Title: Some Exact Results for Conformal Field Theories in d>2

Date: May 27, 2014 04:00 PM

URL: http://pirsa.org/14050137

Abstract:

Convexity and Liberation at Large Spin

Zohar Komargodski

Weizmann Institute of Science, Israel

with Alexander Zhiboedov + in progress with Kulaxizi, Parnachev, and Zhiboedov

see also related work by Fitzpatrick, Kaplan, Poland, Simmons-Duffin

Introduction

- CFTs describe second order phase transitions.
- CFTs appear at the end points of RG flows.
- CFTs can secretly describe quantum gravity in AdS space.

For these and other reasons, they are omnipresent in many areas of physics, including high energy physics.

Introduction

Many CFTs have a useful weak coupling parameter, for example,

- $O(N)$ critical model at large N and similar theories with CS-terms.
- Banks-Zaks-like fixed points.
- QED_3 with many charged fermions.
- Wilson-Fisher fixed points.
- Theories with exactly marginal couplings, e.g. $\mathcal{N} = 4$ at weak coupling.

Introduction

Some of the most interesting systems don't have any known useful weak coupling expansion

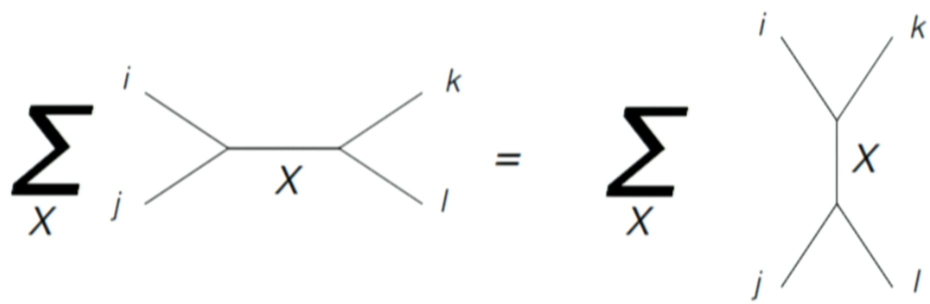
- Many second order phase transition points, such as the 3d Ising model etc.
- QED_3 with a small number of fermions.
- Many theories that appear in theoretical high energy physics, for instance, lots of 4d SCFTs, $d > 4$ CFTs, theories that appear in AdS/CFT etc.

Introduction

CFTs in 2d enjoy an infinite symmetry group and some of them are completely solvable. However, no nontrivial $d > 2$ theory with finite two-point function for the stress tensor has been solved.

Introduction

CFTs are implicitly determined by the constraint of associativity of the operator algebra (bootstrap equations)



These infinite sums have overlapping regions of convergence and the equations are mathematically well defined. However, solving these equations is a formidable task.

Introduction

So in general we have a dreadful set of equations describing a theory without any manifest expansion parameter.

Our main point in this talk is that there is an expansion parameter in any CFT. It is the inverse spin, $1/s$.

Introduction

In a sense that we will make precise, we can perform perturbation theory in $1/s$ in *any* CFT to obtain various analytic results. These results are in principle testable both experimentally and with numeric mehods.

An Unphysical Toy Model

We start with a well-known *unphysical* solution to the bootstrap equations. It is called generalized free fields. This will be our "harmonic oscillator."

We have an operator $\Phi(x)$ of dimension Δ and declare that all the correlation functions are given by Wick contractions using the two-point function

$$\langle \Phi(x)\Phi(0)\rangle = \frac{1}{x^{2\Delta}}$$

In particular

$$\langle \Phi(x)\Phi(0)\Phi(y)\Phi(z)\rangle = \frac{1}{x^{2\Delta}(y-z)^{2\Delta}} + \frac{1}{z^{2\Delta}(x-y)^{2\Delta}} + \frac{1}{y^{2\Delta}(x-z)^{2\Delta}}$$

An Unphysical Toy Model

From this four point function

$$\langle \Phi(x)\Phi(0)\Phi(y)\Phi(z)\rangle = \frac{1}{x^{2\Delta}(y-z)^{2\Delta}} + \frac{1}{z^{2\Delta}(x-y)^{2\Delta}} + \frac{1}{y^{2\Delta}(x-z)^{2\Delta}}$$

