Title: From scale invariance to Lorentz symmetry

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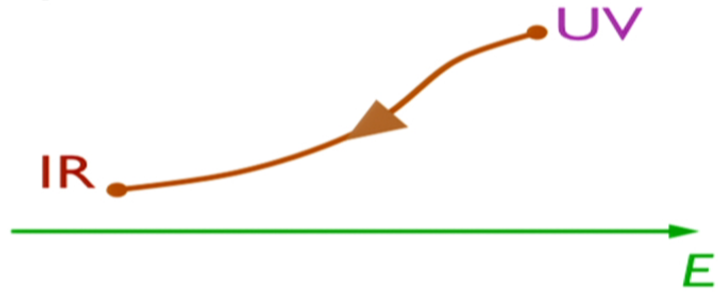
Abstract: I will discuss the enhancement of space-time symmetries to Lorentz (rotation) invariance at the renormalization group fixed points of non-relativistic (anisotropic) field theories. Upon describing examples from the condensed matter physics, I will review the general argument for the stability of the infrared fixed points with the enhanced symmetry. Then I will focus on unitary field theories in (1+1) space-time dimensions which are invariant under translations, isotropic scale transformations and satisfy the requirement that the velocity of signal propagation is bounded from above. No a priori Lorentz invariance will be assumed. Still, I will prove that above properties are sufficient to ensure the existence of an infinite dimensional symmetry given by one or a product of several copies of conformal algebra. In particular, this implies presence of one or several Lorentz groups acting on the operator algebra of the theory. I will conclude by discussing the challenges in extending this result to higher space-time dimensions.

Fixed points of RG flows often enjoy enhanced space-time symmetries



$t \mapsto \lambda t \; , \quad \mathbf{x} \mapsto \lambda \mathbf{x}$

Classic example:

Poincare + dilatations + unitarity

= conformal invariance

Rigorous proof in d=2,
no non-trivial counterexamples in d>2

Proof in 1+1

Zamolodchikov (1986), Polchinski (1988)

Consider the correlators of EMT components in light-cone coordinates $\xi^{\pm} = t \pm x$

$$\left.\begin{aligned} \langle T^{+-}(\xi)T^{+-}(0)\rangle &= \frac{A}{(\xi^{+}\xi^{-})^2} \\ \langle T^{+-}(\xi)T^{++}(0)\rangle &= \frac{B}{\xi^{+}(\xi^{-})^3} \\ \langle T^{++}(\xi)T^{++}(0)\rangle &= \frac{C}{(\xi^{-})^4} \end{aligned}\right\}$$

fixed by Lorentz + dilatations

conservation $\partial_{+}T^{++} + \partial_{-}T^{-+} = 0 \Rightarrow B = 0 \Rightarrow A = 0$

$\xrightarrow{\text{unitarity}} \langle n|T^{+-}(\xi)|0\rangle = 0 \xrightarrow{\text{locality}} T^{+-}(\xi) = 0$

NB. T^{++} depends only on ξ^{-}

Can we make one step more and argue that
Lorentz (rotational) symmetry
follows from translations + dilatations + unitarity ?

NB. General scaling in non-relativistic systems

$$t \mapsto \lambda^z t \ , \quad \mathbf{x} \mapsto \lambda \mathbf{x}$$

$z > 1$ unstable with respect to deformations towards $z = 1$
(isotropic scaling)

$$\mathcal{L}_{Lifshitz} = \dot{\phi}^2 - (\partial_i^2 \phi)^2 - c^2 (\partial_i \phi)^2$$

Counterexample ?

$$\mathcal{L} = \frac{1}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_2^2 - c_1^2(\partial_i \phi_1)^2 - c_2^2(\partial_i \phi_2)^2\right]$$

Obvious scale invariance, but seemingly no relativity ...

In fact, more than relativistic invariance !
Two independent Lorentz groups in non-interacting sectors:

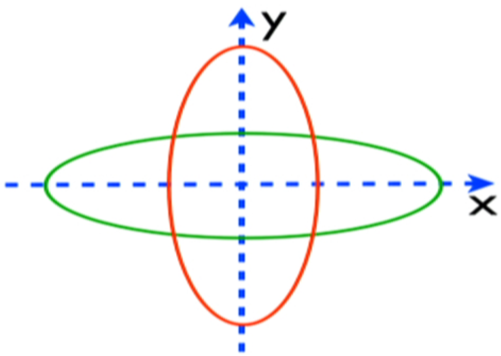
$$L_1 \otimes L_2$$

Interacting examples 1: anisotropic QED in d=2+1 (d-wave superconductors, spin liquids etc.)

