<ij>} J_{ij}S_iS_j}$$

$$= \frac{-1}{2^N\left(e^\beta+e^{-\beta}\right)^{N_B}}\partial_\beta \sum_{\{S_i=\pm1\}}\prod_{<ij>}(e^\beta+e^{-\beta})$$

$$= -N_B \tanh\beta$$

$$[\langle E \rangle] = \frac{1}{2^N} \sum_{\{J_{ij}=\pm 1\}} \frac{\sum_\sigma e^{K_p \sum_{<ij>} J_{ij}\sigma_i\sigma_j}}{\left(e^{K_p} + e^{-K_p}\right)^{N_B}} \frac{-\partial_\beta Z}{\sum_S e^{\beta \sum_{\langle ij \rangle} J_{ij} S_i S_j}}$$

$K_p = \beta$

$$= \frac{1}{2^N} \sum_{\{J_{ij}=\pm 1\}} \frac{-\partial_\beta Z}{\left(e^{\beta} + e^{-\beta}\right)^{N_B}} = \frac{-1}{2^N \left(e^{\beta} + e^{-\beta}\right)^{N_B}} \partial_\beta \sum_{\{S_i=\pm 1\}} \sum_{\{J_{ij}=\pm 1\}} e^{\beta \sum_{<ij>} J_{ij} S_i S_j}$$

$$= \frac{-1}{2^N \left(e^{\beta} + e^{-\beta}\right)^{N_B}} \partial_\beta \sum_{\{S_i=\pm 1\}} \prod_{<ij>} (e^{\beta} + e^{-\beta})$$

$$= -N_B \tanh \beta$$

PART 3

Duality and multicritical point of 2d spin glasses

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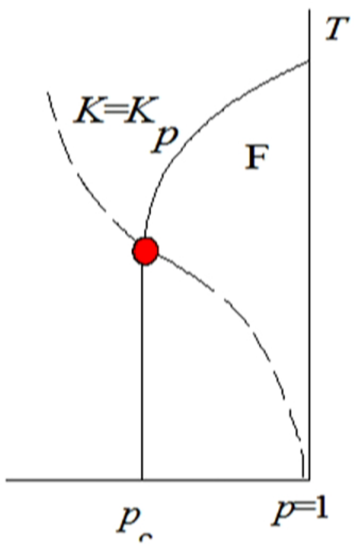
$\pm J$ Ising model on the square lattice



$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$$

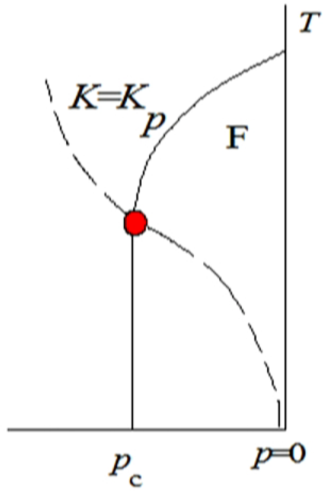
$$-p_c \log p_c - (1-p_c)\log(1-p_c) = \frac{\log 2}{2}$$

$$p_c = 0.889972,\quad 0.110028 (= 1 - 0.889972)$$

Hashing bound of toric code

Nishimori and Nemoto, J. Phys. Soc. Jpn. 71, 1198 (2002)
Maillard, Nemoto, Nishimori, J. Phys. A 36, 9799 (2003)

Potts gauge glass on the square lattice



$$H = -\sum_{\langle ij \rangle} \delta(S_i, S_j + J_{ij})$$

$S_i = 0,1,\cdots,q-1$

δ: mod q

$$J_{ij} = \begin{cases} 0 & : 1-p \\ 1,\dots,q\text{-}1 : & \dfrac{p}{(q-1)} \end{cases}$$

quenched

$$-(1-p_c)\log(1-p_c) - p_c \log p_c + p_c \log(q-1) = \frac{\log q}{2}$$

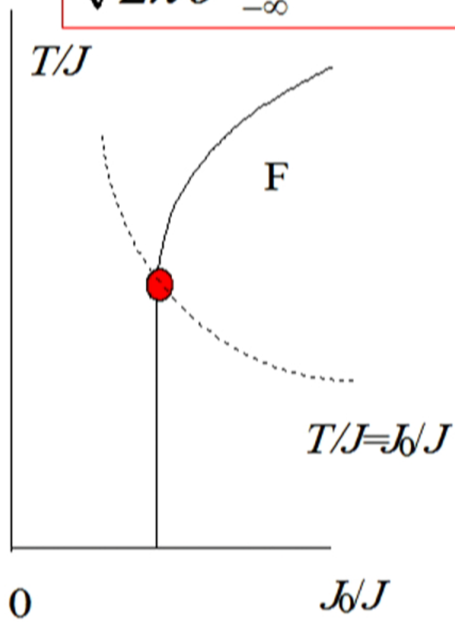
$$q = 3 : p_c = 0.159476$$

Hashing bound

cf. Duclos-Cianti and Poulin (2013), Anwar *et al* (2014), Andrist *et al* (2014)

Gaussian Ising SG on the sq lattice

$$\frac{1}{\sqrt{2\pi}J}\int_{-\infty}^{\infty} dJ_{ij} e^{-(J_{ij}-J_0)^2/2J^2} \log(1+e^{-2J_0 J_{ij}/J^2}) = \frac{\log 2}{2}$$



$$J_0 / J = 1.02177$$

Numerical : 1.02

Derivation

Duality in the ferromagnetic system

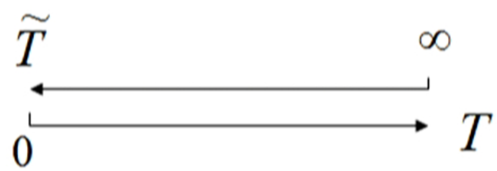
- Duality relation : 2d square lattice ferro Ising

$$H = -J\sum_{\langle ij\rangle} S_i S_j$$

$$Z(K) = c(K)Z(\tilde{K})$$

$$K = \beta J = J/T$$ Coupling constant

$$e^{-2\tilde{K}} = \tanh K$$ Dual coupling

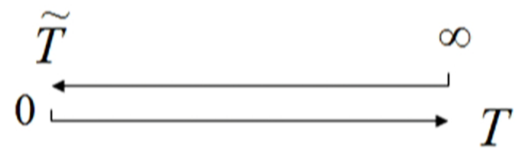


Logic to identify the singularity

$$f(T) \cong f(\tilde{T})$$

- Uniqueness of singularity
  $\rightarrow$ Fixed point of duality = singularity

$$e^{-2\tilde{K}} = \tanh K$$



$$\Longrightarrow \quad e^{-2K_c} = \tanh K_c$$

$$\Longrightarrow \quad 1 + e^{-2K_c} = \sqrt{2}$$

Exact (Onsager) solution

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Invariant subspace
(after replica average)

$$\alpha = \frac{\sum_{l=0}^{q-1} p_l \left(\sum_{\eta=0}^{q-1} e^{V(\eta+l)} \right)^n}{\sum_{l=0}^{q-1} p_l e^{nV(l)}} \quad \leftrightarrow \quad \tilde{\alpha} = \frac{q^n}{\alpha}$$

$$\alpha = q^{n/2} \xrightarrow{n\to 0} \boxed{\sum_{l=0}^{q-1} p_l \log\left(\sum_{\eta=0}^{q-1} e^{V(\eta+l)-V(l)} \right) = \frac{\log q}{2}}$$

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Summary

- Spin glass theory

Mean-field theory: complex SG phase
Finite dimensions: SG phase exists.