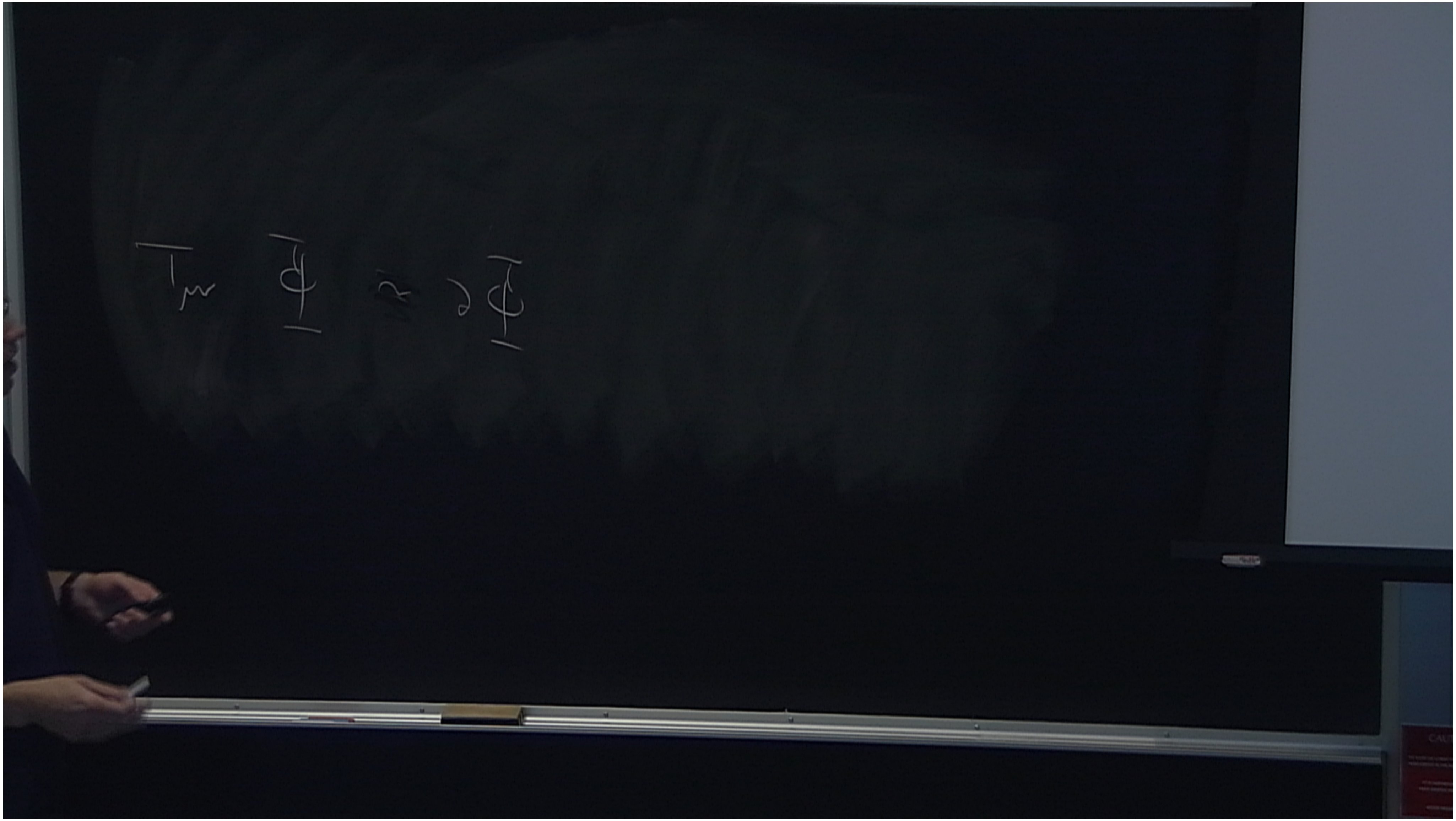
we can read out the spectrum of the theory. It contains the following operators

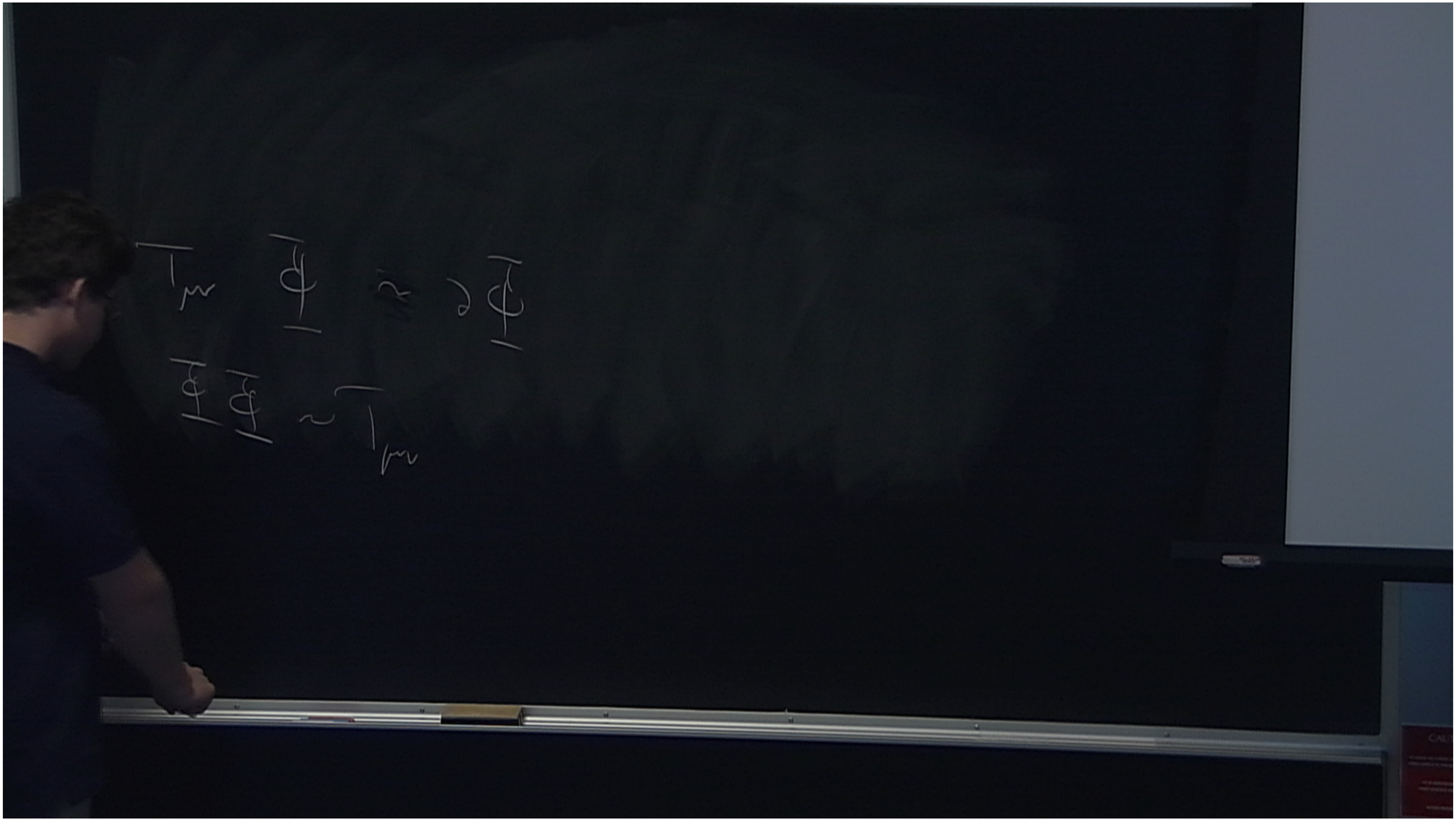
$$\{1, \Phi, \Phi \overleftrightarrow{\partial}^s \Box^n \Phi\}$$

with dimensions

$$\{0, \Delta, 2\Delta + s + 2n\}$$

We see that unless $\Delta = d/2 - 1$ the theory does not contain the energy-momentum operator hence it does not correspond to a physical model. The case of $\Delta = d/2 - 1$ is free field theory (Landau-Ginzburg).




