Title: On Quantum Tunnelling

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Abstract: One of the most basic but intriguing properties of quantum systems is their ability to `tunnel' between configurations which are classically disconnected. That is, processes which are classically impossible, are quantum allowed. In this talk I will outline a new, first-principles approach combining the semiclassical approximation with the concepts of post-selection and weak measurement. Its main virtue is to provide a real-time description within which sharp answers can be given to questions such as 'how long did the tunneling take' and 'where was the particle while it was tunneling?' Potential applications span a vast range, from laboratory tests to understanding black hole evaporation, the stability of the electroweak Higgs vacuum and the future of our universe, and the validity (or otherwise) of the "inflationary multiverse" scenario.

On Quantum Tunneling

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arXiv:1312:1772 [quant-ph]
New Journal of Physics 16 (2014) 063006

work with J. Barnett
and in progress with D. Gaiotto, E. Schnetter, K. Smith

Outline

- elementary approach to quantum tunneling using complex classical paths

- vast range of applications, from foundational questions to quantum chemistry, the Hawking evaporation of black holes and even the validity of the 'inflationary multiverse'

- Classically, tunneling through a barrier is not just hard, it's impossible
- Post-selection + the semiclassical expansion
- Predictions for real-time weak measurements
- Extension to quantum field theory and gravity
- Applications from quantum chemistry to black holes
- Implications for inflationary 'multiverse'

Feynman path integral

$$\Psi(x_f, t_f) = N \int Dx \int dx_i \; e^{\frac{i}{\hbar} S(x_f, t_f, x_i, t_i)} \Psi(x_i, t_i)$$

This incorporates 'pre- and post-selection'
Now take limit $\hbar \longrightarrow 0$
and perform path integral via saddle point method
$\Longrightarrow$ classical solution(s) dominate
Solutions are generically complex

Can introduce weak measuring device to 'see' where the particle was between the initial and final times

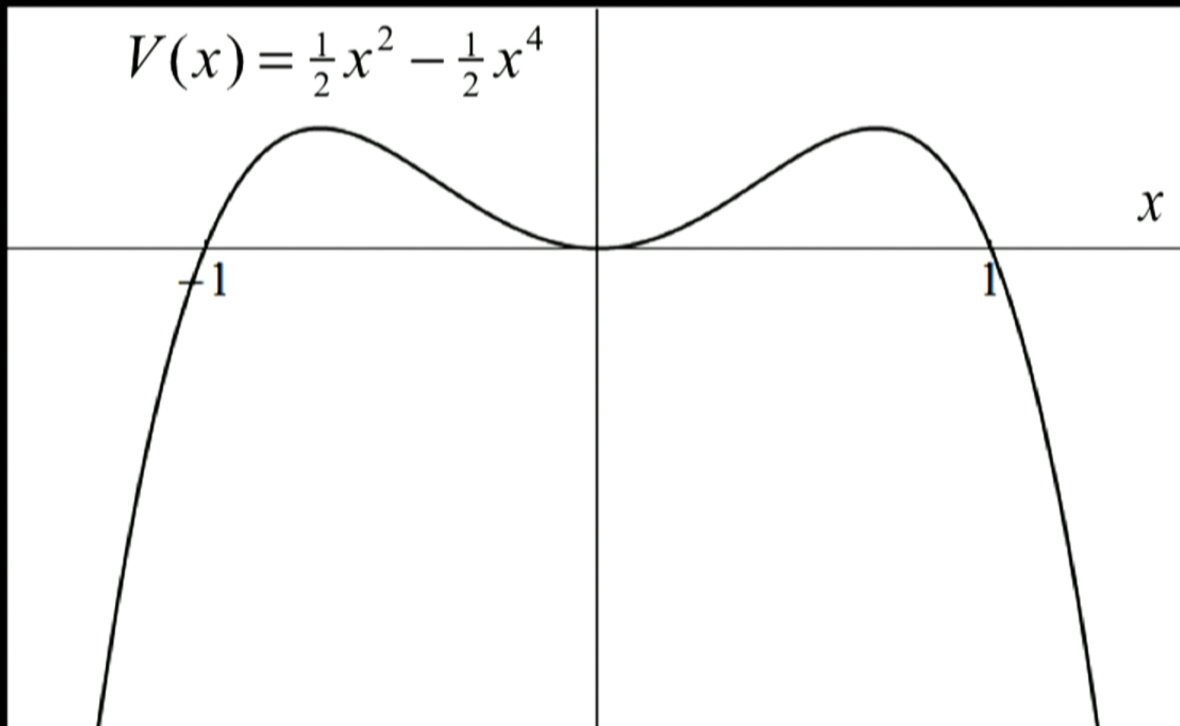
Example: particle in a potential

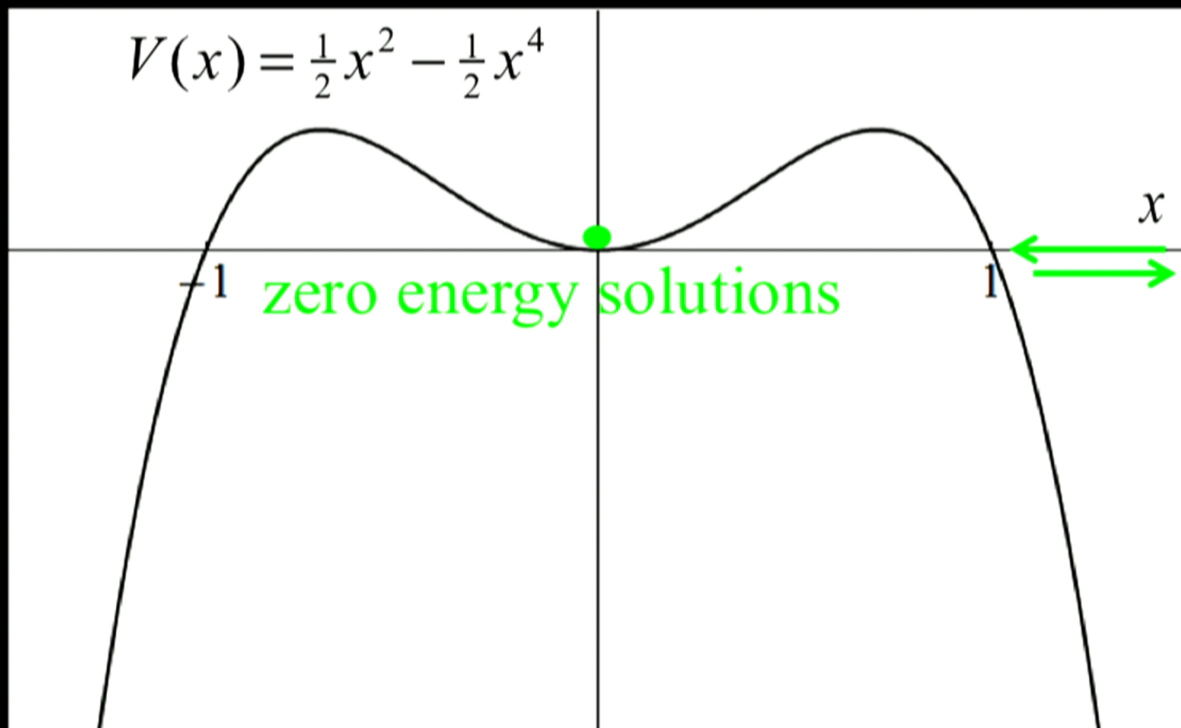
$$\tfrac{i}{\hbar} S = \tfrac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\tfrac{1}{2} m \dot{x}^2 - V(x) \right), \text{ take } V(x) = \tfrac{1}{2} \kappa x^2 - \tfrac{1}{2} \lambda x^4$$

$$t \to \sqrt{\tfrac{m}{\kappa}}\, t; \; x \to \sqrt{\tfrac{\kappa}{\lambda}}\, x$$

$$\tfrac{i}{\hbar} S = i \frac{\kappa^{\frac{3}{2}} m^{\frac{1}{2}}}{\hbar \lambda} \int_{t_i}^{t_f} dt \;\; \tfrac{1}{2} (\dot{x}^2 - x^2 + x^4)$$

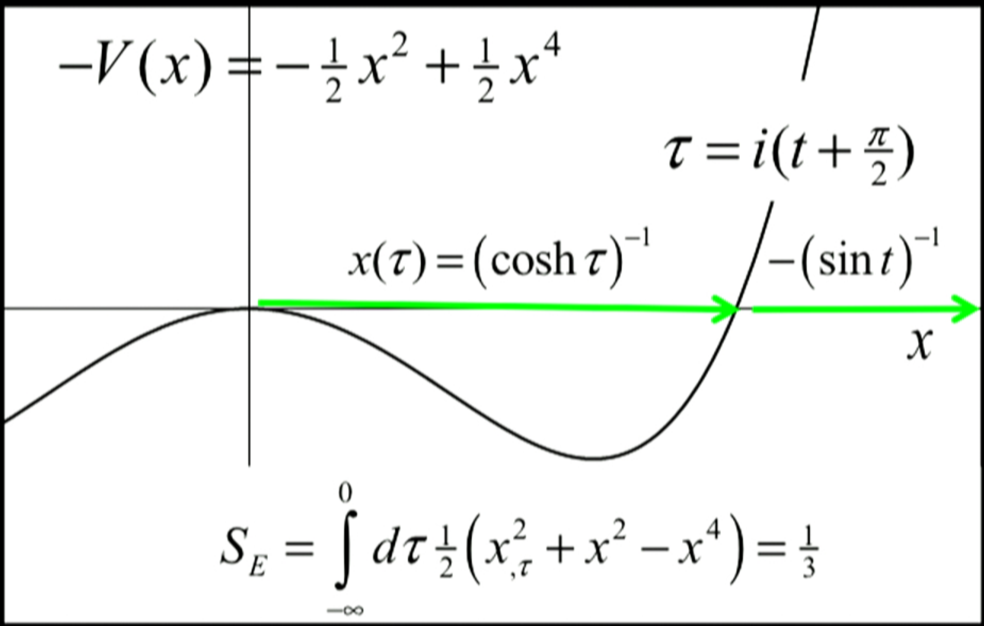
dimensionless

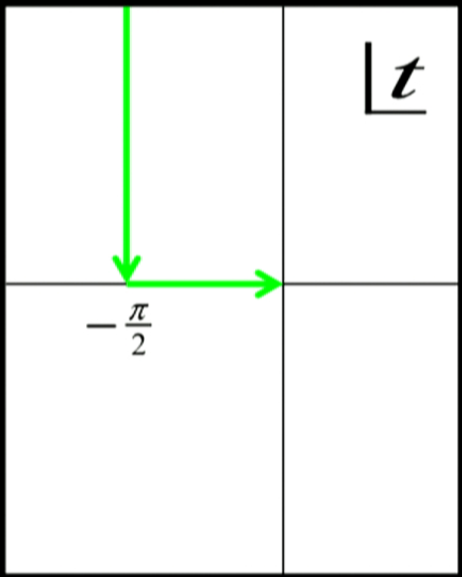

$V(x) = \frac{1}{2}x^2 - \frac{1}{2}x^4$
x
-1
1


$V(x) = \frac{1}{2}x^2 - \frac{1}{2}x^4$
x
-1
zero energy solutions
1

Euclidean "bounce"

Callan/Coleman 70's





Deficiencies of the Euclidean approach:

Dependence on initial state implicit

Cannot easily answer real-time questions
e.g. where was the particle at each moment of time? How did it get through the barrier?

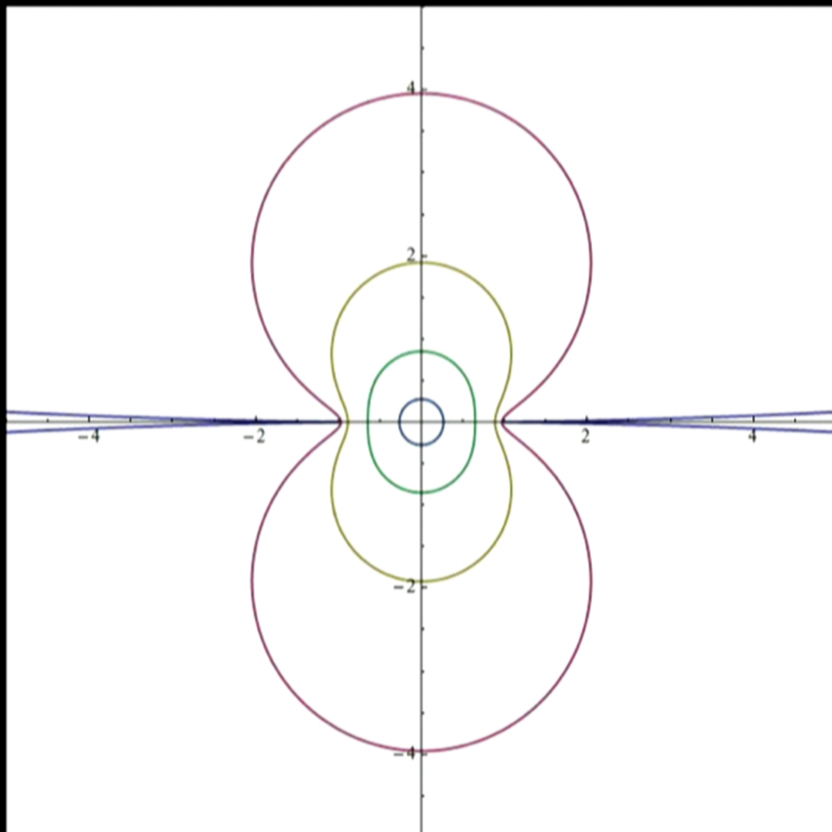
Hard to extend to time-dependent Hamiltonians (such as tunneling in cosmology)

Can we do better?

