Title: Sequential growth dynamics and quantum random walks

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Abstract: Rideout and Sorkin proposed a classical dynamics for causal sets based upon a sequential growth model. Comparing it with models for sequential growth in other systems, and with the dual goals of generating manifold-like causal sets and finding a quantum dynamics for them, we propose some modifications to their model. The resulting, admittedly speculative, proposal is a type of quantum random walk. We explore its properties in some simple cases.

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Sequential growth dynamics and quantum random walks

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Crystal growth

[Temperley 1951, Rost 1981, Krapivsky & Olejarz 2013]

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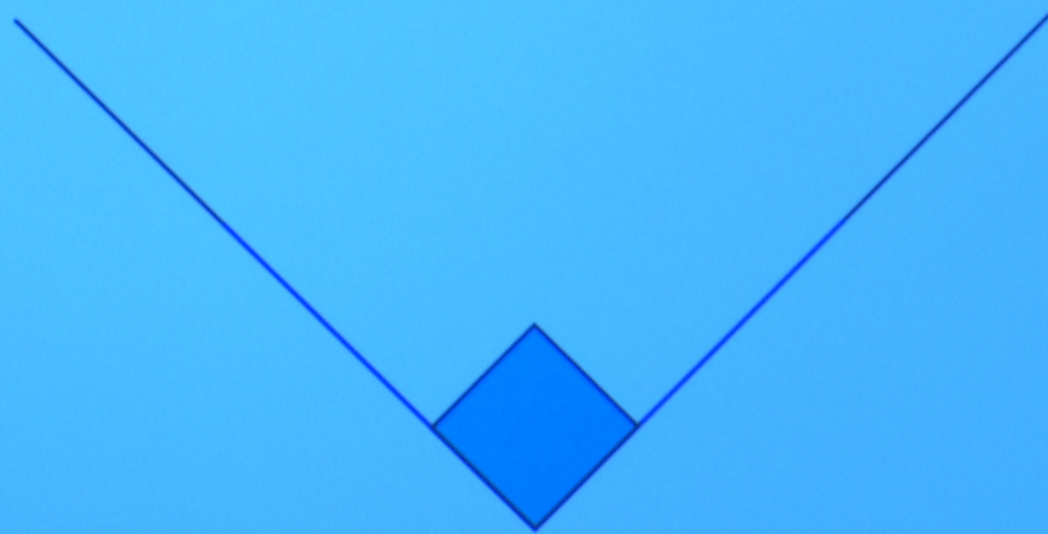
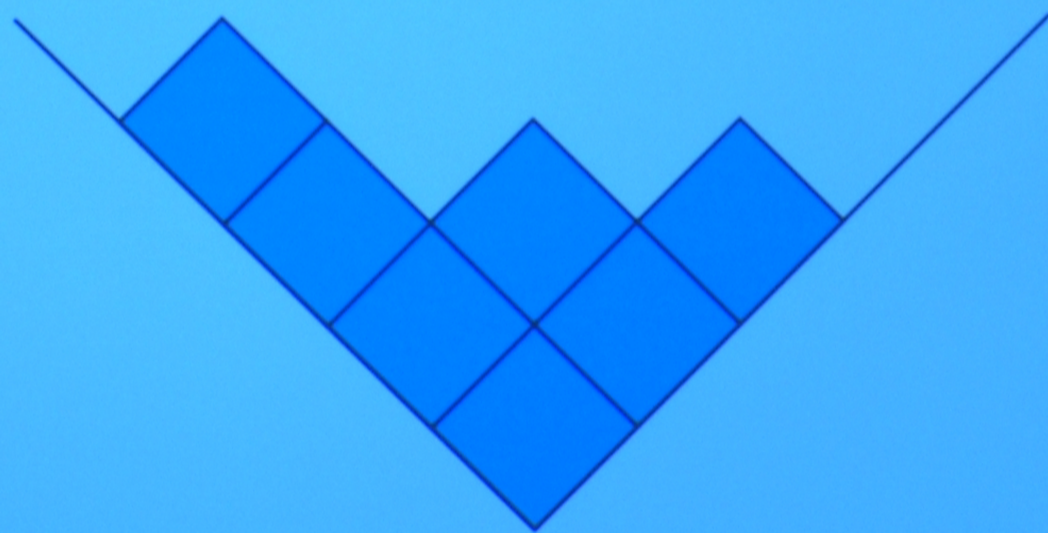


Crystal growth

[Temperley 1951, Rost 1981, Krapivsky & Olejarz 2013]



[movie]

Crystal growth

[Temperley 1951, Rost 1981, Krapivsky & Olejarz 2013]

Classical sequential growth (CSG) model.

Structure is 2 dimensional.

Appears to have limiting shape (it does)
that is Lorentz invariant (not quite; really $\sqrt{u} + \sqrt{v} = (6T)^{1/4}$).

Rideout and Sorkin [1999] invented a CSG process for causal sets.

Causal sets

The causal set approach to quantum gravity takes as the underlying discrete structure locally finite partially ordered sets.

Hawking [1976] and Malament [1977]: **The causal structure of a space-time manifold determines its topology, differentiable structure, and conformal metric.**

Sorkin observed that the local conformal factor (volume) could be approximated by counting if the causal structure—which is a partially ordered set (poset)—were finite.

Causal set kinematics

A causal set C is:

$$\text{reflexive: } x \preccurlyeq x$$
$$\text{antisymmetric: } (x \preccurlyeq y \wedge y \preccurlyeq x) \Rightarrow x = y$$
$$\text{transitive: } (x \preccurlyeq y \wedge y \preccurlyeq z) \Rightarrow x \preccurlyeq z$$

for all $x, y, z \in C$.

Causal set kinematics

A causal set C is:

$$\text{reflexive: } x \preccurlyeq x \text{ and } (x \preccurlyeq y \wedge x \neq y) \Rightarrow x \prec y$$

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$$\text{transitive: } (x \preccurlyeq y \wedge y \preccurlyeq z) \Rightarrow x \preccurlyeq z$$
$$\text{locally finite: } |\{w \mid x \preccurlyeq w \preccurlyeq y\}| \in \mathbb{N}$$

for all $x, y, z \in C$.

The kinematics of causal sets, *e.g.*, dimension, topology, and some geometry, is reasonably well understood.

