Title: The Exact Renormalization Group and Higher Spin Holography.

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Abstract:

ntroduction

- An appealing aspect of holography is its interpretation in terms of the renormalization group of quantum field theories — the 'radial coordinate' is a geometrization of the renormalization scale.

  e.g., [de Boer, Verlinde^2 '99, Skenderis '02, Heemskerk & Polchinski '10, Faulkner, Liu & Rangamani '10 ...]

- I will begin with a free fixed point of a field theory in d dimensions, and examine the Wilson-Polchinski exact RG. This theory certainly does not have a (purely) gravitational holographic dual.

  [Douglas, Mazzucato & Razamat '10]

- I will show that the exact RG equations for both sources and vevs of operators of the field theory give rise to Hamilton-Jacobi dynamics in a space-time of dimension $d + 1$. The corresponding first-order equations can be thought of as the EOM of a higher spin gauge theory, whose geometric interpretation I will explain.
- Some examples of such constructions bear resemblance to the traditional (Vasiliev) higher-spin gauge theories.

Punch Lines

- We will study free field theories perturbed by arbitrary bi-local ‘single-trace’ operators.
  - a playground familiar from the Klebanov-Polyakov conjecture
  - This is a ‘consistent truncation.’
- We identify a formulation in which the operator sources correspond (amongst other things) manifestly to a connection on a *really big* principal bundle (i.e., an infinite jet bundle).
- The ‘gauge group’ can be understood directly in terms of field redefinitions in the path integral, and consequently there are exact Ward identities that correspond to ERG equations.
- The space-time structure extends in a natural way (governed by ERG) to a geometric structure over a spacetime of one higher dimension, and *AdS* emerges as a geometry corresponding to the (relativistic) free fixed point, encoded in a flat connection.

More Punch Lines

- the ERG equations are the first-order equations of motion of a bulk phase space structure (corresponding to 'radial quantization')
- from the bulk point of view, these are equivalent to the equations of motion of a higher spin gauge theory
- identifying this Hamilton-Jacobi structure gives us an action for the higher spin theory
- all of the correlation functions of the free fixed point can be calculated exactly and have a holographic interpretation (corresponding to 'Witten diagrams')
- free fixed points with other symmetries (e.g., the $z = 2$ non-relativistic free theory) also have higher spin duals (with a corresponding background geometry (flat connection))

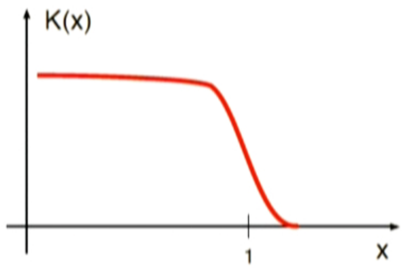
The Exact Renormalization Group

- Polchinski '84: formulated field theory path integral by introducing a regulator given by a cutoff function accompanying the fixed point action (i.e., the kinetic term).
- extracted an exact equation describing the cutoff independence of the partition function by isolating (and discarding) a total derivative in the path integral.
- this equation describes how the couplings must depend on the RG scale in order that the partition function be independent of the cutoff.

he Exact Renormalization Group

$$Z = \int [d\phi] e^{-S_o[M,\phi] - S_{int}[\phi]}$$

$$S_o[M,\phi] = \int \phi K_F^{-1}(-\Box/M^2)\Box\phi$$



$$M\frac{d}{dM}Z = 0$$

mplies

$$M\frac{\partial S_{int}}{\partial M} = -\frac{1}{2}\int M\frac{\partial K_F}{\partial M}\Box^{-1}\left[\frac{\delta S_{int}}{\delta\phi}\frac{\delta S_{int}}{\delta\phi} + \frac{\delta^2 S_{int}}{\delta\phi^2}\right]$$

- can extract equations for each coupling
- can apply similar methods to correlation functions, and thus obtain exact Callan-Symanzik equations as well

Majorana Fermions in $d = 2+1$

- To be specific, it turns out to be convenient to first consider the free Majorana fixed point in $2+1$. This can be described by the regulated action

$$S_0 = \int_x \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}\psi^m(x)$$