General classical solution described by two complex numbers:

Energy E and time delay t_0

For real E, all solutions are periodic and do not represent tunneling e.g. $E = 0 \Rightarrow x(t) = \frac{1}{\sin(t_0 - t)}$; $t_0 = imaginary:$



Small imaginary part of energy will “carry us across” these solutions

General classical solution expressible in terms of a Jacobi elliptic function

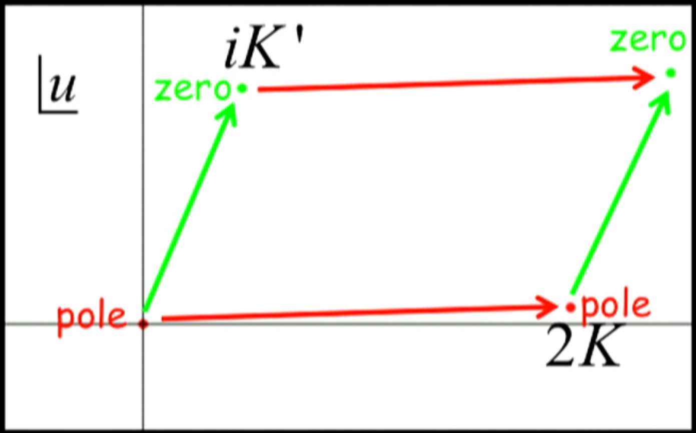
$$x(t) = \frac{1}{\sqrt{1+m}\ sn\left(\frac{t_0 - t}{\sqrt{1+m}}\middle| m\right)};\ E = \frac{m}{2(1+m)^2}$$

(we shall be interested in small complex values of E and t_0)

Double periodicity in complex t-plane

$$x(t) = \frac{1}{\sqrt{1+m}\; sn\left(\frac{t_0 - t}{\sqrt{1+m}} \middle| m\right)};\; u = \frac{t_0 - t}{\sqrt{1+m}}$$

$$K(m) = \text{"quarter period"};\; K'(m) = K(1-m);$$



For small complex energy, i.e. small m

$$K = \tfrac{\pi}{2}(1 + \tfrac{m}{4}\ldots);\; K' = -\tfrac{1}{2}\ln\tfrac{m}{16} + \ldots$$

Expansion in nome $q = e^{-\pi K'/K}$; $q = \frac{m}{16} + \frac{m^2}{32} + \ldots$

Define $U = \frac{\pi}{2K\sqrt{1+m}}(t_0 - t)$

$$x(t) = \frac{\pi}{2K\sqrt{1+m}}\left(\frac{1}{\sin U} + 4\sum_0^\infty \frac{q^{2n+1}}{1-q^{2n+1}}\sin(2n+1)U\right)$$

Initial state: gaussian wavepacket

$$\Psi(x_i, t_i) \propto e^{-\frac{x_i^2}{4L^2}} \Rightarrow \frac{x_i}{L} + i\frac{2Lp_i}{\hbar} = 0$$

For false vacuum "ground state," $L = 1/\sqrt{2}$

Boundary conditions for classical solution

$$x + i\dot{x} = 0, \qquad t = t_i$$

$$x = x_f, \qquad t = t_f$$

Assume $T \equiv t_f - t_i \gg 1,\ \ x_f \gg 1 \Rightarrow t_0 \approx t_f$

Solution has small, nearly imaginary E, $i.e., m = i\varepsilon$

Define $z = e^{iU} = e^{-it-\frac{3\varepsilon}{4}t}$, becomes large for t large, negative

$\Rightarrow x + i\dot{x} \sim -4iz^{-3} + \frac{m}{4i}z \Rightarrow 3\varepsilon T e^{3\varepsilon T} \approx 48iTe^{-4iT}$

General solution $\varepsilon_n = \frac{1}{3}T^{-1}W_n(48iTe^{-4iT}),\ n \in \mathbb{Z}$,

where $W_n(y)$ solves $xe^x = y$ (Lambert)

Principal solution has $\mathrm{Re}(\varepsilon_0) \sim \frac{\ln T}{3T},\ \ \mathrm{Im}(\varepsilon_0) \sim \frac{1}{T}$

For small complex energy, i.e. small m

$$K = \tfrac{\pi}{2}\left(1 + \tfrac{m}{4} \ldots\right);\ K' = -\tfrac{1}{2}\ln\tfrac{m}{16} + \ldots$$

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action $iS_{tot} \equiv -\frac{1}{2}x_i^2 + i\int_{t_i}^{t_f} dt(\frac{1}{2}\dot{x}^2 - V(x)) \approx -\frac{3im^2}{16}T - 2z^{-4}$

stationarise wrt m, calculate real part of exponent

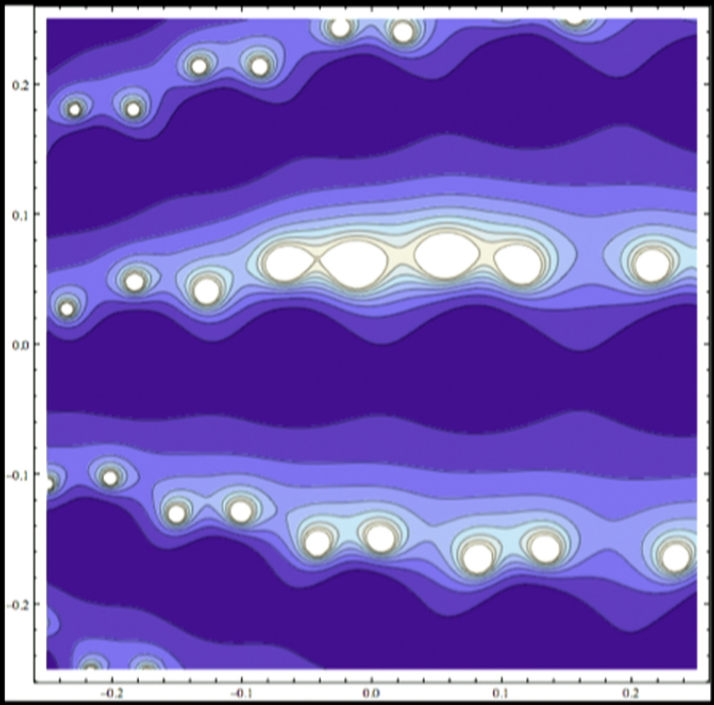
$$\frac{i}{\hbar}(S_{tot} - S_{tot}^*) \approx -\frac{2}{3\hbar} + \frac{3}{8\hbar}\mathrm{Im}\left(m^2\right)T$$

$\left|x+i\dot{x}\right|^{-1}$, in $m-$plane

(for $t_0 = 0, T = -30$)

Each peak represents

a classical solution



$\mathrm{Re}[E] > 0$ solutions represent

transients due to Gaussian initial state

not being local false vacuum state

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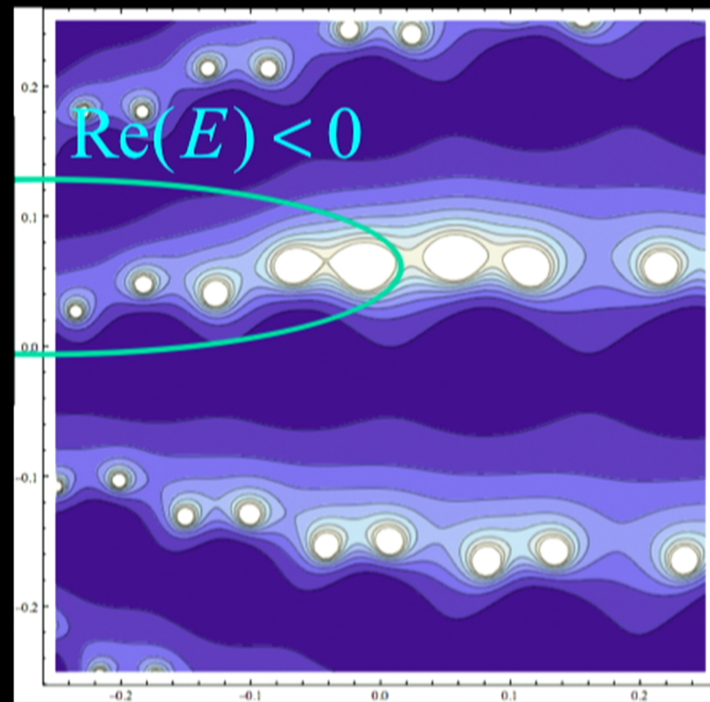

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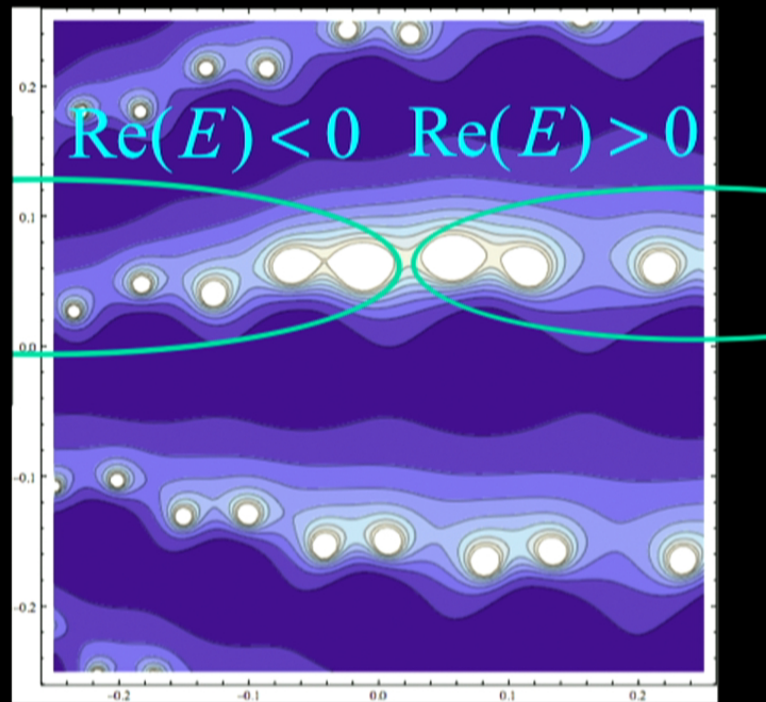

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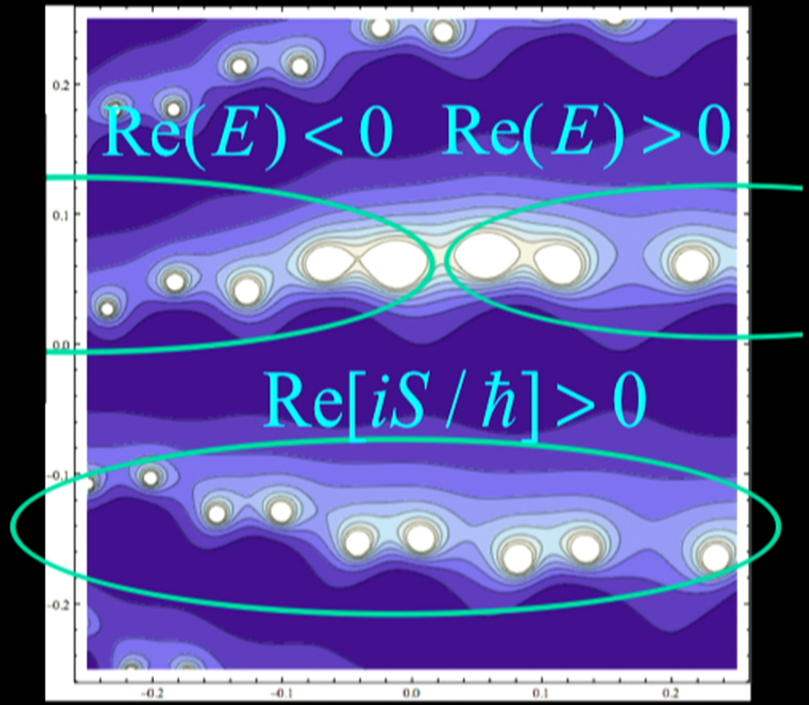