Causal set dynamics

Possible dynamics for causal sets have been harder to determine.

In a sum-over-histories approach one would like an action $A[C] \in \mathbb{R}$ so that the amplitude for (certain) sets of causal sets would be:

$$\sum_{C} e^{-(i/\hbar)A[C]}.$$

More generally, a dynamics is determined by a measure on a sigma algebra over the set of causal sets, where the measure is defined for subsets corresponding to observables.

Classical dynamics for causal sets

Rideout and Sorkin [1999] defined such a measure *via* a stochastic classical sequential growth model:

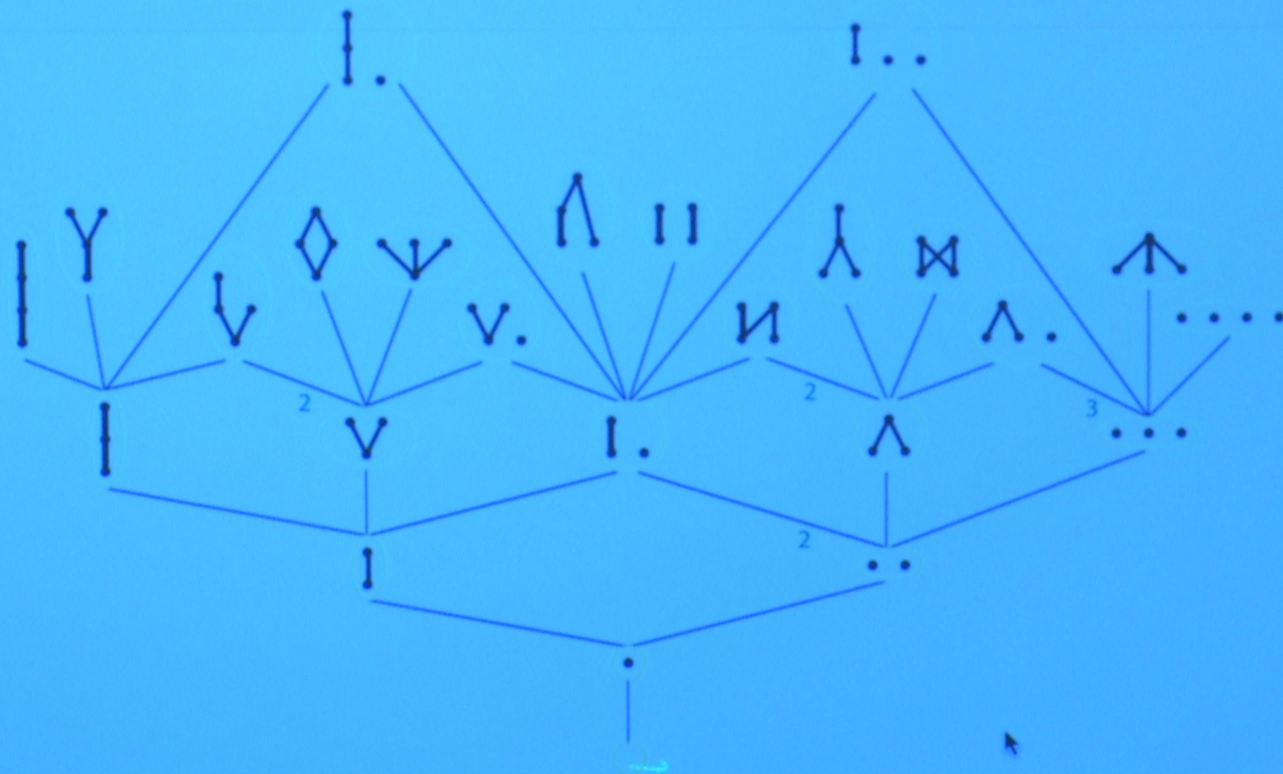
maximal elements accrete one by one,

growing an infinite, but past finite, poset.

So evolution is an infinite upward path from $\emptyset$ in the poset $\mathcal{P}$ of finite causal sets ordered by

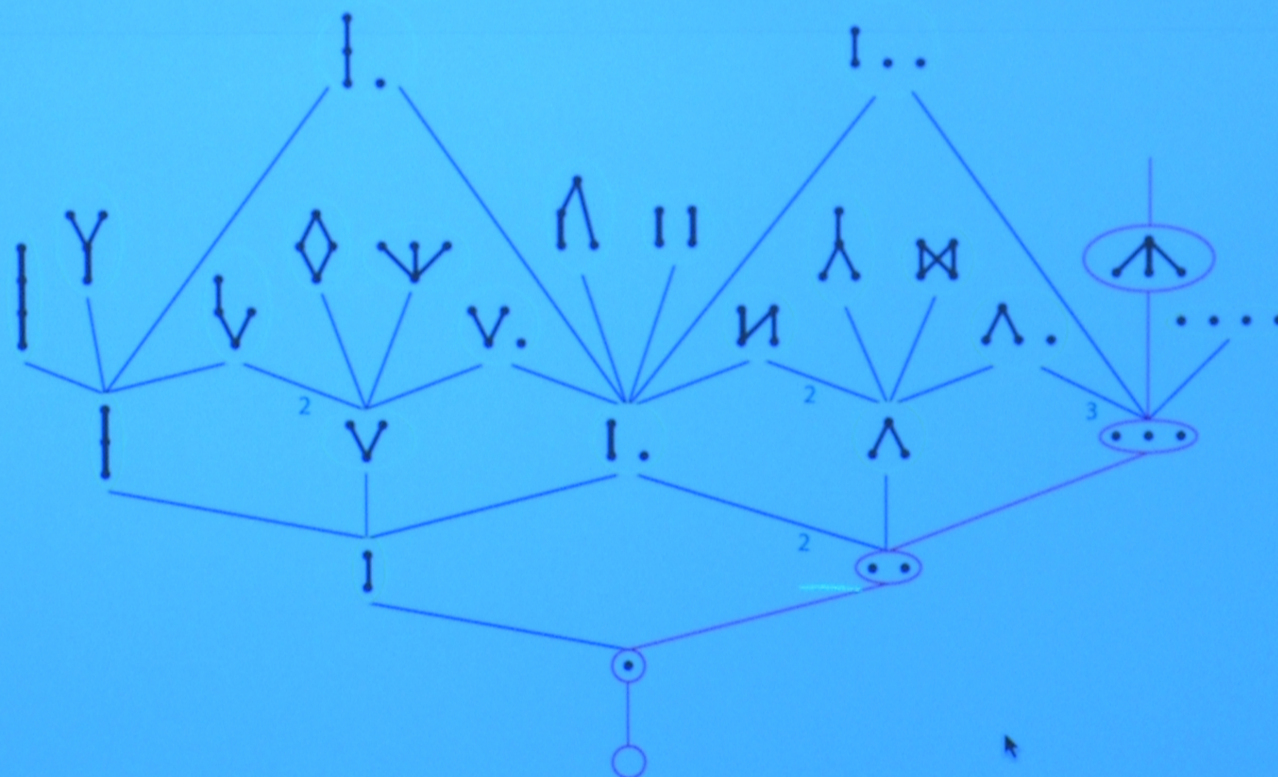
$$C \prec C' \text{ iff } C = C' \setminus \{\text{a maximal element in } C'\}.$$