- Here $P_{F;\mu}$ is a regulated derivative operator

$$P_{F;\mu} = K_F^{-1}(-\Box/M^2)\partial_\mu^{(x)}$$

- It is crucial to write the action in a matrix form

$$S_0 = \int_{x,y} \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}(x,y)\psi^m(y) \equiv \int \widetilde{\psi}^m \cdot \gamma^\mu P_{F;\mu} \cdot \psi^m$$

where

$$P_{F;\mu}(x,y) = K_F^{-1}(-\Box/M^2)\partial_\mu^{(x)}\delta(x-y)$$

Majorana Fermions in $d = 2 + 1$

- In 2+1, a complete basis of 'single-trace' operators consists of

$$\hat{\Pi}(x,y) = \tilde{\psi}^m(x)\psi^m(y), \quad \hat{\Pi}^\mu(x,y) = \tilde{\psi}^m(x)\gamma^\mu\psi^m(y)$$

- We introduce bi-local sources for these operators in the action

$$S_{int} = \frac{1}{2}\int_{x,y} \tilde{\psi}^m(x)\Big(A(x,y) + \gamma^\mu W_\mu(x,y)\Big)\psi^m(y)$$

- One can think of these as collecting together infinite sets of local operators, obtained by expanding near $x \to y$. This quasi-local expansion can be expressed through an expansion of the sources

$$A(x,y) = \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x)\partial^{(x)}_{a_1}\cdots\partial^{(x)}_{a_s}\delta(x-y)$$

(similarly for W_μ). The coefficients are sources for higher spin local operators.

Majorana Fermions in $d = 2 + 1$

- Define the partition function

$$Z[M, g_{(0)}, A, W_\mu] = \mathcal{N} \int [d\psi] \, e^{i(S_0 + S_{int})} \quad \left[\equiv \det(\not{D}_F + A)^{N/2} \right.$$

- The expectation values of the single-trace operators are then given by

$$\Pi(x, y) = -i \frac{\delta}{\delta A(x, y)} \ln Z, \quad \Pi^\mu(x, y) = -i \frac{\delta}{\delta W_\mu(x, y)} \ln Z$$

- Will describe how to introduce true interactions later. It is important to hold off on interactions, as we are exploring the unbroken phase of a gauge theory, and any interactions in the field theory will Higgs this gauge symmetry.

Higher Spin Theories

- Note that this model fits with the conjectured duality between 3d vector models and Vasiliev higher-spin theories on AdS_4

  [Klebanov & Polyakov '02, Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

- Since the conjecture involves free-field theory, such a holographic duality (if true) begs for a geometric understanding in terms of RG

  [Douglas, Mazzucato & Razamat '10, Pando Zayas & Peng '13, Sachs '13]

- The Majorana model is believed to be dual to the B-type Vasiliev HS theory in AdS_4.
  - A and W_μ are in 1–1 correspondence with fields in Vasiliev theory
  - The Vasiliev construction appears to be a specific representation of the higher spin theory
  - That representation masks some of the simplest properties of the corresponding dual field theory.
- The method can be applied to other free fixed points, and other higher spin theories constructed (later).

The $O(L_2)$ symmetry

- the full action takes the form

$$S \equiv \widetilde{\psi}^m \cdot \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \psi^m$$

- Now we consider the following map of elementary fields

$$\psi^m(x) \mapsto \int_y \mathcal{L}(x,y)\psi^m(y) = \mathcal{L} \cdot \psi^m(x)$$

- The ψ^m are not operators; they are just integration variables in the path integral, and so this is just a trivial change of integration variable.
- We ask, what does this do to the partition function?

'he $O(L_2)$ symmetry

- We look at the action

$$\begin{aligned} S &\to \widetilde{\psi}^m \cdot \mathcal{L}^T \cdot \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \mathcal{L} \cdot \psi^m \\ &= \widetilde{\psi}^m \cdot \gamma^\mu \mathcal{L}^T \cdot \mathcal{L} \cdot P_{F;\mu} \cdot \psi^m \\ &\quad + \widetilde{\psi}^m \cdot \left[\gamma^\mu (\mathcal{L}^T \cdot [P_{F;\mu}, \mathcal{L}] + \mathcal{L}^T \cdot W_\mu \cdot \mathcal{L}) + \mathcal{L}^T \cdot A \cdot \mathcal{L}\right] \cdot \psi^m \end{aligned}$$

- Thus, if we take $\mathcal{L}$ to be orthogonal, $\mathcal{L}^T \cdot \mathcal{L}(x,y) = \int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \delta(x,y)$, the kinetic term is invariant, while the sources transform as