$T_{\mu\nu} \, \Phi \approx \partial \Phi$
$\Phi \, \Phi \sim T_{\mu\nu}$

An Unphysical Toy Model

Using the operators $\{\Phi \overleftrightarrow{\partial}^s \Box^n \Phi\}$ one can write an explicit partial wave decomposition of the four point function. It depends on the dimensions of the intermediate operators, i.e. $\{0, 2\Delta + s + 2n\}$ and their OPE coefficients, $c_{s,n}$.
These OPE coefficients are known exactly, but we will not need to quote them here.

An Additivity Theorem

Define the twist of an operator by

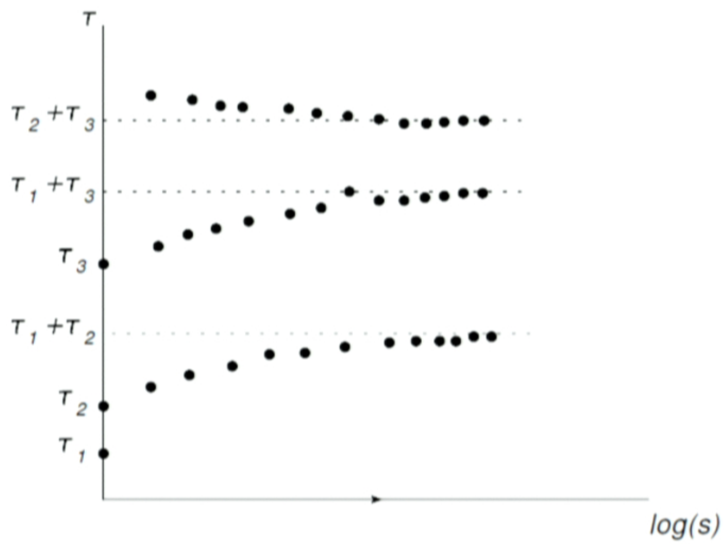
$$\tau = \Delta - s$$

if the twists τ_1 and τ_2 are present in the spectrum then we necessarily have operators with twists arbitrarily close to $\tau_1 + \tau_2$

We refer to this property as "additivity." The proof assumes

- Unitarity
- $d > 2$

An Additivity Theorem



This kind of spectrum is very different from $d = 2$, where we have minimal models, and the spectrum consists of a finite number of twists spaced by integers.

An Additivity Theorem

Consider the OPE of some operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{i,s} f_i^{(s)}(x)\mathcal{O}_i^{(s)}(0)$$

where s is the spin and i is some index that counts operators with the same spin.

The claim is that for large enough s there must always be operators in the OPE with twists arbitrarily close to $\tau_1 + \tau_2$

In this sense, at large s every theory approaches the theory of generalized free fields, where the twist is exactly additive. We can thus always talk about the operators $\Phi \overleftrightarrow{\partial}^s \Box^{n=0} \Phi$ for large enough s.

An Additivity Theorem

Additionally, the OPE coefficients of the operators that approach $\tau_1 + \tau_2$ approach the OPE coefficients of the corresponding generalized free fields operators $c_{s,n=0}$.

One can also study the "daughter trajectories" corresponding to operators with twists $\tau_1 + \tau_2 + 2n$. For fixed n and large enough s, they approach the naive dimensions and OPE coefficients of the operators $\Phi \overleftrightarrow{\partial}^s \Box^n \Phi$. For simplicity, we'll discuss only the main trajectory $n = 0$.

The story for n, s that scale simultaneously to infinity is presumably also of interest, but it is currently not fully developed (see [Cornalba, Costa, Penedones] for connection to the AdS eikonal limit).

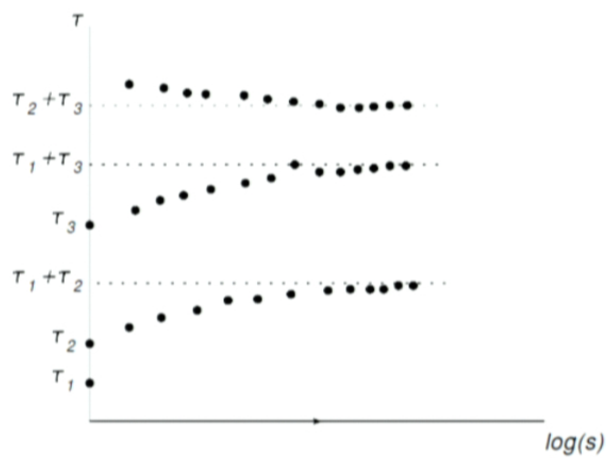
An Additivity Theorem

Rather than to outline the proof, I will now explain how this result can be used as the starting point of a controllable perturbative expansion in any $CFT_{d>2}$.

Note that this additivity theorem is reminiscent of a result by Callan-Gross, who showed that in weakly coupled models consisting of fermions and scalars only, the dimensions of certain operators with large spin get only weakly renormalized.

Perturbation Theory Around $s \to \infty$

Reconsider the typical structure of the spectrum



which we claim becomes "generalized free" as $s \to \infty$. We should be able to determine $\tau(s) - \tau_1 - \tau_2$ for large s in perturbation theory.

Perturbation Theory Around $s \to \infty$

Indeed, since as we claim this is a free limit the bootstrap equations can be studied systematically and we find the following result

$$\lim_{s\to\infty} [\tau(s) - \tau_1 - \tau_2] = -\frac{c}{s^{\tau_*}} ,$$

where τ_* is the smallest twist operator in the theory after the unit operator. (In many theories that would be the stress tensor, thus, $\tau_* = d - 2$).
That the correction around $s = \infty$ is controlled by τ_* was observed in many large N examples in the context of AdS/CFT and it was emphasized in [Alday, Maldacena].