Vafek, Tesanovic, Franz, PRL, 89, 157003 (2002)

$$\mathcal{L} = \sum_{n=1,2} \bar{\psi}^{(n)}[\gamma_\nu e^{(n)}_{\mu\nu}(\partial_\nu + iA_\nu)]\psi^{(n)} + \frac{1}{4e^2}F^2_{\mu\nu}$$

$$e^{(1)}_{00} = e^{(2)}_{00} = 1\,, \quad e^{(1)}_{11} = e^{(2)}_{22} = v_F\,, \quad e^{(1)}_{22} = e^{(2)}_{11} = v_\Delta$$



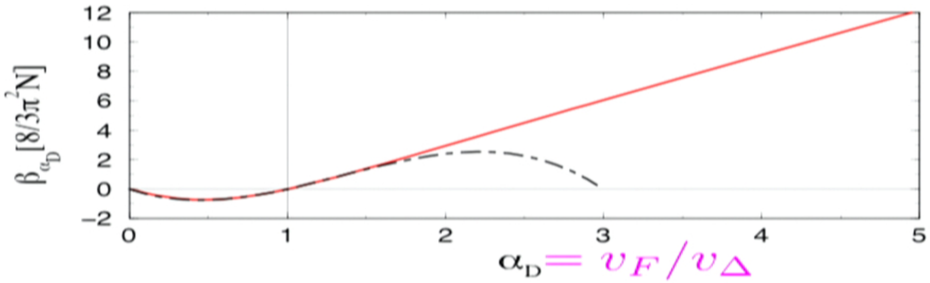


FIG. 1: The RG β-function for the Dirac anisotropy in units of $8/3\pi^2N$. The solid line is the numerical integration while the dash-dotted line is the analytical expansion around the small anisotropy (see Eq. (19-21)). At $\alpha_D = 1$, β_{α_D} crosses zero with positive slope, and therefore at large lengthscales the anisotropic QED_3 scales to an isotropic theory.

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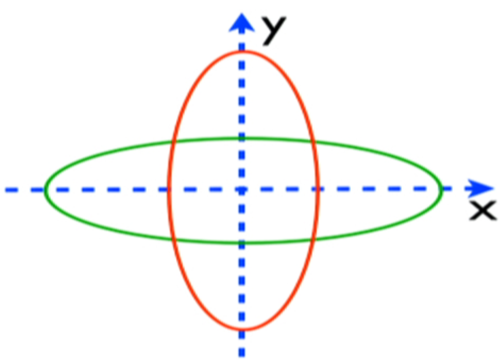

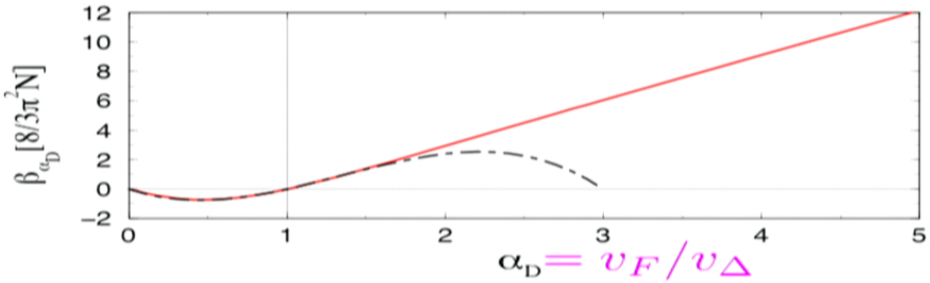

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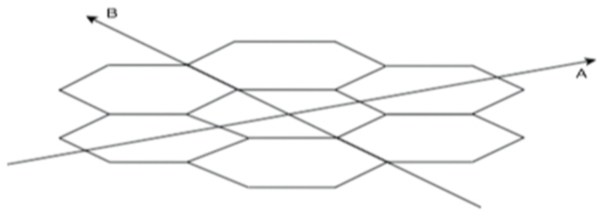
Interacting examples 2: graphene

Herbut, Juricic, Roy, PRB, 79, 185116 (2009)