Classical sequential growth



The bottom of the poset $\mathcal{P}$ of finite causal sets.

Classical sequential growth



Another instance.

Conditions on the classical measure

The elements of the set on which the sigma algebra is defined are equivalence classes of such semi-infinite paths $\gamma : \mathbb{N} \to \mathcal{P}$ with $\gamma \sim \gamma'$ iff $\exists T \in \mathbb{N}$ such that $\forall \tau > T$, $\gamma(\tau) = \gamma'(\tau)$.

Rideout and Sorkin [1999] attach a transition probability to each link in $\mathcal{P}$, satisfying three properties:

Conservation of probability: The sum of the transition probabilities (with multiplicities) from each causal set is 1.

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Rideout and Sorkin [1999] attach a transition probability to each link in $\mathcal{P}$, satisfying three properties:

Discrete covariance: The measure attached to $[\gamma]$ should not depend on a choice of time coordinate, *i.e.*, a representative element of $[\gamma]$. This can be enforced by requiring for all $C \in \mathcal{P}$ that each path from $\emptyset$ to C has the same product of transition probabilities.

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Bell causality: The relative probability of two new elements attaching to past sets in C depends only on the union of those past sets.

Solutions for the classical measure

These three conditions can be satisfied. A generic solution is specified by a countable number of nonnegative reals: $1 = w_0, w_1, w_2, \ldots$.

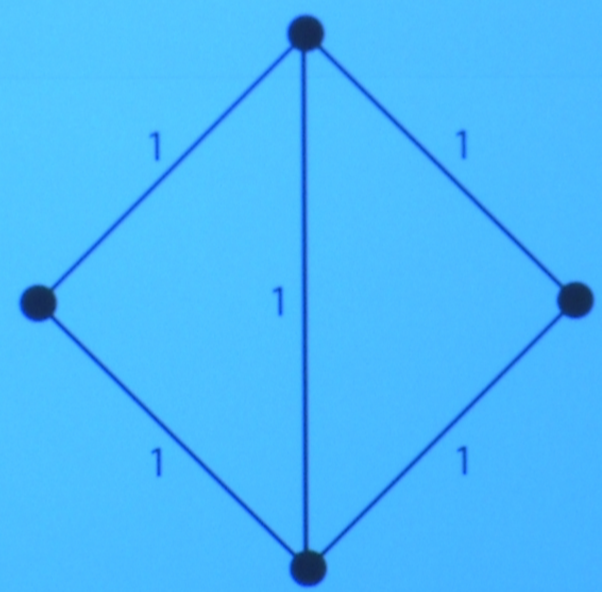
The probability of each path from $\emptyset$ to $C \in \mathcal{P}$ is given by a statistical mechanical model of spins on "relations":

$$\frac{1}{Z} \sum_{\sigma} \prod_{y \in C} w_{r(y;\sigma)},$$

where $\sigma : \{x \prec y\} \to \mathbb{Z}_2$ with $\sigma(x \prec y) = 1$ if there is no $z \in C$ such that $x \prec z \prec y$, $r(y;\sigma) = |\{x \in C \mid x \prec y \wedge \sigma(x \prec y) = 1\}|$, and

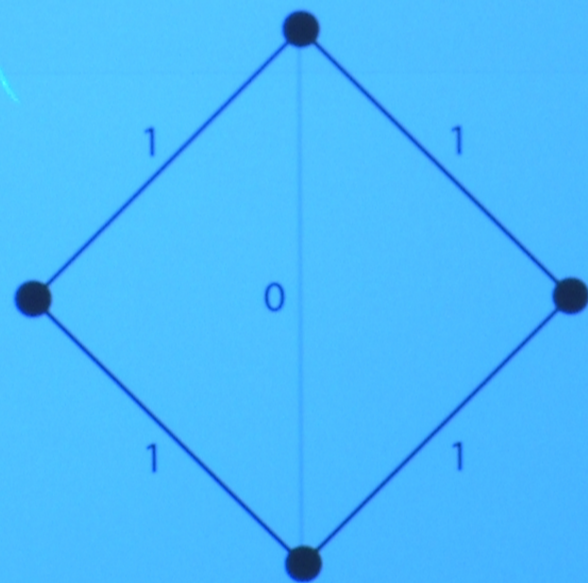
$$Z = \prod_{n=0}^{|C|-1} \binom{n}{k} w_k.$$

Solutions for the classical measure



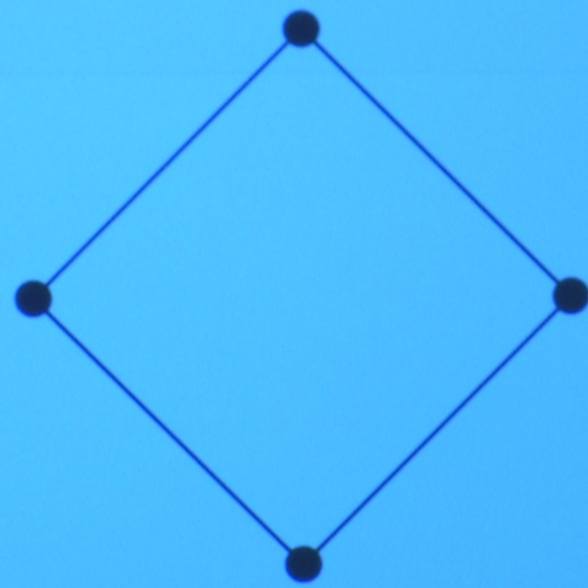
$$\prod_{y \in C} w_{r(y;\sigma)} = w_3 w_1^2$$

Solutions for the classical measure



$$\prod_{y \in C} w_{r(y;\sigma)} = w_2 w_1^2$$

Solutions for the classical measure



So each path to C has probability

$$\sum_{\sigma} \prod_{y \in C} w_{r(y;\sigma)} = (w_3 + w_2) w_1^2 .$$

CSG dynamics for causal sets

Finding and solving this model was a *tour de force* of mathematical physics.