Perturbation Theory Around $s \to \infty$

$$\lim_{s\to\infty}\left[\tau(s)-\tau_1-\tau_2\right] = -\frac{c}{s^{\tau_*}}$$

With our tools we can compute c *exactly*. For simplicity we display the result for the case that $\tau_* = d-2$ coming from the stress tensor and we assume $O_1 = O_2^*$ with dimension denoted by Δ

$$c = \frac{d^2\Gamma(d+2)}{2c_T(d-1)^2\Gamma\left(\frac{d+2}{2}\right)^2}\frac{\Delta^2\Gamma(\Delta)^2}{\Gamma\left(\Delta-\frac{d-2}{2}\right)^2}$$

c_T is the 2-point function of the stress tensors. Note that $c > 0$, hence the spectrum is always *asymptotically convex*.

Perturbation Theory Around $s \to \infty$

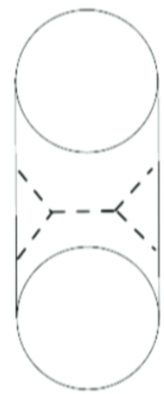
Actually, one can prove that there is some s_* starting from which the spectrum of operators $O_1 \partial^s O_1^*$ is convex. This is correct in all $d > 2$ CFTs. It is harder to prove a general bound on s_*. In all examples we considered we found $s_* = 2$, so we will assume this is indeed the case.

Perturbation Theory Around $s \to \infty$

Actually, one can prove that there is some s_* starting from which the spectrum of operators $O_1 \partial^s O_1^*$ is convex. This is correct in all $d > 2$ CFTs. It is harder to prove a general bound on s_*. In all examples we considered we found $s_* = 2$, so we will assume this is indeed the case.

AdS/CFT

One of the first computations people did in AdS/CFT was the computation of the anomalous dimension at strong coupling of the operator $\mathcal{L}\overleftrightarrow{\partial}^s\mathcal{L}$ where $\mathcal{L}$ is the $\mathcal{N} = 4$ Lagrangian. This corresponds to computing dilaton scattering diagrams in AdS_5.



One finds after computing all the relevant Witten diagrams:

$$\tau_{\mathcal{L}\overleftrightarrow{\partial}^s\mathcal{L}} = 8 - \frac{96}{N^2}\frac{1}{s^2}$$

AdS/CFT

We take our formula for c and remember that c_T is fixed by an anomaly, $c_T = 40N^2$. Plugging this into our formula we find

$$c = \frac{2\Delta^2(\Delta-1)^2}{3N^2}$$

and putting $\Delta = 4$ for the Lagrangian operator we recover precisely the result of the calculation of an AdS_5 diagram!

$$c = \frac{96}{N^2}$$

3d Ising Model

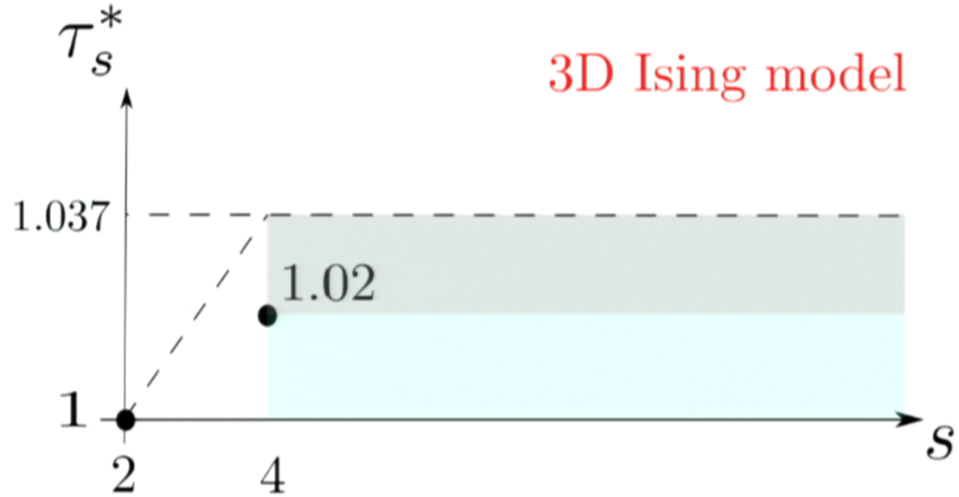
The critical 3d Ising model is a CFT with $\mathbf{Z}_2$ symmetry where we know experimentally that the lowest lying operators are the spin field σ with $\Delta(\sigma) \sim 0.518$ and the energy operator ϵ with $\Delta(\epsilon) = 1.41$. Consider the OPE $\sigma(x)\sigma(0)$. We have shown that there must exist operators with twists arbitrarily close to 1.037 and we can determine how this is approached at large spin

$$\tau^{3d\ Ising}_{\sigma\partial^s\sigma} \sim 1.037 - \frac{0.0028}{s} + \cdots$$

From convexity we learn that we have to have operators with twists smaller than 1.037 for every spin, and indeed, we know 'experimentally' that there is a spin 4 operator with $\Delta = 5.02$ hence $\tau = 1.02$, consistently with our picture. We can now make a prediction

3d Ising Model

$$1.02 < \tau(s) < 1.037\ , \qquad s = 6, 8, ..., \infty$$



There must be operators in the grey shaded region for every spin.

Perturbation Theory Around $s \to \infty$

$$\lim_{s\to\infty} [\tau(s) - \tau_1 - \tau_2] = -\frac{c}{s^{\tau_*}}$$

With our tools we can compute c *exactly*. For simplicity we display the result for the case that $\tau_* = d - 2$ coming from the stress tensor and we assume $O_1 = O_2^*$ with dimension denoted by Δ

$$c = \frac{d^2\Gamma(d+2)}{2c_T(d-1)^2\Gamma\left(\frac{d+2}{2}\right)^2}\frac{\Delta^2\Gamma(\Delta)^2}{\Gamma\left(\Delta - \frac{d-2}{2}\right)^2}$$

c_T is the 2-point function of the stress tensors. Note that $c > 0$, hence the spectrum is always *asymptotically convex*.