)(L_2) gauge symmetry

$$\begin{aligned} W_\mu &\mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_{F;\mu}, \mathcal{L}] \\ A &\mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L} \end{aligned}$$

The $O(L_2)$ symmetry

- But this was a trivial operation from the path integral point of view, and so we conclude that there is an exact Ward identity

$$Z[M, g_{(0)}, W_\mu, A] = Z[M, g_{(0)}, \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot P_{F;\mu} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}]$$

- this is the usual notion of a background symmetry: a transformation of the elementary fields is compensated by a change in background
- if we write $\mathcal{L} \sim \delta + \epsilon$, then we obtain the infinitesimal version

$$\delta W_\mu = [D_\mu, \epsilon]_. \,, \quad \delta A = [\epsilon, A]_.$$

where ϵ is an infinitesimal parameter satisfying $\epsilon(x, y) = -\epsilon(y, x)$, and the Ward identity is just the statement

$$[D_\mu, \Pi^\mu]_. + [\Pi, A]_. = 0$$

where $D_\mu = P_{F;\mu} + W_\mu$.

he $O(L_2)$ symmetry

- Note what is happening here: the $O(L_2)$ symmetry leaves invariant the (regulated) free fixed point action. W_μ is interpreted as a connection for this symmetry, while A transforms tensorily. $D_\mu = P_{F;\mu} + W_\mu$ plays the role of covariant derivative.
- More precisely, the free fixed point corresponds to any configuration

$$(A, W_\mu) = (0, W_\mu^{(0)})$$

  where $W^{(0)}$ is any flat connection, $dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0$
- It is therefore useful to split the full connection as $W_\mu = W_\mu^{(0)} + \widehat{W}_\mu$
- A, $\widehat{W}$ are the operator sources, both transforming tensorially under $O(L_2)$

The $CO(L_2)$ symmetry

- We can generalize the $O(L_2)$ condition to include scale transformations

$$\int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \lambda^{2\Delta_\psi}\delta(x-y)$$

- This is a symmetry (in the previous sense) provided we also transform the metric, the cutoff and the sources

$$g_{(0)} \mapsto \lambda^2 g_{(0)}, \quad M \mapsto \lambda^{-1} M$$

$$A \mapsto \mathcal{L}^{-1}\cdot A\cdot \mathcal{L}$$

$$W_\mu \mapsto \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot\left[P_{F;\mu},\mathcal{L}\right].$$

- A convenient way to keep track of the scale is to introduce the conformal factor $g_{(0)} = \frac{1}{z^2}\eta$. Then $z \mapsto \lambda^{-1}z$. This z should be thought of as the renormalization scale.

'he Renormalization group

- To study RG systematically, we proceed in two steps:

'tep 1: Lower the cutoff $M \mapsto \lambda M$, by integrating out the "fast modes"

$$Z[M, z, A, W] = Z[\lambda M, z, \widetilde{A}, \widetilde{W}] \qquad (\textit{Polchinski})$$

'tep 2: Perform a $CO(L_2)$ transformation to bring the cutoff back to M, ut in the process changing $z \mapsto \lambda^{-1} z$

$$Z[\lambda M, z, \widetilde{A}, \widetilde{W}] = Z[M, \lambda^{-1} z, \mathcal{L}^{-1} \cdot \widetilde{A} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot \widetilde{W} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_F, \mathcal{L}]]$$

- We can now compare the sources at the same cutoff, but different z. Thus, z becomes the natural flow parameter, and we can think of the sources as being z-dependent. (Thus we have the Polchinski formalism extended to include both a cutoff and an RG scale — required for a holographic interpretation).

nfinitesimal version: RG equations

- Infinitesimally, we parametrize the $CO(L_2)$ transformation as

$$\mathcal{L} = \mathbf{1} + \varepsilon z W_z$$

- should be thought of as the z-component of the connection.
- The RG equations become

$$A(z + \varepsilon z) = A(z) + \varepsilon z\, [W_z, A] + \varepsilon z \beta^{(A)} + O(\varepsilon^2)$$

$$W_\mu(z + \varepsilon z) = W_\mu(z) + \varepsilon z \left[P_{F;\mu} + W_\mu, W_z\right] + \varepsilon z \beta_\mu^{(W)} + O(\varepsilon^2)$$