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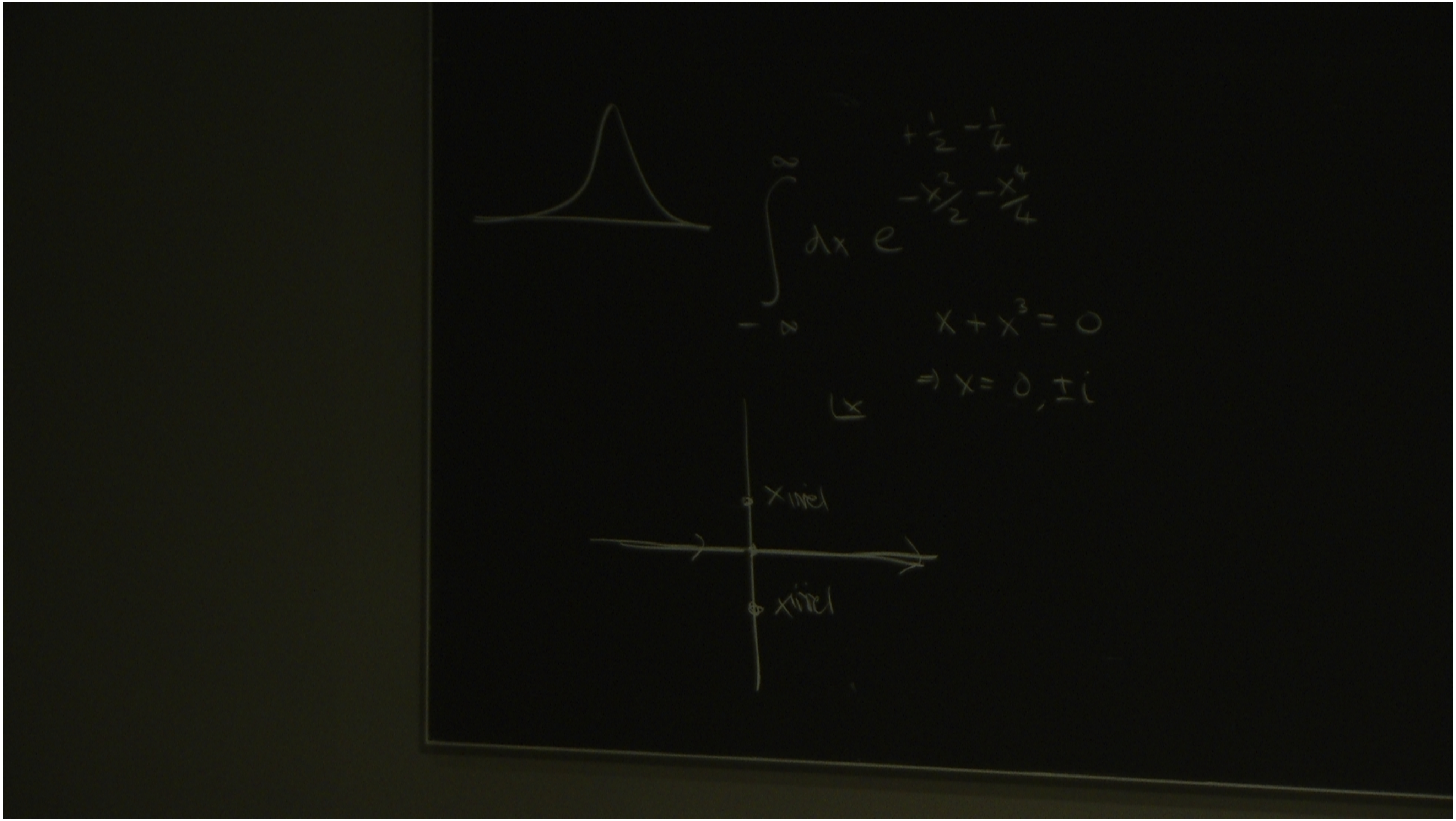
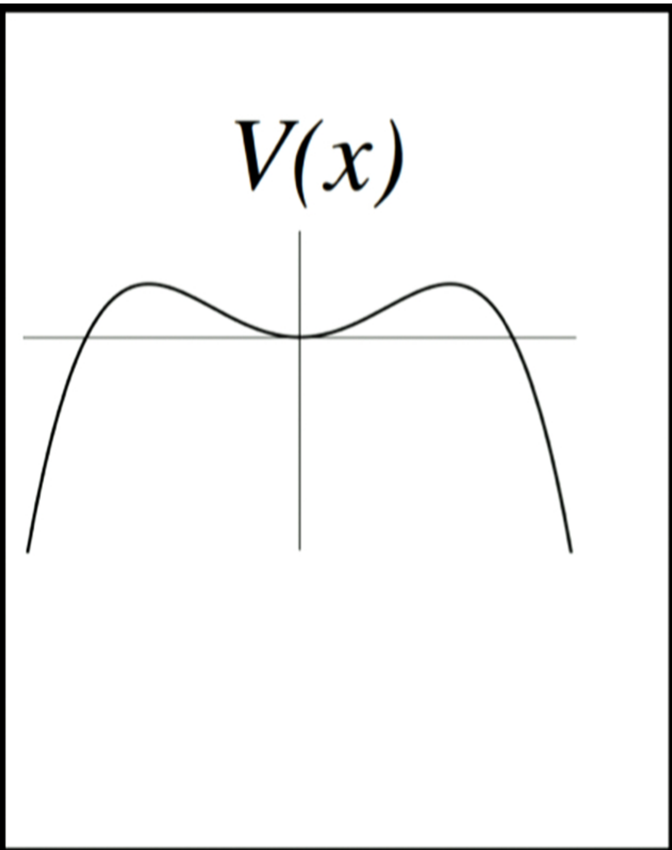
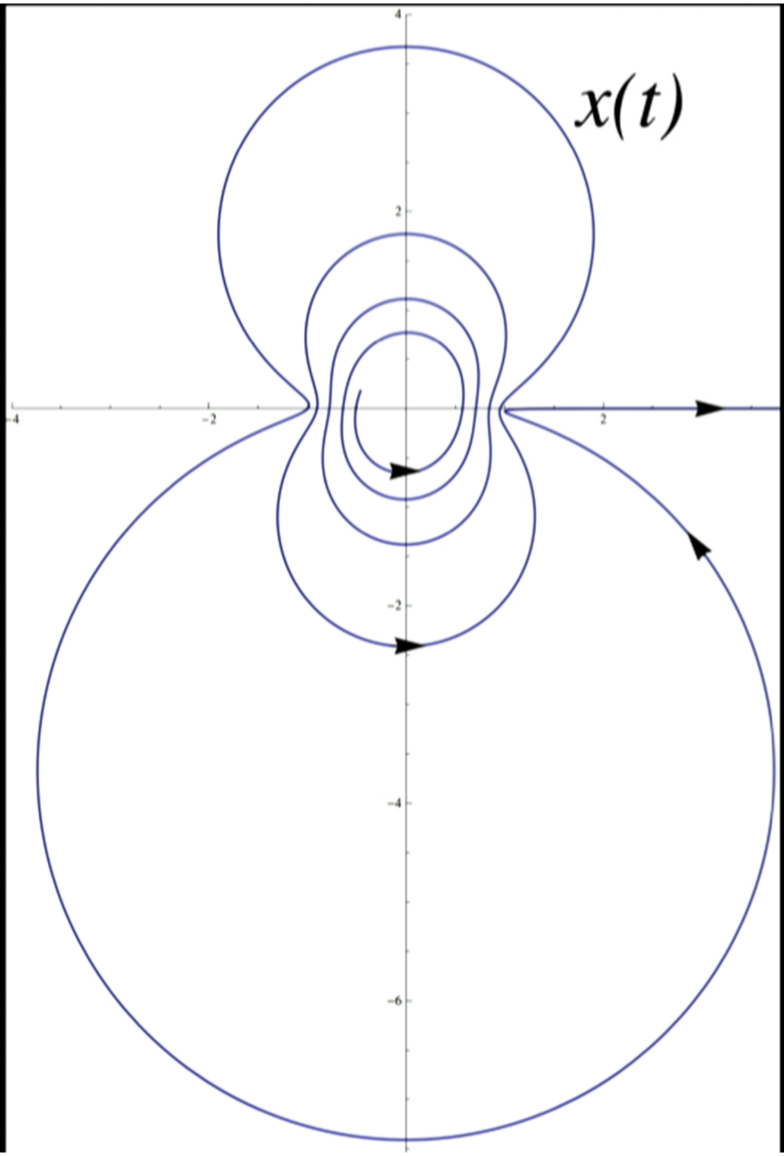

dx e
x + x^3 = 0
⇒ x = 0, ±i

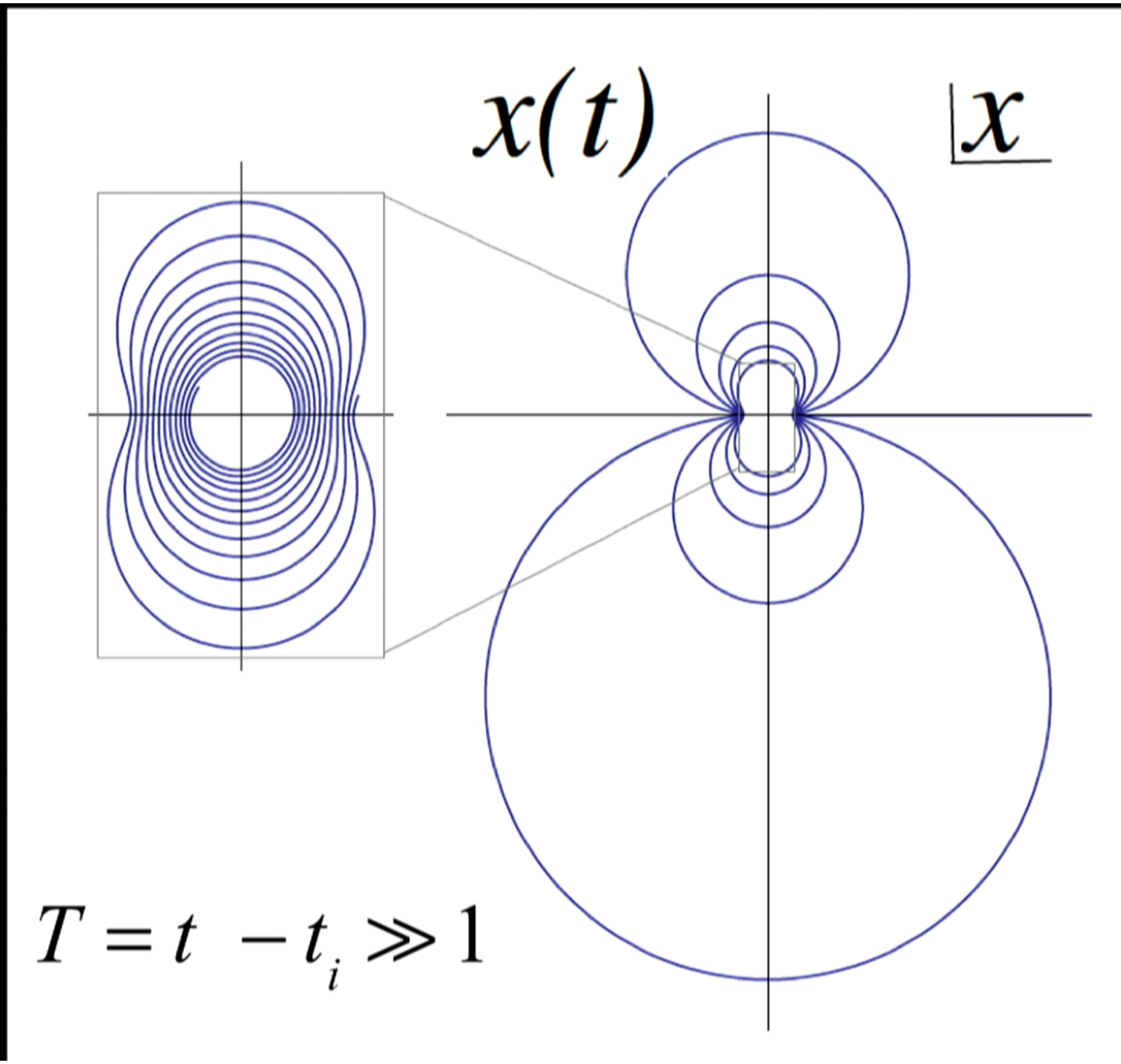
A general technique for deciding which classical solutions actually contribute to the path integral is needed (in many fields).

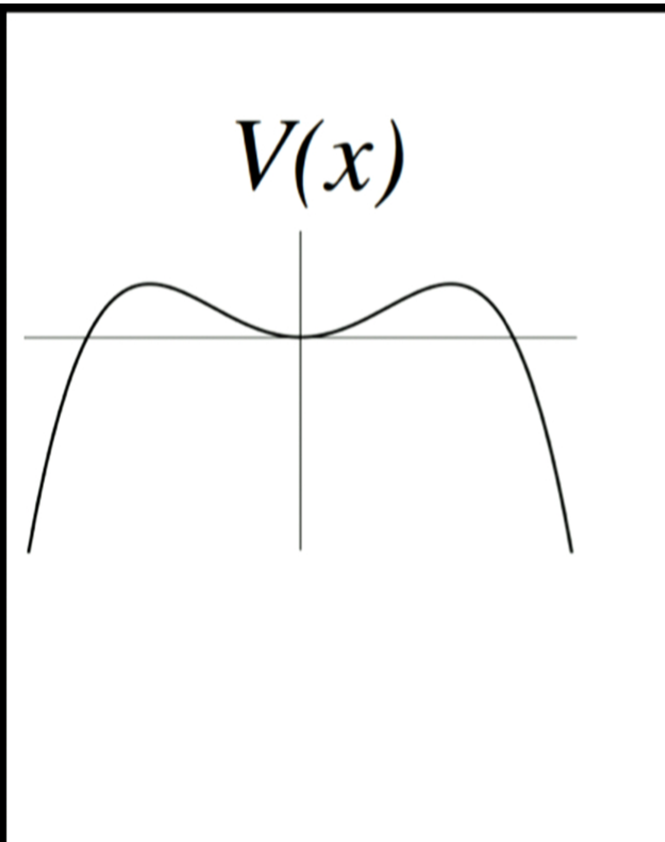
There are rigorous results for finite-dimensional integrals, based on a'flow equation,' which we are now extending to the path integral for quantum mechanics.