Finally, we demonstrate how our general picture for $d > 2$ CFTs is consistent with explicit results from the epsilon expansion.
Consider the $O(N)$ Wilson-Fisher fixed point at $d = 4 - \epsilon$. One can calculate in an expansion in ϵ around $\epsilon = 0$ where the theory is free. Our methods predict that at large s

$$\tau_{\sigma\partial^s\sigma} = 2 + \gamma_\sigma - \frac{c}{s^2} + \cdots$$

and we predict also $c > 0$.
Calculating the anomalous dimensions $\tau_{\sigma\partial^s\sigma}$ in the epsilon expansion one indeed finds [Wilson,Kogut]

$$\tau_{\sigma\partial^s\sigma} = 2 + \gamma_\sigma - \frac{\epsilon^2(3N+6)}{(N+8)^2 s^2} + \cdots$$

Numeric Bootstrap

There has been extremely nice progress on tackling the bootstrap equations via a systematic numeric approach [Rattazzi,Rychkov,Vichi,El-Showk, Paulos, Poland, Simmons-Duffin...]. These authors have generated quite a lot of data, some of which can be directly compared to our claims, and as far as I can tell there is agreement.

Constraints on OPE Coefficients

Let us now turn to the OPE coefficients f_i^s

$$\mathcal{O}(x)\mathcal{O}^*(0) = \sum_{i,s} f_i^{(s)}(x)\mathcal{O}_i^{(s)}(0) \ .$$

Our ideas about $s \to \infty$ apply. For the operators whose twists approach $\tau_1 + \tau_2 + n$ at large s, the OPE coefficients approach the generalized free field ones. One can compute the deviation from the generalized free field value in our large-spin perturbation theory.

However, let us try to say something about the operators with finite spin.

Perturbation Theory Around $s \to \infty$

$$\lim_{s\to\infty} [\tau(s) - \tau_1 - \tau_2] = -\frac{c}{s^{\tau_*}}$$

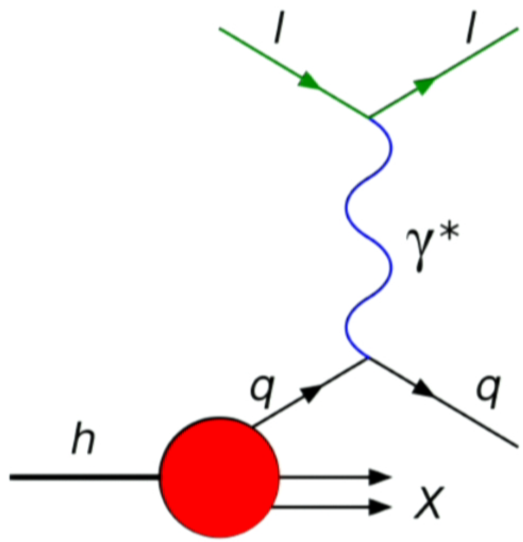
With our tools we can compute c *exactly*. For simplicity we display the result for the case that $\tau_* = d - 2$ coming from the stress tensor and we assume $O_1 = O_2^*$ with dimension denoted by Δ

$$c = \frac{d^2\Gamma(d+2)}{2c_T(d-1)^2\Gamma\left(\frac{d+2}{2}\right)^2} \frac{\Delta^2\Gamma(\Delta)^2}{\Gamma\left(\Delta - \frac{d-2}{2}\right)^2}$$

c_T is the 2-point function of the stress tensors. Note that $c > 0$, hence the spectrum is always *asymptotically convex*.

Constraints on OPE Coefficients

The idea is to consider some field, J, weakly coupled to O, $\int d^dxJO$, and imagine doing deep inelastic scattering with J being like the virtual photon of ordinary deep inelastic



Constraints on OPE Coefficients

Let us now turn to the OPE coefficients f_i^s

$$\mathcal{O}(x)\mathcal{O}^*(0) = \sum_{i,s} f_i^{(s)}(x)\mathcal{O}_i^{(s)}(0) \ .$$

Our ideas about $s \to \infty$ apply. For the operators whose twists approach $\tau_1 + \tau_2 + n$ at large s, the OPE coefficients approach the generalized free field ones. One can compute the deviation from the generalized free field value in our large-spin perturbation theory.

However, let us try to say something about the operators with finite spin.

Constraints on OPE Coefficients

The deep inelastic cross section, which is manifestly positive, is directly related (via the optical theorem) to a two-point function in the state $|h(p)>$:

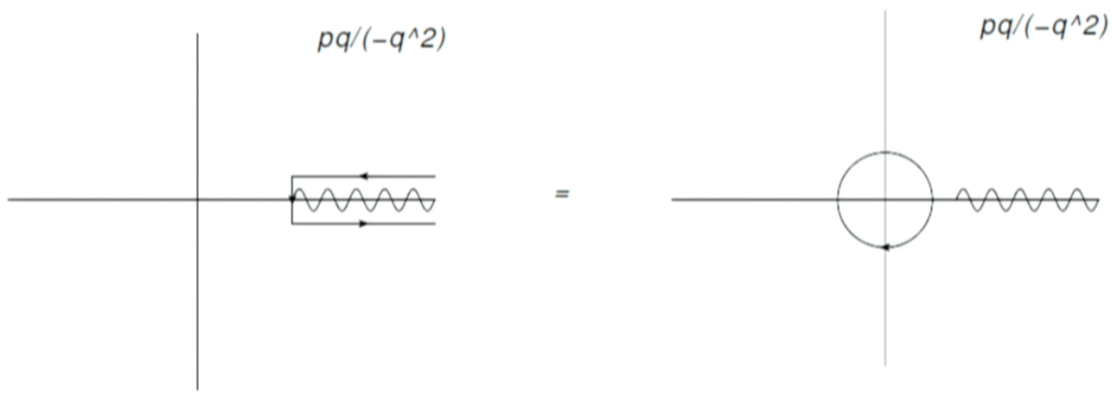
$$\sigma_{DIS} \sim Im \int d^d x e^{iqx} \langle h(p)|\mathcal{O}(x)\mathcal{O}^*(0)|h(p)\rangle \ .$$

$q^2 < 0$ is space like.