Interesting properties have been worked out by Rideout & Sorkin [1999, 2000], by Martín, O'Connor, Rideout & Sorkin [2001], by Rideout [2002], by Varadarajan & Rideout [2006], by Ahmed & Rideout [2010], by Wüthrich [2012], and by Callender & Wüthrich [2014].

But there are some problems ... it seems to be hard to get manifold-like causal sets, and of course, this is not yet a quantum dynamics.

Failure of manifold-like-ness

Suppose the CSG process grows a manifold-like causal set . . .

Failure of manifold-like-ness

. . . then it can just as well bifurcate a second copy of the future of some past set.

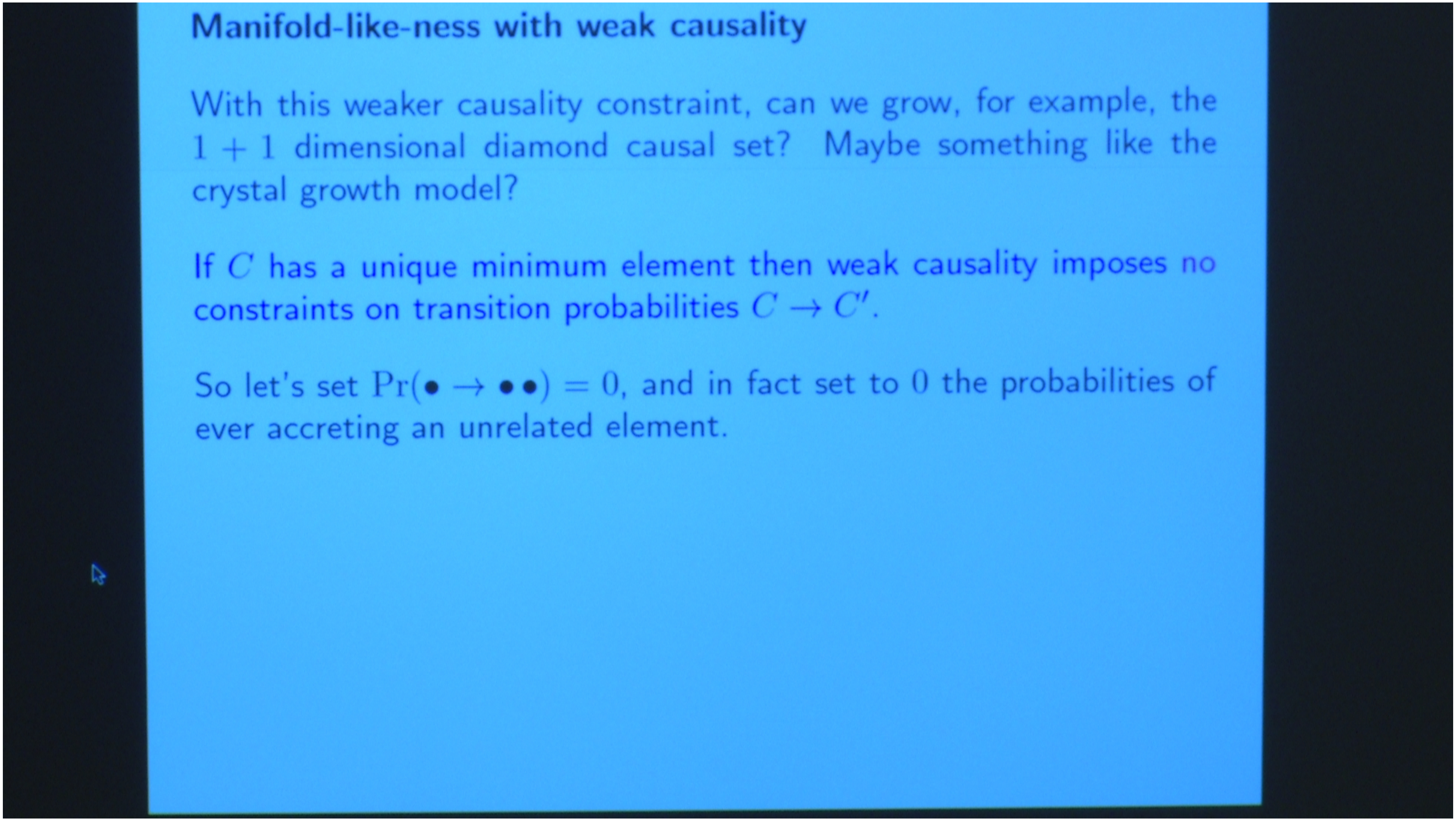
Failure of manifold-like-ness

The problem is that as the second copy starts growing, "it doesn't know" the first copy is there; because of the Bell causality condition it depends only on the past set.