(Spinless) electrons on honeycomb lattice with 4-fermion interactions



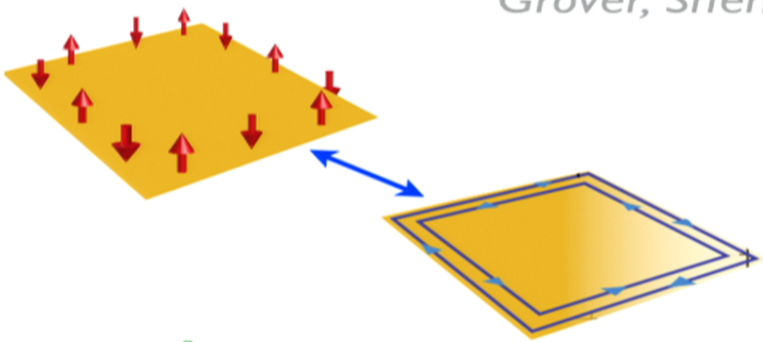
Lattice symmetries ➡ 6 couplings, out of which 3 Lorentz invariant

One-loop calculation ➡ Lorentz violating couplings are always irrelevant

We suspect that this result, although clearly an outcome here of an uncontrolled approximation, may be indicative of the true state of affairs. Hereafter we will assume that the critical points A and C are stable with respect to weak breaking of the Lorentz symmetry in the Lagrangian. It may also be worth mentioning that the

Interacting examples 3: the edge of topological superconductor (B-phase of superfluid He-3)

Grover, Sheng, Vishwanath, Science, 344, 280 (2014)



$$S = \int dt dx (\bar{\chi}\ \partial\!\!\!/ \chi + \dot{\phi}^2 + v_\phi^2 \phi'^2 + r\phi^2 + g\phi\bar{\chi}\chi + u\phi^4)$$

➡ discretize and simulate numerically

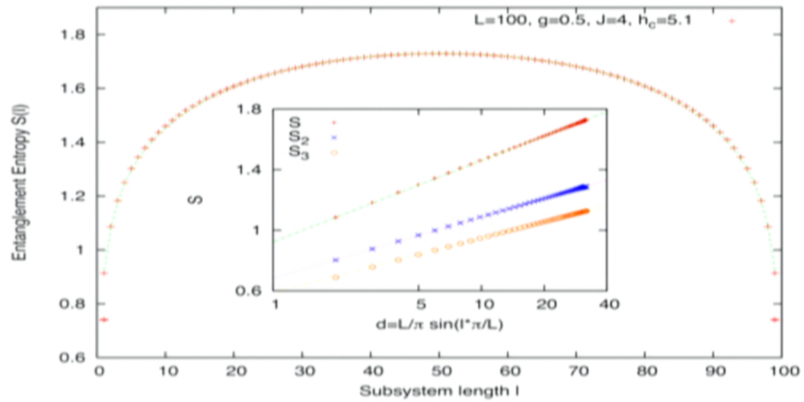


Figure 3: Entanglement entropy at the critical point for the 1+1-D lattice model for the parameter $\mathcal{J} = 4$. The paramater $\mathcal{J}$ equals the ratio of the bare verlocity of the Majorana fermion to that of the boson ϕ. The above curve shows that the supersymmetric critical point with central charge $c = 7/10$ survives even when the velocity anisotropy is four. The red crosses are the numerical data while the green curve is the theoretical expected result for central charge $c = 7/10$. The inset shows the Renyi entropies S_n which also fit perfectly to $c = 7/10$.

Why LI fixed points are attractive in IR ?

Start from a fixed point described by relativistic CFT and deform it by a LV operator