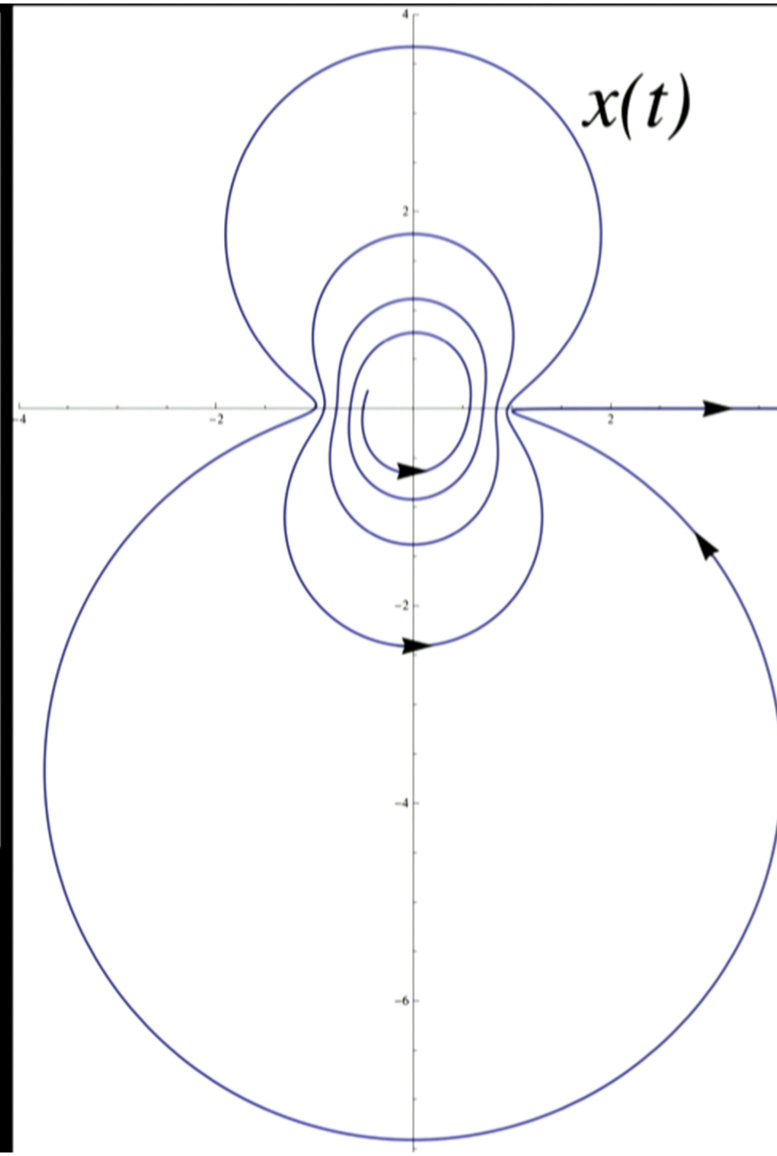


Principal classical solution



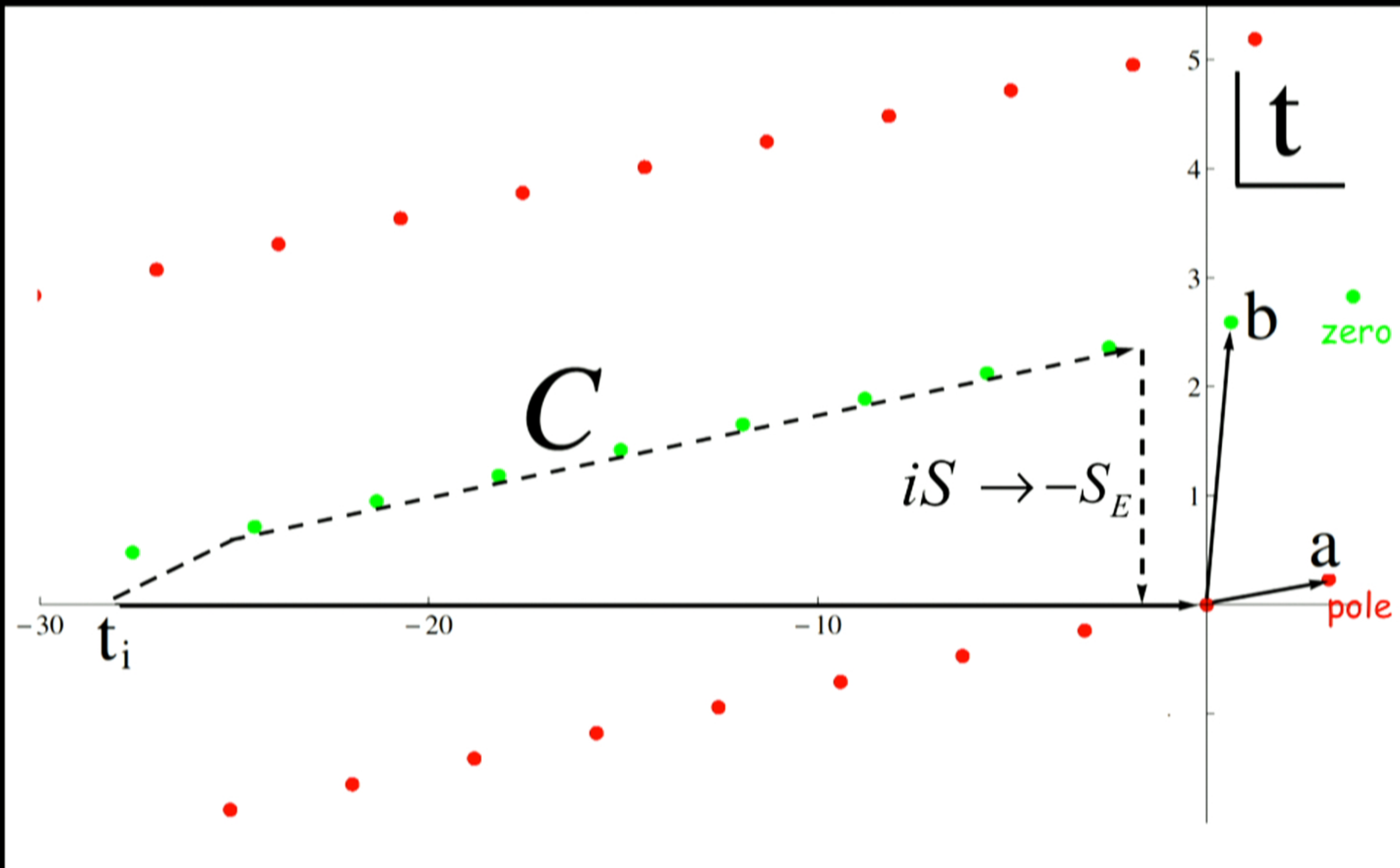

$x(t)$
$\lfloor x$
$T = t - t_i \gg 1$





Principal classical solution

Double periodicity in complex t-plane



$$a = 2K\sqrt{1+m} \approx \pi(1+\tfrac{3}{4}m);\ b = iK'\sqrt{1+m} \approx -\tfrac{i}{2}Log(m)$$

Imaginary part of solution
becomes large just before tunneling!

Cubic potential

$$V(x) = \tfrac{1}{2}\kappa x^2 - \tfrac{1}{3}\lambda x^3 \qquad t \to \sqrt{\tfrac{m}{\kappa}}\,t;\; x \to \tfrac{\kappa}{\lambda}x$$

$$\tfrac{i}{\hbar}S = i\frac{\kappa^{\frac{5}{2}} m^{\frac{1}{2}}}{\hbar\lambda^2}\int_{t_i}^{t_f} dt\ \left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right)$$

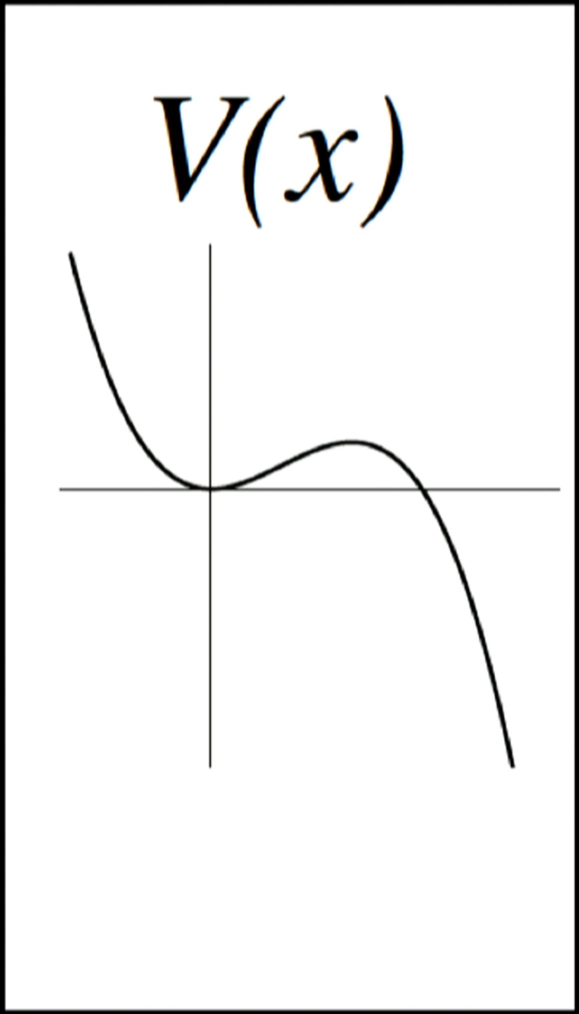
dimensionless

General classical solution (Weierstrass)

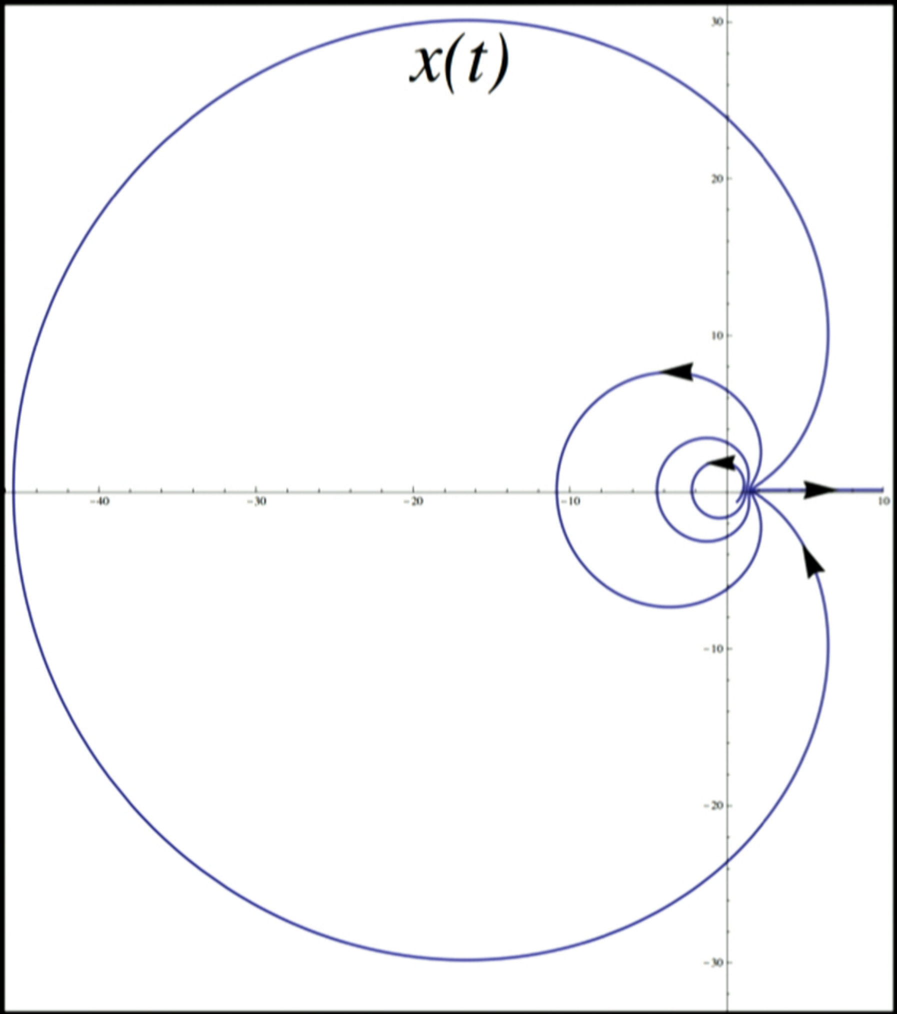
$$x(t) = A + \frac{B}{sn\left(C(t_0 - t)\middle| m\right)^2};$$

$$E = \frac{1}{12} - \frac{(2m-1)(m-2)(m+1)}{24\left[1+m(m-1)\right]^{3/2}} \approx \frac{9}{32}m^2, \quad |m| \ll 1;$$