- The beta functions are *tensorial*, and quadratic in A and $\widehat{W}$.
- The flat connection $W^{(0)}$ also satisfies a "pure-gauge" RG equation

$$W_\mu^{(0)}(z + \varepsilon z) = W_\mu^{(0)}(z) + \varepsilon z \left[P_{F;\mu} + W_\mu^{(0)}, W_z^{(0)}\right] + O(\varepsilon^2)$$

RG equations

- Thus, RG extends the sources A and W to bulk fields $\mathcal{A}$ and $\mathcal{W}$.
- Comparing terms linear in ε gives

$$\partial_z \mathcal{W}_\mu^{(0)} - [P_{F;\mu}, \mathcal{W}_z^{(0)}] + [\mathcal{W}_z^{(0)}, \mathcal{W}_\mu^{(0)}] = 0$$

$$\partial_z \mathcal{A} + [\mathcal{W}_z, \mathcal{A}] = \beta^{(\mathcal{A})}$$

$$\partial_z \mathcal{W}_\mu - [P_{F;\mu}, \mathcal{W}_z] + [\mathcal{W}_z, \mathcal{W}_\mu] = \beta_\mu^{(\mathcal{W})}$$

- These equations are naturally thought of as being part of fully covariant equations (e.g., the first is the $z\mu$ component of a bulk 2-form equation, where $d \equiv dx^\mu P_{F,\mu} + dz\partial_z$.)

$$d\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge \mathcal{W}^{(0)} = 0$$

$$d\mathcal{A} + [\mathcal{W}, \mathcal{A}] = \beta^{(\mathcal{A})}$$

$$d\mathcal{W} + \mathcal{W} \wedge \mathcal{W} = \beta^{(\mathcal{W})}$$

- The resulting equations are then diff invariant in the bulk.

The bulk extensions

- Let me summarize what has happened in going from the field theory spacetime to the bulk

$$
\begin{aligned}
W_\mu(x,y) &\rightarrow \mathcal{W} = W_\mu(z;x,y)dx^\mu + W_z(z;x,y)dz \\
A(x,y) &\rightarrow \mathcal{A}(z;x,y) \\
\Pi^\mu(x,y) &\rightarrow \mathcal{P}^\mu(z;x,y) \\
\Pi(x,y) &\rightarrow \mathcal{P}(z;x,y) \\
\beta_\mu^{(W)} &\rightarrow \boldsymbol{\beta}^{(W)} = \beta_a^{(\mathcal{W})} e^{\underline{z}} \wedge e^a + \beta_{ab}^{(\mathcal{W})} e^a \wedge e^b \\
\beta^{(A)} &\rightarrow \boldsymbol{\beta}^{(A)} = \beta^{(\mathcal{A})} e^{\underline{z}} + \beta_a^{(\mathcal{A})} e^a
\end{aligned}
$$

- The transverse components of the beta functions (that don't appear in the RG equations) satisfy their own flow equations (Bianchi identities)

$$
\mathcal{D}\boldsymbol{\beta}^{(\mathcal{A})} = \left[\boldsymbol{\beta}^{(\mathcal{W})}, \mathcal{A}\right], \quad \mathcal{D}\boldsymbol{\beta}^{(\mathcal{W})} = 0
$$

Hamilton-Jacobi Structure

- If we identify $Z = e^{iS_{HJ}}$, then a fundamental relation in H-J theory is

$$\frac{\partial}{\partial z} S_{HJ} = -\mathcal{H}$$

- We can thus read off the Hamiltonian of the theory, for which the full set of RG equations are the Hamilton equations

$$\begin{aligned} \mathcal{H} &= -\mathrm{Tr}\Big\{ \Big(\Big[\mathcal{A}, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta^{(\mathcal{A})} \Big) \cdot \mathcal{P} \Big\} \\ & \quad -\mathrm{Tr}\Big\{ \Big(\Big[P_{F;\mu} + \mathcal{W}_\mu, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta_\mu^{(\mathcal{W})} \Big) \cdot \mathcal{P}^\mu \Big\} \\ & \quad -\frac{N}{2}\mathrm{Tr}\Big\{ \Big(\Delta^\mu \cdot \widehat{\mathcal{W}}_\mu + \Delta^z \cdot \widehat{\mathcal{W}}_{\underline{e}_z^{(0)}} \Big) \Big\} \end{aligned}$$

- Note that this is linear in momenta — the hallmark of a free theory.