$$\mathcal{L} = \mathcal{L}_{CFT} + \kappa\, O_{\mu_1 \ldots \mu_l}$$

time-component of a symmetric traceless tensor

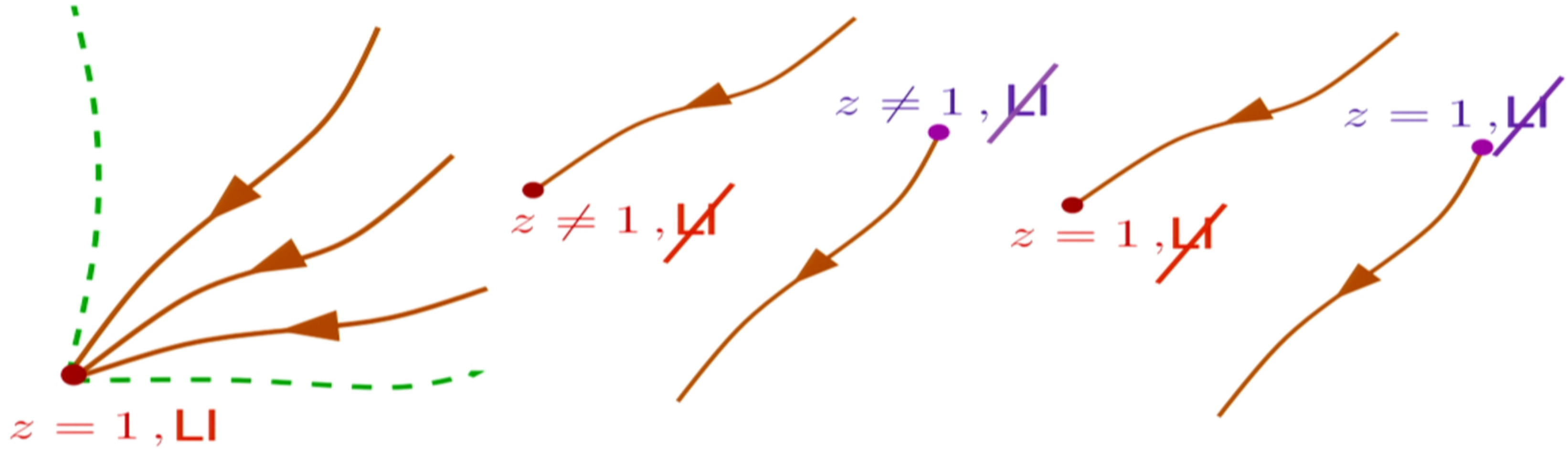
Restrict to the deformations preserving spatial rotations $\Rightarrow$ $\mu_i = 0$

In unitary CFT $\boxed{dim\, O_{\mu_1 \ldots \mu_l} \geq l + d - 2}$

$\Rightarrow$

- all deformations with $l > 2$ are irrelevant
- single marginal $l = 2$ deformation by the stress-energy tensor (can be removed by the redefinition of the metric)
- only danger: vector operators with $d - 1 \leq dim \leq d$ (may be absent due to T or CPT)

Global view: classification of fixed points



strong coupling
accelerates running

emergence of LI
can be fast

Bednik, Pujolas, S.S. (2013)

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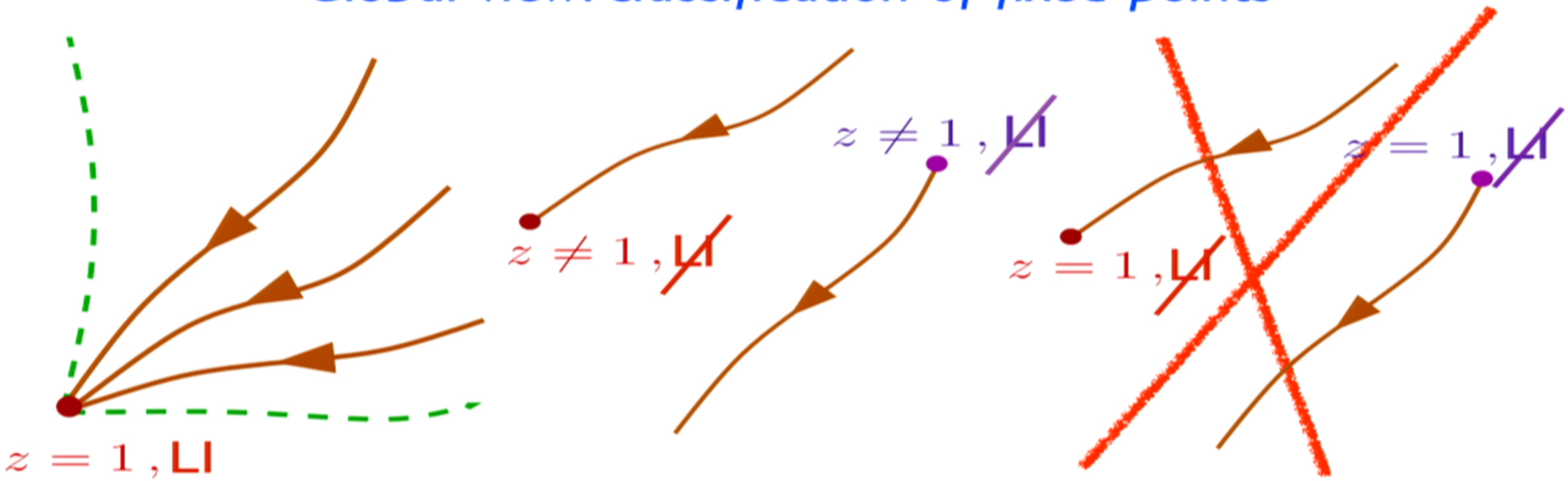