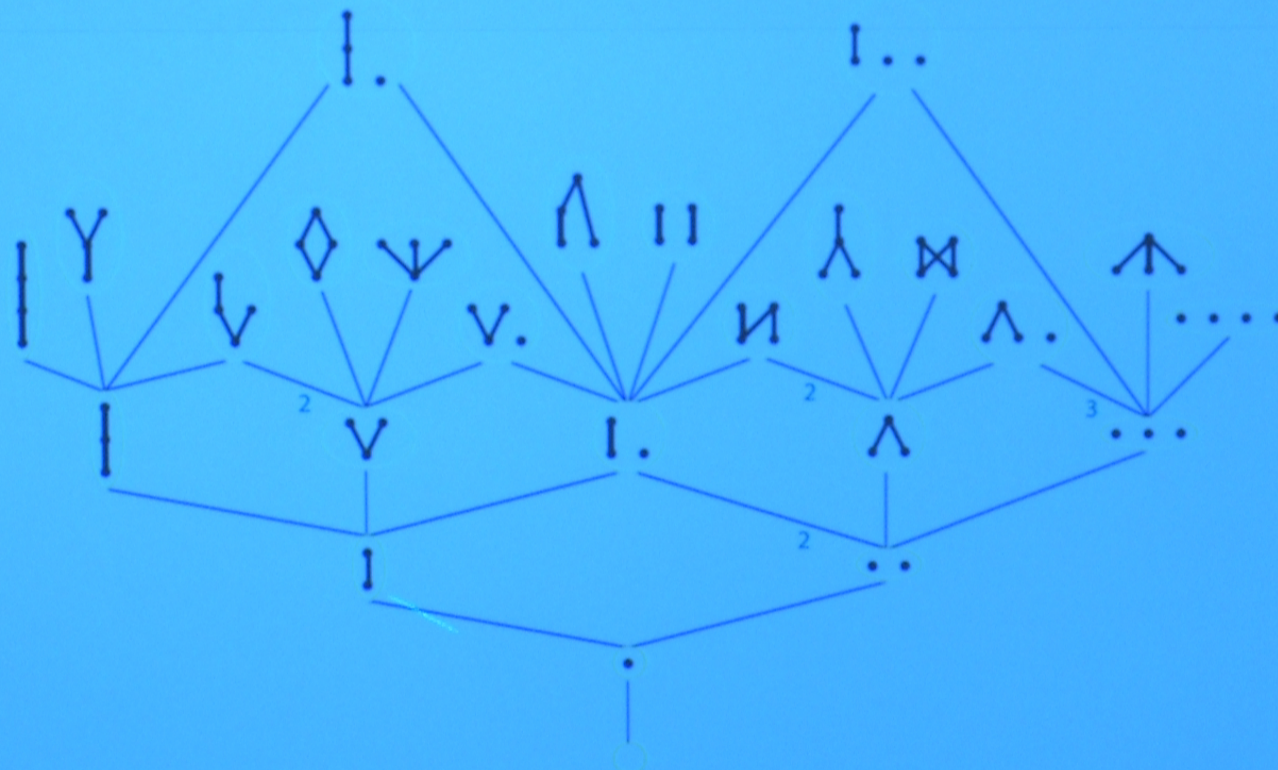
Maybe that condition is too strong: to the extent we take seriously the statistical mechanical model of spins on "relations", perhaps we should take seriously the idea that there is matter in the model, evolving forward in time.

And that matter is present, and evolving into the future, at each non-maximal element of any causal set C.

Weak causality: The relative probability of two new elements attaching to past sets in C depends only on the union of the futures of those past sets.


Manifold-like-ness with weak causality
With this weaker causality constraint, can we grow, for example, the 1 + 1 dimensional diamond causal set? Maybe something like the crystal growth model?
If C has a unique minimum element then weak causality imposes no constraints on transition probabilities $C \to C'$.
So let's set $\mathrm{Pr}(\bullet \to \bullet\bullet) = 0$, and in fact set to 0 the probabilities of ever accreting an unrelated element.

Manifold-like-ness with weak causality



Set all the transition probabilities with new unrelated elements to 0.

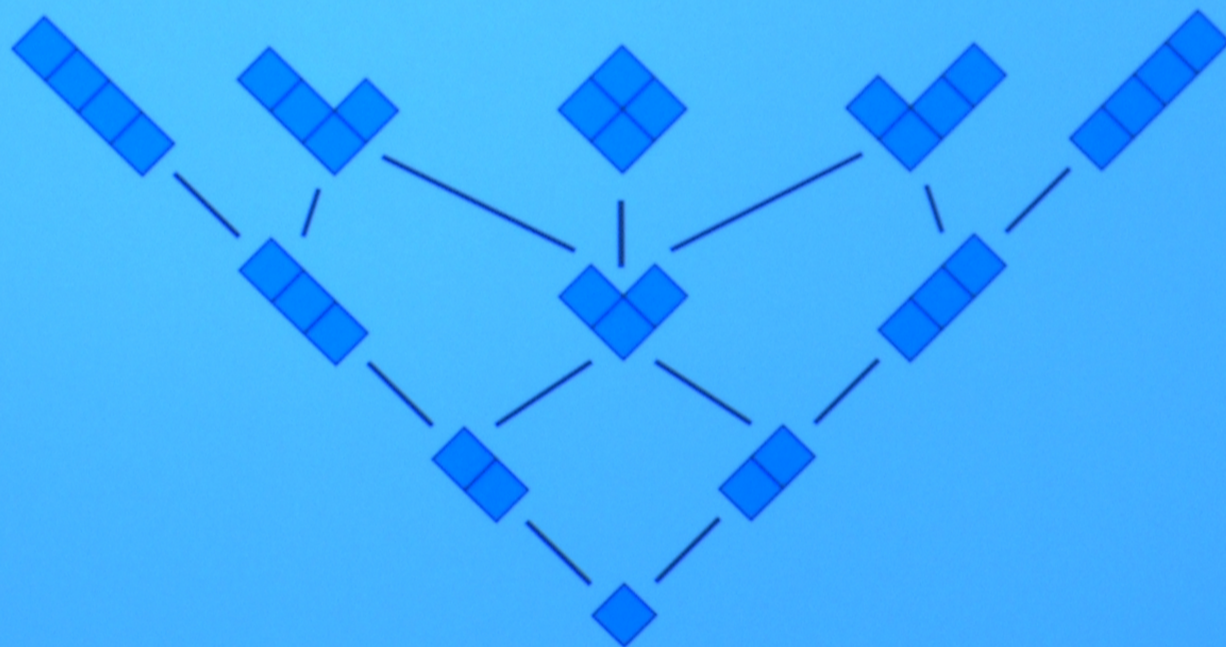
Manifold-like-ness with weak causality

Keep nonzero only transitions to diamond posets.

Discrete covariance

Can we choose transition probabilities that satisfy discrete covariance?

Maybe we don't need to; it's not true in the crystal growth model:



Discrete covariance

Can we choose transition probabilities that satisfy discrete covariance?

Maybe we don't need to; it's not true in the crystal growth model. And remember it was sufficient, but not necessary:

Discrete covariance: The measure attached to $[\gamma]$ should not depend on a choice of time coordinate, *i.e.*, a representative element of $[\gamma]$. This can be enforced by requiring for all $C \in \mathcal{P}$ that each path from $\emptyset$ to C has the same product of transition probabilities.