[see S.-S. Lee]

- there is a corresponding action S_{HJ}, written in terms of phase space variables

Hamilton-Jacobi Structure

- The existence of this Hamiltonian structure is confirmation that $(\mathcal{A}, \mathcal{P})$ etc. are conjugate pairs (with trivial symplectic form), as required by holography.
- The full set of RG equations of the field theory (β-functions and C-S equations) are required to see this structure, and they are 'integrable' in the sense that they can be thought of as the corresponding Hamilton equations.
- In terms of RG, this means that γ-functions that appear in C-S equations are related to (derivatives of) the β-functions.
- Thus the (full set of) ERG equations are holographic in a standard sense.
- This seems to be a feature that is not shared by other attempts at interpreting higher spin theories holographically. Looking just at the connection misses half of the bulk dynamics and by itself does not have a holographic interpretation.

The Bulk Spacetime

- A natural flat background connection is given by

$$\mathcal{W}^{(0)}(x,y) = -\frac{dz}{z} D(x,y) + \frac{dx^\mu}{z} P_\mu(x,y)$$

where $P_\mu(x,y) = \partial_\mu^{(x)}\delta(x-y)$ and $D(x,y) = (x^\mu \partial_\mu^{(x)} + \Delta)\delta(x-y)$.

- (flat because of the commutation relations of D, P_μ)
- This connection is equivalent to the vielbein and spin connection of *AdS*. This is appropriate, since $W^{(0)}$ corresponds to the free fixed point (with $z = 1$), which is conformally invariant.
  - $\mathcal{W}^{(0)}$ is invariant under the conformal algebra $o(2,d) \subset co(L_2)$
- $\mathcal{W}$ and $\mathcal{A}$ correspond to geometric structures over *AdS*.
  - We want to specify more precisely what this geometry is (later).

Remarks

- The fermion is more difficult to deal with in higher dimensions, because of additional single-trace operators like

$$\tilde{\psi}\gamma^{\mu\nu}\psi, \quad \tilde{\psi}\gamma^{\mu\nu\lambda}\psi, \quad \ldots$$

- These will give rise through the ERG construction to higher spin theories that have more than just a connection. (The B-model exists only in $d = 4$).
- the bosonic free fixed point can be understand similarly, although there are a few complications

Bosonic Relativistic Free Fixed Point

- Another example consists of N complex scalar fields. In this case, we formulate the single-trace deformations in terms of the $CU(L_2)$ connection.

$$S = \int \tilde{\phi}_m \cdot \left(\left[D_{F;\mu} + W_\mu \right]^2 + B \right) \cdot \phi^m$$

- The ERG equations give rise to an 'A-model' in any dimension.

[RGL, O. Parrikar, A.B. Weiss, to appear.]

- Here though there is an extra background symmetry

$$Z[M, z, B, W_\mu^{(0)}, \widehat{W}_\mu + \Lambda_\mu] = Z[M, z, B + \{\Lambda^\mu, D_\mu\} + \Lambda_\mu \cdot \Lambda^\mu, W_\mu^{(0)}, \widehat{W}_\mu]$$

- this background symmetry allows for fixing $W_\mu \to W_\mu^{(0)}$, and the corresponding transformed B sources all single-trace currents.

[This was the starting point of Douglas, et al, and so geometry was not manifest.]

Remarks, cont.

- What of standard gravitational holography?
- The standard higher spin lore is expected to kick in here — when interactions are included, the higher spin symmetry breaks (the operators get anomalous dimensions). At strong coupling, all that is left behind is gravity.
- It is an interesting challenge to show that precisely this happens generically (!!).
- Perturbatively nearby fixed points (e.g., large N saddle points) are accessible, and will have an operator spectrum whose anomalous dimensions scale as $1/N^x$.
  - N is insignificant prior to the introduction of field theory interactions

Geometry: The Infinite Jet bundle

- We have been using gauge theory terms — connection, gauge transformations, etc.
- Usually gauge $\equiv$ local, $\psi(x) \mapsto e^{i\alpha(x)}\psi(x)$
- What are we to make of the non-local transformation?

$$\psi(x) \mapsto \int_y \mathcal{L}(x, y)\psi(y)$$

- We have been using matrix notation: space-time coordinates $\equiv$ matrix indices
- If we can regard these really as matrix indices, then we would have an ordinary vector bundle, with W_μ as a connection.
- The proper interpretation here turns out to be in terms of a mathematical construction known as an infinite jet bundle.