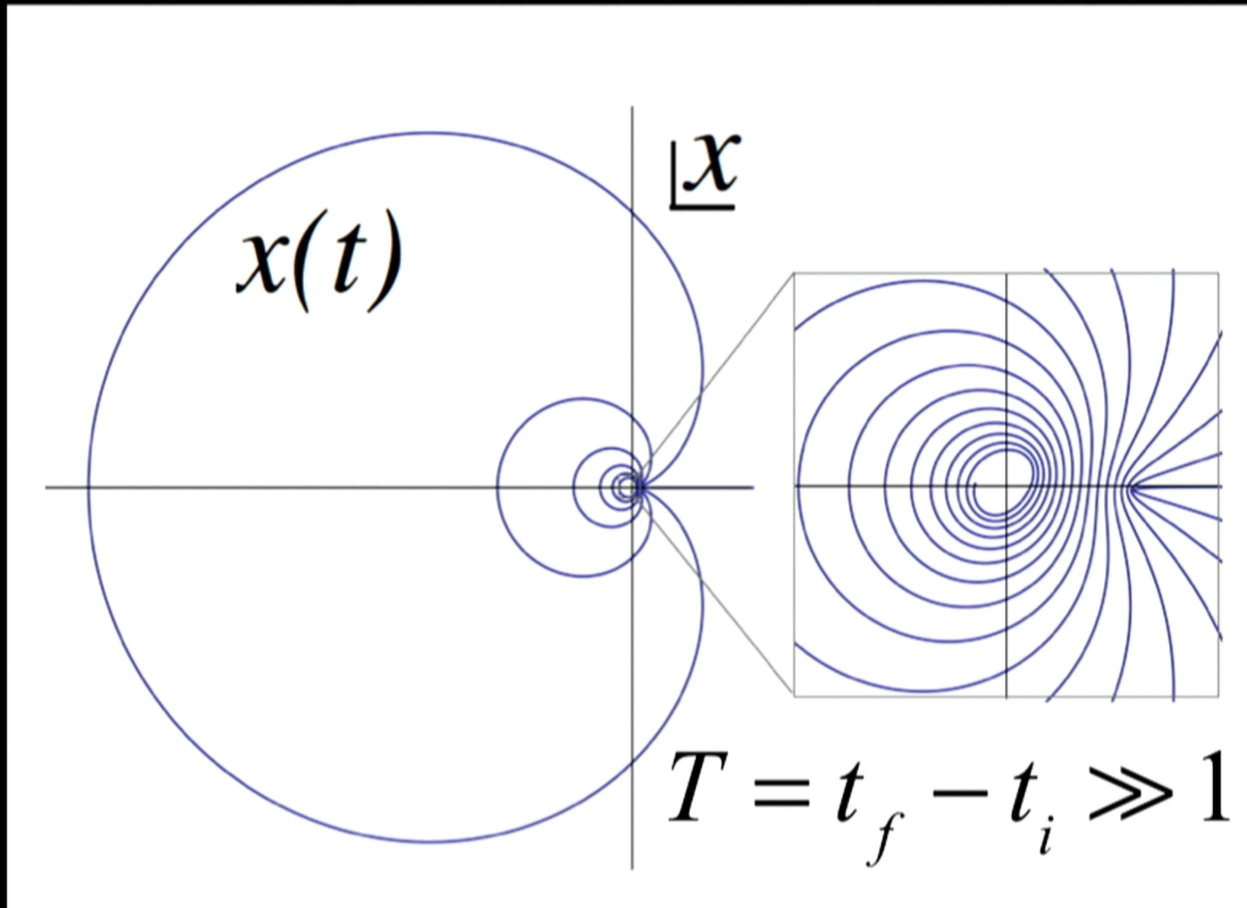
$$A = \frac{1}{2}\left(1 - \frac{1+m}{\sqrt{1+m(m-1)}}\right); \; B = \frac{3}{2\sqrt{1+m(m-1)}}; \; C = \frac{1}{2\sqrt[4]{1+m(m-1)}}$$



$V(x)$

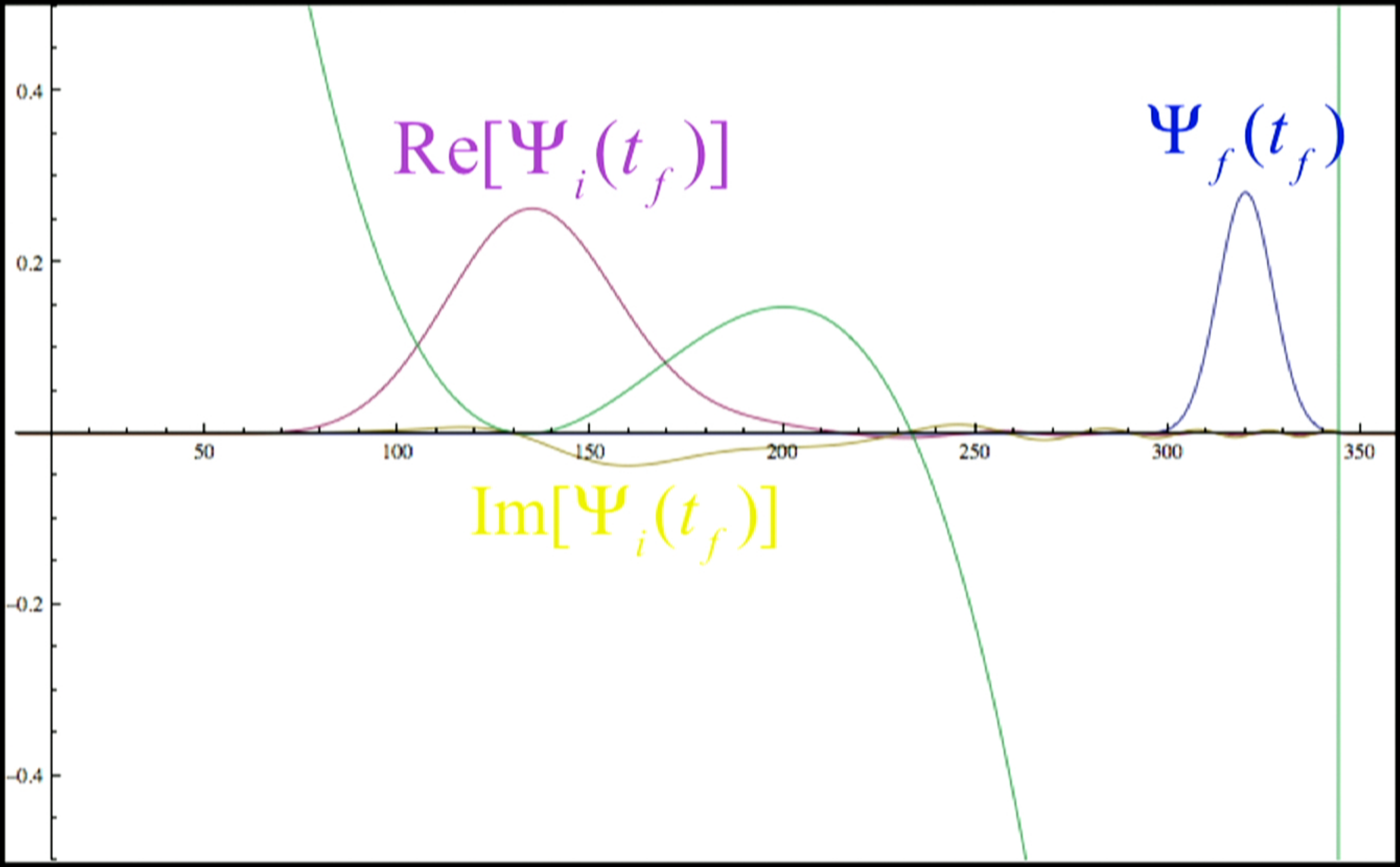


$x(t)$
30
20
10
-10
-20
-30
-40
-30
-20
-10
10


x
x(t)
T = t_f - t_i ≫ 1

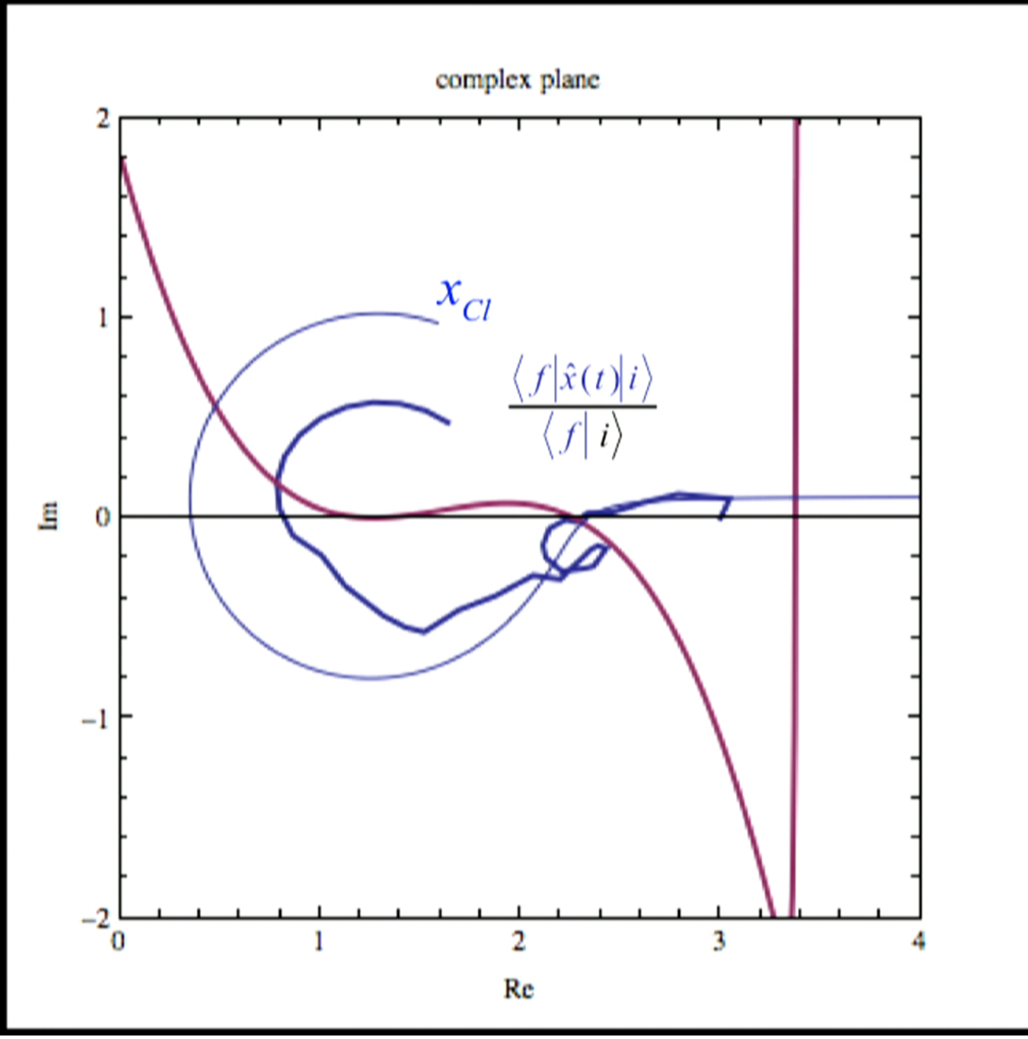
Even larger $\mathrm{Im}(x)$ just before tunneling due to double poles in complex t-plane

Numerical solution of Schrodinger equation



(preliminary)

Post-selected expectation value of x



Prefactors: functional determinants

Expand around saddle point solutions

$$S = S_{sp} + \int dt(\delta x \hat{O}_{sp} \delta x) + \dots$$

and perform Gaussian integrals

Gelfand-Yaglom method

$$\hat{O} \sim \begin{pmatrix} 1+i\varepsilon\frac{\hbar}{2mL^2} & -1 & 0 & \cdots \\ -1 & 2-\varepsilon^2\omega_1^2 & -1 & \cdots \\ 0 & -1 & 2-\varepsilon^2\omega_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

recursion relation for $D = \det \hat{O}$

$$\frac{1}{\varepsilon}\left(\frac{D_N - D_{N-1}}{\varepsilon} - \frac{D_{N-1} - D_{N-2}}{\varepsilon}\right) + \omega_N^2 D_N = 0$$

$\Rightarrow$ continuum limit

$$\ddot{D} + \omega^2(t) D = 0; \quad D(t_i) = 1, \dot{D}(t_i) = i\frac{\hbar}{2mL^2}$$

$\Rightarrow$ *e.g.* usual free particle prefactor

$$\Rightarrow \frac{\left\langle f \middle| \hat{x}(t) \middle| i \right\rangle}{\left\langle f \middle| i \right\rangle} \approx \sum_{sp} \frac{N}{\sqrt{\det \hat{O}_{sp}}} e^{\frac{i}{\hbar} S_{sp}}$$

Can we detect the complex nature of classical trajectories?

i.e. can we test the reality of reality?

Couple to measuring device (pointer):

$$H_x \rightarrow H_x + {P^2}/{2M} + gPx\delta(t - t_m)$$

where $g \ll 1$.

Pointer momentum P commutes with Hamiltonian $\Rightarrow$ work in momentum basis

Interaction has this effect:

$$\Psi(x, P, t_m^+) \approx e^{-igPx_{Cl}(t_m)/\hbar}\Psi(x, P, t_m^-)$$

If Ψ for pointer is Gaussian of width L_{pt}, then for small g, effect on pointer is

$$\langle X \rangle \rightarrow \langle X \rangle + g\,\mathrm{Re}\big(x(t_m)\big)$$
$$\langle P \rangle \rightarrow \langle P \rangle + \frac{g\hbar}{2L_{pt}^2}\,\mathrm{Im}\big(x(t_m)\big)$$

quantum post-selection bias

For $g\,\mathrm{Im}(x) \ll L_{pt}$ this shift in $\langle P \rangle$ is a small fraction of the quantum uncertainty in P, *i.e.*, L_{pt}.

Nevertheless, it can be measured with arbitrary accuracy if the state preparation and weak measurement are repeated a sufficiently large number of times

(Aharonov et al.)