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ruled out in 2d
(up to assumptions)

Theorem:

Consider a local field theory in 1+1 d, which is:

1) translationally invariant
   EMT: $T^t_t\ ,\ T^t_x\ ,\ T^x_t\ , T^x_x$; in general $T^x_t \cancel{\propto} T^t_x$

2) has positive Hamiltonian

3) unitary

4) invariant under $t \mapsto \lambda t\ ,\quad x \mapsto \lambda x$

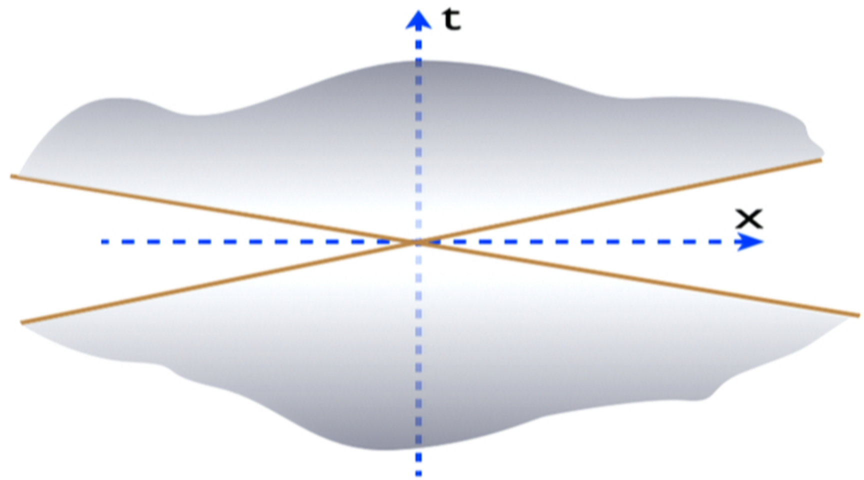
   there is a dilatation current $D^\mu = \xi^\nu T^\mu_\nu + V^\mu$

$$D = \int dx D^t \ : \qquad -i[D, P_\nu] = P_\nu$$

5) possesses discrete diagonalizable positive spectrum of scaling dimensions; finite number of operators of every dimension

6) the velocity of signal propagation is bounded from above:

$[\phi_i(t,x), \phi_j(0,0)] = 0$ if $|x| > v_{max}|t|$



Then the EMT splits into a sum of independent EMT's:

$$T^{\mu}_{\nu} = \sum_{a=1}^{N} T^{(a)\,\mu}_{\quad\;\;\nu}$$

$$\partial_\mu T^{(a)\,\mu}_{\quad\;\;\nu} = 0$$

$$[T^{(a)\,\mu}_{\quad\;\;\nu}, T^{(b)\,\lambda}_{\quad\;\;\rho}] = 0$$

$$T^{(a)\,x}_{\quad\;\;t} = v_a T^{(a)\,t}_{\quad\;\;t} = -v_a T^{(a)\,x}_{\quad\;\;x} = -v_a^2 T^{(a)\,t}_{\quad\;\;x} \;, \qquad v_a \neq 0$$

$$\Rightarrow \quad T^{(a)\,\mu}_{\quad\;\;\mu} = 0$$

$$T^{(a)\,x}_{\quad\;\;t} + v_a^2 T^{(a)\,t}_{\quad\;\;x} = 0$$

Consequences:

N conserved currents

$$m^{(a)\,t} = v_a^{-1} x T^{(a)\,t}_{\,t} + v_a t T^{(a)\,t}_{\,x}\,, \qquad m^{(a)\,x} = v_a^{-1} x\, T^{(a)\,x}_{\,t} + v_a\, t\, T^{(a)\,x}_{\,x}$$

generate Lorentz boosts with the "speeds of light" v_a

In fact, N copies of conformal algebra:

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \ldots \otimes \mathcal{C}_N$$