The Infinite Jet bundle...

- The simple idea is that we can think of a differential operator $\mathcal{L}(x,y)$ as a matrix by "prolongating" the field

$$\psi^m(x) \mapsto \left(\psi^m(x), \frac{\partial\psi^m}{\partial x^\mu}(x), \frac{\partial^2\psi^m}{\partial x^\mu \partial x^\nu}(x)\cdots\right)$$

- The collection of such vectors at a point is called the infinite jet space. The bundle over spacetime of such jet spaces is called the infinite jet bundle.
- Then, differential operators, such as $P_\mu(x,y) = \partial_\mu^{(x)}\delta(x-y)$ are interpreted as matrices $\mathbb{P}_\mu$ that act on these vectors
- Note one effect of this organization: we have a clean demarcation between the vector index on W_μ and the 'higher spin indices' of $W_\mu^{ab\ldots}$.

The Infinite Jet bundle

- The bi-local transformations can be thought of as local gauge transformations of the jet bundle.
- The gauge field W is a connection 1-form on the jet bundle, while A is a section of its endomorphism bundle.
- RG instructs us how to extend the infinite jet bundle of the boundary field theory into the bulk, and then $\mathcal{W}$ and $\mathcal{A}$ are defined correspondingly in the bulk.

The Bulk Action and Correlation Functions

- I presented you with a bulk Hamiltonian. This can be promoted to a bulk phase space action (here, for the bosonic theory)

$$I = \int dz \, \mathrm{Tr} \left\{ \mathcal{P}^I \cdot \left(\mathcal{D}_I \mathfrak{B} - \beta_I^{(\mathfrak{B})} \right) + \mathcal{P}^{IJ} \cdot \mathcal{F}_{IJ}^{(0)} + N \, \Delta_B \cdot \mathfrak{B} \right\}$$

- Here Δ_B is a derivative with respect to M of the cutoff function.
- As in any holographic theory, we solve the bulk equations of motion in terms of boundary data, and obtain the on-shell action, which encodes the correlation functions of the field theory.
- It is straightforward to carry this out exactly for the free fixed point.
- Here we have

$$I_{o.s.} = N \int \Delta_B \cdot \mathfrak{B}$$

where now $\mathfrak{B}$ is the bulk solution

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The Bulk Action and Correlation Functions

- The RG equation

$$\left[\mathcal{D}_z^{(0)}, \mathfrak{B}\right] = \beta_z^{(\mathfrak{B})} = \mathfrak{B} \cdot \Delta_B \cdot \mathfrak{B}$$

can be solved iteratively

$$\mathfrak{B} = \alpha \mathfrak{B}_{(1)} + \alpha^2 \mathfrak{B}_{(2)} + \ldots,$$

$$\begin{aligned}
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(1)}\right] &= 0 \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(2)}\right] &= \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(3)}\right] &= \mathfrak{B}_{(2)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} + \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(2)} \\
&\vdots
\end{aligned}$$

The Bulk Action and Correlation Functions

- The first equation (2) is homogeneous and has the solution

$$\mathfrak{B}_{(1)}(z;x,y) = \int_{x',y'} K^{-1}(z;x,x') b_{(0)}(x',y') K(z;y',y)$$

where we have defined the boundary-to-bulk Wilson line

$$K(z) = P.\exp \int_{\epsilon}^{z} dz'\, \mathcal{W}_z^{(0)}(z')$$

with the boundary being placed at $z = \epsilon$.