Or by defining the probability of C to be the sum of the probabilities of the paths from $\emptyset$ to C.

Such total discrete covariance is no condition at all on the transition probabilities; it only (continues) to control the observables allowed.

CSG with weak causality and total discrete covariance

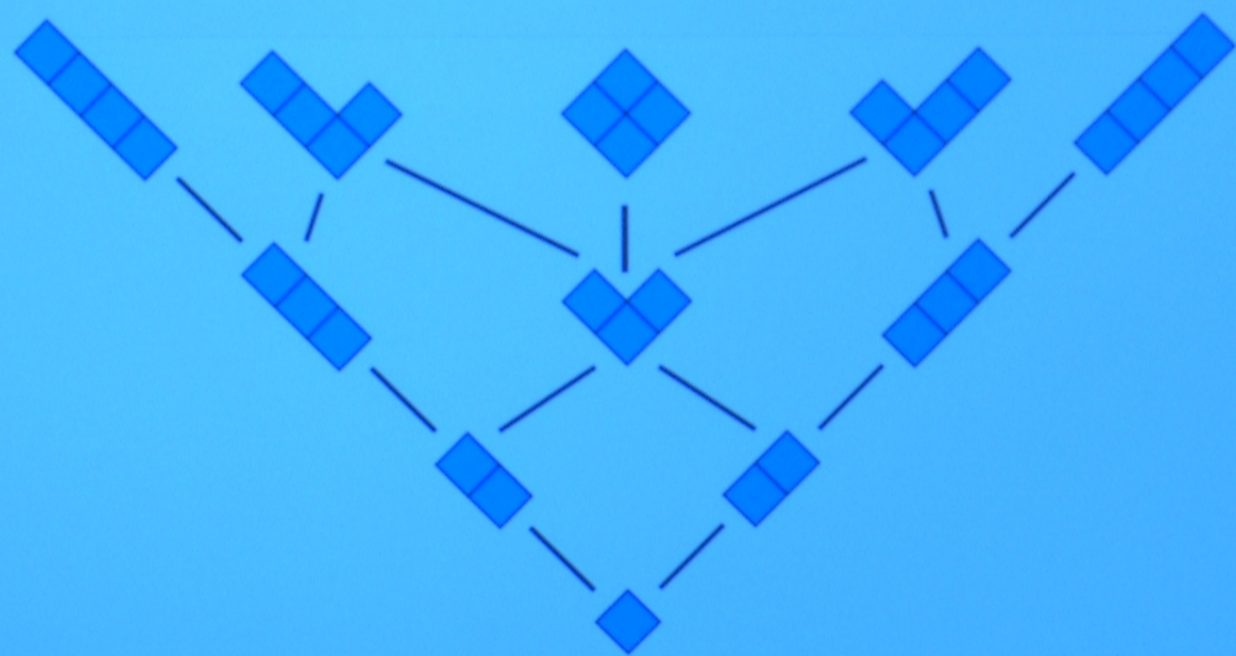
Besides their physical meaning, the original CSG constraints led to a "small" family of models.

By weakening these conditions we lose this, but gain the possibility of evolving causal sets that are manifold-like.

We can also now consider completely different families of models.

Crystal growth at finite temperature

[Temperley 1951, Rost 1981, Krapivsky & Olejarz 2013]



Random walk on this undirected graph. [movie]

Crystal growth at finite temperature

[Temperley 1951, Rost 1981, Krapivsky & Olejarz 2013]

When the crystal only grows, the area is T.

When the transition probabilities from a configuration to each of the adjacent configurations are equal, the expected area grows as $O(\sqrt{T})$.

When the transition probabilities downward are $\epsilon \ll 1$ bigger than the transition probabilities upward, the area has a limiting value, and the top curve is given by

$$e^{-au} + e^{-av} = 1,$$

where $a = \pi/\sqrt{6A}$ and $A \propto \epsilon^{-2}$ is the expected area.

Quantum crystal evolution

Now that the graph Γ of allowed transitions is not directed, we can define a quantum random walk (QRW) on it:

If A is the adjacency "matrix" of Γ and D is the diagonal "matrix" $D_{vv} =$ the degree of vertex v, then the graph Laplacian is $L = A - D$, and

$$H = -L = D - A$$

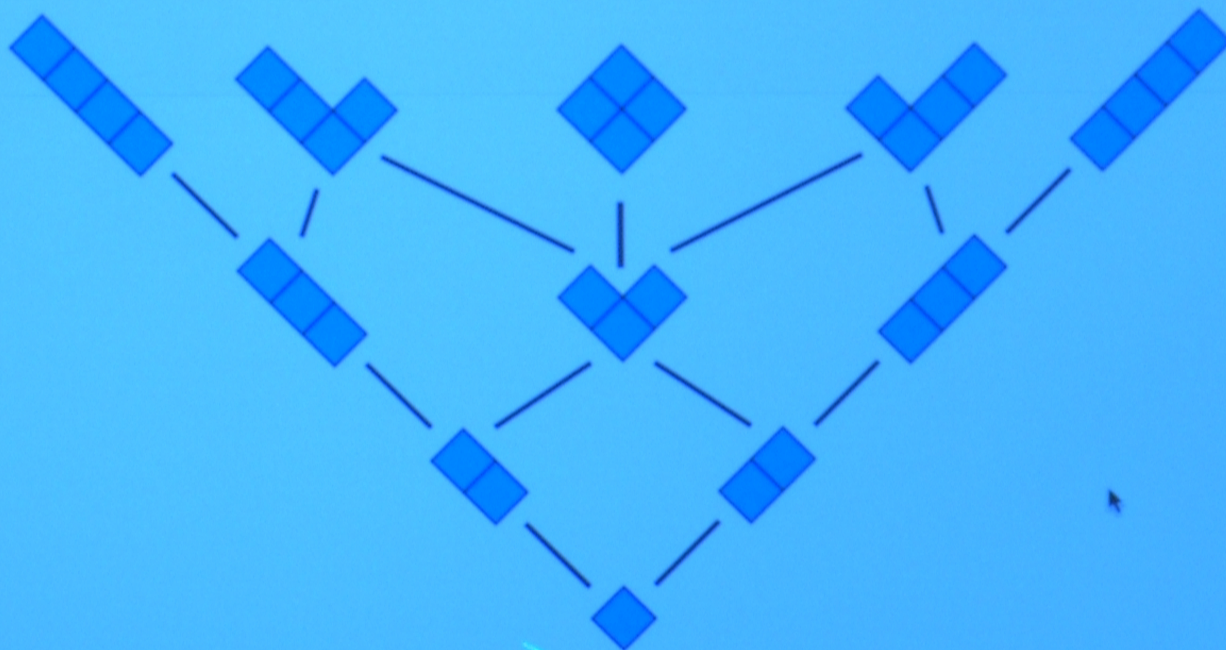
is a Hamiltonian.