Physical deep inelastic scattering can occur as long as $p \cdot q/(-q^2) > 1$. However, the OPE expansion of the two-point function is valid when $-q^2 >> everything$.

Constraints on OPE Coefficients

The amplitude $\int d^dx e^{iqx} \langle h(p)|\mathcal{O}(x)\mathcal{O}^*(0)|h(p)\rangle$ is therefore analytic in the OPE regime. By using the usual Cauchy trick, we can relate, for fixed q^2, the regions of large and small $p \cdot q/(-q^2)$



This assumes some good behavior at infinity, which is the Regge limit.

Constraints on OPE Coefficients

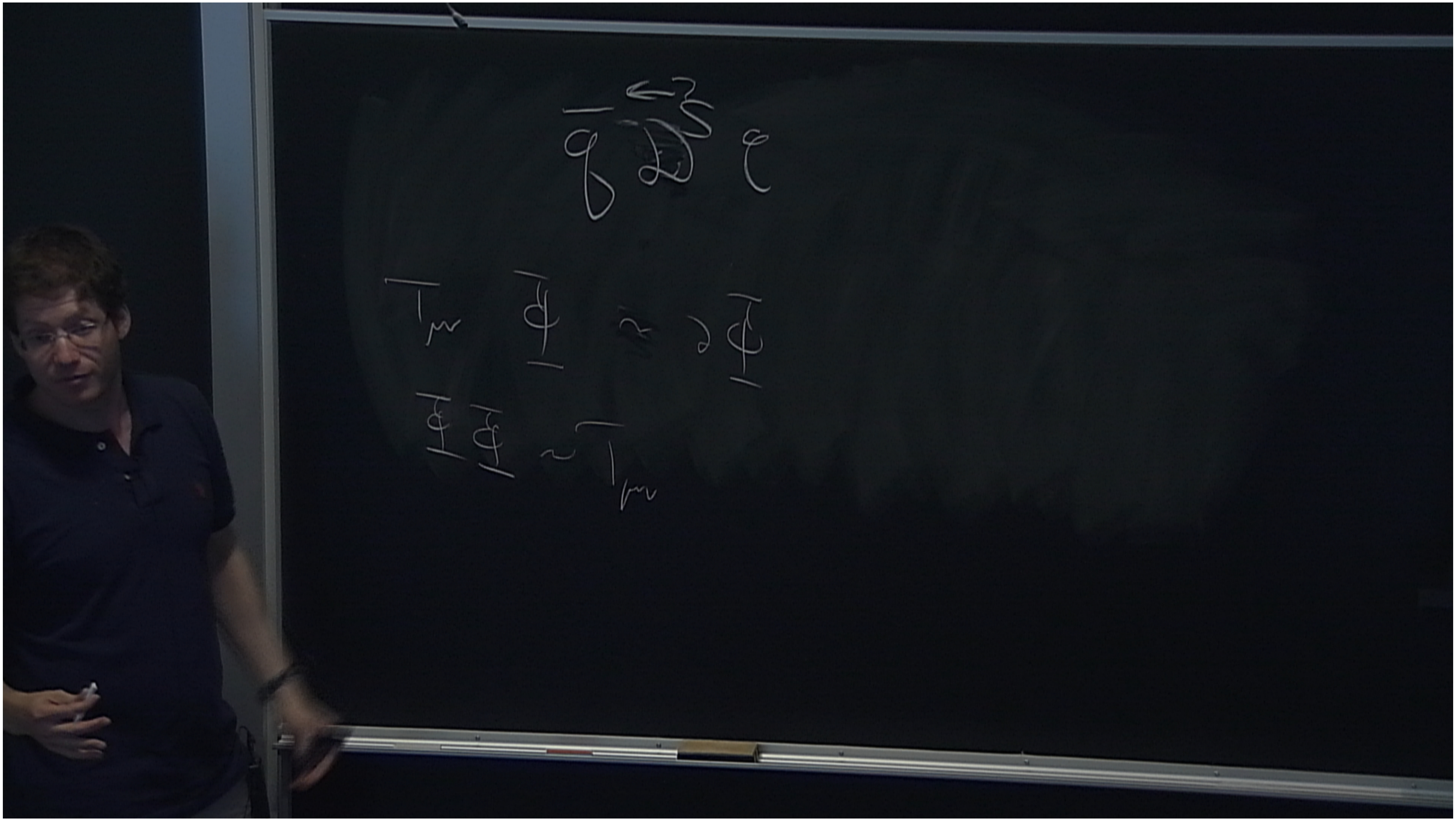
Thus, some OPE coefficients can be related to moments of the deep inelastic cross section. This leads to positivity constraints. In the case of the EM tensor in four dimensions