For a measurement performed a quarter-period before the particle tunnels,

$$\Delta\left\langle P\right\rangle \propto \hbar e^{S_E/\hbar} \rightarrow \infty \text{ as } \hbar \rightarrow 0 \text{ !}$$

Experimental tests may be possible in quantum dots

Extensions and generalisations:

* vary initial Gaussian: L, x_c, p_c
* vary shape of initial wavepacket
* include time-dependent forcing

* higher dimensions
* quantum field theory
* electroweak vacuum stability
* black hole evaporation

* harder: infinite number of degrees of freedom

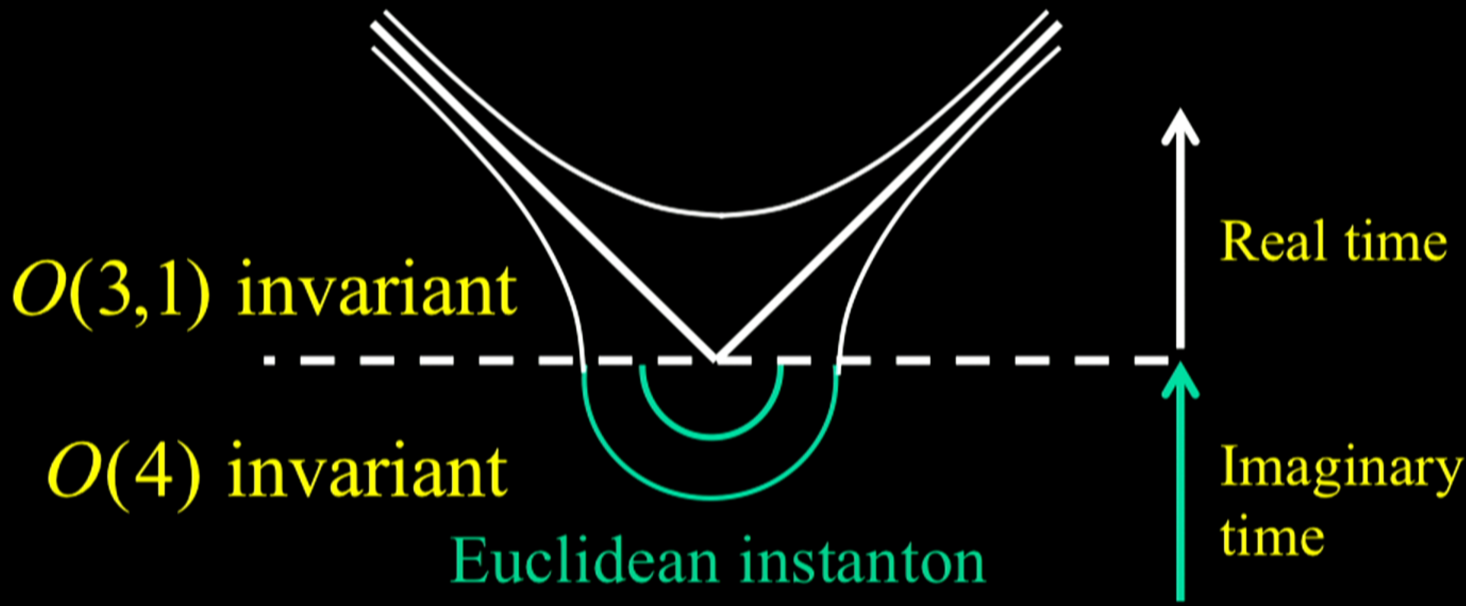
* initial ‘false vacuum’ wavefunctional

* IMPORTANT: this state defines a preferred frame, because it is not the true, Lorentz-invariant ground state

* A Lorentz-invariant solution (of the Callan-Coleman type) is necessarily time-reversal invariant and hence not the semiclassical solution we seek

* Nonetheless, its spatial profile provides a good ansatz for the emerging bubble in the large tunneling time limit.

Bubble Nucleation: Euclidean Approach



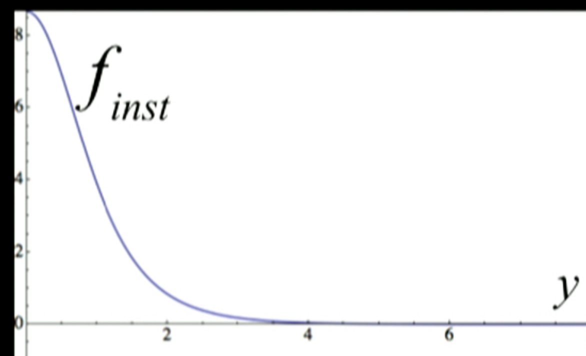
Bubble nucleation in flat spacetime

$$S = \int dt d^3x \left(\tfrac{1}{2}\left(\partial_t \phi\right)^2 - \tfrac{1}{2}\left(\nabla\phi\right)^2 - \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{3}\lambda\phi^3 \right)$$

$$y^\mu = m x^\mu,\ \phi = \tfrac{m^2}{\lambda} f \Rightarrow S_E = \tfrac{m^2}{\lambda^2}\int d^4 y \left(\tfrac{1}{2}\left(\nabla f\right)^2 + \tfrac{1}{2} f^2 - \tfrac{1}{3} f^3 \right)$$

$S =$

Real-time ansatz



$$\phi = \tfrac{m^2}{\lambda} f_{inst}(mr)x(mt);\ \bar{t} \equiv mt$$

$$S = \tfrac{m^2}{\lambda^2}\int d\bar{t}\,\left(\tfrac{1}{2}a\dot{x}^2 - \tfrac{1}{2}bx^2 + \tfrac{1}{3}cx^3\right)$$

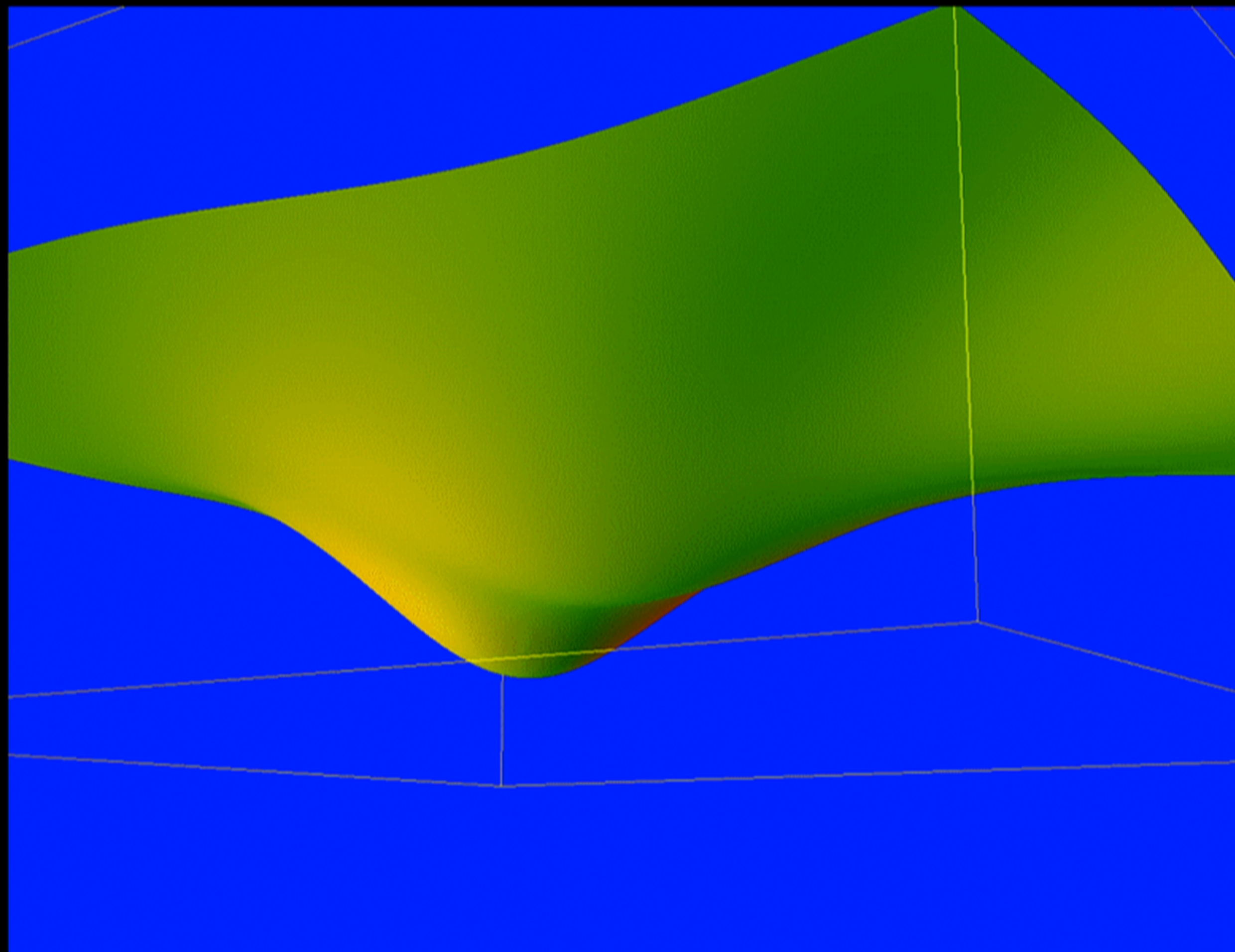
where a, b, c are various moments of $f \Rightarrow S_E \approx 1.04 S_{E,inst}$

Suggests this should be an excellent approximation

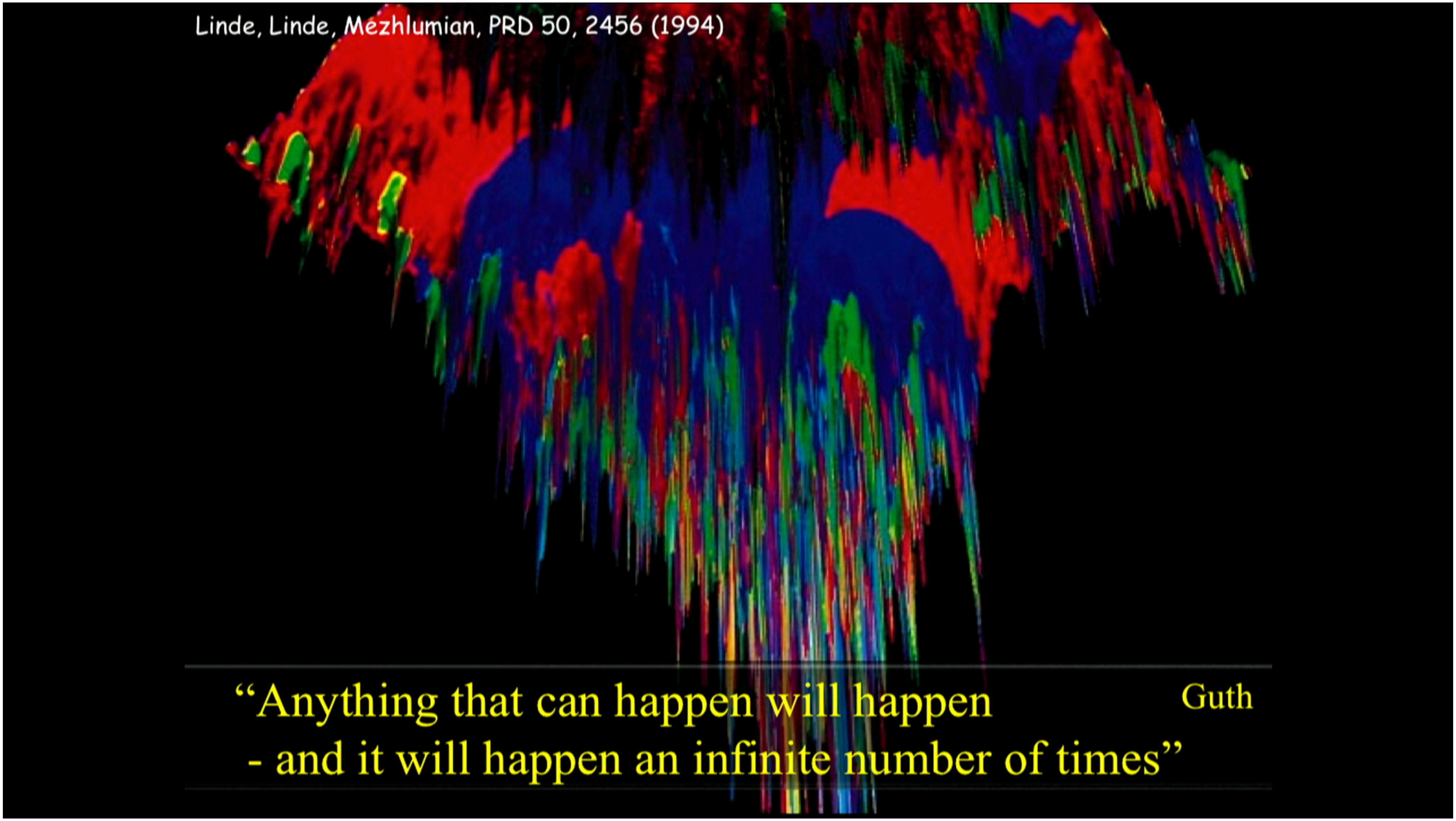
At zeroth order, description reduces to quantum tunneling in a cubic potential

Ansatz may be systematically improved using iterated linear theory response

The Inflationary 'Multiverse'