Quantum random walk evolution is just the Schrödinger equation with this Hamiltonian:

$$i\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = H\psi,$$

for $\psi \in \mathbb{C}[V]$, where V is the vertex set of Γ.

Quantum crystal evolution with width 1



Equivalently, QRW on $\mathbb{N}$. [movie]

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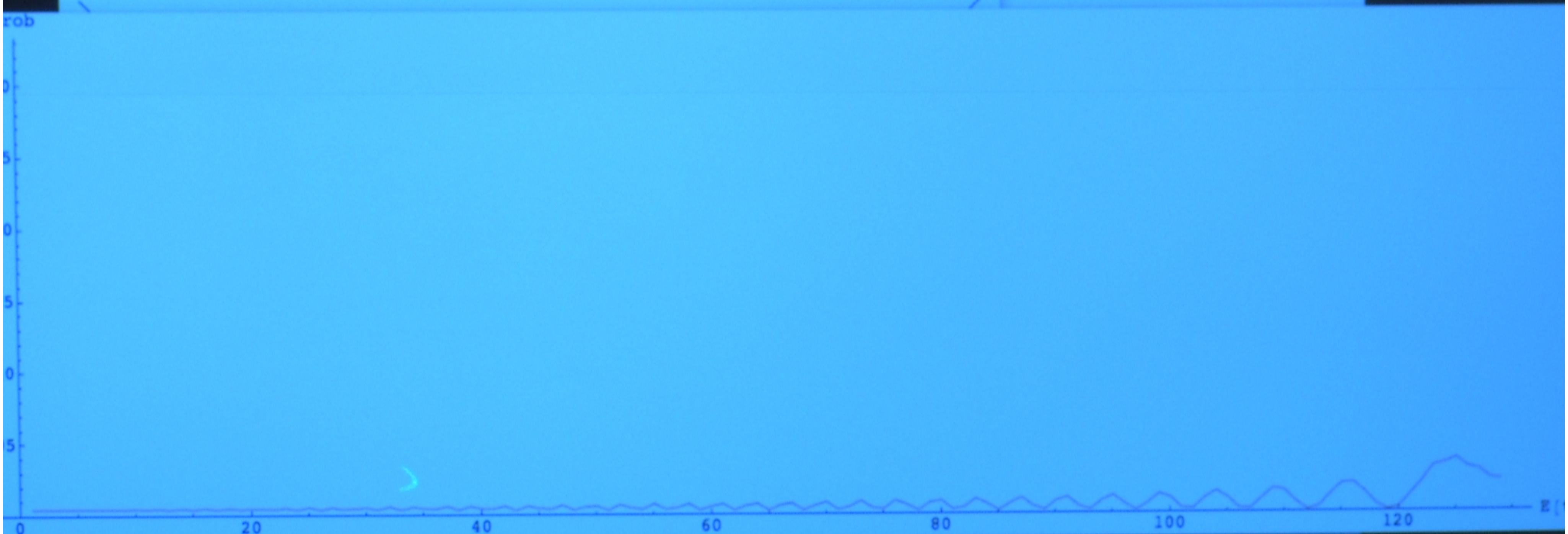