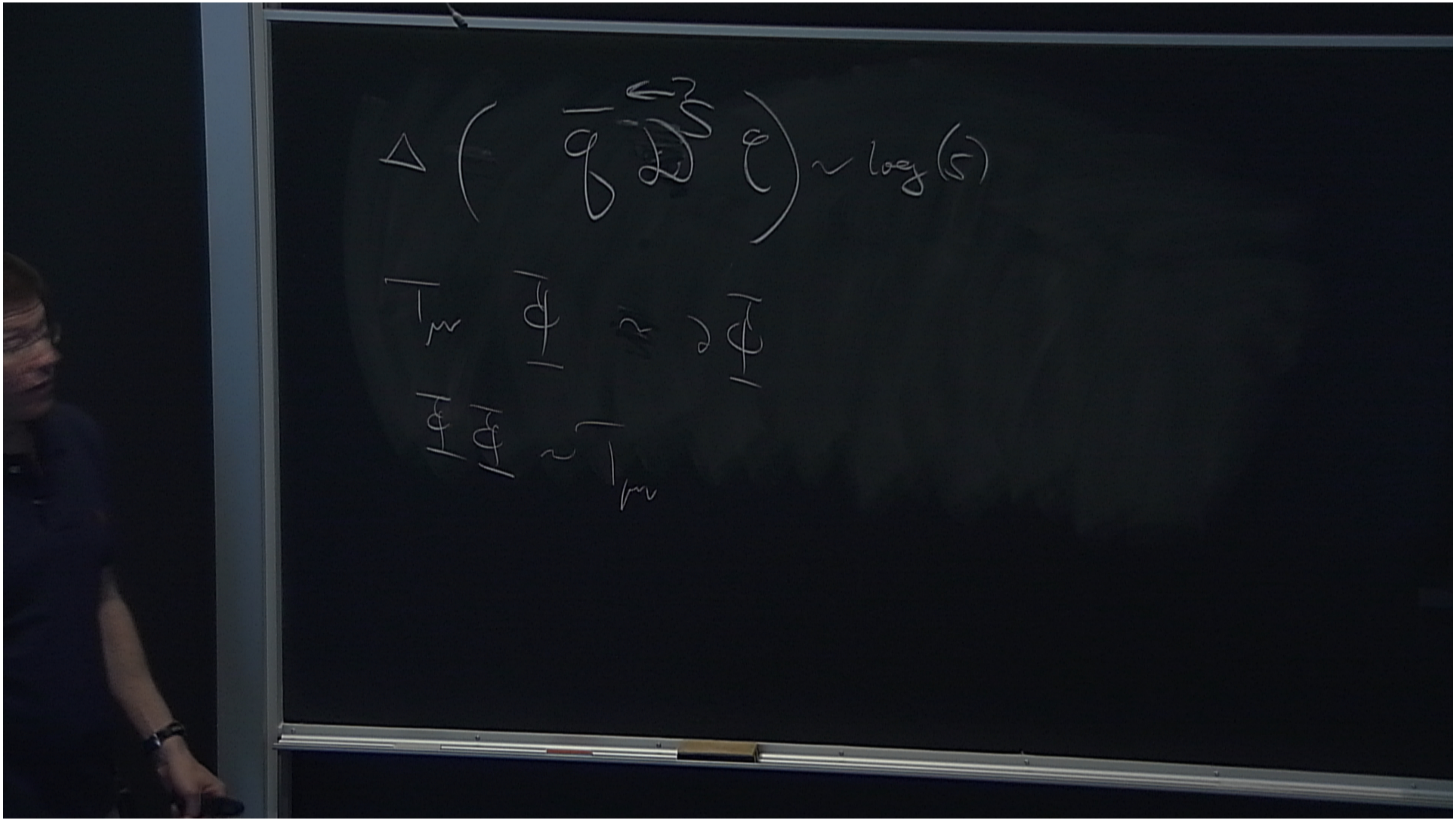
$$T_{\mu\nu}(x)T_{\rho\sigma}(0) \sim \frac{1}{x^8}C_{\mu\nu\rho\sigma} + \frac{1}{x^4}A^{\phi\chi}_{\mu\nu\rho\sigma}(x)T_{\phi\chi}(0) + \ldots$$

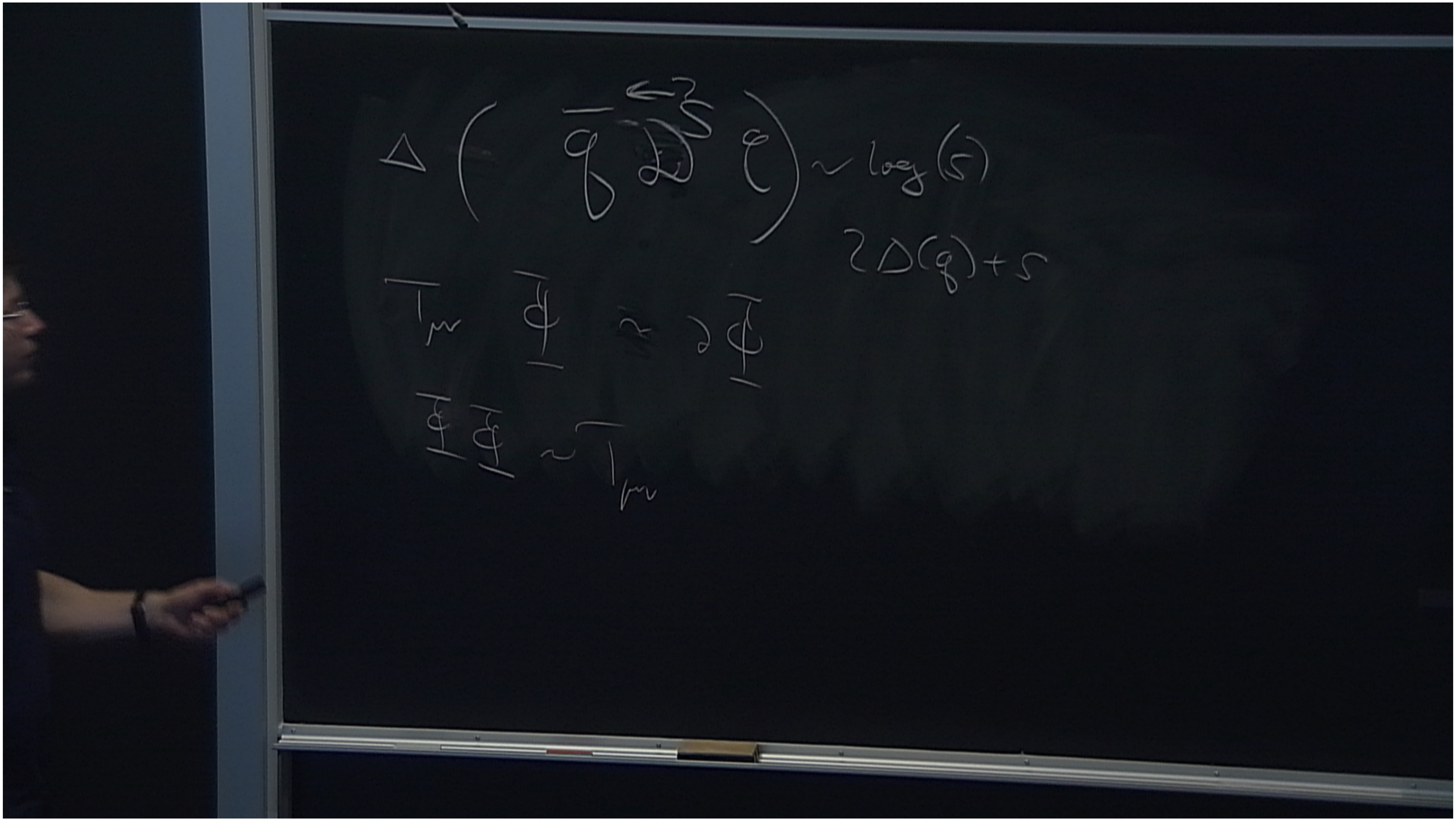
$A^{\phi\chi}_{\mu\nu\rho\sigma}(x)$ is a well known function, that depends on three coefficients a, b, c of which a, c are the trace anomalies. Considering deep inelastic scattering of virtual gravitons with various polarizations, one finds three inequalities, and after eliminating b one remains with $\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}$. This exactly coincides with the result of Hofman-Maldacena that was derived by considering conformal energy correlators! is this general? Other constraints on the OPE coefficients?