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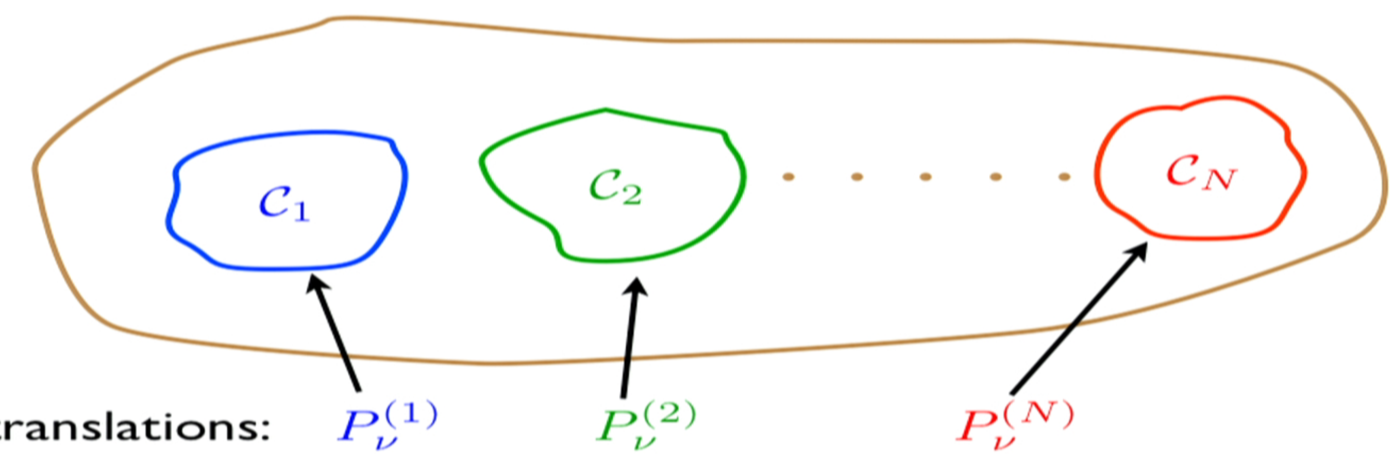
Special cases:

- N=1 ➡ a chiral CFT
- N=2 + parity ➡ $v_1 = -v_2$,"usual" CFT

Non-interacting subsectors:



$$\phi_i^{(a)}(\xi)\phi_j^{(a)}(0) = \sum c_{ij}^k(\xi)\phi_k^{(a)}(0)$$

PROOF


3
Easy Steps
to getting or renewing
your license...
1. Sign Up
2. Take the Class
3. Print your Certificate!

Step 1

- Make T^μ_ν traceless

$\partial_\mu D^\mu = 0$ ➡ $T^\mu_\mu = -\partial_\mu V^\mu$

$T^\mu_\nu \mapsto T^\mu_\nu + \partial_\lambda(\varepsilon^{\lambda\mu}\varepsilon_{\nu\rho}V^\rho)$

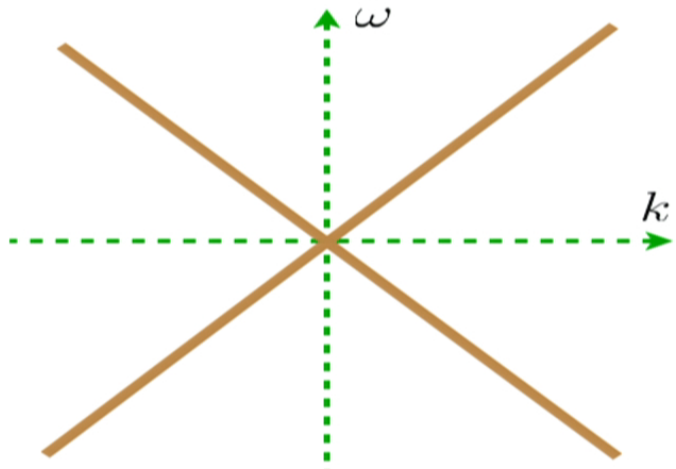
- Bring transformation of T^μ_ν under dilatations to canonical form (use discreteness of the spectrum)