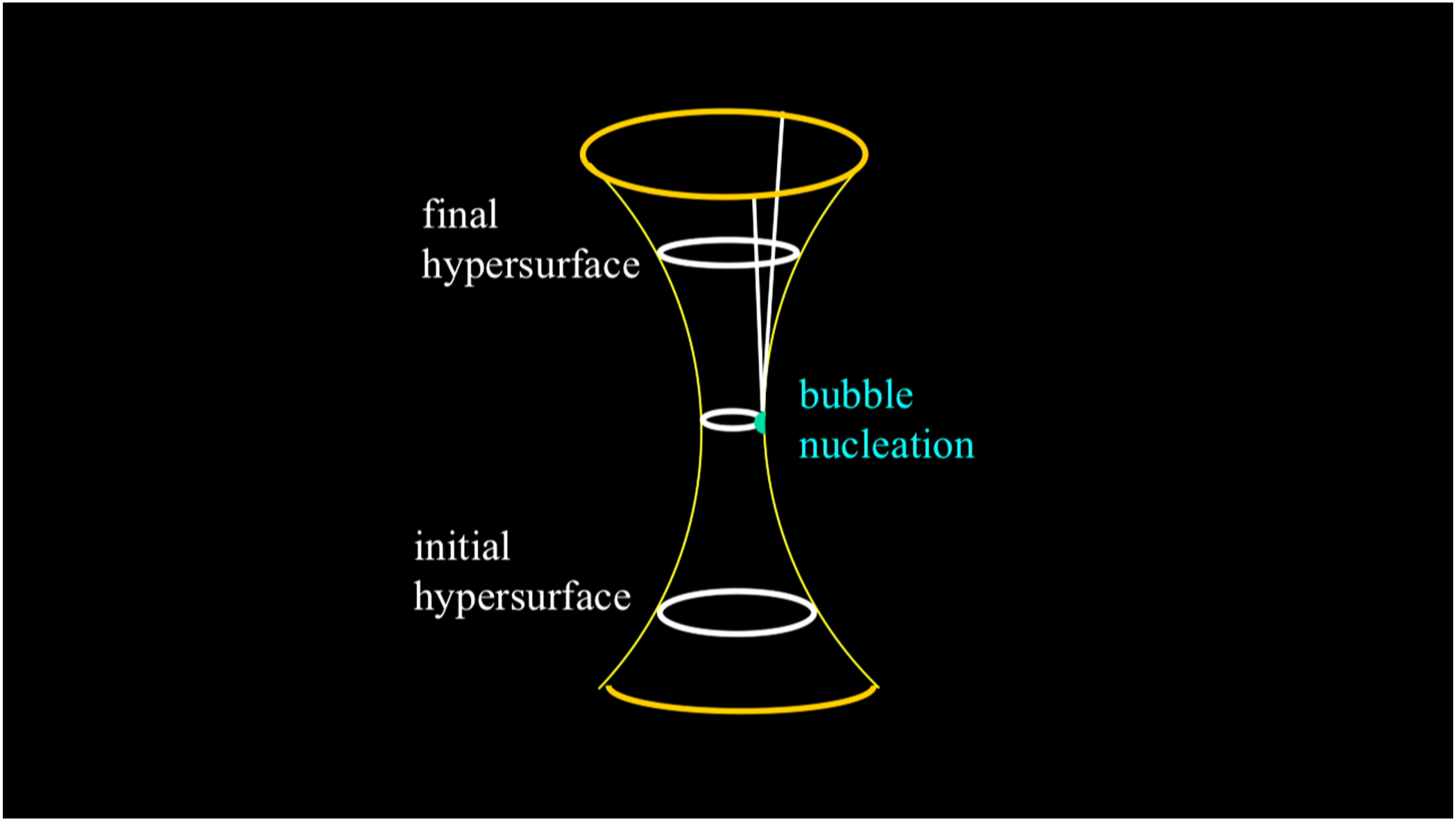


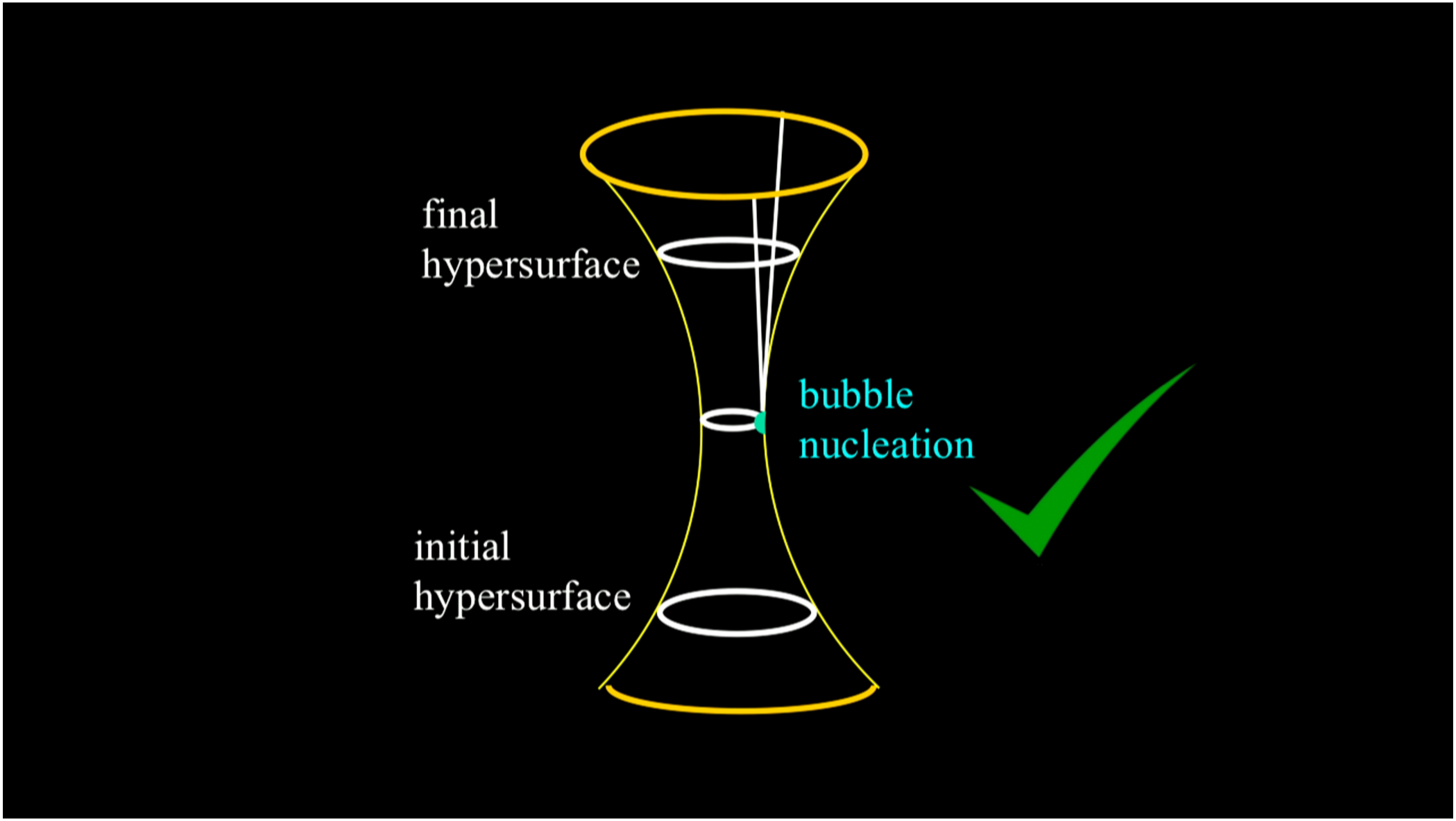
Linde, Linde, Mezhlumian, PRD 50, 2456 (1994)


Linde, Linde, Mezhlumian, PRD 50, 2456 (1994)
“Anything that can happen will happen
- and it will happen an infinite number of times”
Guth

Note: the treatment of quantum effects is only heuristic in this and other discussions of the ‘eternal inflationary multiverse’

Bubble nucleation in de Sitter spacetime provides a minimal setting to explore these questions


final
hypersurface
bubble
nucleation
initial
hypersurface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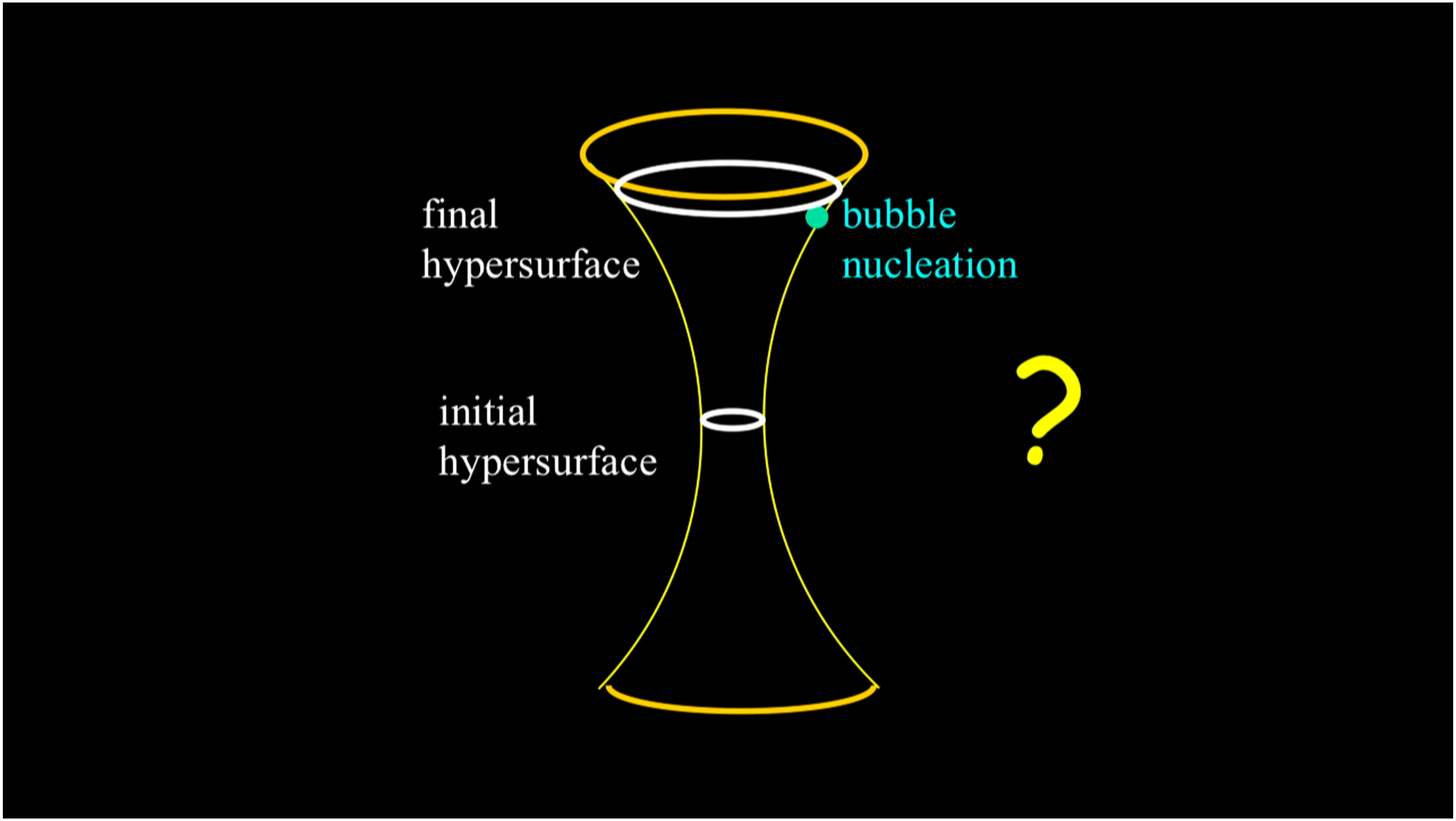
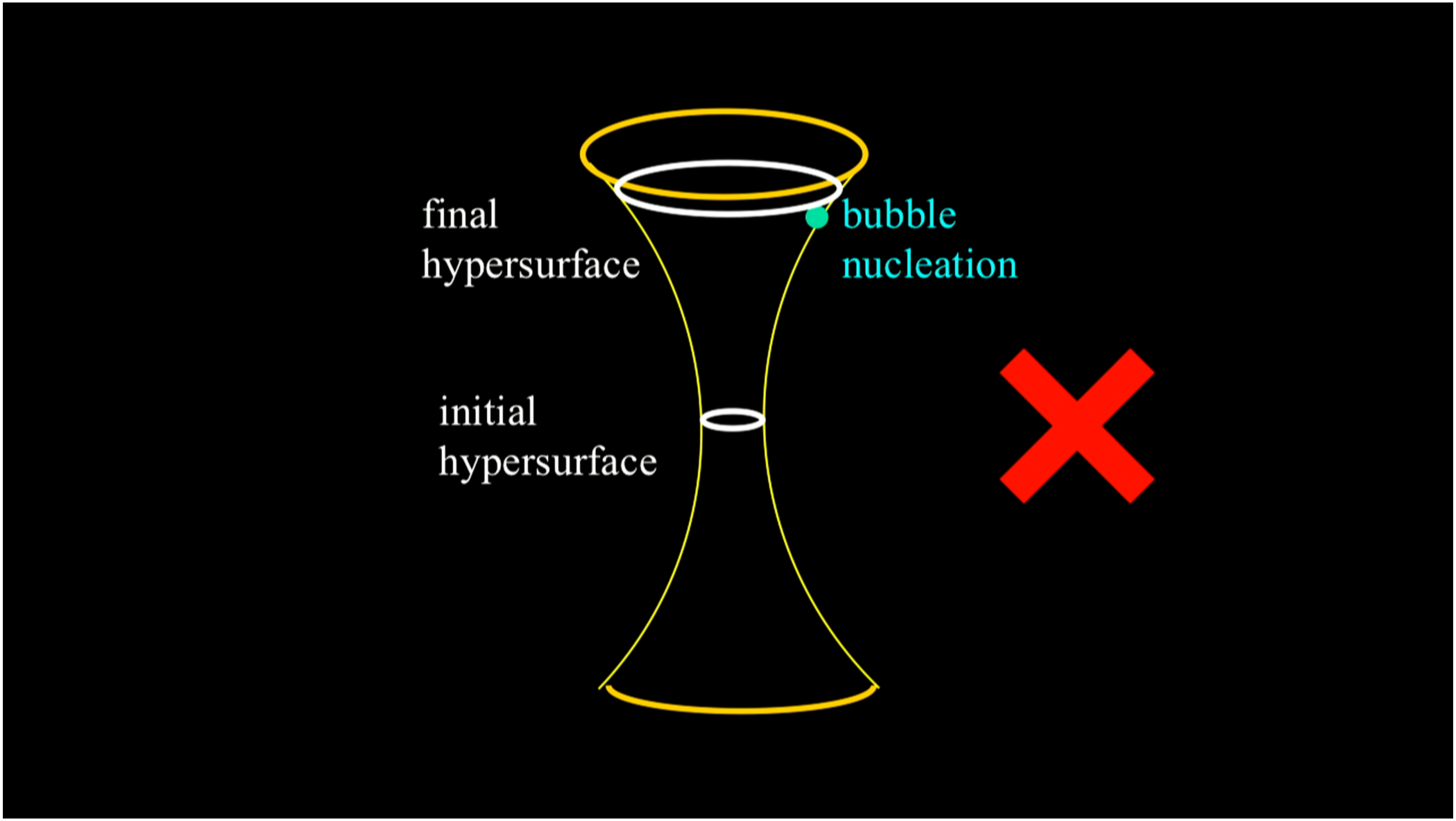

final
hypersurface
bubble
nucleation
initial
hypersurface
?