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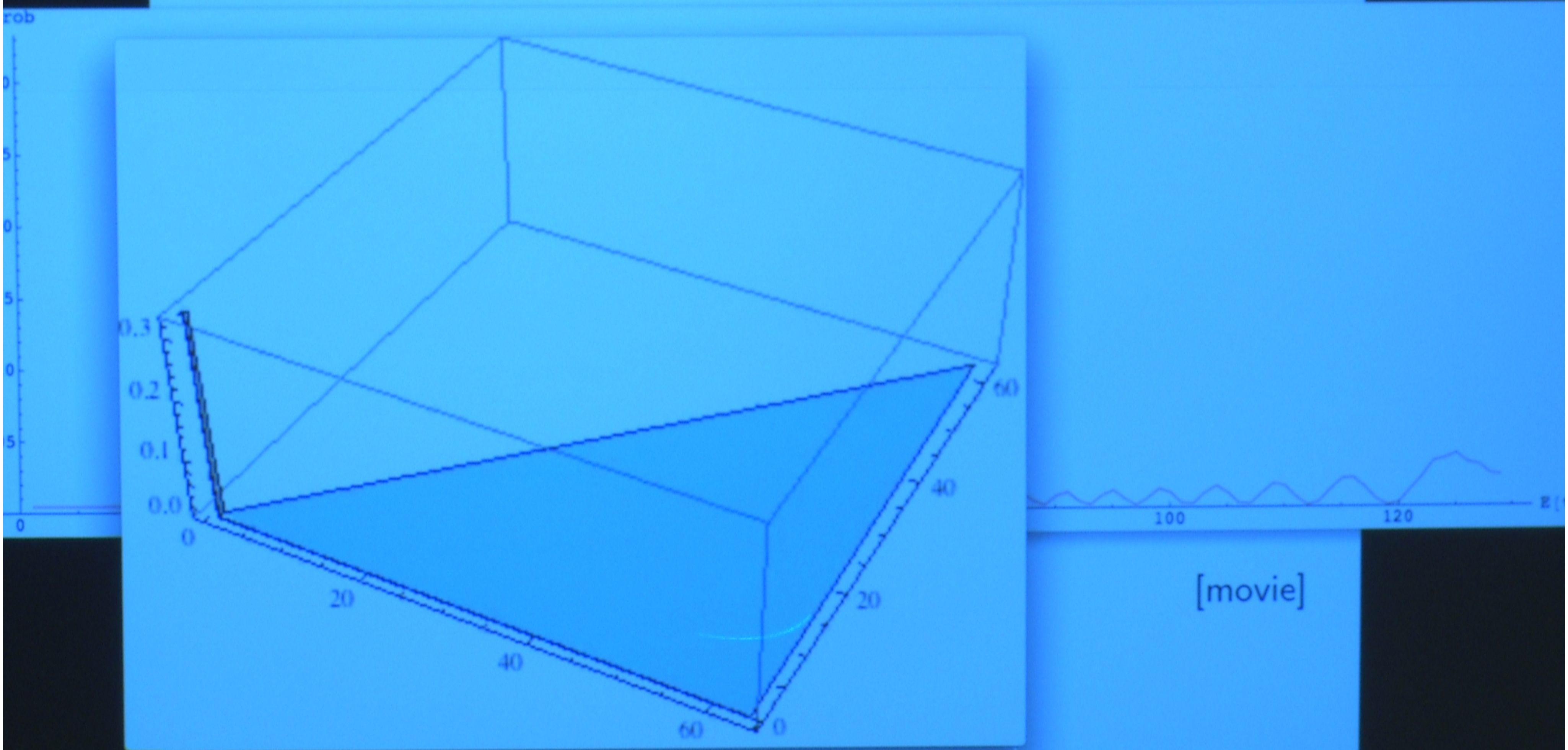
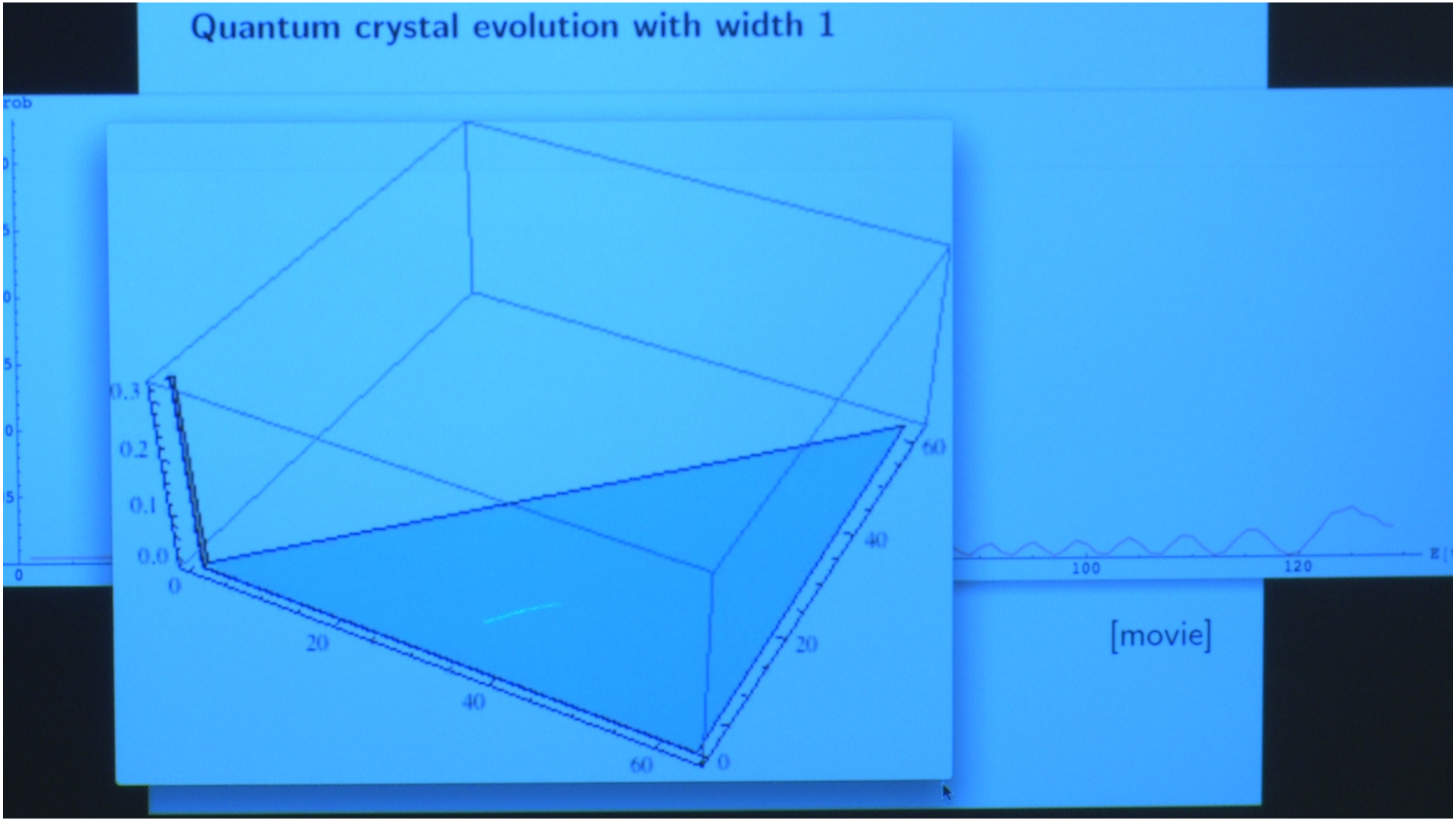


[movie]


Quantum crystal evolution with width 1
0.3
0.2
0.1
0.0
0
20
40
60
0
20
40
60
100
120
[movie]

Summary

The two-dimensionality and approximate Lorentz invariance of the crystal growth model suggests reconsideration of causal set CSG model constraints.

Weakening the causality constraint and totalizing the discrete covariance condition allows sequential growth of the diamond causal set.

These weaker constraints on sequential growth are satisfied by finite temperature crystal growth models, which suggests considering random walks on the **undirected** poset of causal sets.

And it suggests considering **quantum** random walks on the undirected poset of causal sets.

Directions

Simulations of the simplest cases demonstrate that

$$\frac{\mathrm{dE}[\text{past vol}]}{\mathrm{d}\tau} = \frac{\mathrm{dE}[t]}{\mathrm{d}\tau} > 0.$$

Perhaps the true unitarity of evolution in τ is manifested in the classical spacetime regime as very close to, but not exactly, unitarity of evolution in t. This would mislead us into looking for quantum gravity models which are unitary with respect to some internal time, rather than with respect to τ.