$-i[D, T^\mu_\nu] = 2T^\mu_\nu + \xi^\lambda\partial_\lambda T^\mu_\nu$

Step 2

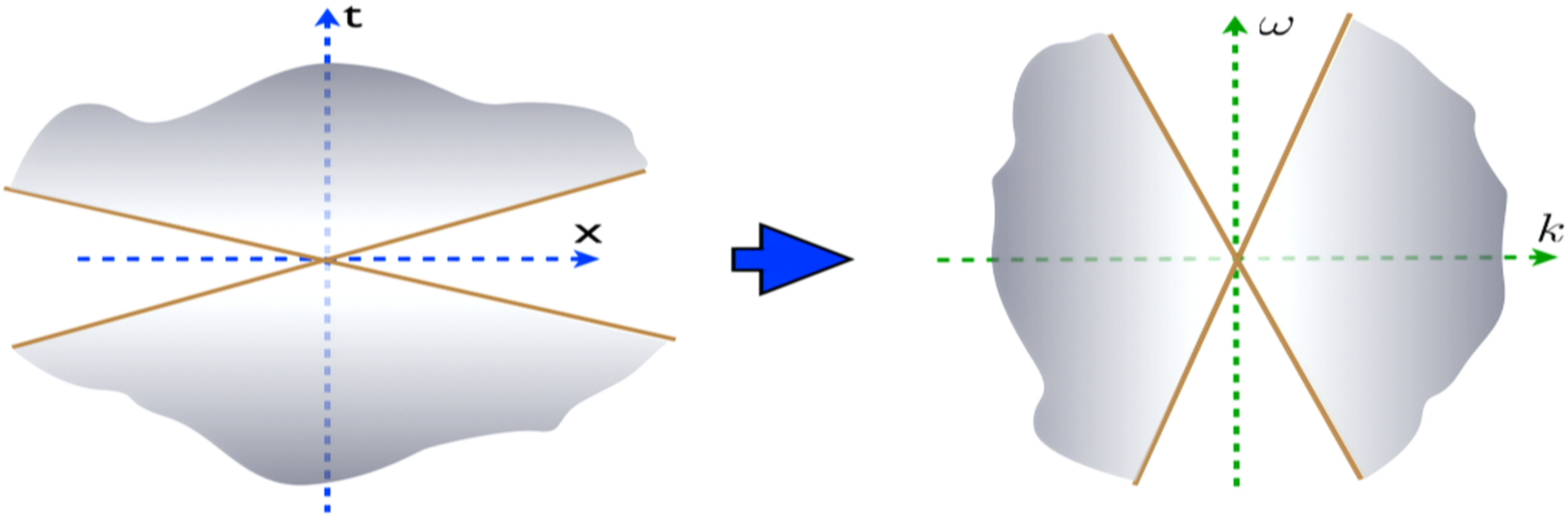
Recall: in LI CFT $T^{++}(\xi_-)$, $T^{--}(\xi^+)$

$\tilde{T}^{\mu}_{\nu}(\omega, k)$ is localized on the light-cone



In particular, vanishes inside the light-cone

Use finite signal velocity to prove this property of EMT in general scale invariant theory



$$F^\mu_\nu(\xi) \equiv \langle 0|[T^\mu_\nu(0),\, T^\mu_\nu(\xi)]|0\rangle = \int d^2p\, \mathrm{e}^{-ipt} \rho^\mu_\nu(\omega, k) - \mathrm{h.c.}$$

scaling + positive energy $\Rightarrow$ $\rho^\mu_\nu = \theta(\omega)\omega^2 \hat\rho^\mu_\nu(k/\omega)$

$$F(t,x) = \int d\omega d\varkappa\, \theta(\omega)\omega^3 \hat\rho(\varkappa) \mathrm{e}^{i\omega(t-\varkappa x)} - \mathrm{h.c.} = \frac{2\pi i}{x^3|x|}\hat\rho'''(t/x)$$

$\Rightarrow$ $\hat\rho'''(k/\omega) = 0$ at $|k/\omega| < 1/v_{max}$

conserved + trace-free EMT

$\Rightarrow$ $\hat\rho^t_x(\varkappa) = \varkappa^2 \hat\rho^x_x(\varkappa) = \varkappa^2 \hat\rho^t_t(\varkappa) = \varkappa^4 \hat\rho^x_t(\varkappa)$

$\Rightarrow$ $\hat\rho^\mu_\nu(k/\omega) = 0$ and $\tilde T^\mu_\nu(\omega, k)|0\rangle = 0$ at