final
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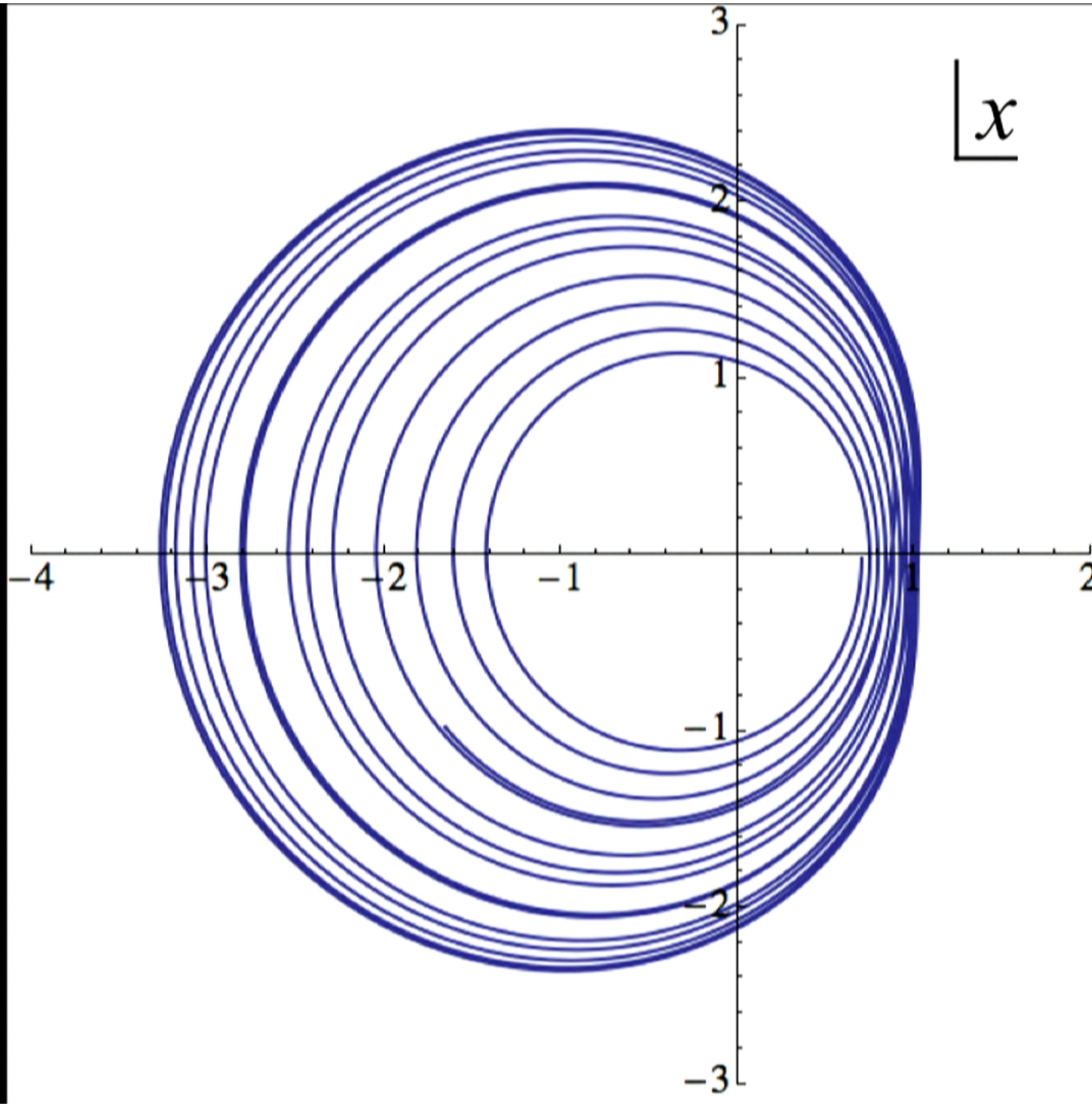
bubble
nucleation
initial
hypersurface


x
3
2
1
−1
−2
−3
−4
−3
−2
−1
1
2

If, instead, the initial hypersurface is chosen ‘at the throat’ and we try to describe a bubble which nucleates much later, then damping of field oscillations due to the exponential expansion of the universe has a big effect, countering the classical instability due to nonlinearity.

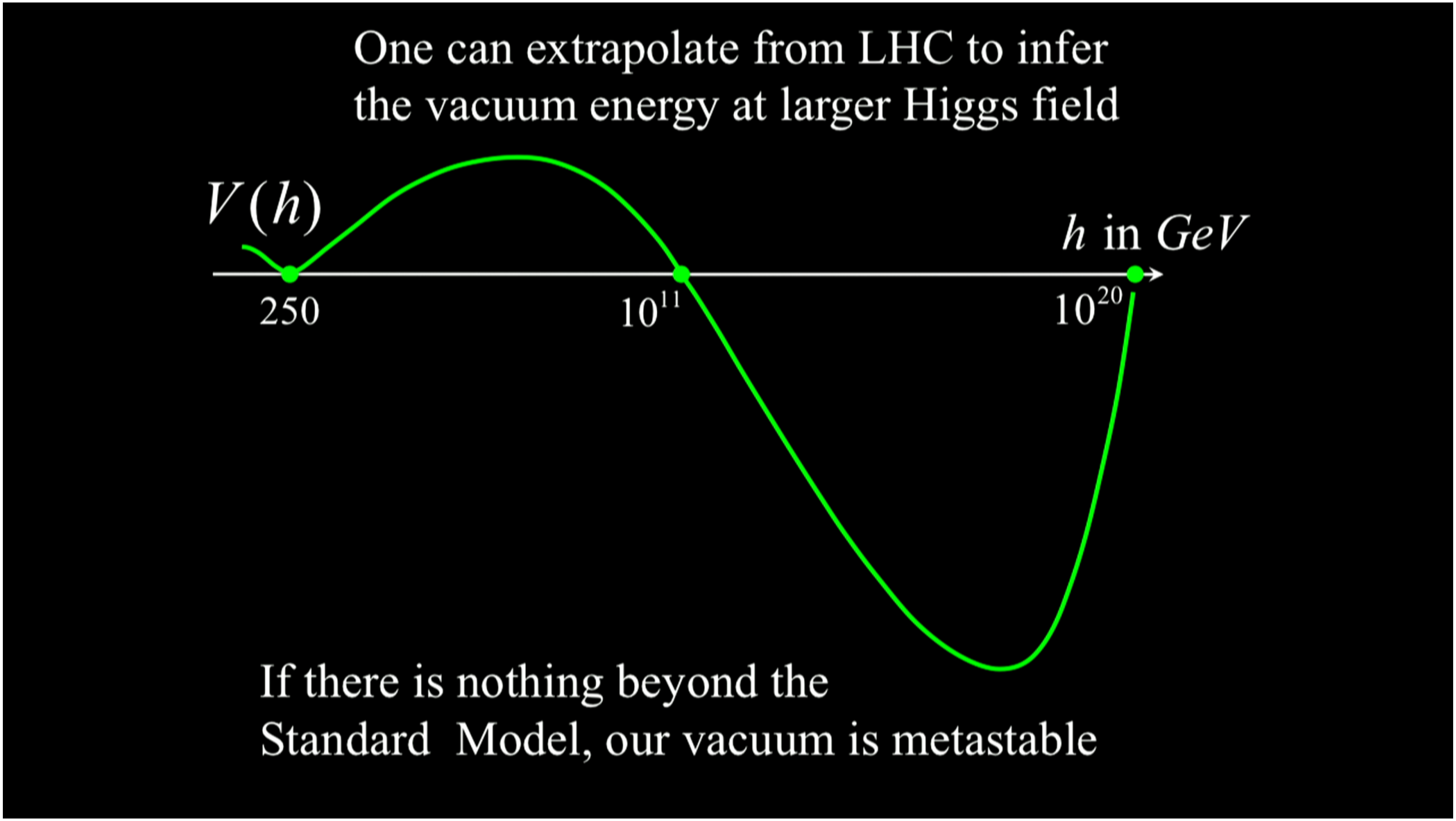
It seems that no complex classical solution of the desired form exists.

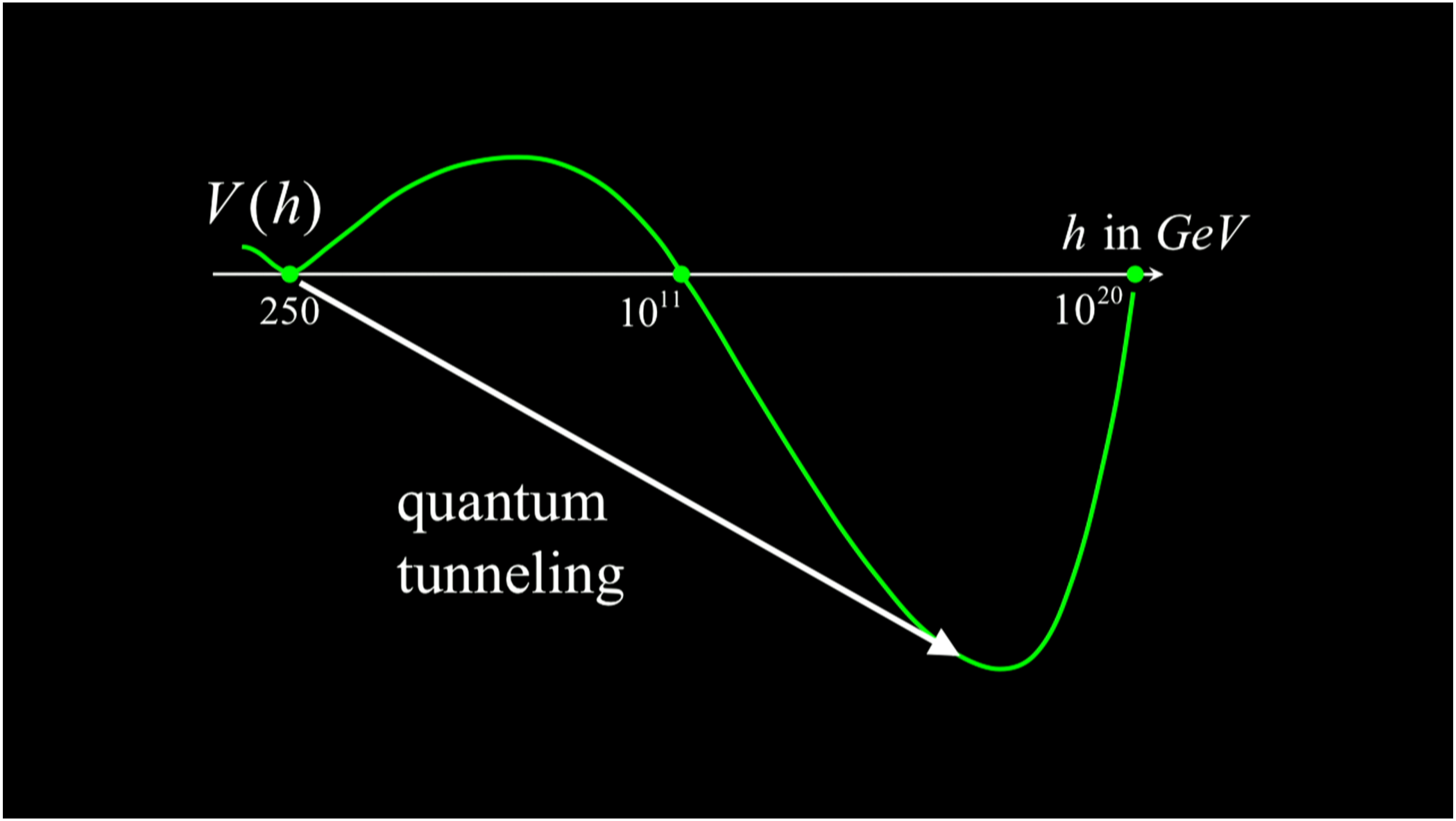
The above discussion suggests that the global description of an ‘inflationary multiverse’ is inconsistent with the semiclassical approximation because there is no complex classical solution describing the nucleation of a bubble long after the initial hypersurface.

This may be a reflection of the fact that the semi-classical Gibbons-Hawking calculation of the entropy of de Sitter spacetime yields a finite number of states.

Interesting implications for today’s metastable Higgs vacuum… and for black holes.

Thank you!


One can extrapolate from LHC to infer
the vacuum energy at larger Higgs field
$V(h)$
h in GeV
250
10^{11}
10^{20}
If there is nothing beyond the
Standard Model, our vacuum is metastable



$V(h)$
h in GeV
250
10^{11}
10^{20}
quantum
tunneling