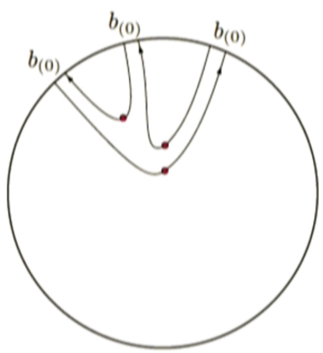
- $b_{(0)}$ has the interpretation of a boundary source
- this can then be inserted into the second order equation and the whole system solved iteratively

The Bulk Action and Correlation Functions

- At k^{th} order, one finds a contribution to the on-shell action

$$\begin{aligned} I^{(k)}_{o.s.} &= N \int_\epsilon^\infty dz_1 \int_\epsilon^{z_1} dz_2 \ldots \int_\epsilon^{z_{k-1}} dz_k \\ &\times \mathrm{Tr}\, H(z_1) \cdot b_{(0)} \cdot H(z_2) \cdot b_{(0)} \cdot \ldots \cdot H(z_k) \cdot b_{(0)} \\ &+ permutations \end{aligned}$$

where $H(z) \equiv K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) = \partial_z g(z)$



The Witten diagram for the bulk on-shell action at third order.

The Bulk Action and Correlation Functions

- The z-integrals can be performed trivially, resulting in

$$I_{o.s.}^{(k)} = \frac{N}{k} \operatorname{Tr} \left(g_{(0)} \cdot b_{(0)} \right)^k$$

where $g_{(0)} = g(\infty)$ is the boundary free scalar propagator

- These can be resummed, resulting in

$$Z[b_{(0)}] = \det{}^{-N} \left(g_{(0)}^{-1} - b_{(0)} \right)$$

which is the exact generating functional for the free fixed point (up to a boundary term).

- Thus, this holographic theory does everything that it can for us.

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Non-Relativistic Higher Spin Theory

- we can use exactly the same methods to construct the $z = 2$ non-relativistic free theory, and its higher spin dual
- use DLCQ, i.e., we change the field theory metric to

$$ds^2 = d\xi dt + d\vec{x}^2$$

with scale transformation acting as

$$\xi \mapsto \xi, t \mapsto \lambda^2 t, \quad \vec{x} \mapsto \lambda \vec{x}$$

- $z = 2$ is special in that ∂_ξ is central, and thus can restrict ϕ^m to a superselection sector with fixed ξ-momentum, n
- corresponding massless higher spin fields can be thought of as having $n = 0$, with n encoded in the flat connection $W^{(0)}$, which here corresponds to the Schrödinger spacetime $Schr_{D+3}$

The Bulk Action and Correlation Functions

- The z-integrals can be performed trivially, resulting in

$$I^{(k)}_{o.s.} = \frac{N}{k}\,\mathrm{Tr}\left(g_{(0)} \cdot b_{(0)}\right)^k$$

  where $g_{(0)} = g(\infty)$ is the boundary free scalar propagator
- These can be resummed, resulting in

$$Z[b_{(0)}] = \det{}^{-N}\left(g^{-1}_{(0)} - b_{(0)}\right)$$

  which is the exact generating functional for the free fixed point (up to a boundary term).
- Thus, this holographic theory does everything that it can for us.

Non-Relativistic Higher Spin Theory

- we can use exactly the same methods to construct the $z = 2$ non-relativistic free theory, and its higher spin dual
- use DLCQ, i.e., we change the field theory metric to

$$ds^2 = d\xi dt + d\vec{x}^2$$

with scale transformation acting as

$$\xi \mapsto \xi, t \mapsto \lambda^2 t, \quad \vec{x} \mapsto \lambda \vec{x}$$

- $z = 2$ is special in that ∂_ξ is central, and thus can restrict ϕ^m to a superselection sector with fixed ξ-momentum, n
- corresponding massless higher spin fields can be thought of as having $n = 0$, with n encoded in the flat connection $W^{(0)}$, which here corresponds to the Schrödinger spacetime $Schr_{D+3}$

nteractions

- one can turn on interactions on the field theory side
- the free fixed point can always be thought of as a vectorial theory, but interactions determine how to think of the field content (depending on what the interactions do to the global symmetries)
- the simplest possibility is to turn on all $O(N)$- (or $U(N)$)-invariant multi-trace interactions
- the corresponding multi-local (tensorial) sources give rise to new canonically conjugate pairs in the bulk
- the ERG equations couple all of these together – there is generically no ‘consistent truncation’ (other than restricting to single-trace ops)
- here, large N plays a crucial role — there is a solution of the ERG equations in which the double-trace coupling is fixed on the boundary. Essentially, we expect B and Π to swap roles. This is the interacting fixed point. [RGL+OP+ABW, to appear]

A Final Remark

- part of the lore of higher spin theory is that you might expect to obtain it from string theory in the tensionless limit (although this is horribly naive)
- consider a typical Witten diagram