But we need $\tilde{T}^{\mu}_{\nu}(\omega, k)|n\rangle = 0$ for any $|n\rangle$

Consider thermal average of the commutator:

$$F_T{}^{\mu}_{\nu}(\xi) \equiv \langle [T^{\mu}_{\nu}(0),\, T^{\mu}_{\nu}(\xi)]\rangle_T = \int d^2p\, \mathrm{e}^{-ipt} \rho_T{}^{\mu}_{\nu}(\omega, k) - \mathrm{h.c.}$$

$$\omega^2 \hat{\rho}_1(\omega/T, k/T)$$

positive in unitary theory, $\rho_T(-\omega, -k) = \mathrm{e}^{-\omega/T} \rho_T(\omega, k)$

$$\mathcal{D}(x; \theta, T) \equiv x^3 \big(F_T(\theta x, x; T) - F_0(\theta x, x; T)\big) = 0$$

in the sense of distributions in x for $|\theta| < 1/v_{max}$

$$\frac{\partial^3}{\partial\theta^3} \int_0^{\infty} \frac{d\hat{\omega}}{\hat{\omega}} (1 - \mathrm{e}^{-\hat{\omega}}) \hat{\rho}_1(\hat{\omega}, \theta) = 0 \quad \text{for}$$

Step 3

$\exists$ local operators $\Phi_n(\xi)$:

$$\partial_t \Phi_n = \partial_x \Phi_{n+1} \ ,$$

$$\Phi_1 = -T^t_x \ , \quad \Phi_2 = T^x_x = -T^t_t \ , \quad \Phi_3 = T^x_t$$

Physical meaning: there are additional conserved charges of dim 1 (as energy and momentum)

Consider

$$\mathcal{M}_3 \equiv \langle n| \int dx\ T_t^x(t,x)|n\rangle = \int \frac{d\omega}{2\pi} \mathrm{e}^{i\omega t} \langle n| \int dx\ \tilde{T}_t^x(\omega, k=0)|n\rangle$$

vanishes for $\omega \neq 0$

$\tilde{T}_t^x(\omega, k=0) \propto \delta(\omega)$ $\mathcal{M}_3 = const$

NB. Derivatives of the δ-function are forbidden:
they would lead to power-law growth of the matrix element, but

$$\mathcal{M}_3 = -\langle n| \int dx\ x\ \partial_x T_t^x(t,x)|n\rangle = \partial_t \underbrace{\langle n| \int dx\ x\ T_t^t(t,x)|n\rangle}_{\sim E\Delta x < E v_{max} \Delta t}$$

$\mathcal{M}_3 = const$

$\exists\ \Phi_4(t,x) \equiv \int_{-\infty}^{x} dx'\ \partial_t T_t^x(t,x')$

$\tilde{\Phi}_4(\omega,k) = \frac{\omega}{k}\tilde{T}_t^x(\omega,k)$

vanishes inside the cone $|k/\omega| < 1/v_{max}$

Continue by induction

All Φ_n have dim 2 $\Rightarrow$ the chain closes

In the linear envelope $\mathcal{F}$ of Φ_n define the map

$$\Phi \mapsto \Phi' : \quad \partial_t \Phi = \partial_x \Phi'$$

This is unitary with respect to a suitable inner product in $\mathcal{F}$

$$\langle \Phi, \Psi \rangle \propto \langle 0 | \Phi^+ \Psi | 0 \rangle$$

➡ can be diagonalized

$$\Phi_n = \sum_a c_{na} \Psi_a \ , \qquad \partial_t \Psi_a = v_a \partial_x \Psi_a \ , \qquad c_{n+1,a} = v_a c_{na}$$

Define:

$$T^{(a)\,t}_{\ \ x} = -c_{1a} \Psi_a \ , \quad T^{(a)\,x}_{\ \ x} = -T^{(a)\,t}_{\ \ t} = c_{2a} \Psi_a \ , \quad T^{(a)\,x}_{\ \ t} = c_{3a} \Psi_a$$

- ☑ traceless
- ☑ symmetric
- ☑ conserved

Outlook: generalization to d>2

Seems tough because:

- it was not easy in already in 2d (recall, no rigorous proof scale+LI=conformal in d>2 after 28 years)
- counterexample:

$$\mathcal{L} = \dot{A}_i^2 - c_1^2(\partial_j A_i)^2 - c_2^2(\partial_i A_i)^2$$

May still be possible with proper definition of "interacting theory"

new ideas are needed ...

Summary

- Effective Lorentz invariance often appears at the fixed points of RG flows
- Under broad assumptions relativistic fixed points are infrared stable
- In 2d rigorous sufficient conditions for:

  isotropic scale invariance $\Rightarrow$ Lorentz symmetry
- Generalizations to higher d ?