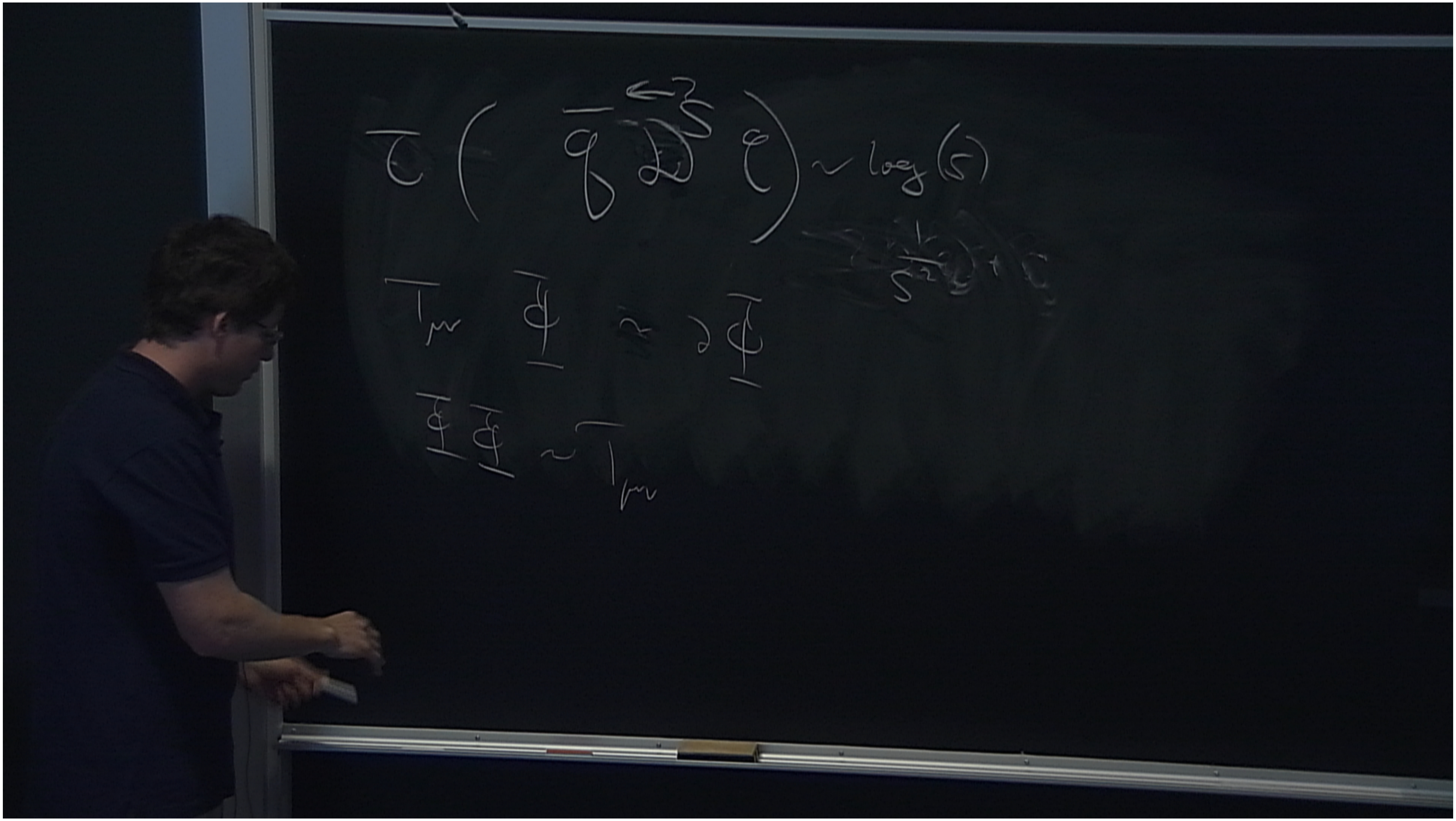
Summary

- $d > 2$ CFTs become free at large spin.
- The spectrum of twists is additive.
- Systematic, non-perturbative computations of corrections to anomalous dimensions and OPE coefficients in an expansion in $1/s$.
- A convex spectrum of operators approaching $2\tau_{\mathcal{O}}$.
- Agreement with explicit perturbative computations, gauge gravity duality, numeric solutions of the bootstrap equations, etc.
- Non-perturbative bounds on various OPE coefficients – the consequences and relation to previous work remain to be explored.




Δ (q̄ D q) ~ log(s)
T_μν Φ ~ ∂Φ
Φ Φ ~ T_μν


~ log(S)
2Δ(q)+S


~ log (s)